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APPROXIMATION SEQUENCES FOR FIXED POINTS OF NON CONTRACTIVE OPERATORS

M.A. NAVASCUÉS

Department of Applied Mathematics, University of Zaragoza, Spain

Abstract. The objective of this paper is two-fold. First, the essential characteristics of the iterative methods of fixed point approximation are extracted, and the convergence of sequences not linked to particular algorithms are studied. The second aim of this paper is to present a new three-steps iterative procedure. The convergence of the procedure for the approximation of fixed points of partial contractivities and nonexpansive partial contractivities is analyzed. In the nonexpansive case, the results of weak and strong convergence of the method are given when the underlying spaces are Banach and enjoy some properties, such as uniform convexity or Opial condition.

Keywords. Fixed point theorems; Fixed point approximation; Nonexpansive maps; Nonlinear dynamics.

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1. Introduction

The method of successive approximations, computing the sequence $(T^n x)$ to approach a fixed point of a map T, is an essential pillar of the numerical mathematics. This procedure is on the basis of important numerical methods such as Newton and Steffensen algorithms for finding the roots of nonlinear equations, or the Picard-Lindelöf method to solve an ordinary differential equation. However, it is known that this iterative scheme may fail to converge if the map is not a contraction of Banach type, for instant, T is nonexpansive. A simple illustrative example is given by the map $T : [0,1] \to [0,1]$ defined as $Tx = 1 - x$. T is nonexpansive and it has a unique fixed point at $x = 1/2$. However, the Picard iterations of a point $x_0 \in [0,1]$ consist of the sequence $(x_0, 1-x_0, x_0, 1-x_0, \ldots)$, only convergent for $x_0 = 1/2$. Krasnoselskii [10] observed this lack of convergence of the Picard method for nonexpansive mappings in normed spaces, and proposed a procedure of averages given by the recurrence $x_{n+1} = (x_n + Tx_n)/2$. Afterwards, he generalized the formula to consider a convex linear combination of x_n and Tx_n :

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \tag{1.1}$$

*Corresponding author.

E-mail address: manavas@unizar.es

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for $0<\lambda<1$. Kirk [9] extended this scheme to consider the recurrence in 1971 $x_{n+1}=\sum_{i=0}^{k}\alpha_i T^i x_n$, where $\sum_{i=0}^{k}\alpha_i=1$, $\alpha_i\geq 0$. In 1953, Mann established the following iterative scheme

$$x_{n+1}=(1-a_n)x_n+a_nTx_n, \tag{1.2}$$

Recently, various iterative weighted averages were proposed. Ishikawa, in [7], studied a two-step algorithm given by:

$$\begin{cases} y_n=(1-\beta_n)x_n+\beta_nT(x_n), \\ x_{n+1}=(1-\alpha_n)x_n+\alpha_nT(y_n), \end{cases}$$

In 2000, Noor proposed the following scheme in the context of variational inequalities

$$\begin{cases} z_n=(1-c_n)x_n+c_nTx_n, \\ y_n=(1-b_n)x_n+b_nTz_n, \\ x_{n+1}=(1-a_n)x_n+a_nTy_n, \end{cases}$$

The so-called S-iteration was presented by Agarwal, O'Regan, and Sahu in [1]:

$$\begin{cases} y_n=(1-b_n)x_n+b_nTx_n, \\ x_{n+1}=(1-a_n)Tx_n+a_nTy_n, \end{cases}$$

In [17], Pheungrattana and Suantai proposed the following SP-algorithm

$$\begin{cases} z_n=(1-c_n)x_n+c_nTx_n, \\ y_n=(1-b_n)z_n+b_nTz_n, \\ x_{n+1}=(1-a_n)y_n+a_nTy_n, \end{cases}$$

In [8], Karakaya et al. studied a particular case of the SP-algorithm for the approximation of a fixed point:

$$\begin{cases} z_n=Tx_n, \\ y_n=(1-\alpha_n)z_n+\alpha_nTz_n, \\ x_{n+1}=Ty_n. \end{cases}$$

In all the cases, the scalar range is in $[0,1]$. Ishikawa and Karakaya iterations were analyzed [11] by establishing value ranges of the parameters for the convergence of the algorithms to a fixed point of a partial contractivity in the framework of a quasi-normed space. The convergence values of the scalars are related to the index of the quasi-norm. The celebrated Kirk iteration was analyzed in [13] and the algorithms were applied to the construction of new types of fractal maps and surfaces in Hilbert and Banach spaces.

The objective of this paper is two-fold. First, we extract the essential characteristics of these methods, and study the convergence capabilities of sequences not linked to a particular algorithm. The sequence merely owns some basic conditions related to the operator T, called the AF and the LE properties, and these conditions provide the possibility of approximating a fixed point of T, specially when it is of nonexpansive type. The second aim of this paper is the presentation of a new three-steps iterative procedure for fixed-point approaching, where the sequence generated enjoys the related properties. Section 2 deals with the sequences belonging to normed spaces that enjoy two properties related to a map T, giving sufficient conditions on the underlying space and the operator T to obtain convergence (weak or strong) to a fixed point of T. In Section 3, we study a particular case where map T is a nonexpansive partial contractivity, and we provide sufficient conditions of the convergence of these sequences in Banach spaces satisfying the Opial condition [15]. Section 4 introduces a new three-steps iterative procedure with two map evaluations and studies the properties of the sequences generated by our algorithm, giving sufficient conditions for weak and strong convergence to a critical point. Section 5 is devoted to a particular type of contraction, introduced in [14], called $(\varphi-\psi)$-partial contractivity, and considers the convergence of Picard iterations. This type of self-maps contains many others as particular cases (see [11, Corollary 2.2]). Section 6 analyzes the convergence of the proposed iterative scheme in the case that T is a partial contractivity (particular case of $(\varphi-\psi)$-partial contractivity) in a normed space. Finally, a potential application of the new method to the approximation of repelling fixed points of nonlinear mappings is presented with two examples.

2. Approximative Sequences of Fixed Points of Nonexpansive Operators

Let us consider the following definition about a sequence in a normed space X and a self-map T (see, e.g., [2, 18]).

Definition 2.1. Let E be a normed space, $C \subseteq E$ a subset, $T : C \to C$ a map, and $(f_n) \subseteq C$ a sequence. Then (f_n) has the approximate fixed point property (AF property) if $\lim_{n\to\infty} ||f_n - Tf_n|| = 0$. The sequence (f_n) has the limit existence property (LE property) if $\lim_{n\to\infty} ||f_n - f^*||$ exists and is finite for any $f^* \in Fix(T)$, provided that $Fix(T) \neq \emptyset$.

In Sections 4, 5, and 6 of this article, we studied a class of sequences that enjoy these properties for nonexpansive operators.

Example 2.2. Let us consider the sequence of Picard iterates of the map $T : [0,1] \to [0,1]$ defined as $Tx = 1-x$, whose single fixed point is $x^* = 1/2$. Picard iterates $x_n = T^n x_0$ satisfy the LE property since $(|x_n - x^*|)$ is the constant sequence $(|x_0 - \frac{1}{2}|)$. However, it does not have the AF property since $(|x_n - Tx_n|) = (|2x_0 - 1|)$, for $n \geq 0$, does not converge to zero except if $x_0 = 1/2$. The n-th Krasnoselskii iteration (1.1) is given explicitly by $x_n = \frac{1}{2} + (1-2\lambda)^n (x_0 - \frac{1}{2})$ and it has the LE property for $0 < \lambda < 1$. In fact, for these values of the scalar, (x_n) converges strongly to the fixed point. The Krasnoselskii iterates own the AF property for $0 < \lambda < 1$ as well because $(|x_n - Tx_n|) = (|(1-2\lambda)^n (2x_0 - 1)|)$ converges to zero for any $x_0 \in [0,1]$.

In the following, we study the implications of the properties LE and AF for the approximation of fixed points of the operators of nonexpansive type. Let us start with a couple of definitions.

Definition 2.3. Let X,Y be Banach spaces over the same field. Then $S : X \to Y$ is closed if $x_n \to x$ and $Sx_n \to y$ imply that $y = Sx$.

Let us mention the definition of weak convergence of a sequence in a normed space.

Definition 2.4. Let X be normed space. A sequence $(x_n) \subseteq X$ is said to converge weakly to $x \in X$ if $(f(x_n))$ converges to $(f(x))$ for any $f \in X^*$, where X^* denotes the topological dual of X.

From now on, $\rightharpoonup$ is used to denote the weak convergence of a sequence

Definition 2.5. Let X,Y be Banach spaces over the same field. Then $S : X \to Y$ is said to be demiclosed if $x_n \rightharpoonup x$ and $Sx_n \to y$ imply that $y = Sx$.

Theorem 2.6. *Let X be a Banach space, C be a compact and convex subset of X, and $T : C \to C$ be a closed operator. If $(x_n) \subseteq C$ has the AF property, then $Fix(T) \neq \emptyset$. If T has a single fixed point, then (x_n) converges strongly to it. Otherwise any strongly convergent subsequence of (x_n) tends to a fixed point of T.*

Proof. Since C is compact, (x_n) has a convergent subsequence (x_{n_j}). Let $x = \lim_{j\to\infty} x_{n_j}$. Since (x_n) has the AF property, $\lim_{j\to\infty}(I-T)x_{n_j} = 0$. The hypothesis on T implies that $0 = (I-T)x$ and x is a fixed point of T. Thus, the critical point x is a subsequential limit of (x_n). If $Fix(T) = \{x^*\}$, all the subsequences of (x_n) converge to x^* for the same reasons, and consequently (x_n) converges strongly to it. □

Theorem 2.7. *Let X be a reflexive Banach space, C be a bounded, closed, and convex subset of X, and $T : C \to C$ be such that $I - T$ is a demiclosed operator. If $(x_n) \subseteq C$ has the AF property, then every subsequential weak limit of (x_n) is a fixed point of T, that is, $Fix(T) \neq \emptyset$. If T has a single fixed point, then (x_n) converges weakly to it. Otherwise, any weakly convergent subsequence of (x_n) tends weakly to a fixed point of T.*

Proof. In a reflexive Banach space, a closed, bounded, and convex set is weakly compact. This is a consequence of Theorem of Eberlein–Smulian (see, e.g., [5]). Then, (x_n) has a weakly convergent subsequence (x_{n_j}). Let x be the weak limit of (x_{n_j}). Since (x_n) has the AF property, $\lim_{j\to\infty}(I-T)x_{n_j} = 0$. The hypothesis on T implies that $0 = (I-T)x$, and x is a fixed point of T. The critical point x is a subsequential weak limit of (x_n). If $Fix(T) = \{x^*\}$, all the subsequences converge weakly to it. Consequently, $x_n \rightharpoonup x$. □

An example of reflexive Banach spaces is the Bochner space $\mathscr{B}^p(\Omega, E)$, where Ω is a measure space and E is a Banach space, whenever E is reflexive and $1 < p < \infty$. This includes the usual Lebesgue spaces for $1 < p < \infty$ (see [3]). The following theorem is called Demiclosedness Principle (see [6, Theorem 10.4]).

Theorem 2.8. *Let X be a uniformly convex Banach space, C be a closed and convex subset of X, and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed on C.*

Theorem 2.9. *Let X be a uniformly convex Banach space, C be a bounded, closed and convex subset of X and $T : C \to C$ be a nonexpansive operator. If $(x_n) \subseteq C$ has the AF property, then every subsequential weak limit of it is a fixed point of T. If T has a single fixed point then (x_n) converges weakly to it. Otherwise any weakly convergent subsequence of (x_n) tends weakly to a fixed point of T.*

Proof. According to Theorem 2.8, one has that $I - T$ is demiclosed. On the other hand, a uniformly convex Banach space is reflexive ([16]), and we have the hypotheses of Theorem 2.7. □

For the next definition, we refer to [15].

Definition 2.10. A normed space X satisfies the Opial's condition if, for any sequence $(x_n) \subseteq X$ such that (x_n) converges weakly to $x \in X$, $\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||$. for any $y \neq x$.

There is another demiclosedness principle for reflexive Banach spaces (see, e.g., [6]).

Theorem 2.11. *Let X be a reflexive Banach space satisfying the Opial condition, C be a closed and convex subset of X, and $T : C \to X$ be a nonexpansive mapping. Then $I - T$ is demiclosed on C.*

Then Theorem 2.9 is also true for reflexive Banach spaces with the Opial condition.

Theorem 2.12. *Let X be a reflexive Banach space with the Opial condition, C be a bounded, closed, and convex subset of X, and $T : C \to C$ be a nonexpansive operator. If $(x_n) \subseteq C$ has the AF property, then every subsequential weak limit of it is a fixed point of T. If T has a single fixed point, then (x_n) converges weakly to it. Otherwise, any weakly convergent subsequence of (x_n) tends weakly to a fixed point of T.*

Proof. According to Theorem 2.11, we have the hypotheses of Theorem 2.7, and the result holds immediately. □

3. Fixed Points of Nonexpansive Partial Contractivities

Definition 3.1. Let X be a normed space. $T : X \to X$ is said to be a nonexpansive partial contractivity if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(0) = 0$, such that for any $x, y \in X$,

$$||Tx - Ty|| \leq ||x - y|| + \psi(\min\{||x - Tx||, ||y - Ty||\}).$$

For $\psi = 0$, we have a nonexpansive mapping.

In the following, we give some properties of the set of fixed points of a nonexpansive partial contractivity. The proofs of these results are similar to the given in [12, Corollaries 1, 2, 3, 4].

Proposition 3.2. *Let X be a normed space, and T be a continuous nonexpansive partial contractivity. Then the set of fixed points $Fix(T)$ is a closed set.*

Proposition 3.3. *Let X be a strictly convex Banach space, and let $C \subseteq X$ be a nonempty, closed, and convex set. Let $T : C \to C$ be a nonexpansive partial contractivity. If $Fix(T) \neq \emptyset$, then it is closed and convex, and T is continuous on it. Consequently, $Fix(T)$ is empty, a singleton or infinite.*

Proposition 3.4. *Let X be a uniformly convex Banach space, and let $C \subseteq X$ be a nonempty, closed, and convex set. Let $T : C \to C$ be a nonexpansive partial contractivity. If $Fix(T) \neq \emptyset$, then, for any $y \in X$, there exists a unique $x^* \in Fix(T)$ such that $||y - x^*|| = d(y, Fix(T))$.*

Remark 3.5. In particular, all these propositions are true for nonexpansive partial contractivities defined on Lebesgue spaces $X = \mathscr{L}^p(I)$ with $1 < p < \infty$.

Proposition 3.6. *Let X be a Banach space satisfying the Opial condition, $C \subseteq X$ be nonempty, closed and convex, and $T : C \to C$ be a nonexpansive partial contractivity, where ψ is right continuous at zero and non-decreasing. If a sequence $(x_n) \subseteq X$ converges weakly to $x \in C$ and it satisfies the AF property with respect to T, then $x \in Fix(T)$.*

Proof. Let (x_n) be a weakly convergent sequence satisfying the condition AF. Then $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Let x be its weak limit, and let us assume that $x \neq Tx$. Since X satisfies the Opial condition,

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - Tx|| \leq \liminf_{n\to\infty} (||x_n - Tx_n|| + ||Tx_n - Tx||).$$

The fact that T is a nonexpansive partial contractivity implies that

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty}(||x_n - x|| + \psi(||Tx_n - x_n||)) = \liminf_{n\to\infty} ||x_n - x||.$$

Consequently, $x = Tx$. □

Theorem 3.7. *Let X be a reflexive Banach space satisfying the Opial condition. Let $C \subseteq X$ be a nonempty, closed, bounded and convex set, and let $T: C \to C$ be a nonexpansive partial contractivity, where ψ is right continuous at zero and non-decreasing and $Fix(T) \neq \emptyset$. Any sequence having the AF and LE properties converges weakly to a fixed point x^* of T. If X is a uniformly convex Banach space and $(||x_n||)$ converges to $||x^*||$, then (x_n) converges strongly to x^*.*

Proof. As mentioned before, in a reflexive Banach space, a nonempty, closed, bounded, and convex set is weakly compact. Let (x_n) a sequence with the LE and AF properties. Since C is weakly compact, there exists a weakly convergent subsequence (x_{n_j}). Let x^* be the weak limit of (x_{n_j}). Then $x^* \in C$. By Proposition 3.6, one has $x^* \in Fix(T)$. Thus all the subsequential weak limits of (x_n) are fixed points. If there exists another subsequence (x_{m_k}) converging to $y^* \in X$, and $y^* \neq x^*$, by the LE property and the Opial condition,

$$\lim_{n\to\infty} ||x_n - x^*|| = \lim_{n\to\infty} ||x_{n_j} - x^*|| < \lim_{n\to\infty} ||x_{n_j} - y^*|| = \lim_{n\to\infty} ||x_n - y^*||,$$

and

$$\lim_{n\to\infty} ||x_n - y^*|| = \lim_{n\to\infty} ||x_{m_k} - y^*|| < \lim_{n\to\infty} ||x_{m_k} - x^*|| = \lim_{n\to\infty} ||x_n - x^*||.$$

Hence $x^* = y^*$ and (x_n) converges weakly to x^*. If (x_n) converges weakly to x^*, $(||x_n||)$ converges to $||x^*||$, and X is uniformly convex, then (x_n) converges strongly to x^* (see, e.g., [3, Proposition 3.32]). □

Corollary 3.8. *Let X be a uniformly convex Banach space with the Opial condition, and let $T: C \to C$, where C is nonempty, bounded, closed, and convex subset of X, be a nonexpansive mapping. Then $Fix(T) \neq \emptyset$. If (x_n) enjoys the AF and LE properties, (x_n) converges weakly to a fixed point of T.*

Proof. Browder's Fixed Point Theorem ([4]) implies that $Fix(T) \neq \emptyset$ (see, e.g., [4]) by the conditions imposed on X and T. Then Theorem 3.7 yields the result immediately. □

4. A New Three Steps Algorithm

In this section, we introduce a three-step algorithm with two map evaluations. The new procedure may be advantageous when the evaluation of T is time-consuming, regarding the three-steps methods present in the current literature

$$z_n = (1 - c_n)x_n + c_n T x_n, \tag{4.1}$$

$$y_n = (1 - b_n)x_n + b_n z_n, \tag{4.2}$$

$$x_{n+1} = (1 - a_n)y_n + a_n T y_n, \tag{4.3}$$

where $a_n, b_n, c_n \in [0,1]$ and $x_0 \in X$. We call this iterative scheme N-algorithm.

This procedure generalizes Mann iteration (1.2), obtained for $a_n = 0$ and $b_n = 1$, and consequently the Krasnoselskii algorithm (1.1) as well, where c_n is constant.

Proposition 4.1. *Let X be a normed space, C a closed and convex subset of X, and $T : C \to C$ be a nonexpansive partial contractivity such that $Fix(T) \neq \emptyset$. Then the N-iteration has the LE property for any $x_0 \in X$.*

Proof. For $x^* \in Fix(T)$, from (4.1) and the definition of nonexpansive partial contractivity, one has

$$||z_n - x^*|| = ||(1-c_n)(x_n - x^*) + c_n(Tx_n - x^*)|| \leq ||x_n - x^*||, \tag{4.4}$$

bearing in mind that, for any $x \in C$ and $x^* \in Fix(T)$, $||Tx - x^*|| \leq ||x - x^*||$. From (4.2) and (4.4), one has

$$||y_n - x^*|| = ||(1-b_n)(x_n - x^*) + b_n(z_n - x^*)|| \leq ||x_n - x^*||. \tag{4.5}$$

Using (4.3) and (4.5), one has $||x_{n+1} - x^*|| \leq (1-a_n)||x_n - x^*|| + a_n||x_n - x^*|| = ||x_n - x^*||$. Consequently, $||x_n - x^*||$ is bounded and decreasing, and there exists $l := \lim_{n\to\infty} ||x_n - x^*|| \in \mathbb{R}$. □

Next lemma can be found in [19].

Lemma 4.2. *Let X be a uniformly convex Banach space, and a sequence $(\lambda_n) \subseteq X$ be such that there exist $p, q \in \mathbb{R}$ satisfying the condition $0 < p \leq \lambda_n \leq q < 1$ for all $n \in \mathbb{N}$. Let (x_n), (y_n) be sequences of X such that $\limsup_{n\to\infty} ||x_n|| \leq r$, $\limsup_{n\to\infty} ||y_n|| \leq r$, and $\limsup_{n\to\infty} ||\lambda_n x_n + (1-\lambda_n)y_n|| = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

Theorem 4.3. *Let X be a uniformly convex Banach space. Let C be a closed and convex subset of X, and let $T : C \to C$ a nonexpansive partial contractivity. Then the N-iteration such that $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, has the AF property for any $x_0 \in C$.*

Proof. By (4.5), one has

$$\limsup_{n\to\infty} ||y_n - x^*|| \leq l = \lim_{n\to\infty} ||x_n - x^*|| \tag{4.6}$$

and $\limsup_{n\to\infty} ||Ty_n - x^*|| \leq \limsup_{n\to\infty} ||y_n - x^*|| \leq l$. Moreover, by (4.3), we obtain that

$$l = \lim_{n\to\infty} ||x_{n+1} - x^*|| = \lim_{n\to\infty} ||(1-a_n)(y_n - x^*) + a_n(Ty_n - x^*)||.$$

Lemma 4.2 implies that $\lim_{n\to\infty} ||Ty_n - y_n|| = 0$. From (4.3), we see that

$$||x_{n+1} - x^*|| \leq ||y_n - x^*|| + a_n||Ty_n - y_n||.$$

Consequently, $l \leq \liminf_{n\to\infty} ||y_n - x^*||$, since $\lim_{n\to\infty} ||y_n - Ty_n|| = 0$. Bearing in mind (4.6), we have $l = \lim_{n\to\infty} ||y_n - x^*||$. Then

$$||y_n - x^*|| \leq (1-b_n)||x_n - x^*|| + b_n||z_n - x^*||,$$

$$||y_n - x^*|| - ||x_n - x^*|| \leq -b_n||x_n - x^*|| + b_n||z_n - x^*||,$$

$$||y_n - x^*|| - ||x_n - x^*|| \leq \frac{||y_n - x^*|| - ||x_n - x^*||}{b_n} \leq ||z_n - x^*|| - ||x_n - x^*||,$$

and $||y_n - x^*|| \leq ||z_n - x^*||$. It follows from $l = \lim_{n\to\infty} ||y_n - x^*||$ that

$$l = \lim_{n\to\infty} ||y_n - x^*|| \leq \liminf_{n\to\infty} ||z_n - x^*||.$$

By (4.4), we have $\limsup_{n\to\infty}||z_n - x^*|| \leq l = \lim_{n\to\infty}||x_n - x^*||$. Consequently, $l = \lim_{n\to\infty}||z_n - x^*||$. The equality

$$l = \lim_{n\to\infty}||z_n - x^*|| = \lim_{n\to\infty}||(1-c_n)(x_n - x^*) + c_n(Tx_n - x^*)||,$$

along with $l = \lim_{n\to\infty}||x_n - x^*||$ and $\limsup_{n\to\infty}||Tx_n - x^*|| \leq l$ imply that, by Lemma 4.2, $\lim_{n\to\infty}||Tx_n - x_n|| = 0$ and (x_n) has the AF property, for any $x_0 \in C$. □

Let us consider now the convergence of the N-iteration for nonexpansive partial contractivities. It is very relevant to the Schauder's Fixed Point Theorem (see, e.g., [3] for this result and its practical applications).

Theorem 4.4. *Let X be a Banach space, and let C be a nonempty closed convex set in X. Let $F : C \to C$ be a continuous map such that $F(C) \subseteq K$, where K is a compact subset of C. Then F has a fixed point in K.*

Regarding the convergence of the N-iteration, we have the following results.

Theorem 4.5. *Let X be a uniformly convex Banach space. Let K be a compact and convex subset of X, and let $T : K \to K$ be a continuous nonexpansive partial contractivity. Then $Fix(T) \neq \emptyset$. If T has a single fixed point x^*, then the N-iteration, where $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, converges strongly to x^* for any $x_0 \in C$. Otherwise, any strongly convergent subsequence of (x_n) tends to a fixed point of T.*

Proof. The statement about the set of fixed points is a consequence of Schauder's Fixed Point Theorem. By Theorem 4.3, the N-iteration has the AF property in a uniformly convex Banach space, and Theorem 2.6 yields the desired result immediately. □

For the next result, we use the following definition.

Definition 4.6. Let X be a Banach space. A self-map $S : X \to X$ is said to be completely continuous if, for any sequence (x_n) converging weakly to $x \in X$, the sequence of images (Sx_n) converges strongly to Sx.

Theorem 4.7. *Let X be a uniformly convex Banach space. Let C be a bounded, closed, and convex subset of X, and let $T : C \to C$ be a nonexpansive partial contractivity such that $I - T$ is completely continuous. Let $(x_n) \subseteq C$ be the N-iteration where $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$. Then every subsequential weak limit of (x_n) is a fixed point of T and hence $Fix(T) \neq \emptyset$. If T has a single fixed point, then (x_n) converges weakly to it. Otherwise, any weakly convergent subsequence of (x_n) tends weakly to a fixed point of T.*

Proof. In a uniformly convex Banach space, the N-iteration, where $0 < \inf a_n \leq \sup a_n < 1$, $0 < \inf b_n \leq \sup b_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, has the AF property according to Theorem 4.3. If $I - T$ is completely continuous, then $I - T$ is demiclosed, and the result is a straightforward consequence of Theorem 2.7. □

Definition 4.8. Let X be a normed space. A map $T : X \to X$ is called a strict nonexpansive partial contractivity if there exists $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, $\psi(0) = 0$, and, for $x \neq y$, $x, y \in X$,

$$||Tx - Ty|| < ||x - y|| + \psi(\min\{||x - Tx||, ||y - Ty||\}). \tag{4.7}$$

Proposition 4.9. *If T is a strict nonexpansive partial contractivity, then $Fix(T) = \emptyset$ or $Fix(T)$ is a singleton.*

Proof. If $x^* \in Fix(T)$, then, it follows from (4.7) that, for any $x \neq x^*$, $||Tx - x^*|| < ||x - x^*||$. Consequently, x^* is unique. □

Then we have the following results.

Corollary 4.10. *Let X be a uniformly convex Banach space. Let K be a compact and convex subset of X, and let $T : K \to K$ be a continuous strict nonexpansive partial contractivity. Then $Fix(T) = \{x^*\}$ and the N-iteration where $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, converges strongly to it.*

Proof. It is a consequence of Theorem 4.5 and Proposition 4.9. □

Corollary 4.11. *Let X be a uniformly convex Banach space. Let C be a bounded, closed, and convex subset of X, and let $T : C \to C$ be a strict nonexpansive partial contractivity such that $I - T$ is completely continuous. Then the N-iteration where $0 < \inf a_n \leq \sup a_n < 1$ and $0 < \inf c_n \leq \sup c_n < 1$, converges weakly to the fixed point of T.*

Proof. It is a consequence of Theorem 4.7 and Proposition 4.9. □

For the spaces with the Opial condition, based on Theorem 3.7, we have the following result.

Theorem 4.12. *Let X be a uniformly convex Banach space with the Opial condition. Let $C \subseteq X$ be nonempty, closed, bounded and convex set, and let $T : C \to C$ be a nonexpansive partial contractivity such that ψ is non-decreasing and right-continuous at zero, and $Fix(T) \neq \emptyset$. The N-iteration with intermediate scalars a_n, c_n converges weakly to a fixed point x^* of T for any $x_0 \in C$. If $(||x_n||)$ converges to $||x^*||$, then (x_n) converges strongly to x^*.*

Proof. A uniformly convex Banach space is reflexive. In a uniformly convex Banach space, the N-iteration with intermediate values of a_n and c_n has the LE and AF properties, and the result is a consequence of Theorem 3.7. □

Corollary 4.13. *Let X be a uniformly convex Banach space with the Opial condition, and $T : C \to C$, where C is nonempty, bounded, closed, and convex, be a nonexpansive mapping. Then $Fix(T) \neq \emptyset$ and the N-iteration described converges weakly to a fixed point of T for any $x_0 \in C$.*

Proof. It is a consequence of Corollary 3.8. □

Remark 4.14. Theorem 4.12 and Corollary 4.13 are true in particular for Hilbert spaces since they are uniformly convex and satisfy the Opial condition.

5. Fixed Point Theorems for $(\varphi - \psi)$-partial Contractivities

In [11], the concept of $(\varphi - \psi)$-partial contractivity was introduced, in the context of a b-metric space. We recall the definition in a normed space. Then we give sufficient conditions for the existence of fixed points of this kind of maps, which are different from the ones given in previous articles.

Definition 5.1. Let X be a normed space. $T : X \to X$ is a $(\varphi - \psi)$-partial contractivity if there exist a comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, e.g., φ is increasing and $\varphi^n(t) \to 0$ when n tends to infinity for any $t > 0$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that, for all $x, y \in X$,

$$||Tx - Ty)|| \leq \varphi(||x - y||) + \psi(\min\{||x - Tx||, ||y - Ty||\}).$$

Definition 5.2. Let X be a normed space and $C \subseteq X$. $T : C \to C$ is said asymptotically regular if $\lim_{n\to\infty} ||T^{n+1}x - T^n x|| = 0$ for any $x \in C$.

Theorem 5.3. *Let X be a Banach space, and let C be a compact and convex subset of X. Let $T : C \to C$ be closed and asymptotically regular. Then $Fix(T) \neq \emptyset$ and a subsequence of the Picard iterates converges to a fixed point.*

Proof. Let (x_n) be the sequence of Picard iterations. Since C is compact, (x_n) has a convergent subsequence (x_{n_j}). Let $x = \lim_{j\to\infty} x_{n_j}$. The asymptotic regularity of T implies that $(I-T)(x_{n_j}) \to 0$, since $I-T$ is closed, and $x \in Fix(T)$. □

Theorem 5.4. *Let X be a Banach space, and let C be a compact and convex subset of X. Let $T : C \to C$ be a $(\varphi - \psi)$-partial contractivity, where φ is sublinear and $\psi(t) = Bt$. If $\varphi + BI$ is a comparison function, then T is asymptotically regular. If further T is closed, then $Fix(T) = \{x^*\}$ and the Picard iterations of any point converge strongly to x^*.*

Proof. From [13, Proposition 2], one has, for any $x, y \in X$,

$$||T^n x - T^n y|| \leq \varphi^n(||x-y||) + ((\varphi + BI)^n - \varphi^n)(||x - Tx||). \tag{5.1}$$

Substituting $y = Tx$ in (5.1), one has $||T^n x - T^{n+1}x|| \leq (\varphi + BI)^n(||x - Tx||)$. As $\varphi + BI$ is a comparison function, one sees that $\lim_{n\to\infty} ||T^n x - T^{n+1}x|| = 0$. If T is closed, we have the hypotheses of Theorem 5.3 and $Fix(T) \neq \emptyset$. Let $x^* \in Fix(T)$. Substituting x by x^* in (5.1), one has $||T^n y - x^*|| \leq \varphi^n(||y - x^*||) \to 0$ as n tends to infinity. This implies that the fixed point is unique, and the Picard iterates of any element of X converge to it. □

Theorem 5.5. *Let X be a Banach space, and let K be a compact and convex subset of X. Let $T : K \to K$ be a continuous $(\varphi - \psi)$-partial contractivity. Then $Fix(T) = \{x^*\}$ and the Picard iterates are convergent to it.*

Proof. The existence of fixed points is ensured by Schauder's Theorem 4.4. The uniqueness and convergence of Picard iterates come from Definition 5.1, due to $||Tx - x^*|| \leq \varphi(||x - x^*||)$. In general, $||T^n x - x^*|| \leq \varphi^n(||x - x^*||) \to 0$. □

Theorem 5.6. *Let X be a reflexive Banach space, and let C be a bounded, closed, and convex subset of X. Let $T : C \to C$ be a mapping such that $I - T$ is demiclosed and T is asymptotically regular. Then $Fix(T) \neq \emptyset$ and a subsequence of the Picard iterates converges weakly to a fixed point.*

Proof. Let (x_n) be the sequence of Picard iterations of $x_0 \in C$. Since C is weakly compact, there is a weakly convergent subsequence (x_{n_j}). Let x be the weak limit of (x_{n_j}). The asymptotic regularity of T implies that $(I-T)(x_{n_j}) \to 0$. Since $I - T$ is demiclosed $0 = (I-T)x$, $x \in Fix(T)$ and a subsequence of the Picard iterates converges to it. □

Theorem 5.7. *Let X be a reflexive Banach space, and let C be a bounded, closed, and convex subset of X. Let $T : C \to C$ be a $(\varphi - \psi)$-partial contractivity, where φ is sublinear and $\psi(t) = Bt$. If $\varphi + BI$ is a comparison function, then T is asymptotically regular. If further $I - T$ is demiclosed, then $Fix(T) = \{x^*\}$ and the Picard iterations of any point converge strongly to x^*.*

Proof. The proof of the asymptotic regularity of T is similar to that given in Theorem 5.4. Then we have the hypotheses of Theorem 5.6 and hence $Fix(T) \neq \emptyset$. In a $(\varphi - \psi)$-partial contractivity, the fixed point is unique (in case of the existence) and the Picard iterations are strongly convergent to it as mentioned previously. □

Iteration	Approximate value	Error
0	0.000000	0.353553
1	0.359211	0.005658
2	0.353557	0.000004
3	0.353553	0.000000
4	0.353553	0.000000
5	0.353553	0.000000

TABLE 1. T_1-fixed point approximations for $x_0 = 0$.

6. Convergence of the N-iterative Algorithm to the Fixed Point of a Partial Contractivity

In this section, the convergence and stability of the N-iteration applied to a partial contractivity (introduced in ([11])), is studied.

Definition 6.1. Let X be a normed space, C be a closed and convex subset of X, $k < 1$, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\psi(0) = 0$. $T : C \to C$ is said to be a partial contractivity if, for any $x, y \in X$, $||Tx - Ty|| \leq k||x - y|| + \psi(\min\{||x - Tx||, ||y - Ty||\})$.

Theorem 6.2. *Let X be a normed space. Let C be a closed and convex subset of X, and let $T : C \to C$ be a partial contractivity with a fixed point $x^* \in C$. Choose $r \in \mathbb{R}$ such that $k < r < 1$, and a_n, b_n, and c_n satisfy*

$$\frac{1-r}{1-k} < a_n \leq 1 \text{ and } \frac{1-r}{1-k} < b_n c_n \leq 1. \tag{6.1}$$

Then the N-iteration is convergent to x^* for any $x_0 \in C$ with rate of convergence $O(r^{2n})$ and asymptotic stability.

Proof. The selection of a_n, b_n, and c_n implies that

$$1 - (1-k)a_n < r < 1 \text{ and } 1 - (1-k)b_n c_n < r < 1, \tag{6.2}$$

The first step of Algorithm (4.1) implies that $||z_n - x^*|| \leq (1 - (1-k)c_n)||x_n - x^*||$. It follows from (4.2) that

$$||y_n - x^*|| \leq (1 - b_n c_n(1-k))||x_n - x^*||. \tag{6.3}$$

On the other hand, (4.3) yields $||x_{n+1} - x^*|| \leq (1 - (1-k)a_n)||y_n - x^*||$. From (6.3), we have

$$||x_{n+1} - x^*|| \leq (1 - (1-k)a_n)(1 - b_n c_n(1-k))||x_n - x^*||$$

In view of (6.2), we have $||x_{n+1} - x^*|| \leq r^2||x_n - x^*||$. In general, $|x_n - x^*|| \leq r^{2n}||x_0 - x^*||$. Thus, for the ranges of values of the scalars given by (6.1), the algorithm is convergent with asymptotic stability and order of convergence $O(r^{2n})$ at least. □

7. Application of the N-iteration to the Approximation of Repelling Fixed Points

In this section, we present two examples of the N-iteration to the approximation of repelling fixed points of nonlinear mappings.

Iteration	Approximate value	Error
0	0.700000	0.346447
1	0.369132	0.015579
2	0.353582	0.000029
3	0.353553	0.000000
4	0.353553	0.000000
5	0.353553	0.000000

TABLE 2. T_1-fixed point approximations for $x_0 = 0.7$.

Iteration	Approximate value	Error
0	0.000000	0.381966
1	0.406250	0.024284
2	0.380744	0.001222
3	0.38203	0.000064
4	0.381963	0.000000
5	0.381966	0.000000

TABLE 3. T_2-fixed point approximations for $x_0 = 0$.

Iteration	Approximate value	Error
0	0.700000	0.318034
1	0.376128	0.005838
2	0.382272	0.000306
3	0.381950	0.000016
4	0.381967	0.000001
5	0.381966	0.000000

TABLE 4. T_2-fixed point approximations for $x_0 = 0.7$.

Example 7.1. Let us consider the map $T_1 : [0,1] \to [0,1]$, defined by $T_1(x) = (1-x^{2/3})^{3/2}$. This function has a fixed point at $x^* = \frac{1}{2\sqrt{2}} \simeq 0.353553$. However, this point is a repellor for the Picard iteration. Starting at $x_0 = 0$, we obtain an oscillating sequence of Picard iterates: $(0,1,0,1,\ldots)$. For the rest of the initial values, the algorithm does not converge either except if $x_0 = x^*$. Let us apply the N-algorithm, presented in Section 4, by taking $a_n = b_n = c_n = 1/2$ for all n. The iterative scheme to be computed is the following:

$$z_n = \frac{x_n + (1-x_n^{2/3})^{3/2}}{2},\ y_n = \frac{x_n + z_n}{2},\ x_{n+1} = \frac{y_n + (1-y_n^{2/3})^{3/2}}{2},$$

and $x_0 \in [0,1]$. For the N-operator, the fixed point acts like an attractor (see Tables 1 and 2). Table 1 represents the first N-iterations (x_n) for the starting point $x_0 = 0$. The second column collects the approximate values of the fixed point, and the third one, the error committed. Table 2 gathers the same data, but in the case that $x_0 = 0.7$.

Example 7.2. Consider the map $T_2 : [0,1] \to [0,1]$, defined by $T_2(x) = (1-x)^2$. This function has a fixed point in $[0,1]$ at $x^* = \frac{3-\sqrt{5}}{2} \simeq 0.381966$. This point is also repelling for the Picard iteration. Starting at $x_0 = 0$, we obtain the sequence of Picard iterates $(0,1,0,1,\ldots)$. For the rest of the initial values the algorithm does not converge either, except in the case $x_0 = x^*$. Let $a_n = b_n = c_n = 1/2$ for all n in the N-algorithm, defined in Section 4. The iterative scheme is

$$z_n = \frac{x_n + (1-x_n)^2}{2},\ y_n = \frac{x_n + z_n}{2},\ x_{n+1} = \frac{y_n + (1-y_n)^2}{2},$$

and $x_0 \in [0,1]$. For the N-operator, the fixed point is attracting (see Tables 3 and 4). Table 3 represents the first N-iterations for the initial value $x_0 = 0$. The second column collects the approximate values of the fixed point, and the third one, the error committed. Table 4 gathers the same data, but in the case $x_0 = 0.7$.

Both examples demonstrate that the N-algorithm could be used to find ”hidden” fixed points in nonlinear dynamics.

REFERENCES

[1] R. Agarwal, O. Regan, D. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Analysis 8 (2007) 61-79.

[2] R.P. Agarwal, D. O’Regan, D.R. Sahu, Fixed Point Theory for Lipschitzian-Type Mappings with Applications, Springer, New York, 2009.

[3] H. Brézis, Functional Analysis, Sobolev spaces and Partial Differential Equations, Springer, New York, 2011.

[4] F.E. Browder, Nonexpansive nonlinear operators in a Banach space. Proc. Nat. Acad. Sci. USA 54 (1965) 1041-1044.

[5] T. Bühler, D.A. Salamon, Functional Analysis. Graduate Studies in Mathematics, 191, AMS, 2018.

[6] K. Goebel, W.A. Kirk, Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge, 1990.

[7] S. Ishikawa, Fixed points by a new iteration method. Proc. Amer. Math. Soc. 44 (1974) 147-150.

[8] V. Karakaya, Y. Atalan, K. Dogan, El Houda, N. Bouzara, Some fixed point results for a new three steps iteration process in Banach spaces, Fixed Point Theory 18 (2017) 625-640.

[9] W.A. Kirk, On successive approximation for nonexpansive mappings in Banach spaces, Glasgow Math. J. 12(1) (1971) 6-9.

[10] M.A. Krasnoselskij, Two remarks on the method of successive approximations, (Russian) Uspehi Mat. Nauk. 10-1 (1955) 123–127.

[11] M.A Navascués, Approximation of fixed points and fractal functions by means of different iterative algorithms, Chaos, Solitons, Fractals 180 (2024) 114535.

[12] M.A. Navascués. Nonexpansiveness and fractal maps in Hilbert spaces, Symmetry, 16 (2024) 738.

[13] M.A. Navascués, Stability of fixed points of partial contractivities and fractal surfaces, Axioms 13 (2024) 474.

[14] M.A. Navascués, R.N. Mohapatra, Fixed point dynamics in a new type of contracfion in b-metric spaces, Symmetry, 16 (2024) 506.

[15] Z. Opial, Weak convergence theorems for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591-597.

[16] B.J. Pettis, A proof that every uniformly convex space is reflexive, Duke Math. J. 5 (1939) 249-253.

[17] W. Pheungrattana, S. Suantai, On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 235 (2011) 3006-3014.

[18] D.R. Sahu, Applications of the S-iteration process to constrained minimization problems and split feasibility problems, Fixed Point Theory 12 (2011) 187–204.

[19] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, Bull. Australian Math. Soc. 43 (1991) 153-159.