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TURING INSTABILITY FOR A SPACE AND TIME DISCRETE DELAY LOTKA-VOLTERRA COMPETITIVE MODEL WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. In this paper, a space and time discrete delay Lotka-Volterra competitive model with periodic boundary conditions is considered. The stability analysis is investigated for the model by means of Schur theorem, and the Turing instability conditions are obtained. Numerical simulations are performed to verify the theoretical results.

Keywords. Discrete delay Lotka-Volterra model; Turing instability; Schur theorem.

1. Introduction

According to the theory of evolution, competition within a species and between species plays a fundamental and crucial role in natural selection. The study of competitive mechanism is important to understand the behavior and survival mechanism of natural selection. Lotka-Volterra competition systems are hot and celebrated ecological models that can describe the interaction among various competing species and have been extensively investigated; see, e.g., [1, 2, 3, 4, 5] and the references therein. In earlier literature, the two-competing species competition models were often formulated in the form of ordinary differential systems as follows:

$$\begin{cases} u'(t) = u(t)\left(r_1 - a_{11}u(t) - a_{12}v(t)\right), \\ v'(t) = v(t)\left(r_2 - a_{21}v(t) - a_{22}v(t)\right), \end{cases} \tag{1.1}$$

for $t \in [0,+\infty)$ $a_{ij} \geq 0, i, j = 1,2,$ where $u(t)$ and $v(t)$ are the quantities of the two species at time t, $r_1 > 0$ and $r_2 > 0$ are growth rates of the respective species, a_{11} and a_{22} represent the strength of the intraspecific competition, and a_{12} and a_{21} represent the strength of the interspecific competition.

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In the field of biology, numerous movements and changes for organisms occur in discrete forms, and the data collected and recorded is also in discrete forms; the populations have non-overlapping generations or the population statistics are compiled from given time intervals and not continuously. Then it is reasonable to study discrete time models governed by difference equations, and the discrete time models can also provide efficient computational models of continuous models for numerical simulations. By considering a variation with piecewise constant arguments for certain terms on the right side for (1.1), the following difference equation can be obtained:

$$\begin{cases} u(k+1) = u(k)\exp(r_1 - a_{11}u(k) - a_{12}v(k)), \\ v(k+1) = v(k)\exp(r_2 - a_{21}u(k) - a_{22}v(k)). \end{cases} \tag{1.2}$$

There are some reasons for introducing time delay into the real biological systems, such as the maturation delay and digestion delay in the population system, which means that the evolution of system depends on not only the present but also the historical information. Thus a kind of delay discrete form from (1.2) can be listed as follows [6]

$$\begin{cases} u(k+1) = u(k)\exp(r_1 - a_{11}u(k-\tau) - a_{12}v(k-\tau)), \\ v(k+1) = v(k)\exp(r_2 - a_{21}u(k-\tau) - a_{22}v(k-\tau)), \end{cases}$$

where τ is the delay, and $\tau \le k, \tau \in Z^+$.

In modeling population dynamics and interactions of biological species, it is significant to investigate the interactions happened at different locations. The effect of spatial factors play a crucial role in the stability of populations; see, e.g., [7, 8, 9, 10]. Then we can have a space and time discrete delay Lotka-Volterra competitive model as follows.

$$\begin{cases} u_{ij}^{k+1} = u_{ij}^{k}\exp\{r_1 - a_{11}u_{ij}^{k-\tau} - a_{12}v_{ij}^{k-\tau}\} + D_1\nabla^2 u_{ij}^k, \\ v_{ij}^{k+1} = v_{ij}^{k}\exp\{r_2 - a_{21}u_{ij}^{k-\tau} - a_{22}v_{ij}^{k-\tau}\} + D_2\nabla^2 v_{ij}^k, \end{cases} \tag{1.3}$$

with the periodic boundary conditions

$$u_{i,0}^k = u_{i,m}^k, u_{i,1}^k = u_{i,m+1}^k, u_{0,j}^k = u_{m,j}^k, u_{1,j}^k = u_{m+1,j}^k, \tag{1.4}$$

$$v_{i,0}^k = v_{i,m}^k, v_{i,1}^k = v_{i,m+1}^k, v_{0,j}^k = v_{m,j}^k, v_{1,j}^k = v_{m+1,j}^k, \tag{1.5}$$

where $i, j \in \{1, 2, \cdots, m\} = [1, m], m \in Z^+, k \in Z^+, u_{ij}^k$ is the density of first population in (i, j) lattice at time kth generation, v_{ij}^k is the density of second population in (i, j) lattice at time kth generation, and

$$\begin{aligned} \nabla^2 u_{ij}^k &= u_{i+1,j}^k + u_{i,j+1}^k + u_{i-1,j}^k + u_{i,j-1}^k - 4u_{ij}^k, \\ \nabla^2 v_{ij}^k &= v_{i+1,j}^k + v_{i,j+1}^k + v_{i-1,j}^k + v_{i,j-1}^k - 4v_{ij}^k. \end{aligned}$$

After the pioneering work by Alan Turing on chemical morphogenesis [11], diffusion has been identified as a source of the spontaneous creation of ordered structures, known as patterns, which is connected to the occurrence of what he called a diffusion driven instability (Turing instability). Turing instability has become an important mechanism for the emergence of interesting patterns in many discrete or continuous reaction diffusion models, and has been widely studied; see, e.g., [12, 13, 14, 15, 16, 17, 18] and the references therein.

In studying the pattern formation of the competitive systems, the reaction-diffusion model and Turing instability theory have been widely employed and numerous results have been obtained; see, e.g., [10, 19, 20, 21, 22, 23, 24, 25, 26]. For example, in [19], the Turing bifurcation critical value and the condition of the occurrence of Turing pattern were obtained when control parameters are selected by means of the linear stability analysis. By using the multiple scale method on a Lotka-Volterra competitive system with nonlocal delay, the amplitude equations of the different Turing patterns were obtained and spots pattern and stripes pattern arise in [20]. Linear stability analysis was applied to an exponential discrete Lotka-Volterra system, which describes the competition between two identical species, and the conditions for the Turing instability were obtained in [26]. However, as far as we know, there is no relevant research on the Turing instability of space and time discrete delay models, including Lotka-Volterra competitive ones. We, in this work, focus on the Turing instability analysis of system (1.3) - (1.5).

The rest of the paper is organized as follows. We state some basic preliminaries in Section 2. The stability of the positive equilibrium and Turing instability analysis are discussed in Sections 3. In Section 4, some numerical simulations are done to verify our theoretical results. Section 5, which is also the last section of this paper, presents some concluding conclusions.

2. Preliminaries

To guarantee that system (1.3) has always a positive equilibrium, throughout this paper, we assume that the coefficients of system (1.3) satisfies $r_1a_{22}-r_2a_{12}>0$ and $r_2a_{11}-r_1a_{21}>0$, and the unique positive equilibrium $E^*(u^*,v^*)$ is

$$u^*=\frac{r_1a_{22}-r_2a_{12}}{a_{11}a_{22}-a_{12}a_{21}},v^*=\frac{r_2a_{11}-r_1a_{21}}{a_{11}a_{22}-a_{12}a_{21}}.$$

To prove the main results in this paper, we transform system (1.3) into the following equivalent system of $2\tau+2$ equations without delays

$$\begin{cases} u_{ij}^{(0)}(k+1)=u_{ij}^{(0)}(k)\exp\{r_1-a_{11}u_{ij}^{(\tau)}(k)-a_{12}v_{ij}^{(\tau)}(k)\}+D_1\nabla^2u_{ij}^{(0)}(k),\\ v_{ij}^{(0)}(k+1)=v_{ij}^{(0)}(k)\exp\{r_2-a_{21}u_{ij}^{(\tau)}(k)-a_{22}v_{ij}^{(\tau)}(k)\}+D_2\nabla^2v_{ij}^{(0)}(k),\\ u_{ij}^{(p)}(k+1)=u_{ij}^{(p-1)}(k),\\ v_{ij}^{(p)}(k+1)=v_{ij}^{(p-1)}(k),p=1,2,\dots,\tau, \end{cases} \tag{2.1}$$

where

$$\begin{cases} u_{ij}^{(0)}(k)=u_{ij}^{k},\\ v_{ij}^{(0)}(k)=v_{ij}^{k}, \end{cases}$$

whose positive equilibrium can be written as $(u^*,v^*,u^*,v^*,\dots,u^*,v^*)\in R^{2\tau+2}$.

We also need some lemmas for our work.

Lemma 2.1. [25] *For eigenvalue problem*

$$\begin{cases} -\nabla^2x_{ij}=\lambda x_{ij},\\ x_{0,j}=x_{m,j},\ x_{1,j}=x_{m+1,j},\\ x_{i,0}=x_{i,m},\ x_{i,1}=x_{i,m+1},i,j\in[1,m], \end{cases}$$

one has

$$\lambda_{kl}=4(\sin^2\frac{(k-1)\pi}{m}+\sin^2\frac{(l-1)\pi}{m}),k,l\in[1,m],$$

and the corresponding eigenvector

$$
\begin{aligned}
\varphi_{ij1}^{kl} &= \sin\frac{2i(k-1)\pi}{m}\sin\frac{2j(l-1)\pi}{m},\\
\varphi_{ij2}^{kl} &= \cos\frac{2i(k-1)\pi}{m}\cos\frac{2j(l-1)\pi}{m},\\
\varphi_{ij3}^{kl} &= \sin\frac{2i(k-1)\pi}{m}\cos\frac{2j(l-1)\pi}{m},\\
\varphi_{ij4}^{kl} &= \cos\frac{2i(k-1)\pi}{m}\sin\frac{2j(l-1)\pi}{m}.
\end{aligned}
$$

Lemma 2.2. [6, 27] *The polynomial $P(\lambda)=\lambda^{\tau+1}-a\lambda^{\tau}+b$, where $a,b\in R$, is of Schur type (i.e., all its eigenvalues are inside the unit circle) if and only if one of the following conditions hold*

(1) if $ba^{\tau+1}\le 0$, then $|a|+|b|<1$;

(2) if $ba^{\tau+1}>0$, then $|a|\le\frac{\tau+1}{\tau}$ $|a|-1<|b|<(a^2+1-2|a|\cos\phi)^{\frac{1}{2}}$,

where $\phi\in(0,\frac{\tau+1}{\tau})$ is the solution to $|a|\sin\tau\phi=\sin(\tau+1)\phi$.

3. Turing Instability

In order to study the Turing instability, we first recall the stability conditions for the discrete reaction-diffusion system without diffusion part of the form

$$
\begin{cases}
u^{(0)}(k+1)=u^{(0)}(k)\exp\{r_1-a_{11}u^{(\tau)}(k)-a_{12}v^{(\tau)}(k)\},\\
v^{(0)}(k+1)=v^{(0)}(k)\exp\{r_2-a_{21}u^{(\tau)}(k)-a_{22}v^{(\tau)}(k)\},\\
u^{(p)}(k+1)=u^{(p-1)}(k),\\
v^{(p)}(k+1)=v^{(p-1)}(k), p=1,2,\dots,\tau,
\end{cases}
\tag{3.1}
$$

where

$$
\begin{cases}
u^{(0)}(k)=u^{k},\\
v^{(0)}(k)=v^{k},
\end{cases}
$$

which is the equivalent system of the following equation

$$
\begin{cases}
u^{k+1}=u^{k}\exp\{r_1-a_{11}u^{k-\tau}-a_{12}v^{k-\tau}\},\\
v^{k+1}=v^{k}\exp\{r_2-a_{21}u^{k-\tau}-a_{22}v^{k-\tau}\}.
\end{cases}
\tag{3.2}
$$

From [6, Theorem 3.1], system (3.1) or (3.2) is asymptotically stable if and only if one of the following conditions hold

$$
(1)\quad \beta \le \alpha^2, \beta>2c_\tau\alpha-c_\tau^2, \beta>0, c_\tau<\alpha<0. \tag{3.3}
$$

$$
(2)\quad \beta > \alpha^2, \beta<h_\tau(\alpha), c_\tau<\alpha<0. \tag{3.4}
$$

where $c_\tau=-(2-2\cos\phi_\tau)^{\frac{1}{2}}$, $\phi_\tau\in(0,\frac{\pi}{\tau+1})$ is the unique solution to $\sin\tau\phi=\sin(\tau+1)\phi$, $h_\tau=(g_\tau\mid_{[0,c_\tau^2]})^{-1}, g_\tau(\alpha)=\cos((\tau+1)\arccos\frac{2-\alpha}{2})-\cos(\tau\arccos\frac{2-\alpha}{2})$, $\alpha=\frac{1}{2}tr(B)$, and $\beta=\det(B)$ with $B=\begin{bmatrix}-a_{11}u^* & -a_{12}u^*\\ -a_{21}v^* & -a_{22}v^*\end{bmatrix}$.

If $\beta \leq \alpha^2$, one sets

$$S_{11}(\alpha,\beta) = \{(\alpha,\beta) \mid \beta > 2c_\tau\alpha - c_\tau^2, \beta > 0, c_\tau < \alpha < 0\}.$$

If $\beta > \alpha^2$, one sets

$$S_{21}(\alpha,\beta) = \{(\alpha,\beta) \mid \beta < h_\tau(\alpha), c_\tau < \alpha < 0\}.$$

Next, we discuss the conditions for instability in systems with diffusion terms. We linearise system (2.1) at the steady state $E^*(u^*,v^*)$, and obtain

$$\begin{cases} u_{ij}^{(0)}(k+1) = u_{ij}^{(0)}(k) - a_{11}u^*u_{ij}^{(\tau)}(k) - a_{12}u^*v_{ij}^{(\tau)}(k) + D_1\nabla^2 u_{ij}^{(0)}(k), \\ v_{ij}^{(0)}(k+1) = v_{ij}^{(0)}(k) - a_{21}v^*u_{ij}^{(\tau)}(k) - a_{22}v^*v_{ij}^{(\tau)}(k) + D_2\nabla^2 v_{ij}^{(0)}(k), \\ u_{ij}^{(p)}(k+1) = u_{ij}^{(p-1)}(k), \\ v_{ij}^{(p)}(k+1) = v_{ij}^{(p-1)}(k), p = 1,2,\ldots,\tau, \end{cases} \tag{3.5}$$

Taking the inner product of (3.5) with the corresponding eigenfunction φ_{ij}^{ls} of the eigenvalue λ_{ls}, respectively, we see that

$$\begin{cases} U^{(0)}(k+1) = (1-D_1k_{ls}^2)U^{(0)}(k) - a_{11}u^*U^{(\tau)}(k) - a_{12}u^*V^{(\tau)}(k), \\ V^{(0)}(k+1) = (1-D_2k_{ls}^2)V^{(0)}(k) - a_{21}v^*U^{(\tau)}(k) - a_{22}v^*V^{(\tau)}(k), \\ U^{(p)}(k+1) = U^{(p-1)}(k), \\ V^{(p)}(k+1) = V^{(p-1)}(k), p = 1,2,\ldots,\tau, \end{cases} \tag{3.6}$$

where

$$\begin{cases} U^{(0)}(k) = \sum_{i,j=1}^{m} \varphi_{ij}^{ls} u_{ij}(k), \\ V^{(0)}(k) = \sum_{i,j=1}^{m} \varphi_{ij}^{ls} v_{ij}(k), \end{cases}$$

whose Jacobian matrix is:

$$J_{ls} = \begin{bmatrix} A & 0 & 0 & \cdots & 0 & B \\ E_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & E_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E_2 & 0 \end{bmatrix}_{(2\tau+2)\times(2\tau+2)},$$

where

$$A = \begin{bmatrix} 1-D_1k_{ls}^2 & 0 \\ 0 & D_2k_{ls}^2 \end{bmatrix}, B = \begin{bmatrix} -a_{11}u^* & -a_{12}u^* \\ -a_{21}v^* & -a_{22}v^* \end{bmatrix}, \text{ and } E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For simplicity, let $D_1 = D_2 = D$. Then we can obtain the characteristic equation for J_{ls}

$$\lambda^{2\tau}(\lambda - 1 + Dk_{ls}^2)^2 - 2\alpha\lambda^\tau(\lambda - 1 + Dk_{ls}^2) + \beta = 0.$$

Therefore, the eigenvalues of the matrix J_{ls} are the solutions to the following two equations

$$\lambda^{\tau+1} - (1 - Dk_{ls}^2)\lambda^\tau = \eta_i, i = 1,2, \tag{3.7}$$

where η_1 and η_2 are the eigenvalues of matrix B.

If $1 - Dk_{ls}^2 = 0$, then $\lambda^{\tau+1} = \eta_i$, and $|\lambda| > 1$ holds if and only if $|\eta_i| > 1$.

We have the following results on the instability of the positive equilibrium.

Theorem 3.1. *$|\eta_i| > 1$ and $\{\alpha, \beta, D\} \in S_{11}(\alpha, \beta)$ or $S_{21}(\alpha, \beta)$ mean or show that the problem (1.3) - (1.5) is diffusion-driven unstable or Turing unstable.*

Next, let us discuss the case that $1 - Dk_{ls}^2 \neq 0$.

Case 1. $\beta \leq \alpha^2$.

In this case, the eigenvalues $\eta_i = \alpha \pm \sqrt{\alpha^2 - \beta}$ of matrix B are real. Hence, we have the following results.

Theorem 3.2. *If $\beta \leq \alpha^2$, then the positive equilibrium of (3.6) is asymptotically stable if and only if $\eta_1,\ \eta_2 \in (c_{D,\tau}, 1 - \left|1 - Dk_{ls}^2\right|)$, where $c_{D,\tau} = -(2 - 2\cos\phi_{D,\tau})^{\frac{1}{2}}$, and $\phi_{D,\tau}$ is the unique solution to the equation $\left|1 - Dk_{ls}^2\right| \sin\tau\phi = \sin(\tau+1)\phi$.*

Proof. Based on (3.7), one sees that the positive equilibrium of (3.6) is asymptotically stable if and only if both polynomials $P_i(\lambda) = \lambda^{\tau+1} - (1 - Dk_{ls}^2)\lambda^{\tau} + \eta_i, i = 1,2$ are Schur polynomials. Lemma 2.2 presents that this is true if and only if $c_{D,\tau} < \eta_i < 1 - \left|1 - Dk_{ls}^2\right|$, or $c_{D,\tau} < \eta_i < Dk_{ls}^2 (1 - Dk_{ls}^2 > 0), c_{D,\tau} < \eta_i < 2 - Dk_{ls}^2 (1 - Dk_{ls}^2 < 0)$. This completes the proof. □

Remark 3.3. It can be easily verified that the eigenvalues η_1 and η_2 of matrix B belong to the interval $(c_{D,\tau}, 1 - \left|1 - Dk_{ls}^2\right|)$ if and only if the half-trace α and the determinant β of matrix B verify the following set of inequalities

$$\beta > 2c_{D,\tau}\alpha - c_{D,\tau}^2, \beta > 2(1 - \left|1 - Dk_{ls}^2\right|)\alpha - (1 - \left|1 - Dk_{ls}^2\right|)^2, c_{D,\tau} < \alpha < 1 - \left|1 - Dk_{ls}^2\right|. \tag{3.8}$$

Hence, Theorem 3.2 provides that the positive equilibrium is asymptotically stable if and only if inequalities (3.8) hold.

Let

$$\begin{aligned}
S_{12}(\alpha, \beta, D) &= \{(\alpha, \beta, D) \mid \beta > 2c_{D,\tau}\alpha - c_{D,\tau}^2, \beta > 2(1 - \left|1 - Dk_{ls}^2\right|)\alpha - (1 - \left|1 - Dk_{ls}^2\right|)^2, \\
c_{D,\tau} &< \alpha < 1 - \left|1 - Dk_{ls}^2\right|\}, \\
C_{11}(\alpha, \beta, D) &= \{(\alpha, \beta, D) \mid Dk_{ls}^2 = \pm(\alpha \pm \sqrt{\alpha^2 - \beta})\},
\end{aligned}$$

We can obtain the following result.

Theorem 3.4. *If $\beta \leq \alpha^2$, $\{\alpha, \beta, D\} \in S_{11}$ and $\{\alpha, \beta, D\} \notin \{S_{12}(\alpha, \beta, D) \cup C_{11}(\alpha, \beta, D)\}$, then problem (1.3) - (1.5) is diffusion-driven unstable or Turing unstable.*

Case 2. $\beta > \alpha^2$

In this case, the eigenvalues $\eta_i = \alpha \pm i\sqrt{\beta - \alpha^2}$ of matrix B are complex. The following results hold.

Lemma 3.5. *If $\beta > \alpha^2$, then matrix J has eigenvalues on the unit circle if and only if $\beta \in [D^2k_{ls}^4, (2 - Dk_{ls}^2)^2]$ or $\beta \in [(2 - Dk_{ls}^2)^2, D^2k_{ls}^4]$ and*

$$\begin{aligned}
\alpha &= \cos((\tau+1)\arccos\frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)}) \\
&\quad -(1 - Dk_{ls}^2)\cos(\tau\arccos\frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)})
\end{aligned} \tag{3.9}$$

hold.

Proof. In this case, $\beta > \alpha^2$, $\eta_1 = \overline{\eta_2} = \alpha + i\sqrt{\beta - \alpha^2}$. Suppose that the polynomial $P_1(\lambda) = \lambda^{\tau+1} - (1 - Dk_{ls}^2)\lambda^\tau + \eta_1$ has a root on the unit circle, $\lambda = e^{i\theta}$, where $\theta \in [0, \pi]$. Therefore, $e^{i\tau\theta}(e^{i\theta} - 1 + Dk_{ls}^2) = \eta_1$ and $\left|e^{i\theta} - 1\right|^2 = |\eta_1|^2 = \beta$. Hence,

$$(1 - \left|1 - Dk_{ls}^2\right|)^2 \le \beta \le (1 + \left|1 - Dk_{ls}^2\right|)^2,$$

that is, $\beta \in [D^2k_{ls}^4, (2 - Dk_{ls}^2)^2]$ or $\beta \in [(2 - Dk_{ls}^2)^2, D^2k_{ls}^4]$, and $\theta = \arccos \frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)}$.

On the other hand, we also have that $Re(e^{i\tau\theta}(e^{i\theta} - 1 + Dk_{ls}^2)) = Re(\eta_1) = \alpha$. Then we obtain (3.9). This completes the proof. □

Lemma 3.6. *The* $(\tau + 1)$ *degree polynomial function* $g_{D,\tau} : [(1 - \left|1 - Dk_{ls}^2\right|)^2, (1 + \left|1 - Dk_{ls}^2\right|)^2] \to R$ *defined by*

$$\begin{aligned} g_{D,\tau}(\beta) &= \cos((\tau + 1)\arccos \frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)}) \\ &\quad - (1 - Dk_{ls}^2)\cos(\tau \arccos \frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)}), \end{aligned}$$

is strictly decreasing on $[(1 - \left|1 - Dk_{ls}^2\right|)^2, c_{D,\tau}^2]$, *and* $g_{D,\tau}[(1 - \left|1 - Dk_{ls}^2\right|)^2, c_{D,\tau}^2] = [c_{D,\tau}, 1 - \left|1 - Dk_{ls}^2\right|]$. *Moreover,* $|g_{D,\tau}(\beta)| \le \sqrt{\beta}$ *and* $(1 - \left|1 - Dk_{ls}^2\right|)^2 \le \beta \le (1 + \left|1 - Dk_{ls}^2\right|)^2$.

Based on the lemmas above, the stability results of the positive equilibrium can be obtained immediately.

Theorem 3.7. *If* $\beta > \alpha^2$, *then the positive equilibrium of (3.6) is asymptotically stable if and only if the following inequalities hold*

$$\beta < g_{D,\tau}(\alpha), c_{D,\tau} < \alpha < 1 - \left|1 - Dk_{ls}^2\right|.$$

Let

$$S_{22}(\alpha, \beta, D) = \{(\alpha, \beta, D) \mid \beta < g_{D,\tau}(\alpha), c_{D,\tau} < \alpha < 1 - \left|1 - Dk_{ls}^2\right|\},$$

and

$$\begin{aligned} C_{21}(\alpha, \beta, D) &= \{(\alpha, \beta, D) \mid \alpha = \cos((\tau + 1)\arccos \frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)}) \\ &\quad - (1 - Dk_{ls}^2)\cos(\tau \arccos \frac{1 + (1 - Dk_{ls}^2)^2 - \beta}{2(1 - Dk_{ls}^2)}). \end{aligned}$$

We have the following result immediately.

Theorem 3.8. *If* $\beta > \alpha^2$, $\{\alpha, \beta, D\} \in S_{21}$, *and* $\{\alpha, \beta, D\} \notin \{S_{22}(\alpha, \beta, D) \cup C_{21}(\alpha, \beta, D)\}$, *then problem (1.3) - (1.5) is diffusion-driven unstable or Turing unstable.*

4. A Numerical Example

In this section, numerical simulation is presented to verify the efficiency of our theoretical analysis.

Let $\tau = 1, r_1 = 1, r_2 = 0.9, a_{11} = 4, a_{12} = 0.5, a_{21} = 1, a_{22} = 2$, and $D = 0.25$. Then $r_1a_{22} - r_2a_{12} = 1.55 > 0, r_2a_{11} - r_1a_{21} = 2.6 > 0$, $u^* = \frac{r_1a_{22} - r_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = 0.2067 > 0$, and $v^* = \frac{r_2a_{11} - r_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} = 0.3467 > 0$.

We first verify the stability conditions (3.3) or (3.4) for system (3.2). In this case, $\alpha = \frac{1}{2}tr(B) = -0.7601, \beta = \det(B) = 0.5375$, and $\phi = \frac{\pi}{3}$, $c_\tau = -(2-2\cos\phi_\tau)^{\frac{1}{2}} = -1$, then $\beta = 0.5375 \leq \alpha^2 = (-0.7601)^2 = 0.5778, \beta = 0.5375 > 2c_\tau\alpha - c_\tau^2 = 2\times(-1)\times(-0.7601) - 1^2 = 0.5202, \beta > 0$, and $c_\tau = -1 < \alpha = -0.7601 < 0$. Thus condition (3.3) is satisfied, so the positive equilibrium of system (3.2) is asymptotically stable.

Letting $m = 200, l = 101, s = 100$, and $k_{ls}^2 = \lambda_{ls} = 4(\sin^2\frac{(l-1)\pi}{m} + \sin^2\frac{(s-1)\pi}{m}) = 7.999$, we can obtain $\alpha = -0.7601 > 1 - \left|1 - Dk_{ls}^2\right| = -0.99975$, that is, condition (3.8) is not satisfied, so the positive equilibrium of the system (1.3)-(3.1) is not asymptotically stable, and Turing stability will emerge.

Let

$$\begin{aligned} u_{ij}^{-1} &= u_{ij}^0 = 0.2067 + 0.001\cos\frac{2i(101-1)\pi}{200}\cos\frac{2j(100-1)\pi}{200}, \\ v_{ij}^{-1} &= v_{ij}^0 = 0.3467 + 0.001\cos\frac{2i(101-1)\pi}{200}\cos\frac{2j(100-1)\pi}{200}. \end{aligned}$$

We performed simulations for the discrete reaction-diffusion system, a stable pattern of square shapes, namely, stationary wave, as Figure 1.

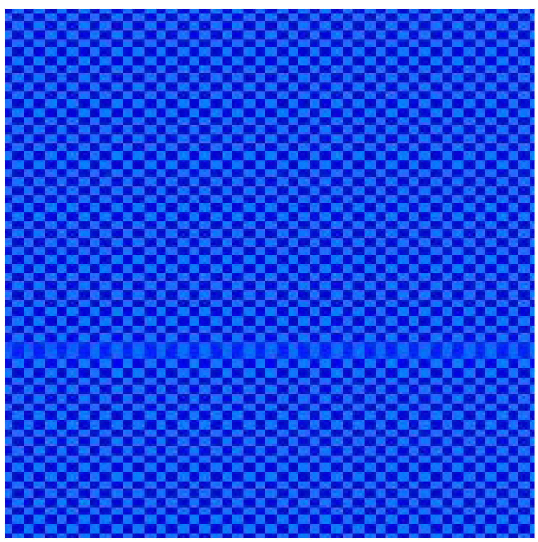

FIGURE 1. Stationary pattern snapshots of contour pictures of the time evolution on u, when $\tau = 1, r_1 = 1, r_2 = 0.9, a_{11} = 4, a_{12} = 0.5, a_{21} = 1, a_{22} = 2, D = 0.25$, and iteration times $k = 100000$.

5. CONCLUSIONS

The stability analysis of a space and time discrete delay Lotka-Volterra competitive model with periodic boundary conditions reveals the existence of Turing instability for certain parameter values that are chosen. A concrete example is given to verify our theoretical results.

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