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ON SOME NONLINEAR ELLIPTIC PROBLEMS WITH BOUNDARY CONDITIONS IN GENERALIZED SOBOLEV SPACES

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Abstract. The aim of this paper is to establish existence results for the following anisotropic elliptic equation $-\operatorname{div} B(x,\vartheta,\nabla\vartheta) = f - \operatorname{div} F$, where the datum $f \in L^1(\Omega)$ and $F \in \prod_{i=1}^{N} L^{p_i'}(\Omega, w_i^*)$. Furthermore, only the large monotonicity conditions are assumed on $B(x,s,\xi)$. To overcome this difficulty, we use the approach of Minty's lemma in the anisotropic weighted Sobolev spaces.

Keywords. Anisotropic weighted Sobolev spaces; Elliptic problem; Entropy solutions; Measure data; Truncation.

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1. Introduction

The anisotropic weighted Sobolev spaces introduce a novel framework incorporating directional derivatives with varying weights. Initially, we consider Ω as an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, and let p_i denote $N+1$ exponents, where $1 < p_i < N$ for $i \in 0,\dots,N$. In this paper, we aim to investigate the existence of entropy solutions for the given problem

$$\begin{cases} -\operatorname{div}(B(x,\vartheta,\nabla\vartheta)) = \mu \text{ in } \Omega \\ \vartheta = 0 \quad \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where $w = \{w_i, 0 \leq i \leq N\}$ is the set of weight functions on Ω, and $w^* = \left\{ w_i^{1-p_i'}, 0 \leq i \leq N \right\}$, and μ belongs to a measure space $L^1(\Omega) + W^{-1,p_i'}(\Omega, w^*)$.

Recently, existence results on some qualitative properties and regularity of nonlinear anisotropic elliptic equations where the data belonns to L^1- were extensively proved. In [9], Boccardo examined the problem (1.1) within the conventional Sobolev space $W_0^{1,p}(\Omega)$. In [13], El Haji et

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al. proved a existence result on entropy solutions in weighted-Orlicz spaces. In [1], Akdim, Azroul and Rhoudaf studied degenerated problem (1.1) via Minty's Lemma in weigthed Orlicz-Sobolev spaces. In [16], Faria et al. investigated a similar problem as (1.1), where the solution ϑ of the elliptic problem depends on the gradient.

The goal of this paper is to investigate several elliptic problems for which the classical monotone operators, (see, e.g., [2, 17, 18]) do not apply. We employ a new tool to address the challenge posed by the absence of strict monotonicity (which does not ensure the almost everywhere convergence of the gradient). This tool involves exploring certain techniques derived from Minty's lemma. The pseudo-monotonicity approach cannot be applied because of the presence of $f \in L^1(\Omega)$. To verify the almost everywhere convergence of $\nabla\vartheta_n$, the authors in [8] demonstrated the boundedness of ϑ_n in the Marcinkiewicz space. In this investigation, we establish the local convergence in measure of ϑ_n (as outlined in Section 4). For recent related results on the existence of solutions to some parabolic and elliptic problems, we refer to [3, 5, 6, 7, 10, 12, 13, 15] and the references therein.

The outline of this paper is as follows. In Section 2, we recall the new anisotropic weighted sobolev spaces. Section 3 is devoted to some essential assumptions which are necessary to guarantee the existence of solution. Finally, Section 4 presents the main proof.

2. Preliminaries

In this section, we present the enlargement of Sobolev spaces to a novel class of anisotropic weighted Sobolev spaces. Let Ω be a bounded open set of $\mathbf{R}^N$. Let $p_0, p_1, \ldots, p_N$ be $N+1$ exponents with $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$ and $w = \{w_i(x), 0 \le i \le N\}$ be a vector of weight functions which is measurable and strictly positive a.e. in Ω. Furthermore, we assume the following hypothesis: There exist $w_i \in L^1_{\text{loc}}(\Omega)$ and $w_i^{\frac{-1}{p_i-1}} \in L^1_{\text{loc}}(\Omega)$ for any $0 \le i \le N$. The anisotropic weighted orlicz space $L^{p_i}(\Omega, \gamma)$, where γ is a weight function on Ω is formulated by the following expression

$$L^{p_i}(\Omega, \gamma) = \left\{ \vartheta = \vartheta(x), \vartheta\gamma^{\frac{1}{p_i}} \in L^{p_i}(\Omega) \right\}$$

and endowed by the norm

$$\|\vartheta\|_{L^{p_i}(\Omega,\gamma)} = \|\vartheta\|_{p_i,\gamma} = \left(\int_\Omega |\vartheta(x)|^{p_i} \gamma(x) dx \right)^{\frac{1}{p_i}}.$$

Putting

$$(p_i) = (p_0, \ldots, p_N), \quad D^0\vartheta = \vartheta, \quad \text{and} \quad D^i\vartheta = \frac{\partial\vartheta}{\partial x_i} \quad \text{for } i = 1, \ldots, N,$$

we consider $\underline{p_i} = \min\{p_0, p_1, \ldots, p_N\}$ $(\underline{p_i} > 1)$ and

$$\|\vartheta\|_{1,(p_i),w} = \|\vartheta\|_{p_0,w_0} + \sum_{i=1}^N \left\| \frac{\partial\vartheta}{\partial x_i} \right\|_{p_i,w_i}. \tag{2.1}$$

The anisotropic weighted Sobolev space $W^{1,(p_i)}(\Omega, w)$ is defined as

$$\vartheta \in L^{p_0}(\Omega, w_0) = \left\{ u(x), uw_0^{\frac{1}{p_0}} \in L^{p_0}(\Omega) \right\}$$

such that $D^i\vartheta \in L^{p_i}(\Omega, w_i)$ for $i = 1, \ldots, N$.

We use the technical lemmas introduced in [2, 4].

Lemma 2.1. [4] *Let $(\vartheta_n)_n$ be a bounded sequence in $W_0^{1,(p_i)}(\Omega, w)$. If $\vartheta_n \to \vartheta$ in $W_0^{1,(p_i)}(\Omega, w)$, then $T_k(\vartheta_n) \to T_k(\vartheta)$ in $W_0^{1,(p_i)}(\Omega, w)$, where $T_k(.)$ is the truncation function.*

3. Basic Assumptions and Existence Results

We suppose that the norm

$$\|\vartheta\| = \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \vartheta}{\partial x_i} \right|^{p_i} w_i(x)dx \right)^{\frac{1}{p_i}} \tag{3.1}$$

is equivalent to (2.1), and there exists $\sigma(x)$(a weight function) on Ω and $1 < q_i < \infty$ such that the Hardy inequality

$$\left(\int_\Omega |\vartheta(x)|^{q_i} \sigma(x)dx \right)^{\frac{1}{q_i}} \leq c \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial \vartheta}{\partial x_i} \right|^{p_i} w_i(x)dx \right)^{\frac{1}{p_i}} \tag{3.2}$$

is verified for every $\vartheta \in W_0^{1,(p_i)}(\Omega, w)$ with a constant $c > 0$ independent of ϑ. Furthermore,

$$W_0^{1,(p_i)}(\Omega, w) \hookrightarrow\hookrightarrow L^{q_i}(\Omega, \sigma) \text{ is compact .} \tag{3.3}$$

Let $\mathbb{B}$ denote a nonlinear operator mapping from $W_0^{1,(p_i)}(\Omega, w)$ to its dual space $W^{-1,p_i'}(\Omega, w^*)$. This operator is defined as $\mathbb{B}(\vartheta) = -\operatorname{div}(B(x, \vartheta, \nabla\vartheta))$, where $B(x, s, \xi) : \Omega \times R \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the hypothesises listed bellow. For $i = 1, \ldots, N$,

$$|B_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p_i}}(x) \left[k(x) + \sigma^{\frac{1}{p_i}} |s|^{\frac{q_i}{p_i'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p_i'}}(x) \left|\xi_j\right|^{p_i - 1} \right], \tag{3.4}$$

for a.e., $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $k(x) \in L^{p_i'}(\Omega)$ $\left(\frac{1}{p_i} + \frac{1}{p_i'} = 1\right)$ and $\beta > 0$, where σ and q_i are as in (3.2),

$$\text{for all } (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, 0 < \langle B(x, s, \xi) - B(x, s, \eta), \xi - \eta \rangle, \tag{3.5}$$

and

$$\langle B(x, s, \xi), \xi \rangle \geq \alpha \sum_{i=1}^{N} w_i |\xi_i|^{p_i}, \quad \alpha > 0. \tag{3.6}$$

Let us consider, the truncation function :

$$T_k(s) = \begin{cases} s \text{ if } |s| \leq k \\ k\dfrac{s}{|s|} \text{ if } |s| > k \end{cases} \qquad \text{for } k > 1 \text{ and } s \text{ in } \mathbb{R}.$$

Our goal is to study the problem:

$$\begin{cases} \mathbb{B}(\vartheta) + \operatorname{div}(F) = f & \text{in } \Omega, \\ \vartheta = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where $\mathbb{B}(\vartheta) = -\operatorname{div}(B(x, \vartheta, \nabla\vartheta))$, the datum f belongs to $L^1(\Omega)$, and $F \in \prod_{i=1}^{N} L^{p_i'}(\Omega, w_i^*)$.

Definition 3.1. A measurable function ϑ is said to be an entropy solution of (3.7) if $T_k(\vartheta)$ belongs to $W_0^{1,(p_i)}(\Omega, w)$, for every $k > 0$,

$$\int_\Omega \langle B(x,\vartheta,\nabla\vartheta), \nabla T_k[\vartheta-\Psi]\rangle\,dx = \int_\Omega f T_k[\vartheta-\Psi]dx + \int_\Omega \langle F, \nabla T_k[\vartheta-\Psi]\rangle\,dx$$

for every $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$.

Theorem 3.2. *Let the assumptions (3.1)-(3.6) hold. Then ϑ is a solution to (3.7) in the sense of the definition 3.1 with*

$$\int_\Omega \langle B(x,\vartheta,\nabla\vartheta), \nabla T_k[\vartheta-\Psi]\rangle\,dx = \int_\Omega f T_k[\vartheta-\Psi]dx + \int_\Omega \langle F, \nabla T_k[\vartheta-\Psi]\rangle\,dx$$

for every $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$, and for every $k > 0$.

4. Proof of the Existence Theorem

Lemma 4.1. *Let ϑ be a measurable function such that $T_k(\vartheta) \in W_0^{1,(p_i)}(\Omega, w)$ for every $k > 0$. Then*

$$\int_\Omega \langle B(x,\vartheta,\nabla\vartheta), \nabla T_k[\vartheta-\Psi]\rangle\,dx \le \int_\Omega f T_k[\vartheta-\Psi]dx + \int_\Omega \langle F, \nabla T_k[\vartheta-\Psi]\rangle\,dx.$$

$\Leftrightarrow$

$$\int_\Omega \langle B(x,\vartheta,\nabla\vartheta), \nabla T_k[\vartheta-\Psi]\rangle\,dx = \int_\Omega f T_k[\vartheta-\Psi]dx + \int_\Omega \langle F, \nabla T_k[\vartheta-\Psi]\rangle\,dx.$$

for every Ψ in $W^{1,p_i}(\Omega, w) \cap L^\infty(\Omega)$, and for every $k > 0$.

Following the same way as in the proof in [14], we have this lemma immediately. Next, we present the Proof of existence theorem 3.2

4.1. A priori estimate.

Let $f_n \in L^\infty(\Omega) \longrightarrow f$ strongly in $L^1(\Omega)$ such that $\|f_n\|_{L^1} \le \|f\|_{L^1}$, and let ϑ_n be a solution in $W_0^{1,(p_i)}(\Omega, w)$ of the problem

$$\begin{cases} -\operatorname{diva} B(x,\vartheta_n,\nabla\vartheta_n) = f_n - \operatorname{div}(F) & \text{in } \Omega, \\ \vartheta_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

which exists thanks to [11].

Now we take $T_k(\vartheta_n)$ as thetest function in (4.1). Then

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n), \nabla T_k(\vartheta_n)\rangle\,dx = \int_\Omega f_n T_k(\vartheta_n)\,dx + \int_\Omega \langle F, \nabla T_k(\vartheta_n)\rangle\,dx.$$

Using $\nabla T_k(\vartheta_n) = \nabla\vartheta_n \chi_{\{|\vartheta_n|\le k\}}$ and hypothesis (3.6), we have

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n), \nabla T_k(\vartheta_n)\rangle\,dx \ge \alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i} dx.$$

Thus

$$\begin{aligned} \alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i} dx &\le k\|f\|_{L^1} + \sum_{i=1}^N \int_\Omega F_i \left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right| dx \\ &\le k\|f\|_{L^1} + \sum_{i=1}^N \int_\Omega F_i w_i^{\frac{-1}{p_i}} \left(\frac{\alpha}{2}\right)^{\frac{-1}{p_i}} \left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right| w_i^{\frac{1}{p_i}} \left(\frac{\alpha}{2}\right)^{\frac{1}{p_i}} dx. \end{aligned}$$

Then Young's inequality yields

$$\alpha\sum_{i=1}^{N}\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx \le k\|f\|_{L^1}+\frac{c(\alpha)}{p_i'}\|F\|_{\Pi^{p_i'}(\Omega,w_i^*)}+\frac{\alpha}{2}\sum_{i=1}^{N}\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx$$

It follows that

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_\Omega w_i\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}dx \le k\left(\|f\|_{L^1}+\frac{c(\alpha)}{p_i'}\|F\|_{\Pi^{p_i'}(\Omega,w_i^*)}\right)$$

for $k>1$, which implies that

$$\left(\sum_{i=1}^{N}\int_\Omega\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}w_i(x)dx\right)^{\frac{1}{p_i}} \le ck^{\frac{1}{p_i}},\quad \forall k>1.$$

4.2. **Locally convergence of ϑ_n in measure.** Let $k>0$ be large enough. By using (3.3), we obtain

$$\begin{aligned}
k \text{ meas } (\{|\vartheta_n|>k\}\cap B_R) &= \int_{\{|\vartheta_n|>k\}\cap B_R}|T_k(\vartheta_n)|\,dx \le \int_{B_R}|T_k(\vartheta_n)|\,dx\\
&\le \left(\int_\Omega |T_k(\vartheta_n)|^{p_i}w_0dx\right)^{\frac{1}{p_i}}\cdot\left(\int_{B_R}w_0^{1-p_i'}dx\right)^{\frac{1}{q_i}}\\
&\le c\left(\int_\Omega\sum_{i=1}^{N}\left|\frac{\partial T_k(\vartheta_n)}{\partial x_i}\right|^{p_i}w_i(x)dx\right)^{\frac{1}{p_i}}\\
&\le c_1k^{\frac{1}{p_i}}.
\end{aligned}$$

which gives

$$\text{meas}(\{|\vartheta_n|>k\}\cap B_R) \le \frac{c_1}{k^{1-\frac{1}{p_i}}}\quad \forall k>1. \tag{4.2}$$

Thus, for every $\delta>0$,

$$\begin{aligned}
\text{meas}(\{|\vartheta_n-\vartheta_m|>\delta\}\cap B_R) \le \text{meas}(\{|\vartheta_n|>k\}\cap B_R) &+ \text{ meas } (\{|\vartheta_m|>k\}\cap B_R)\\
&+ \text{ meas } \{|T_k(\vartheta_n)-T_k(\vartheta_m)|>\delta\}.
\end{aligned} \tag{4.3}$$

Since $T_k(\vartheta_n)$ is bounded in $W_0^{1,(p_i)}(\Omega,w)$, we see that there exists some $v_k\in W_0^{1,(p_i)}(\Omega,w)$ such that

$$\begin{cases} T_k(\vartheta_n)\rightharpoonup v_k & \text{weakly in } W_0^{1,(p_i)}(\Omega,w),\\ T_k(\vartheta_n)\to v_k & \text{strongly in } L^{q_i}(\Omega,\sigma) \text{ and a.e. in } \Omega.\end{cases}$$

Consequently, it can be deduced that $T_k(\vartheta_n)$ forms a Cauchy sequence in measure in Ω. Let $\varepsilon>0$. By utilizing (4.2) and (4.3), we see that there exists a specific $k(\varepsilon)>0$ such that meas $(\{|\vartheta_n-\vartheta_m|>\ \delta\}\cap B_R)<\varepsilon$ for all $n,m\ge n_0(k(\varepsilon),\delta,R)$. This implies that (ϑ_n) is a Cauchy sequence in measure within B_R, thus converges almost everywhere to a measurable function ϑ. We have

$$\begin{cases} T_k(\vartheta_n)\rightharpoonup T_k(\vartheta) \text{ weakly in } W_0^{1,(p_i)}(\Omega,w),\\ T_k(\vartheta_n)\to T_k(\vartheta) \quad \text{strongly in } L^{q_i}(\Omega,\sigma) \text{ and a.e in } \Omega.\end{cases} \tag{4.4}$$

4.3. **An intermediate inequality.** We show that, for $\Psi \in W_0^{1,(p_i)}(\Omega,w) \cap L^\infty(\Omega)$,

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\Psi), \nabla T_k[\vartheta_n-\Psi]\rangle\,dx \le \int_\Omega f_n T_k[\vartheta_n-\Psi]\,dx + \int_\Omega \langle F, \nabla T_k[\vartheta_n-\Psi]\rangle\,dx. \quad (4.5)$$

Furthermore, by taking $T_k(\vartheta_n-\Psi)$ as test in (4.1), with Ψ in $W_0^{1,(p_i)}(\Omega,w)\cap L^\infty(\Omega)$, we have

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n), \nabla T_k[\vartheta_n-\Psi]\rangle\,dx = \int_\Omega f_n T_k[\vartheta_n-\Psi]\,dx + \int_\Omega \langle F, \nabla T_k[\vartheta_n-\Psi]\rangle\,dx.$$

Thus

$$\begin{aligned}&\int_\Omega \langle B(x,\vartheta_n,\nabla\vartheta_n), \nabla T_k[\vartheta_n-\Psi]\rangle\,dx + \int_\Omega \langle B(x,\vartheta_n,\nabla\Psi), \nabla T_k[\vartheta_n-\Psi]\rangle\,dx\\ &-\int_\Omega \langle B(x,\vartheta_n,\nabla\Psi), \nabla T_k[\vartheta_n-\Psi]\rangle\,dx \\ &= \int_\Omega f_n T_k[\vartheta_n-\Psi]\,dx + \int_\Omega \langle F, \nabla T_k[\vartheta_n-\Psi]\rangle\,dx.\end{aligned} \quad (4.6)$$

Using (3.5) and the truncation function, we see that

$$\int_\Omega \langle [B(x,\vartheta_n,\nabla\vartheta_n) - B(x,\vartheta_n,\nabla\Psi)], \nabla T_k[\vartheta_n-\Psi]\rangle\,dx \ge 0. \quad (4.7)$$

According to (4.6) and (4.7), we obtain (4.5) immediately.

4.4. **Passing to the limit.** Here we prove that, for $\Psi \in W_0^{1,(p_i)}(\Omega,w)\cap L^\infty(\Omega)$,

$$\int_\Omega \langle B(x,\vartheta,\nabla\Psi), \nabla T_k[\vartheta-\Psi]\rangle\,dx \le \int_\Omega f T_k[\vartheta-\Psi]dx + \int_\Omega \langle F, \nabla T_k[\vartheta-\Psi]\rangle\,dx.$$

First, we verify that

$$\int_\Omega \langle B(x,\vartheta_n,\nabla\Psi), \nabla T_k[\vartheta_n-\Psi]\rangle\,dx \to \int_\Omega \langle B(x,\vartheta(\nabla\Psi), \nabla T_k[\vartheta-\Psi]\rangle\,dx \text{ as } n\to+\infty.$$

As $T_M(\vartheta_n) \rightharpoonup T_M(\vartheta)$ weakly in $W_0^{1,(p_i)}(\Omega,w)$, with $M = k + \|\Psi\|_\infty$, we obtain by Lemma 2.1 that

$$T_k(\vartheta_n-\Psi) \rightharpoonup T_k(\vartheta-\Psi) \text{ in } W_0^{1,(p_i)}(\Omega,w), \quad (4.8)$$

which gives

$$\frac{\partial T_k}{\partial x_i}(\vartheta_n-\Psi) \rightharpoonup \frac{\partial T_k}{\partial x_i}(\vartheta-\Psi) \text{ weakly in } L^{p_i}(\Omega,w_i)\,\forall i=1,..,N. \quad (4.9)$$

We now show that $B_i(x,T_M(\vartheta_n),\nabla\Psi) \to B_i(x,T_M(\vartheta),\nabla\Psi)$ strongly in $L^{p_i'}(\Omega,w_i^*)$. It follows from (3.4) that

$$\begin{aligned}|B_i(x,T_M(\vartheta_n),\nabla\Psi)|^{p_i'} w_i^{\frac{-p_i'}{p_i}} &\le \beta\left[k(x) + |T_M(\vartheta_n)|^{\frac{q_i}{p_i'}}\sigma^{\frac{1}{p_i'}} + \sum_{j=1}^N \left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i-1} w_i^{\frac{1}{p_i'}}\right]^{p_i'}\\ &\le \gamma\left[k(x)^{p_i'} + |T_M(\vartheta_n)|^{q_i}\sigma + \sum_{j=1}^N \left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i} w_i\right],\end{aligned}$$

where $\beta, \gamma > 0$. Since $T_M(\vartheta_n) \rightharpoonup T_M(\vartheta)$ weakly in $W_0^{1,(p_i)}(\Omega, w)$ and $W_0^{1,(p_i)}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega, \sigma)$, then $T_M(\vartheta_n) \to T_M(\vartheta)$ strongly in $L^q(\Omega, \sigma)$ and a.e. in Ω. Consequently,

$$|B_i(x, T_M(\vartheta_n), \nabla\Psi)|^{p_i'} w_i^* \to |B_i(x, T_M(\vartheta), \nabla\Psi)|^{p_i'} w_i^* \text{ a.e.in } \Omega.$$

and

$$\gamma\left[k(x)^{p_i'} + |T_M(\vartheta_n)|^{q_i}\sigma + \sum_{j=1}^{N}\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i} w_i\right] \to \gamma\left[k(x)^{p_i'} + |T_M(\vartheta)|^{q_i}\sigma + \sum_{j=1}^{N}\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i} w_i\right] \text{ a.e. in } \Omega.$$

From Vitali's theorem, we see that

$$B_i(x, T_M(\vartheta_n), \nabla\Psi) \to B_i(x, T_M(\vartheta), \nabla\Psi) \text{ strongly in } L^{p_i'}(\Omega, w_i^*), \text{ as } n \to +\infty. \tag{4.10}$$

According to (4.8) and (4.9), we obtain

$$\int_\Omega \langle B(x, \vartheta_n, \nabla\Psi), \nabla T_k[\vartheta_n - \Psi]\rangle\, dx \to \int_\Omega \langle B(x, \eta, \nabla\Psi), \nabla T_k[\vartheta - \Psi]\rangle\, dx, \text{ as } n \to +\infty \tag{4.11}$$

Finally, we prove that

$$\int_\Omega f_n T_k[\vartheta_n - \Psi]\, dx \to \int_\Omega f T_k[\vartheta - \Psi] dx. \tag{4.12}$$

Observe that $f_n T_k[\vartheta_n - \Psi] \to f T_k[\vartheta - \Psi]$ a.e. in Ω and $|f_n T_k[\vartheta_n - \Psi]| \le k|f_n|$ and $k|f_n| \to k|f|$ in $L^1(\Omega)$. Using Vitali's theorem, we find(4.12) immediately.

We now show that

$$\int_\Omega \langle F, \nabla T_k[u_n - \Psi]\rangle\, dx \to \int_\Omega \langle F, \nabla T_k[\vartheta - \Psi]\rangle\, dx. \tag{4.13}$$

From (4.9) and $F \in \prod^N L^{p_i'}(\Omega, w_i^*)$, we obtain (4.13). According to (4.11), (4.12), and (4.13), we pass to the limit in (4.5) to yield that, for all $\Psi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$,

$$\int_\Omega \langle B(x, \vartheta(\nabla\Psi), \nabla T_k[\vartheta - \Psi]\rangle\, dx \le \int_\Omega f T_k[\vartheta - \Psi] dx + \int_\Omega \langle F, \nabla T_k[\vartheta - \Psi]\rangle\, dx.$$

Thus we conclude that ϑ is a solution to the problem (3.7) in the sense of the Definition 3.1 .

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