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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A COUPLED SYSTEM OF ψ-HILFER FRACTIONAL DIFFERENTIAL EQUATIONS UNDER UNCOUPLED NON-LOCAL MULTI POINT CONDITIONS INVOLVING FIXED POINT THEOREMS

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Abstract. In this paper we study and investigate the existence and uniqueness of solutions for a coupled system of the ψ-Hilfer fractional differential equation under uncoupled non-local multi-point conditions. By using Schaefer and Banach fixed point theorems, we prove the existence and uniqueness of solutions to the problem. An example is provided to support our main theorems.

Keywords. Coupled system; Hilfer fractional derivative; Hilfer fractional differential equation; Fixed point theorem; Uncoupled non-local multi point conditions.

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1. Introduction

As systems of fractional differential equations arise in various areas of pure and applied science, they have become a hot research filed; see, e.g., [5, 7, 12, 14, 15, 16]). Coupled systems involving fractional differential equations are also important to study because they arise in a wide range of real-world problems and phenomena, such as biology, chemistry, physics and so on; see, e.g., [9, 11, 13, 20]) and the references.

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Borisut et al. [6] studied the $\psi-$Hilfer fractional differential equation with nonlocal multi point conditions of the form:

$$\begin{aligned} D_{a^+}^{q,p;\psi} u(t) &= f(t, u(t), D_{a^+}^{q,p;\psi} u(t)),\ t \in [a,b], \\ I_{a^+}^{1-r;\psi} u(a) &= \sum_{i=1}^{m} \beta_i u(\eta_i), \end{aligned}$$

where $q \leq r = q+p-qp < 1,\ \eta_i \in [a,b], 0 < q < 1,\ 0 \leq p \leq 1,\ m \in \mathbb{N},\ \beta_i \in \mathbb{R},\ i = 1,2,\ldots,m,$ $-\infty < a < b < \infty$, $D_{a^+}^{q,p;\psi}$ is the $\psi-$ Hilfer fractional derivatives, $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, and $I_{a^+}^{1-r;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $1-r$. The existence, uniqueness, and stability of a solution to this problem can be proved by using Schaefer's fixed point theorem. A coupled system of Hadamard type fractional differential equations with uncoupled Hadamard integral boundary conditions was investigated by Ahmad et al. [3] in 2017. The following describes the system:

$$\begin{aligned} D^{\alpha} u(t) &= f(t, u(t), v(t)),\ 1 < t < e,\ 1 < \alpha \leq 2, \\ D^{\beta} u(t) &= f(t, u(t), v(t)),\ 1 < t < e,\ 1 < \beta \leq 2, \\ u(1) = 0,\ u(e) = I^{\gamma} u(\sigma_1) &= \tfrac{1}{\gamma} \textstyle\int_1^{\sigma_1} \left(\log \tfrac{\sigma_1}{s} \right)^{\gamma-1} \tfrac{u(s)}{s} ds,\ \gamma > 0,\ 1 < \sigma_1 < e, \\ v(1) = 0,\ v(e) = I^{\gamma} v(\sigma_2) &= \tfrac{1}{\gamma} \textstyle\int_1^{\sigma_2} \left(\log \tfrac{\sigma_2}{s} \right)^{\gamma-1} \tfrac{v(s)}{s} ds,\ \gamma > 0,\ 1 < \sigma_2 < e, \end{aligned}$$

where $D^{\delta},\ \delta = \alpha,\ \beta$ is the Hadamard fractional derivative of fractional order δ, I^{γ} is the Hadamard fractional integral order γ and $f, g : [1,e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

The motivation of this paper is from [3, 6]. Our goal is to derive the existence and uniqueness of a positive solution to a nonlinear fractional differential equation under nonlocal multipoint conditions:

$$\begin{cases} D_{a^+}^{q,p;\psi} u_1(t) = f_1(t, u_1(t), u_2(t), \ldots, u_n(t)) \\ D_{a^+}^{q,p;\psi} u_2(t) = f_2(t, u_1(t), u_2(t), \ldots, u_n(t)) \\ \qquad\qquad \vdots \\ D_{a^+}^{q,p;\psi} u_n(t) = f_n(t, u_1(t), u_2(t), \ldots, u_n(t), t \in (a,b). \\ \text{with the conditions} \\ I_{a^+}^{1-r;\psi} u_1(a) = \sum_{i=1}^{m_1} c_i^1 u_1(\eta_i^1),\ I_{a^+}^{1-r;\psi} u_2(a) = \sum_{i=1}^{m_2} c_i^1 u_2(\eta_i^2), \ldots, \\ I_{a^+}^{1-r;\psi} u_n(a) = \sum_{i=1}^{m_n} c_i^n u_n(\eta_i^n), \end{cases} \tag{1.1}$$

where $0 < q < 1,\ 0 \leq p \leq 1,\ q \leq r = q+p-qp < 1\ m_1, m_2, \ldots, m_n \in \mathbb{N},\ c_i^1, c_i^2, \ldots, c_i^n \in \mathbb{R},$ $\eta_i^1, \eta_i^2, \ldots, \eta_i^n \in (a,b),\ i = 1,2,\ldots,m,\ -\infty < a < b < \infty$, $D_{a^+}^{q,p;\psi}$ is the $\psi-$ Hilfer fractional derivative, $f_1, f_2, \ldots, f_n : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $I_{a^+}^{1-r;\psi}$ is the ψ-Riemann-Liouville fractional integral of order $1-r$. With the help of Schaefer's and Banach's fixed point theorems, we establish the existence and uniqueness of the problem. Application of our research results is demonstrated with the aid of an example.

2. PRELIMINARIES

In this section, we review some fundamental concepts, including notations, definitions, lemmas, and theorems that are used to prove our main results.

Definition 2.1. [3] The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : (0,\infty) \to \mathbb{R}$ is defined by

$$_{RL}I^{q}_{0^+} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds,$$

where $\Gamma(\cdot)$ stands for the Gamma function.

Definition 2.2. [10] The left-sided Hilfer fractional derivative operator of function $f \in C^n[a,b]$ of order $n-1 < q < n$ and type $0 \le p \le 1$ is given by $D^{q,p}_{a^+} f(t) = I^{r-q}_{a^+} D^n I^{(1-p)(n-q)}_{a^+} f(t)$, $t > a$, where $r = q + np - qp$. Particularly, if $0 < q < 1$ and $f \in C^1[a,b]$, then $D^{q,p}_{a^+} f(t) = I^{r-q}_{a^+} D I^{(1-p)(1-q)}_{a^+} f(t), t > a$.

Definition 2.3. [17] For an integrable function $f : [a,b] \to \mathbb{R}$ with respect to another function $\psi : [a,b] \to \mathbb{R}$ which is an increasing function such that $\psi'(t) \neq 0$, for all $t \in [a,b]$, $(-\infty \le a < b \le +\infty)]$, the left-sided ψ-Riemann-Liouville fractional integral and fractional derivative of order q, $(n-1 < q < n)$ are defined as

$$I^{q;\psi}_{a^+} f(t) = \frac{1}{\Gamma(q)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{q-1} f(s)ds \tag{2.1}$$

and

$$\begin{aligned} D^{q;\psi}_{a^+} f(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I^{n-q;\psi}_{a^+} f(t) \\ &= \frac{1}{\Gamma(n-q)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-q-1} f(s)ds. \end{aligned}$$

Definition 2.4. [17] The left-sided ψ-Caputo fractional derivative of function $f \in C^n[a,b]$, $n-1 < q < n$ with respect to another function ψ is defined by

$$\begin{aligned} {}^cD^{q;\psi}_{a^+} f(t) &= I^{n-q;\psi}_{a^+} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t) \\ &= \frac{1}{\Gamma(n-q)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-q-1} f^{[n]}_{\psi}(s)ds, \end{aligned}$$

where n is the smallest integer greater than or equal to q, $f^{[n]}_{\psi}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds}\right)^n f(s)$ and ψ as in Definition 2.3.

Definition 2.5. [18] Let $n-1 < q < n$, $n \in \mathbb{N}$, and $[a,b]$ be an interval, where $-\infty \le a < b \le +\infty$ and $\psi \in C^n[a,b]$ is a function such that $\psi(t)$ is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$. The ψ-Hilfer fractional derivative (left-sided) of function $f \in C^n[a,b]$ with order q and type $0 \le p \le 1$ is determined as $D^{q,p;\psi}_{a^+} f(t) = I^{p(n-q);\psi}_{a^+} \left[\frac{1}{\psi'(t)} \frac{d}{dt}\right]^n I^{(1-p)(n-q);\psi}_{a^+} f(t)$, $t > a$. Observe

$$D^{q,p;\psi}_{a^+} f(t) = I^{p(n-q);\psi}_{a^+} D^{r;\psi}_{a^+} f(t), \; t > a, \tag{2.2}$$

where $D^{r;\psi}_{a^+} = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I^{(1-p)(n-q);\psi}_{a^+} f(t)$. In particular, the ψ-Hilfer fractional derivative of $0<q<1$ and type $0 \le p \le 1$ can be expressed by

$$D^{q,p;\psi}_{a^+} f(t) = \frac{1}{\Gamma(r-q)} \int_a^t (\psi(t)-\psi(s))^{r-q-1} D^{r;\psi}_{a^+} f(s)ds,$$

where $r = q+p-qp$, $I^{1-r;\psi}_{a^+}(\cdot)$ is defined in (2.1), $D^{r;\psi}_{a^+} f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right] I^{1-r;\psi}_{a^+} f(t)$.

Now, we introduce the space $E_1 = \{u_1 \in C_{1-r;\psi}[a,b]\}$ endowed with the norm $\|u_1\|_{C_{1-r;\psi}[a,b]} = \max\left\{|(\psi(t)-\psi(a))^{1-r}u_1(t)|: t \in (a,b]\right\}$, $E_2 = \{u_2 \in C_{1-r;\psi}[a,b]\}$ endowed with the norm $\|u_2\|_{C_{1-r;\psi}[a,b]} = \max\left\{|(\psi(t)-\psi(a))^{1-r}u_2(t)|: t \in (a,b]\right\}$, $\cdots$, $E_n = \{u_n \in C_{1-r;\psi}[a,b]\}$ endowed with the norm $\|u_n\|_{C_{1-r;\psi}[a,b]} = \max\left\{|(\psi(t)-\psi(a))^{1-r}u_n(t)|: t \in (a,b]\right\}$. The product space $(E_1\times E_2\times E_3\times\ldots\times E_n, \|(u_1,u_2,\ldots,u_n)\|_{C_{1-r,\psi}[a,b]})$ is also a Banach space equipped with norm

$$\|(u_1,u_2,\ldots,u_n)\|_{C_{1-r,\psi}[a,b]} = \|u_1\|_{C_{1-r,\psi}[a,b]} + \|u_2\|_{C_{1-r,\psi}[a,b]} \cdots \|u_n\|_{C_{1-r,\psi}[a,b]}.$$

Lemma 2.6. [8] *Let $q>0$, $0\le r<1$ and $f \in L^1(a,b)$. Then $I^{q;\psi}_{a^+} I^{p;\psi}_{a^+} f(t) = I^{q+p;\psi}_{a^+} f(t)$, a.e. $t \in [a,b]$. In particular,*

(1) If $f \in C_{r;\psi}[a,b]$, then $I^{q;\psi}_{a^+} I^{p;\psi}_{a^+} f(t) = I^{q+p;\psi}_{a^+} f(t)$, $t\in[a,b]$.
(2) If $f \in C[a,b]$, then $I^{q;\psi}_{a^+} I^{p;\psi}_{a^+} f(t) = I^{q+p;\psi}_{a^+} f(t)$, $t\in[a,b]$.

Lemma 2.7. [1] *Let $0<q<1$, $0\le p\le 1$, and $r=q+p-qp$. If $f(t)\in C^r_{1-r;\psi}[a,b]$, then $I^{r;\psi}_{a^+} D^{r;\psi}_{a^+} f(t) = I^{q;\psi}_{a^+} D^{q,p;\psi}_{a^+} f(t)$ and $D^{r;\psi}_{a^+} I^{q;\psi}_{a^+} f(t) = D^{p(1-q);\psi}_{a^+} f(t)$.*

Proposition 2.8. [19] *Let $t>a$, $q\ge 0$ and $\rho>0$. Then, ψ-fractional derivative and integral of a power function are given by*

$$D^{q;\psi}_{a^+}(\psi(t)-\psi(a))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho-q)}(\psi(t)-\psi(a))^{q+\rho-1}$$

and

$$I^{q;\psi}_{a^+}(\psi(t)-\psi(a))^{\rho-1} = \frac{\Gamma(\rho)}{\Gamma(\rho+q)}(\psi(t)-\psi(a))^{q+\rho-1}.$$

Moreover, if $0<q<1$, then $D^{q;\psi}_{a^+}(\psi(t)-\psi(a))^{q-1} = 0$.

Lemma 2.9. [4] *Let $0<q<1$, $0\le p\le 1$ and $r=q+p-qp$, and let $\psi\in C^1([a,b],\mathbb{R})$ be a function such that ψ is increasing and $\psi(t)\neq 0$, for all $t\in[a,b]$. Then*

(1) $I^{q;\psi}_{a^+}(\cdot)$ maps $C[a,b]$ into $C[a,b]$.
(2) $I^{q;\psi}_{a^+}(\cdot)$ is bounded from $C_{1-r;\psi}[a,b]$ into $C_{1-r;\psi}[a,b]$.
(3) If $r\le q$, then $I^{q;\psi}_{a^+}(\cdot)$ is bounded from $C_{1-r;\psi}[a,b]$ into $C[a,b]$.

Lemma 2.10. [4] *Let $q>0$, $0\le r<1$ and $f\in C_{r;\psi}[a,b]$. If $q>r$, then $I^{q;\psi}_{a^+} f \in C[a,b]$ and $I^{q;\psi}_{a^+} f(a) = \lim_{t\to a^+} I^{q;\psi}_{a^+} f(t) = 0$.*

Theorem 2.11. [4] *If $f\in C^n[a,b]$, $n-1<q<n$, $0\le p\le 1$, and $r=q+p-qp$. Then*

$$I^{q;\psi}_{a^+} D^{q,p;\psi}_{a^+} f(t) = f(t) - \sum_{i=1}^{n} \frac{[\psi(t)-\psi(a)]^{r-i}}{\Gamma(r-i+1)} f^{[n-i]}_{\psi} I^{(1-p)(n-q);\psi}_{a^+} f(a),$$

for all $t \in (a,b]$. In particular, if $0<q<1$, then

$$I_{a^+}^{q;\psi} D_{a^+}^{q,p;\psi} f(t) = f(t) - \frac{[\psi(t)-\psi(a)]^{r-1}}{\Gamma(r)} I_{a^+}^{(1-p)(1-q);\psi} f(a).$$

Moreover, if $f \in C_{1-r;\psi}[a,b]$ and $I_{a^+}^{1-r;\psi} f \in C_{1-r}^1[a,b]$ such that $0<r<1$, then

$$I_{a^+}^{r;\psi} D_{a^+}^{r;\psi} f(t) = f(t) - \frac{[\psi(t)-\psi(a)]^{r-1}}{\Gamma(r)} I_{a^+}^{(1-r);\psi} f(a), \text{ for all } t \in (a,b].$$

Let us end this section with the following two essential fixed point theorems.

Theorem 2.12. [3]*(Schaefer's fixed point theorem) Let $A : E \to E$ be a completly continuous operator. If the set $F(A) = \{u \in E : u = \lambda^* Au \text{ for some } \lambda^* \in [0,1]\}$ is bounded, then A has fixed points.*

Theorem 2.13. [3] *(Contraction Mapping Principle) Let E be a Banach space, $E_1 \subset E$ be closed, and $A : E_1 \to E_1$ be a contraction mapping (i.e., $\|Ax - Ay\| \leq k\|x-y\|$ for some $k \in (0,1)$) and for all $x, y \in E_1$. Then, A has a unique fixed point.*

3. Main Results

In this section, we investigate the existence of solutions to problem (1.1) by using Schaefer's fixed point theorem and the Banach principle contraction as the main tools.

Lemma 3.1. [2] *Let $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ be a function such that $f \in C_{1-r;\psi}[a,b]$ for any $u \in C_{1-r;\psi}[a,b]$. A function $u \in C_{1-r;\psi}^r[a,b]$ is solution of fractional initial value problem:*

$$\begin{cases} D_{a^+}^{q,p;\psi} u(t) = f(t,u(t)), \ 0<q<1, \ 0 \leq p \leq 1, \\ I_{a^+}^{1-r;\psi} u(a) = u_a, \ r = q+p-qp, \end{cases} \tag{3.1}$$

if and only if u satisfies the following Volterra integral equation:

$$u(t) = \frac{(\psi(t)-\psi(a))^{r-1}}{\Gamma(r)} u_a + I_{a^+}^{q;\psi} f(t,u(t)).$$

Lemma 3.2. *Let $0<q<1$, $0 \leq p \leq 1$ and $r = q+p-qp$, and let $f : (a,b) \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $f_1, f_2, \ldots f_n \in C_{1-r;\psi}[a,b]$ for any $u_1, u_2, \ldots, u_n \in C_{1-r;\psi}[a,b]$. If $u_1, u_2, \ldots, u_n \in C_{1-r;\psi}^r[a,b]$, then $u_1, u_2, \ldots, u_n$ satisfies problem (1.1) if and only if $u_1, u_2, \ldots, u_n$ satisfies the Volterra integral equation:*

$$\begin{cases} u_1(t) &= \frac{(\psi(t)-\psi(a))^{r-1}}{\Lambda_1} \sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi} f_1(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)) \\ & \quad + I_{a^+}^{q;\psi} f_1(t, u_1(t), u_2(t), \ldots, u_n(t)), \end{cases} \tag{3.2}$$

$$\begin{cases} u_2(t) &= \frac{(\psi(t)-\psi(a))^{r-1}}{\Lambda_2} \sum_{i=1}^{m_2} c_i^2 I_{a^+}^{q;\psi} f_2(\eta_i^2, u_1(\eta_i^2), u_2(\eta_i^2), \ldots, u_n(\eta_i^2)) \\ & \quad + I_{a^+}^{q;\psi} f_2(t, u_1(t), u_2(t), \ldots, u_n(t)), \end{cases} \tag{3.3}$$

$$\vdots$$

$$\begin{cases} u_n(t) &= \frac{(\psi(t)-\psi(a))^{r-1}}{\Lambda_n} \sum_{i=1}^{m_n} c_i^n I_{a^+}^{q;\psi} f_n(\eta_i^n, u_1(\eta_i^n), u_2(\eta_i^n), \ldots, u_n(\eta_i^n)) \\ & \quad + I_{a^+}^{q;\psi} f_n(t, u_1(t), u_2(t), \ldots, u_n(t)), \end{cases} \tag{3.4}$$

where $\Lambda_1 = \Gamma(r) - \sum_{i=1}^{m_1} c_i^1 (\psi(\eta_i^1) - \psi(a))^{r-1} \neq 0,\ \Lambda_2 = \Gamma(r) - \sum_{i=1}^{m_2} c_i^2 (\psi(\eta_i^2) - \psi(a))^{r-1} \neq 0, \ldots, \Lambda_n = \Gamma(r) - \sum_{i=1}^{m_n} c_i^n (\psi(\eta_i^n) - \psi(a))^{r-1} \neq 0.$

Proof. Consider $D_{a^+}^{q,p;\psi} u_1(t) = f_1(t, u_1(t), u_2(t), \ldots, u_n(t))$ and $I_{a^+}^{1-r;\psi} u_1(a) = \sum_{i=1}^{m_1} c_i^1 u_1(\eta_i^1)$. Lemma 3.1 yields that following solution to this problem

$$u_1(t) = \frac{u_0}{\Gamma(r)} (\psi(t) - \psi(a))^{r-1} + I_{a^+}^{q;\psi} f_1(t, u_1(t), u_2(t), \ldots, u_n(t))$$
$$u_1(t) = \frac{I_{a^+}^{1-r;\psi} u_1(a)}{\Gamma(r)} (\psi(t) - \psi(a))^{r-1} + I_{a^+}^{q;\psi} f_1(t, u_1(t), u_2(t), \ldots, u_n(t)).$$

After making the substitution $t = \eta_i^1$, we have

$$u_1(\eta_i^1) = \frac{I_{a^+}^{1-r;\psi} u_1(a)}{\Gamma(r)} (\psi(\eta_i^1) - \psi(a))^{r-1} + I_{a^+}^{q;\psi} f_1(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)).$$

This implies

$$\begin{aligned} \sum_{i=1}^{m_1} c_i^1 u_1(\eta_i^1) &= \frac{I_{a^+}^{1-r;\psi} u_1(a)}{\Gamma(r)} \sum_{i=1}^{m_1} c_i^1 (\psi(\eta_i^1) - \psi(a))^{r-1} \\ &\quad + \sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi} f_1(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)), \end{aligned}$$

so

$$\begin{aligned} I_{a^+}^{1-r;\psi} u_1(a) &= \frac{\Gamma(r)}{\Lambda_1} \sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi} f_1(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)) \\ u_1(t) &= \frac{(\psi(t) - \psi(a))^{r-1}}{\Lambda_1} \sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi} f_1(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)) \\ &\quad + I_{a^+}^{q;\psi} f_1(t, u_1(t), u_2(t), \ldots, u_n(t)). \end{aligned}$$

Thus we can deduce that $u_1(t)$ satisfies (3.2). By applying $I_{a^+}^{1-r;\psi}$ to both sides and making use of Lemma 2.6 and Proposition 2.8, we arrive at

$$\begin{aligned} I_{a^+}^{1-r;\psi} u_1(t) &= \frac{\Gamma(r)}{\Lambda_1} \sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi} f(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)) \\ &\quad + I_{a^+}^{1-p(1-q);\psi} f_1(t, u_1(t), u_2(t), \ldots, u_n(t)). \end{aligned}$$

From the fact that $1 - r + q > 1 - r$, Lemma 2.10 yields by taking the limit as $t \to a$ that

$$I_{a^+}^{1-r;\psi} u_1(a) = \frac{\Gamma(r)}{\Lambda_1} \sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi} f_1(\eta_i^1, u_1(\eta_i^1), u_2(\eta_i^1), \ldots, u_n(\eta_i^1)). \tag{3.5}$$

Replacing $t=\eta_i^1$ into (3.2), we obtain

$$\begin{aligned}u_1(\eta_i^1)=&\frac{(\psi(\eta_i^1)-\psi(a))^{r-1}}{\Lambda_1}\sum_{i=1}^{m_1}c_i^1 I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\\&+I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1)).\end{aligned}\tag{3.6}$$

Substitute $\sum_{i=1}^{m_1}c_i^1$ into both sides, we have

$$\begin{aligned}\sum_{i=1}^{m_1}c_i^1u_1(\eta_i^1)=&\sum_{i=1}^{m_1}c_i^1\frac{(\psi(\eta_i^1)-\psi(a))^{r-1}}{\Lambda_1}\sum_{i=1}^{m_1}c_i^1 I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\\&+\sum_{i=1}^{m_1}c_i^1I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\\\sum_{i=1}^{m_1}c_i^1u_1(\eta_i^1)=&\frac{\Gamma(r)}{\Lambda_1}\sum_{i=1}^{m_1}c_i^1I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1)).\end{aligned}$$

In view of (3.5) and (3.6), we have $I_{a^+}^{1-r;\psi}u_1(a)=\sum_{i=1}^{m_1}c_i^1u_1(\eta_i^1)$. Suppose $u_1\in C_{1-r;\psi}^r[a,b]$. Applying $D_{a^+}^{r;\psi}$ to both sides of (3.2), it follows Lemma 2.7 and Proposition 2.8 that

$$\begin{aligned}D_{a^+}^{r;\psi}u_1(t)=&D_{a^+}^{r;\psi}\frac{(\psi(t)-\psi(a))^{r-1}}{\Lambda_1}\sum_{i=1}^{m_1}c_i^1I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\\&+D_{a^+}^{r;\psi}I_{a^+}^{q;\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t)).\end{aligned}$$

Hence,

$$D_{a^+}^{r;\psi}u_1(t)=D_{a^+}^{p(1-q);\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t)).\tag{3.7}$$

Since $D_{a^+}^{r;\psi}u_1\in C_{1-r;\psi}[a,b]$, (3.7) implies that

$$D_{a^+}^{r;\psi}u(t)=\Big[\frac{1}{\psi'(t)}\frac{d}{dt}\Big]I_{a^+}^{1-p(1-q);\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t)).$$

From $f_1(t,u_1(t),u_2(t),\ldots,u_n(t))\in C_{1-r;\psi}[a,b]$ and Lemma 2.9 , we have

$$I_{a^+}^{1-p(1-q);\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t))\in C_{1-r;\psi}[a,b].\tag{3.8}$$

Based on (3.8) and Definition of $C_{r;\psi}^n[a,b]$, we have $I_{a^+}^{1-p(1-q)}f_1(t,u_1(t),u_2(t),\ldots,u_n(t))\in C_{1-r;\psi}^1[a,b]$. Using Lemma 2.10 and Theorem 2.11, we apply the operator $I_{a^+}^{p(1-q);\psi}$ to both sides of (3.7) to obtain

$$\begin{aligned}I_{a^+}^{p(1-q);\psi}D_{a^+}^{r;\psi}u(t)&=I_{a^+}^{p(1-q);\psi}D_{a^+}^{p(1-q);\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t)),\\I_{a^+}^{p(1-q);\psi}D_{a^+}^{r;\psi}u(t)&=f_1(t,u_1(t),u_2(t),\ldots,u_n(t)).\end{aligned}\tag{3.9}$$

By comparing (3.9) with (2.2), we determine $D_{a^+}^{q,p}u_1(t)=f_1(t,u_1(t),u_2(t),\ldots,u_n(t))$. This indicates that equation ((3.2) is valid. Define the operator $A:E_1\times E_2\times\ldots\times E_n\to E_1\times E_2\times\ldots\times E_n$ by

$$A=\begin{pmatrix}A_1(u_1,u_2,\ldots,u_n(t))\\A_2(u_1,u_2,\ldots,u_n(t))\\\vdots\\A_n(u_1,u_2,\ldots,u_n(t))\end{pmatrix}$$

with the norm

$$\|(u_1,u_2,\ldots,u_n)\|_{C_{1-r;\psi}[a,b]}= \|u_1\|_{C_{1-r;\psi}[a,b]}+\|u_2\|_{C_{1-r;\psi}[a,b]}\cdots+\|u_n\|_{C_{1-r;\psi}[a,b]}.$$

Then A has a fixed point if and only if problem (1.1) has a solution. Therefore, the proof is completed. $\square$

Next, we investigate the existence and uniqueness of solutions for problem (1.1) via Schaefer's fixed point theorem and Banach's fixed point theorem.

3.1. **Existence result via Schaefer's fixed point theorem.** The existence of solutions to system (1.1) is the subject of our following investigation, which is guaranteed by Schaefer's fixed point theorem.

Theorem 3.3. *Assume that there exist real constants $k_i^1,k_i^2,\ldots,k_i^n \geq 0$, $\forall x_i \in \mathbb{R}$, $i=1,2,\ldots,n$ such that*

$$\begin{aligned}
|f_1(t,x_1,x_2,\ldots,x_n)| &\leq k_0^1+k_1^1|x_1|+k_2^1|x_2|+\ldots+k_n^1|x_n|\\
|f_2(t,x_1,x_2,\ldots,x_n)| &\leq k_0^2+k_1^2|x_1|+k_2^2|x_2|+\ldots+k_n^2|x_n|\\
&\cdots\\
|f_n(t,x_1,x_2,\ldots,x_n)| &\leq k_0^n+k_1^n|x_1|+k_2^n|x_2|+\ldots+k_n^n|x_n|.
\end{aligned}$$

Further, it is assumed that

$$\begin{aligned}
M_1k_1^1+M_2k_1^2+\ldots+M_nk_1^n &< 1,\\
M_1k_2^1+M_2k_2^2+\ldots+M_nk_2^n &< 1,\\
&\cdots\\
M_1k_n^1+M_2k_n^2+\ldots+M_nk_n^n &< 1,
\end{aligned}$$

where $M_1,M_2,\ldots,M_n$ are give by

$$\begin{aligned}
M_1 &= \Big(\frac{1}{\Lambda_1}\sum_{i=1}^{m_1}|c_i^1|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)},\\
M_2 &= \Big(\frac{1}{\Lambda_2}\sum_{i=1}^{m_2}|c_i^2|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)},\\
&\cdots\\
M_n &= \Big(\frac{1}{\Lambda_2}\sum_{i=1}^{m_n}|c_i^n|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)}, \quad \textit{respectively.}
\end{aligned}$$

Then, system (1.1) has at least one solution on (a,b).

Proof. The proof is divided into four steps. Consider $A: E_1\times E_2\times\ldots\times E_n \to E_1\times E_2\times\ldots\times E_n$.

Step 1. Show that A is continuous which is sufficient to prove that $A_1,A_2,\ldots,A_n$ are continuous. We first consider A_1 is continuous. Let $\{\mathbf{u}_k\}=\{u_{1k},u_{2k},\ldots,u_{nk}\}$ and $\{\mathbf{u}\}=\{u_1,u_2,\ldots,u_n\}$

be sequences such that $\mathbf{u}_k \to \mathbf{u}$ in Ω, where $k \to \infty$. Then, for each $t \in (a,b)$,

$$
\begin{aligned}
&\Big|\Big(\psi(t)-\psi(a)\Big)^{1-r}\Big[(A_1\mathbf{u}_k)(t)-(A_1\mathbf{u})(t)\Big]\Big| \\
&\quad\le \Big|\frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1} c_i^1 I_{a^+}^{q;\psi}\Big(f_1(\eta_i^1,u_{1k}(\eta_i^1),u_{2k}(\eta_i^1),\ldots,u_{nk}(\eta_i^1)) \\
&\qquad - f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\Big)\Big| \\
&\qquad + \Big|\Big(\psi(t)-\psi(a)\Big)^{r-1} I_{a^+}^{q;\psi}\Big(f_1(t,u_{1k}(t),u_{2k}(t),\ldots,u_{nk}(t)-f_1(t,u_1(t),u_2(t),\ldots,u_n(t))\Big)\Big| \\
&\quad\le \frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|\Big|I_{a^+}^{q;\psi}\Big(\psi(t)-\psi(a)\Big)^{r-1}\Big| \\
&\qquad \times\Big|f_1(\eta_i^1,u_{1k}(\eta_i^1),u_{2k}(\eta_i^1),\ldots,u_{nk}(\eta_i^1)-f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\Big)\Big| \\
&\qquad + \Big|I_{a^+}^{q;\psi}\Big(\psi(t)-\psi(a)\Big)^{r-1}\Big|\Big|f_1(t,u_{1k}(t),u_{2k}(t),\ldots,u_{nk}(t)-f_1(t,u_1(t)),u_2(t),\ldots,u_n(t))\Big|.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\Big\|A_1\mathbf{u}_k - A_1\mathbf{u}\Big\|_{C_{1-r;\psi}[a,b]} \\
&\quad\le \Big|\Big(\frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|+1\Big)\Big(\frac{\Gamma(r)}{\Gamma(r+q)}(\psi(b)-\psi(a))^{q+r-1}\Big)\Big\|f_1u_k(\cdot)-f_1u(\cdot)\Big\|_{C_{1-r;\psi}[a,b]}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\Big\|A_2\mathbf{u}_k - A_2\mathbf{u}\Big\|_{C_{1-r;\psi}[a,b]} \\
&\quad\le \Big|\Big(\frac{1}{|\Lambda_2|}\sum_{i=1}^{m_2}|c_i^2|+1\Big)\Big(\frac{\Gamma(r)}{\Gamma(r+q)}(\psi(b)-\psi(a))^{q+r-1}\Big)\Big\|f_2u_k(\cdot)-f_2u(\cdot)\Big\|_{C_{1-r;\psi}[a,b]} \\
&\quad\vdots \\
&\Big\|A_n\mathbf{u}_k - A_n\mathbf{u}\Big\|_{C_{1-r;\psi}[a,b]} \\
&\quad\le \Big|\Big(\frac{1}{|\Lambda_n|}\sum_{i=1}^{m_n}|c_i^n|+1\Big)\Big(\frac{\Gamma(r)}{\Gamma(r+q)}(\psi(b)-\psi(a))^{q+r-1}\Big)\Big\|f_nu_k(\cdot)-f_nu(\cdot)\Big\|_{C_{1-r;\psi}[a,b]}.
\end{aligned}
$$

Since $f_1,f_2,\ldots,f_n$ are continuous, we have $\|A_1\mathbf{u}_k-A_1\mathbf{u}\|_{C_{1-r;\psi}[a,b]}\to 0$, $\|A_2\mathbf{u}_k-A_2\mathbf{u}\|_{C_{1-r;\psi}[a,b]}\to 0$, $\ldots$, $\|A_n\mathbf{u}_k-A_n\mathbf{u}\|_{C_{1-r;\psi}[a,b]}\to 0$ as $n\to\infty$. In this case, it is clear that operator A is continuous.

Step 2. Show that $A(\Omega)\subset\Omega$. Let $\Omega=\{\mathbf{u}:=(u_1,u_2,\ldots,u_n)\in E_1\times E_2\times\ldots\times E_n : \|\mathbf{u}\|_{C_{1-r;\psi}[a,b]} \le L_M\}$, where $L_M=L_1M_1+L_2M_2+\ldots+L_nM_n$ and the positive constants $L_1,L_2,\ldots,L_n$ are such that

$$
\begin{aligned}
|f_1(t,u_1(t),u_2(t),\ldots,u_n(t))| &< L_1 \\
|f_2(t,u_1(t),u_2(t),\ldots,u_n(t))| &< L_2 \\
&\cdots \\
|f_n(t,u_1(t),u_2(t),\ldots,u_n(t))| &< L_n,
\end{aligned}
$$

for all $(u_1, u_2, \ldots, u_n) \in \Omega$.

$$
\begin{aligned}
M_1 &= \Big(\frac{1}{\Lambda_1}\sum_{i=1}^{m_1}|c_i^1|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)}\\
M_2 &= \Big(\frac{1}{\Lambda_2}\sum_{i=1}^{m_2}|c_i^2|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)}\\
&\cdots\\
M_n &= \Big(\frac{1}{\Lambda_2}\sum_{i=1}^{m_n}|c_i^n|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)}.
\end{aligned}
$$

Let $\mathbf{u} \in \Omega$. We show that $A\mathbf{u} \in \Omega$, that is, $A_1\mathbf{u} \in \Omega$, $A_2\mathbf{u} \in \Omega, \ldots, A_n\mathbf{u} \in \Omega$. It suffices to show that $|(\psi(t)-\psi(a))^{1-r}(A_1\mathbf{u})(t)| \leq L_M$ for $t \in (a,b)$. Observe that

$$
\begin{aligned}
\Big|(\psi(t)-\psi(a))^{1-r}(A_1\mathbf{u})(t)\Big| &\leq \Big|\frac{1}{|\Lambda_1}\sum_{i=1}^{m_1}c_i^1 I_{a^+}^{q;\psi}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))|\\
&\quad + |(\psi(t)-\psi(a))^{1-r}I_{a^+}^{q;\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t))|\\
&\leq L_1\Big(\frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|+(\psi(b)-\psi(a))^{1-r}\Big)I_{a^+}^{q;\psi}(1).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\Big|(\psi(t)-\psi(a))^{1-r}(A_1\mathbf{u})(t)\Big| &\leq L_1\Big(\frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)}\\
\|A_1\mathbf{u}\|_{C_{1-r;\psi}[a,b]} &\leq L_1 M_1.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\|A_2\mathbf{u}\|_{C_{1-r;\psi}[a,b]} &\leq L_2\Big(\frac{1}{|\Lambda_2|}\sum_{i=1}^{m_2}|c_i^2|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)} \leq L_2 M_2,\\
&\cdots\\
\|A_n\mathbf{u}\|_{C_{1-r;\psi}[a,b]} &\leq L_n\Big(\frac{1}{|\Lambda_n|}\sum_{i=1}^{m_n}|c_i^n|+(\psi(b)-\psi(a))^{1-r}\Big)\frac{(\psi(b)-\psi(a))^q}{\Gamma(q+1)} \leq L_n M_n,
\end{aligned}
$$

$$
\|A\mathbf{u}\|_{C_{1-r;\psi}[a,b]} = \|A_1\mathbf{u}\|_{C_{1-r;\psi}[a,b]} + \|A_2\mathbf{u}\|_{C_{1-r;\psi}[a,b]} + \ldots + \|A_n\mathbf{u}\|_{C_{1-r;\psi}[a,b]} \leq L_M.
$$

Thus $A(\Omega) \subset \Omega$.

Step 3. Prove that $A(\Omega)$ is uniformly bounded and equicontinuous. From Step 2, we have $A(\Omega)=\{A\mathbf{u}:\mathbf{u}\in\Omega\}\subset\Omega$. Thus $\|A\mathbf{u}\|\leq L_M$, for all $\mathbf{u}\in\Omega$. Next, we show that A_1 is equicontionuous. Let $\tau_1,\tau_2\in(a,b)$ with $\tau_1<\tau_2$ and choose $\mathbf{u}\in\Omega$

$$
\begin{aligned}
&\Big|\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}(A_1\mathbf{u}(\tau_2))-\Big(\psi(\tau_1)-\psi(a)\Big)^{1-r}(A_1\mathbf{u}(\tau_1))\Big|\\
&\quad\leq\Big|\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}I^{q;\psi}_{a^+}f_1(\tau_2,u_1(\tau_2),u_2(\tau_2)\ldots,u_n(\tau_2))\\
&\qquad-\Big(\psi(\tau_1)-\psi(a)\Big)^{1-r}I^{q;\psi}_{a^+}f_1(\tau_1,u_1(\tau_1),u_2(\tau_1)\ldots,u_n(\tau_1))\Big|\\
&\quad\leq\Big|\frac{1}{\Gamma(q)}\int_a^{\tau_1}\psi'(s)\Big\{\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}\Big(\psi(\tau_2)-\psi(s)\Big)^{q-1}\\
&\qquad-\Big(\psi(\tau_1)-\psi(a)\Big)^{1-r}\Big(\psi(\tau_1)-\psi(s)\Big)^{q-1}\Big\}\times f_1(s,u_1(s),u_2(s),\ldots,u_n(s))ds\Big|\\
&\qquad+\Big|\frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}\psi'(s)\Big\{\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}\Big(\psi(\tau_2)-\psi(s)\Big)^{q-1}\Big\}\\
&\qquad\times f_1(s,u_1(s),u_2(s),\ldots,u_n(s))ds\Big|.
\end{aligned}
$$

Since the right hand side of above inequality tends to zero as $\tau_1\to\tau_2$, we have

$$
\begin{aligned}
&\Big|\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}(A_n\mathbf{u}(\tau_2))\Big(\psi(\tau_1)-\psi(a)\Big)^{1-r}(A_n\mathbf{u}(\tau_1))\Big|\\
&\quad\leq\Big|\frac{1}{\Gamma(q)}\int_a^{\tau_1}\psi'(s)\Big\{\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}\Big(\psi(\tau_2)-\psi(s)\Big)^{q-1}\\
&\qquad-\Big(\psi(\tau_1)-\psi(a)\Big)^{1-r}\Big(\psi(\tau_1)-\psi(s)\Big)^{q-1}\Big\}\times f_n(s,u_1(s),u_2(s),\ldots,u_n(s))ds\Big|\\
&\qquad+\Big|\frac{1}{\Gamma(q)}\int_{\tau_1}^{\tau_2}\psi'(s)\Big\{\Big(\psi(\tau_2)-\psi(a)\Big)^{1-r}\Big(\psi(\tau_2)-\psi(s)\Big)^{q-1}\Big\}\\
&\qquad\times f_n(s,u_1(s),u_2(s),\ldots,u_n(s))ds\Big|.
\end{aligned}
$$

The right-hand side of the inequality approaches zero as $\tau_1\to\tau_2$. Thus $A(\Omega)$ is equicontinuous.

Step 4. Show that $E=\{\mathbf{u}:=(u_1,u_2,\cdot,u_n)\in E_1\times E_2\times\cdot\times E_n:\mathbf{u}=\lambda A\mathbf{u},\ 0\leq\lambda\leq 1\}$ is bounded. Let $\mathbf{u}:=(u_1,u_2,\ldots,u_n)\in E$. Then $\mathbf{u}=\lambda A\mathbf{u}$. Given that $t\in(a,b)$, we have $u_1(t)=\lambda A_1(u_1,u_2,\cdot,u_n)$, $u_2(t)=\lambda A_2(u_1,u_2,\cdot,u_n),\cdots$, $u_n(t)=\lambda A_n(u_1,u_2,\cdot,u_n)$. Thus

$$
\begin{aligned}
\Big|\Big(\psi(t)-\psi(a)\Big)^{1-r}u_1(t)\Big|&\leq\Big|\frac{1}{\Lambda}\sum_{i=1}^{m_1}c_i^1I^{q;\psi}_{a^+}f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))\Big|\\
&\quad+\Big|I^{q;\psi}_{a^+}f_1(t,u_1(t),u_2(t),\ldots,u_n(t))\Big|\\
&\leq M_1\Big((\psi(b)-\psi(a))^{1-r}k_0^1+k_1^1\|u_1\|_{C_{1-r;\psi}[a,b]}\\
&\quad+k_2^1\|u_2\|_{C_{1-r;\psi}[a,b]}+\ldots+k_n^1\|u_n\|_{C_{1-r;\psi}[a,b]}\Big)
\end{aligned}
$$

and

$$
\|u_1\|_{C_{1-r;\psi}[a,b]}\leq M_1\Big(k_0^1+k_1^1\|u_1\|_{C_{1-r;\psi}[a,b]}+k_2^1\|u_2\|_{C_{1-r;\psi}[a,b]}+\ldots+k_n^1\|u_n\|_{C_{1-r;\psi}[a,b]}\Big),
$$

where $M_1 = \Big(\frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|+(\psi(b)-\psi(a)^{1-r})\Big)\dfrac{(\psi(t)-\psi(a))^q}{\Gamma(q+1)}$ and $0<\psi(b)-\psi(a)\leq 1$. In a similar way, we see that

$$\begin{aligned}
\|u_2\|_{C_{1-r;\psi}[a,b]} &\leq M_2\Big(k_0^2+k_1^2\|u_1\|_{C_{1-r;\psi}[a,b]}+k_2^2\|u_2\|_{C_{1-r;\psi}[a,b]}+\ldots+k_n^2\|u_n\|_{C_{1-r;\psi}[a,b]}\Big),\\
&\ldots\\
\|u_n\|_{C_{1-r;\psi}[a,b]} &\leq M_2\Big(k_0^2+k_1^n\|u_1\|_{C_{1-r;\psi}[a,b]}+k_2^n\|u_2\|_{C_{1-r;\psi}[a,b]}+\ldots+k_n^n\|u_n\|_{C_{1-r;\psi}[a,b]}\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\|u_1\|_{C_{1-r;\psi}[a,b]}+\|u_2\|_{C_{1-r;\psi}[a,b]}+\ldots+\|u_n\|_{C_{1-r;\psi}[a,b]}\\
&\quad\leq M_1k_0^1+M_2k_0^2+\ldots+M_nk_0^n\\
&\qquad+(M_1k_1^1+M_2k_1^2+\ldots+M_nk_1^n)\|u_1\|_{C_{1-r;\psi}[a,b]}\\
&\qquad+(M_1k_2^1+M_2k_2^2+\ldots+M_nk_2^n)\|u_2\|_{C_{1-r;\psi}[a,b]}\\
&\qquad\ldots\\
&\qquad+(M_1k_n^1+M_2k_n^2+\ldots+M_nk_n^n)\|u_n\|_{C_{1-r;\psi}[a,b]},
\end{aligned}$$

which implies that

$$\|(u_1,u_2,\ldots,u_n)\|_{C_{1-r;\psi}[a,b]}\leq \frac{M_1k_0^1+M_2k_0^2+\ldots+M_nk_0^n}{M_0},$$

for every $t\in(a,b)$, where $M_0=\min\{1-(M_1k_1^1+M_2k_1^2+\ldots+M_nk_1^n),\ 1-(M_1k_2^1+M_2k_2^2+\ldots+M_nk_2^n),\ldots,1-(M_1k_n^1+M_2k_n^2+\ldots+M_nk_n^n)\}$. This clearly shows that $E(A)$ is bounded. According to Schaefer's fixed point theorem, there must be at least one fixed point for A. This means that system (1.1) has at least one solution on (a,b). This completes the proof. □

3.2. **Existence and uniqueness results via Banach fixed point theorem.** In this section, we mainly investigate the existence and uniqueness of solutions based on Banach fixed point theorem.

Theorem 3.4. *Assume that $f_1,f_2,\ldots,f_n:(a,b)\times\mathbb{R}^n\to\mathbb{R}$ are continuous functions and there exist positive constants $l_i^1,l_i^2,\ldots,l_i^n\geq 0$, $i=0,1,2,\ldots,n$ such that, for all $t\in(a,b)$ and $u_1,u_2,\ldots,u_n$, $u_1^*,u_2^*,\ldots,u_n^*\in\mathbb{R}$,*

$$\begin{aligned}
|f_1(t,u_1,u_2,\ldots,u_n)-f_1(t,u_1^*,u_2^*,\ldots,u_n^*)| &\leq l_1^1|u_1-u_1^*|+l_2^1|u_2-u_2^*|+\ldots+l_n^1|u_n-u_n^*|,\\
|f_2(t,u_1,u_2,\ldots,u_n)-f_2(t,u_1^*,u_2^*,\ldots,u_n^*)| &\leq l_1^2|u_1-u_1^*|+l_2^2|u_2-u_2^*|+\ldots+l_n^2|u_n-u_n^*|,\\
&\ldots\\
|f_n(t,u_1,u_2,\ldots,u_n)-f_n(t,u_1^*,u_2^*,\ldots,u_n^*)| &\leq l_1^n|u_1-u_1^*|+l_2^n|u_2-u_2^*|+\ldots+l_n^n|u_n-u_n^*|.
\end{aligned}$$

Then system (1.1) has a unique solution on (a,b), if $N_1(l_1^1+l_2^1+\ldots+l_n^1)+N_2(l_1^2+l_2^2+\ldots+l_n^2)+\ldots+N_n(l_1^n+l_2^n+\ldots+l_n^n)<1$, where $N_1,N_2,\ldots,N_n$ are defined by

$$
\begin{aligned}
N_1 &= \Big(\frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|+1\Big)\Big(\frac{\Gamma(r)}{\Gamma(r+q)}(\psi(b)-\psi(a)^{q+r-1})\Big),\\
N_2 &= \Big(\frac{1}{|\Lambda_2|}\sum_{i=1}^{m_2}|c_i^2|+1\Big)\Big(\frac{\Gamma(r)}{\Gamma(r+q)}(\psi(b)-\psi(a)^{q+r-1})\Big),\\
&\ldots\\
N_n &= \Big(\frac{1}{|\Lambda_n|}\sum_{i=1}^{m_n}|c_i^n|+1\Big)\Big(\frac{\Gamma(r)}{\Gamma(r+q)}(\psi(b)-\psi(a)^{q+r-1})\Big).
\end{aligned}
$$

Proof. For $\mathbf{u}:=(u_1,u_2,\ldots,u_n)$, $\mathbf{u}^*:=(u_1^*,u_2^*,\ldots,u_n^*)\in E_1\times E_2\times\ldots\times E_n$ and for any $t\in(a,b)$, we have

$$
\begin{aligned}
&\Big|\Big(\psi(t)-\psi(a)\Big)^{1-r}\Big\{A_1(u_1,u_2,\ldots,u_n)(t)-A_1(u_1^*,u_2^*,\ldots,u_n^*)\Big\}\Big|\\
&\quad\le \frac{1}{|\Lambda_1|}\sum_{i=1}^{m_1}|c_i^1|I_{a^+}^{q;\psi}\Big|f_1(\eta_i^1,u_1(\eta_i^1),u_2(\eta_i^1),\ldots,u_n(\eta_i^1))-f_1(\eta_i^1,u_1^*(\eta_i^1),u_2^*(\eta_i^1),\ldots,u_n^*(\eta_i^1))\Big|\\
&\qquad+\Big(\psi(t)-\psi(a)\Big)^{1-r}I_{0^+}^{q;\psi}f_1(t,u_1(t),u_2(t),\ldots,u_n(t))-f_1(t,u_1^*(t),u_2^*(t),\ldots,u_n^*(t))\Big|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big\|A_1(u_1,u_2,\ldots,u_n)-A_1(u_1^*,u_2^*,\ldots,u_n^*)\Big\|_{C_{1-r;\psi}[a,b]}\\
&\quad\le N_1\Big(l_1^1+l_2^1+\ldots+l_n^1\Big)\Big(\|u_1-u_1^*\|_{C_{1-r;\psi}[a,b]}+\|u_2-u_2^*\|_{C_{1-r;\psi}[a,b]}+\ldots\\
&\quad+\|u_n-u_n^*\|_{C_{1-r;\psi}[a,b]}\Big).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\Big\|A_2(u_1,u_2,\ldots,u_n)-A_2(u_1^*,u_2^*,\ldots,u_n^*)\Big\|_{C_{1-r;\psi}[a,b]}\\
&\quad\le N_2\Big(l_1^2+l_2^2+\ldots+l_n^2\Big)\Big(\|u_1-u_1^*\|_{C_{1-r;\psi}[a,b]}+\|u_2-u_2^*\|_{C_{1-r;\psi}[a,b]}+\ldots+\|u_n-u_n^*\|_{C_{1-r;\psi}[a,b]}\Big).\\
&\qquad\vdots\\
&\Big\|A_n(u_1,u_2,\ldots,u_n)-A_n(u_1^*,u_2^*,\ldots,u_n^*)\Big\|_{C_{1-r;\psi}[a,b]}\\
&\quad\le N_n\Big(l_1^n+l_2^n+\ldots+l_n^n\Big)\Big(\|u_1-u_1^*\|_{C_{1-r;\psi}[a,b]}+\|u_2-u_2^*\|_{C_{1-r;\psi}[a,b]}+\ldots+\|u_n-u_n^*\|_{C_{1-r;\psi}[a,b]}\Big).
\end{aligned}
$$

By combining the inequalities above, we obtain

$$\begin{aligned}
&\left\|A_1(u_1,u_2,\ldots,u_n)-A_1(u_1^*,u_2^*,\ldots,u_n^*)\right\|_{C_{1-r;\psi}[a,b]}+\left\|A_2(u_1,u_2,\ldots,u_n)-A_2(u_1^*,u_2^*,\ldots,u_n^*)\right\|_{C_{1-r;\psi}[a,b]}\\
&+\ldots+\left\|A_n(u_1,u_2,\ldots,u_n)-A_n(u_1^*,u_2^*,\ldots,u_n^*)\right\|_{C_{1-r;\psi}[a,b]}\\
&\quad\le\Big\{N_1(l_1^1+l_2^1+\ldots+l_n^1)+N_2(l_1^1+l_2^1+\ldots+l_n^1)+\ldots+N_n(l_1^1+l_2^1+\ldots+l_n^1)\Big\}\times\\
&\qquad\Big\{\|u_1-u_1^*\|_{C_{1-r;\psi}[a,b]}+\|u_2-u_2^*\|_{C_{1-r;\psi}[a,b]}+\ldots+\|u_n-u_n^*\|_{C_{1-r;\psi}[a,b]}\Big\}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|A\mathbf{u}-A\mathbf{u}^*\right\|_{C_{1-r;\psi}[a,b]}\le&\Big\{N_1(l_1^1+l_2^1+\ldots+l_n^1)+N_2(l_1^2+l_2^2+\ldots+l_n^2)+\ldots\\
&+N_n(l_1^n+l_2^n+\ldots+l_n^n)\Big\}\left\|\mathbf{u}-\mathbf{u}^*\right\|_{C_{1-r;\psi}(a,b)}.
\end{aligned}$$

We have that A is a contraction. According to the Banach fixed point theorem, A has a unique fixed point. ☐

Next, we give an example to support our results.

Example 3.5. Consider the following fractional boundary value problems

$$\begin{aligned}
D_{2^+}^{\frac{2}{7},\frac{1}{5};\sqrt[3]{t}}u_1(t)&=\frac{\sin^2(t)}{19+9^t}\cdot\frac{|u_1(t)|}{1+|u_1(t)|}+\frac{\cos^2(t)}{1+7^t}\cdot\frac{|u_2(t)|}{1+|u_2(t)|}+\frac{(t^2+1)}{200}|u_3(t)|+1\\
D_{2^+}^{\frac{2}{7},\frac{1}{5};\sqrt[3]{t}}u_2(t)&=\frac{|\cos(t)|}{100t}u_1(t)+\frac{|\sin(t)|}{5+5^t}u_2(t)+\frac{e^{-t}}{2+t^3}\cdot\frac{|u_3(t)|}{1+|u_3(t)|}+\sqrt{2}\\
D_{2^+}^{\frac{2}{7},\frac{1}{5};\sqrt[3]{t}}u_3(t)&=\frac{t^2+1}{150}\cdot\frac{|u_1(t)|}{1+|u_1(t)|}+\frac{1}{19+9^t}u_2(t)+\frac{\cos^2(t)}{44+4^t}\cdot\frac{|u_3(t)|}{1+|u_3(t)|}+\sqrt{3},\quad t\in(2,3),
\end{aligned}$$

with the conditions

$$\begin{aligned}
I_{2^+}^{\frac{4}{7};\sqrt[3]{t}}u_1(2)&=u_1(\frac{5}{2})+2u_1(\frac{7}{3})\\
I_{2^+}^{\frac{4}{7};\sqrt[3]{t}}u_2(2)&=\frac{1}{2}u_2(\frac{9}{4})+\frac{1}{3}u_2(\frac{8}{3})+\frac{1}{4}u_2(\frac{7}{3})\\
I_{2^+}^{\frac{4}{7};\sqrt[3]{t}}u_3(2)&=u_3(\frac{7}{3})-2u_3(\frac{9}{4}),
\end{aligned}$$

where $q=2/7,\ p=1/5,\ r=3/7,\ c_1^1=1,\ c_2^1=2,\ c_1^2=1/2,\ c_2^2=1/3,\ c_3^2=1/4,\ c_1^3=1,\ c_2^3=-2,\ \eta_1^1=5/2,\ \eta_2^1=7/3,\ \eta_1^2=9/4,\ \eta_2^2=8/3,\ \eta_3^2=7/3,\ \eta_1^3=7/3,\ \eta_2^3=9/4$, and $\psi(t)=\sqrt[3]{t}$. With the given data, we find that $\Lambda_1=-11.708,\ \Lambda_2=-2.9576,\ \Lambda_3=8.3728$,

$$\begin{aligned}
\Big|f_1(t,u_1(t),u_2(t),u_3(t))-f_1(t,v_1(t),v_2(t),v_3(t))\Big|&\le\frac{1}{100}|u_1-v_1|+\frac{1}{50}|u_2-v_2|+\frac{1}{20}|u_3-v_3|,\\
\Big|f_2(t,u_1(t),u_2(t),u_3(t))-f_2(t,v_1(t),v_2(t),v_3(t))\Big|&\le\frac{1}{200}|u_1-v_1|+\frac{1}{30}|u_2-v_2|+\frac{1}{10e^2}|u_3-v_3|\\
\Big|f_3(t,u_1(t),u_2(t),u_3(t))-f_3(t,v_1(t),v_2(t),v_3(t))\Big|&\le\frac{1}{15}|u_1-v_1|+\frac{1}{100}|u_2-v_2|+\frac{1}{60}|u_3-v_3|,
\end{aligned}$$

where $N_1 = 3.2812$, $N_2 = 3.5687$, $N_3 = 0.9359$ and $N_1(l_1^1 + l_2^1 + l_3^1) + N_2(l_1^2 + l_2^2 + l_3^2) + N_n(l_1^n + l_2^n + l_3^n) = 0.5348 < 1$. Thus, the requirements of Theorem 3.4 are met, and its conclusion is applicable to Example 3.5.

4. Conclusions

In this paper, we, based on the Schaefer and Banach fixed point theorems, proved that there exists a solution to a coupled system of the $\psi-$Hilfer fractional differential equation under uncoupled non-local multi point conditions. Moreover, we also provided an example to demonstrate our results.

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