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# A VISCOSITY ALTERNATING RESOLVENT ALGORITHM WITH TWO MAXIMAL MONOTONE OPERATORS

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**Abstract.** In the paper, we study a viscosity alternating resolvent algorithm with over relaxed factors for finding a common zero point of two maximal monotone operators. We give the strong convergence of the algorithm under some mild conditions and more error criteria. This algorithm can be seen as a generalization of the results in some current literature.

**Keywords.** Alternating resolvent algorithm; Monotone operators; Over relaxed factor; Proximal point algorithm; Viscosity.

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## 1. Introduction

Convex feasibility problems, which consist of find a common point in finite convex sets, were studied extensively in Hilbert spaces due to its wide applications in various problem, such as signal processing, image recovery and so on; see, e.g., [16, 17, 18]. Recently, various resolvent-based iterative algorithms were introduced and studied; see, e.g., [8, 12, 13, 14] and the references therein.

In this paper, we consider the following convex feasibility problem, which consists of finding $x \in H$ such that

$$x \in A^{-1}(0) \cap B^{-1}(0), \tag{1.1}$$

where $A : D(A) \subset H \to H$, and $B : D(B) \subset H \to H$ are maximal monotone operators.

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To solve problem (1.1), the alternating resolvent algorithm (abbreviated as ARA) was proposed: For any initial guess $x_0 \in H$,

$$x_{2n+1} = J^A_{\beta_n}(x_{2n} + e_n), \quad n = 0, 1, 2, \dots,$$
$$x_{2n} = J^B_{\mu_n}(x_{2n-1} + e'_n), \quad n = 1, 2, 3, \dots,$$

where $J^A_{\beta_n} = (I + \beta_n A)^{-1}$ and $J^B_{\mu_n} = (I + \mu_n B)^{-1}$ are resolvents of $A$ and $B$, respectively, with $\beta_n,\ \mu_n > 0$, and $\{e_n\}$ and $\{e'_n\}$ are error sequences.

The ARA is an extension of the celebrated proximal point algorithm (abbreviated as PPA), which was used to solve problem (1.1) with only one maximal monotone operator. However, PPA generally only has weak convergence [10] in the framework of infinite dimensional spaces. In order to improve the strong convergence (the convergence in norm) of PPA, researchers proposed several modifications; see, e.g., [3, 4, 5, 7, 11, 15, 20, 22, 23, 25]. One modification is the contraction proximal point algorithm (CPPA), which was introduced by Xu [23],

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J^A_{\beta_n}(x_n) + e_n,\ n \geq 0, \tag{1.2}$$

where $u,\ x_0 \in H$, $\alpha_n \in (0,1),\ \beta_n \in (0,\infty)$, and $\{e_n\}$ is an error sequence. He proved the strong convergence of algorithm (1.2) under the following error criterion

$$\sum_{n=0}^{\infty} \|e_n\| < \infty. \tag{1.3}$$

In 2010, Boikanyo and Morosanu [3] generalized the results and further discussed the strong convergence of algorithm (1.2) under the error criterion

$$\|e_n\|/\alpha_n \to 0. \tag{1.4}$$

In 2008, Yao and Noor [25] generated PPA by the rule

$$x_{n+1} = \alpha_n u + \delta_n x_n + \gamma_n J^A_{\beta_n} x_n + e_n, \quad n \geq 0, \tag{1.5}$$

where $u,\ x_0 \in H$, $\alpha_n,\ \delta_n,\ \gamma_n \in (0,1),\ \beta_n \in (0,\infty)$, $\alpha_n + \delta_n + \gamma_n = 1$, and $\{e_n\}$ is an error sequence. They proved the strong convergence of algorithm (1.5) under error criterion (1.3).

In 2017, Cui and Ceng [7] extended (1.5) as

$$x_{n+1} = \alpha_n f(x_n) + \delta_n x_n + \gamma_n J^A_{\beta_n} x_n + e_n, \quad n \geq 0, \tag{1.6}$$

where $x_0 \in H$, $\alpha_n \in (0,1),\ \delta_n \in (-1,1),\ \gamma_n \in (0,2),\ \beta_n \in (0,\infty)$, $\alpha_n + \delta_n + \gamma_n = 1$, $f : H \to H$ is a $q$-contraction for some $q \in [0,1)$ and $\{e_n\}$ is an error sequence. They proved the strong convergence of algorithm (1.6) under error criterion (1.3) and (1.4).

Similar to PPA, ARA is also weakly convergent. Inspired by the modifications of PPA, researchers [1, 2, 6, 19] modified the ARA to make it also have strong convergence.

In 2012, Boikanyo and Morosanu [3] modified ARA in the following form

$$x_{2n+1} = \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n} x_{2n} + e_n, \quad n = 0, 1, \dots,$$
$$x_{2n} = \lambda_n u + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n} x_{2n-1} + e'_n, \quad n = 1, 2, \dots,$$

where $u,\ x_0 \in H,\ \alpha_n, \delta_n, \gamma_n, \lambda_n, \rho_n, \sigma_n \in (0,1),\ \alpha_n + \delta_n + \gamma_n = 1,\ \lambda_n + \rho_n + \sigma_n = 1\ ,\ \beta_n,\ \mu_n > 0$ and $\{e_n\}$, $\{e'_n\}$ are error sequences that satisfy any of the following conditions:

$$\sum_{n=0}^{\infty} \|e_n\| < \infty,\ \sum_{n=1}^{\infty} \|e'_n\| < \infty \tag{1.7}$$

$$\text{and } \|e_n\|/\alpha_n \to 0,\ \|e'_n\|/\lambda_n \to 0. \tag{1.8}$$

They also gave the strong convergence of the algorithm. This algorithm unifies the results in [5, 22, 25].

In 2013, another modification was proposed by Boikanyo and Morosanu in [6]. They modified the ARA into the following form

$$\begin{aligned} x_{2n+1} &= J^A_{\beta_n}(\alpha_n u + (1-\alpha_n)x_{2n} + e_n), \quad n \geq 0, \\ x_{2n} &= J^B_{\mu_n}(\lambda_n u + (1-\lambda_n)x_{2n-1} + e'_n), \quad n \geq 1, \end{aligned}$$

where $u,\ x_0 \in H,\ \alpha_n,\ \lambda_n \in (0,1),\ \beta_n,\ \mu_n \in (0,\infty)$, $\{e_n\}$ and $\{e'_n\}$ are error sequences that satisfy (1.7) and (1.8). They proved the strong convergence of the sequences generated by the above algorithm.

In 2015, Wang and Xu [19] modified the ARA into the following form

$$\begin{aligned} x_{2n+1} &= \alpha_n u + \delta_n x_{2n} + \gamma_n J^A_{\beta_n}(x_{2n} + e_n), \quad n = 0,1,\ldots, \\ x_{2n} &= \lambda_n u + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n}(x_{2n-1} + e'_n), \quad n = 1,2,\ldots, \end{aligned} \tag{1.9}$$

where $u,\ x_0 \in H,\ \alpha_n, \delta_n, \gamma_n, \lambda_n, \rho_n, \sigma_n \in (0,1),\ \alpha_n + \delta_n + \gamma_n = 1,\ \lambda_n + \rho_n + \sigma_n = 1$ and $\beta_n,\ \mu_n > 0$, and $\{e_n\}$ and $\{e'_n\}$ are error sequences which satisfy the following error criteria

$$\begin{aligned} \|e_n\| &\leq \eta_n \|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\| \text{with} \sum_{n=0}^{\infty} {\eta_n}^2 < \infty, \\ \|e'_n\| &\leq \eta'_n \|J^B_{\mu_n}(x_{2n-1} + e'_n) - x_{2n-1}\| \text{with} \sum_{n=0}^{\infty} \eta'^2_n < \infty \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} \|e_n\| &\leq \eta_n \|J^A_{\beta_n}(x_{2n} + e_n) - x_{2n}\|, \quad \lim_{n\to\infty} \frac{\eta_n^2}{\alpha_n} = 0, \\ \|e'_n\| &\leq \eta'_n \|J^B_{\mu_n}(x_{2n-1} + e'_n) - x_{2n-1}\|, \lim_{n\to\infty} \frac{(\eta'_n)^2}{\lambda_n} = 0. \end{aligned} \tag{1.11}$$

They proved the strong convergence of algorithm (1.9) under error criteria (1.10) and (1.11).

In 2019, Boikanyo and Makgoeng [1] modified ARA into the following form

$$\begin{aligned} x_{2n+1} &= \alpha_n f(x_{2n}) + \gamma_n x_{2n} + \delta_n J^A_{\beta_n}(x_{2n} + e_n), \quad n = 0,1,..., \\ x_{2n} &= \lambda_n f(x_{2n-1}) + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n}(x_{2n-1} + e'_n), \quad n = 1,2,..., \end{aligned}$$

where $f : H \to H$ is a $q$-contraction for some $q \in [0,1)$, $x_0 \in H$, $\alpha_n, \lambda_n \in (0,1), \gamma_n, \rho_n \in (-1,1)$, $\delta_n, \sigma_n \in (0,2)$,and $\alpha_n + \gamma_n + \delta_n = 1,\ \lambda_n + \rho_n + \sigma_n = 1$, $\{e_n\}$ and $\{e'_n\}$ are error sequences. They proved that this algorithm is strongly convergent under the error criterion (1.10).

In this paper, inspired by the results of Boikanyo and Morosanu [2], we consider a viscosity of ARA with over relaxed factors. The algorithm that we study in this paper is stated as below:

$$x_{2n+1} = \alpha_n f(x_{2n}) + \delta_n x_{2n} + \gamma_n J^A_{\beta_n} x_{2n} + e_n, \quad n = 0, 1, ..., \tag{1.12}$$

$$x_{2n} = \lambda_n f(x_{2n-1}) + \rho_n x_{2n-1} + \sigma_n J^B_{\mu_n} x_{2n-1} + e'_n, \quad n = 1, 2, ..., \tag{1.13}$$

where $f : H \to H$ is a $q$-contraction for some $q \in [0,1)$, $\alpha_n,\ \lambda_n \in (0,1)$, $\delta_n,\ \rho_n \in (-2,1)$, $\gamma_n,\ \sigma_n \in (0,2)$ satisfying $\alpha_n + \delta_n + \gamma_n = 1,\ \lambda_n + \rho_n + \sigma_n = 1$, and $\{e_n\}$ and $\{e'_n\}$ are error sequences. We will give the strong convergence of the algorithm under more error criteria and some mild conditions.

## 2. Preliminaries

Throughout this paper, $H$ is borrowed to denote a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. The symbol $x_n \rightharpoonup x$ presents that $\{x_n\}$ weakly converges to $x$ in $H$ and $x_n \to x$ denotes that $\{x_n\}$ strongly converges to $x$ in $H$. One uses $C$ to stand for a nonempty, convex, and closed subset of $H$, and uses $P_C$ to denote the nearest point projection onto $C$ from $H$. That is, for any $x$ in $H$, its projection point satisfies:

$$\|x - P_C\| = \min_{y \in C} \|x - y\|.$$

Recall a mapping $A$ is called monotone if

$$\langle x - x', y - y' \rangle \geq 0, \quad \forall (x,y), (x',y') \in G(A),$$

where $G(A) = \{(x,y) \in H \times H : x \in D(A), y \in Ax\}$. We say an operator $A$ is said to be maximal monotone if, in addition to being monotone, its graph is not properly contained in the graph. Note that if $A$ is maximal monotone, then $A^{-1}$ is maximal monotone. One says that a mapping $T$ is said to be nonexpansive if, for all $x$ and $y$ in $H$, $\|Tx - Ty\| \leq \|x - y\|$; and $T$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2. \tag{2.1}$$

Let $x \in H, y \in C$. Then $y = P_C x$ if and only if $\langle y - x, y - z \rangle \leq 0$ for all $z \in C$.

Next, we present some lemmas that are helpful in the subsequent analysis.

**Lemma 2.1.** *Let$x, y \in H$, and $\alpha \in R$. Then*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$,
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$.

**Lemma 2.2.** [24] *Let $A$ be a maximal monotone operator in $H$. Then $\|x - J^A_\beta x\| \leq 2\|x - J^A_{\beta'} x\|$, for all $0 < \beta \leq \beta'$ and for all $x \in H$.*

**Lemma 2.3.** [9] *Let $C$ be a nonempty, convex, and closed subset of $H$. Let $T : C \to H$ be an nonexpansive mapping and $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in $C$ such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, i.e., $x \in Fix(T)$.*

**Lemma 2.4.** [23] *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n b_n + c_n,\ n \geq 0,$$

*where $\{\alpha_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy the conditions:*

(i) $\alpha_n \in (0,1)$, with $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$;

(iii) $c_n \geq 0$, *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} c_n < \infty$.

*Then* $\lim_{n\to\infty} s_n = 0$.

## 3. Main Results

In this section, we prove the convergence of the sequence generated by (1.12) and (1.13) under more error criteria. We assume that problem (1.1) is consistent, and use $S$ to denote its solution set. We begin by proving the strong convergence of the algorithm which associated with the following exact iterative process

$$v_{2n+1} = \alpha_n f(v_{2n}) + \delta_n v_{2n} + \gamma_n J^A_{\beta_n} v_{2n}, \quad n = 0,1,..., \tag{3.1}$$

$$v_{2n} = \lambda_n f(v_{2n-1}) + \rho_n v_{2n-1} + \sigma_n J^B_{\mu_n} v_{2n-1}, \quad n = 1,2,..., \tag{3.2}$$

where $f : H \to H$ is a $q$-contraction for some $q \in [0,1)$, $\alpha_n,\ \lambda_n \in (0,1)$, $\delta_n,\ \rho_n \in (-2,1)$, $\gamma_n,\ \sigma_n \in (0,2)$ satisfying $\alpha_n + \delta_n + \gamma_n = 1,\ \lambda_n + \rho_n + \sigma_n = 1$.

**Theorem 3.1.** *Let $A : D(A) \subset H \to H$ and $B : D(B) \subset H \to H$ be maximal monotone operators with $S = A^{-1}(0) \cap B^{-1}(0)$. If $\{v_n\}$ is the sequence generated by (3.1) and (3.2). Then $\{v_n\}$ converges strongly to $P_S f(z)$, which is also the unique solution of the variational inequality:*

$$z \in S,\ \langle (I - f)z, v - z\rangle \geq 0, \quad v \in S,$$

*provided that the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \lambda_n = 0,\ \sum_{n=0}^{\infty} \lambda_n = \infty$;

(ii) $\beta_n \geq \beta > 0,\ \mu_n \geq \mu > 0$, *for all* $n > 0$;

(iii) $\limsup_{n\to\infty} \gamma_n < 2$ *and* $\limsup_{n\to\infty} \sigma_n < 2$;

(iv) $\lim_{n\to\infty} \frac{\alpha_n+\lambda_n}{\gamma_n} = 0$ *and* $\lim_{n\to\infty} \frac{\alpha_n+\lambda_n}{\sigma_n} = 0$.

*Proof.* Let $z = P_S f(z)$ and denote $C_n = \frac{\gamma_n}{1-\alpha_n}$, $C'_n = \frac{\sigma_n}{1-\lambda_n}$, $y_n = (1-C_n)v_{2n} + C_n J^A_{\beta_n} v_{2n}$, and $y'_n = (1-C'_n)v_{2n-1} + C'_n J^B_{\mu_n} v_{2n-1}$. It follows that

$$v_{2n+1} = \alpha_n f(v_{2n}) + (1-\alpha_n)y_n,$$
$$v_{2n} = \lambda_n f(v_{2n-1}) + (1-\lambda_n)y'_n.$$

In order to prove the strong convergence, we next divide it into the following steps.

*Step 1*. Prove $\{v_n\}$ is bounded.

By (2.1), we have

$$\left\|J^A_{\beta_n} v_{2n} - z\right\|^2 \leq \|v_{2n} - z\|^2 - \left\|v_{2n} - J^A_{\beta_n} v_{2n}\right\|^2. \tag{3.3}$$

From Lemma 2.1 (ii), we have

$$\begin{aligned}\| y_n - z\|^2 =& (1-C_n)\|v_{2n} - z\|^2 + C_n\|J^A_{\beta_n} v_{2n} - z\|^2 \\ & - C_n(1-C_n)\|J^A_{\beta_n} v_{2n} - v_{2n}\|^2.\end{aligned} \tag{3.4}$$

Substituting (3.3) into (3.4) yields

$$\|y_n - z\|^2 \leq \|v_{2n} - z\|^2 - C_n(2-C_n)\|J^A_{\beta_n} v_{2n} - v_{2n}\|^2. \tag{3.5}$$

From the definition of $C_n$ and $C_n'$ and condition (iii), we deduce that

$$\limsup_{n\to\infty} C_n < 2,\ \limsup_{n\to\infty} C_n' < 2. \tag{3.6}$$

Hence, $\|y_n - z\|^2 \leq \|v_{2n} - z\|^2$. In particular, we have $\|y_n - z\| \leq \|v_{2n} - z\|$. In view of (3.1), we find that

$$\begin{aligned}\|v_{2n+1} - z\| \leq& \alpha_n \|f(v_{2n}) - f(z)\| + (1-\alpha_n)\|y_n - z\| + \alpha_n \|f(z) - z\| \\ \leq& (1-\alpha_n(1-q))\|v_{2n} - z\| + \alpha_n \|f(z) - z\|.\end{aligned} \tag{3.7}$$

Take a similar approach, we have

$$\|v_{2n} - z\| \leq (1-\lambda_n(1-q))\|v_{2n-1} - z\| + \lambda_n \|f(z) - z\|. \tag{3.8}$$

Substitute (3.8) into (3.7) yields

$$\begin{aligned}\|v_{2n+1} - z\| \leq& (1-\alpha_n(1-q))[(1-\lambda_n(1-q))\|v_{2n-1} - z\| \\ &+ \lambda_n \|f(z) - z\|] + \alpha_n \|f(z) - z\| \\ \leq& (1-\alpha_n(1-q))(1-\lambda_n(1-q))\|v_{2n-1} - z\| + (1-\alpha_n(1-q))\lambda_n \|f(z) - z\| \\ &+ \alpha_n \|f(z) - z\| \\ =& (1-\alpha_n(1-q))(1-\lambda_n(1-q))\|v_{2n-1} - z\| \\ &+ [\,1 - (1-\alpha_n(1-q))(1-\lambda_n(1-q))\,]\frac{\|f(z) - z\|}{1-q} \\ \leq& \max\left\{\|v_1 - z\|,\ \frac{\|f(z) - z\|}{1-q}\right\}.\end{aligned} \tag{3.9}$$

Thus, by induction, we obtain that $\{v_{2n+1}\}$ is bounded. By inequality (3.8), we see that $\{v_{2n}\}$ is bounded too. Theus $\{v_n\}$ is bounded.

*Step 2.* Prove the following inequality

$$\|v_{2n+1} - z\|^2 \leq (1-\theta_n)\|v_{2n-1} - z\|^2 + \theta_n b_n, \tag{3.10}$$

where

$$\theta_n = \alpha_n(1-q) + \lambda_n(1-q) - \alpha_n\lambda_n(1-q)^2$$

and

$$\begin{aligned}b_n =& -\frac{\gamma_n(2-C_n)}{\theta_n}\,\|\,J_{\beta_n}^A v_{2n} - v_{2n}\,\|^2 - \frac{\sigma_n(2-C_n')}{\theta_n}(1-\alpha_n(1-q))\,\|\,J_{\mu_n}^B v_{2n-1} - v_{2n-1}\,\|^2 \\ &+ 2\frac{\alpha_n}{\theta_n}\langle f(z) - z, v_{2n+1} - z\rangle + 2\frac{\lambda_n}{\theta_n}(1-\alpha_n(1-q))\langle f(z) - z, v_{2n} - z\rangle.\end{aligned}$$

It follows from (3.1), (3.5) and Lemma 2.1 (i) that

$$\begin{aligned}\|v_{2n+1} - z\|^2 \leq& \|\alpha_n(f(v_{2n}) - f(z)) + (1-\alpha_n)(y_n - z)\|^2 + 2\alpha_n \langle f(z) - z, v_{2n+1} - z\rangle \\ \leq& \alpha_n \|f(v_{2n}) - f(z)\|^2 + (1-\alpha_n)\|y_n - z\|^2 + 2\alpha_n \langle f(z) - z, v_{2n+1} - z\rangle \\ \leq& (1-\alpha_n(1-q))\|v_{2n} - z\|^2 + 2\alpha_n \langle f(z) - z, v_{2n+1} - z\rangle \\ &- \gamma_n(2-C_n)\|J_{\beta_n}^A v_{2n} - v_{2n}\|^2.\end{aligned} \tag{3.11}$$

Taking a similar approach, one also has

$$\begin{aligned}\|v_{2n}-z\|^2 \leq (1-\lambda_n(1-q))\|v_{2n-1}-z\|^2+2\lambda_n\langle f(z)-z, v_{2n}-z\rangle \\ -\sigma_n(2-C_n')\|J^B_{\mu_n}v_{2n-1}-v_{2n-1}\|^2.\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11), one can deduce that

$$\begin{aligned}&\|v_{2n+1}-z\|^2\\ \leq&(1-\alpha_n(1-q))[(1-\lambda_n(1-q))\|v_{2n-1}-z\|^2+2\lambda_n\langle f(z)-z,v_{2n}-z\rangle\\ &-\sigma_n(2-C_n')\|J^B_{\mu_n}v_{2n-1}-v_{2n-1}\|^2]\\ &+2\alpha_n\langle f(z)-z,v_{2n+1}-z\rangle-\gamma_n(2-C_n)\|J^A_{\beta_n}v_{2n}-v_{2n}\|^2\\ =&(1-\theta_n)\|v_{2n-1}-z\|^2+\theta_n[-\frac{\gamma_n(2-C_n)}{\theta_n}\parallel J^A_{\beta_n}v_{2n}-v_{2n}\parallel^2\\ &-\frac{\sigma_n(2-C_n')}{\theta_n}(1-\alpha_n(1-q))\parallel J^B_{\mu_n}v_{2n-1}-v_{2n-1}\parallel^2\\ &+2\frac{\alpha_n}{\theta_n}\langle f(z)-z,v_{2n+1}-z\rangle+2\frac{\lambda_n}{\theta_n}(1-\alpha_n(1-q))\langle f(z)-z,v_{2n}-z\rangle].\end{aligned} \tag{3.13}$$

*Step 3.* Prove that $\{b_n\}$ is a bounded sequence in $H$. In particular, it satisfies

$$-\delta \leq \limsup_{n\to\infty} b_n < +\infty.$$

In view of the definition of $b_n$, the definition of $\theta_n$, (3.6), and the boundedness of $\{v_n\}$, we have $\frac{\alpha_n}{\theta_n}, \frac{\lambda_n}{\theta_n} \in (0, \frac{1}{1-q})$, and

$$\sup_{n\geq 0} b_n \leq \sup_{n\geq 0}[2\frac{\alpha_n}{\theta_n}\|f(z)-z\|\|v_{2n+1}-z\|+2\frac{\lambda_n}{\theta_n}(1-\alpha_n(1-q))\|f(z)-z\|\|v_{2n}-z\|] < \infty.$$

Now we prove that $-\delta \leq \limsup_{n\to\infty} b_n$. If $\limsup_{n\to\infty} b_n < -\delta$, then there exits $n_0$ such that, for all $n \geq n_0$, $b_n < -\delta$. According to (3.10), it can be concluded that

$$\|v_{2n+1}-z\|^2 \leq (1-\theta_n)\|v_{2n-1}-z\|^2 - \theta_n\delta.$$

From the condition (i) and the definition of $\theta_n$, it can be inferred that $\sum_{n=0}^{\infty}\theta_n = +\infty$. Thus

$$\begin{aligned}\|v_{2n+1}-z\|^2 \leq&\|v_{2n-1}-z\|^2-\theta_n(\|v_{2n-1}-z\|^2+\delta)\\ \leq&\|v_{2n-1}-z\|^2-\theta_n\delta\\ \leq&\|v_1-z\|^2-\sum_{i=1}^{n}\theta_i\delta.\end{aligned}$$

We immediately have $\limsup_{n\to\infty}\parallel v_{2n+1}-z\parallel^2 \leq \parallel v_1-z\parallel^2 - \sum_{i=1}^{\infty}\theta_i\delta = -\infty$, which contradicts the fact that $\|v_{2n+1}-z\|^2$ is nonnegative. Thus $\limsup_{n\to\infty} b_n$ is finite.
*Step 4.* Prove that $\{v_n\}$ converges to $z = P_S f(z)$.

By Step 3, we can take a subsequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty}[-&\frac{\gamma_{n_k}(2-C_{n_k})}{\theta_{n_k}} \parallel J^A_{\beta_{n_k}} v_{2n_k} - v_{2n_k} \parallel^2 \\ &-\frac{\sigma_{n_k}(2-C'_{n_k})}{\theta_{n_k}}(1-\alpha_{n_k}(1-q)) \parallel J^B_{\mu_{n_k}} v_{2n_k-1} - v_{2n_k-1} \parallel^2 \\ &+2\frac{\alpha_{n_k}}{\theta_{n_k}}\langle f(z)-z, v_{2n_k+1}-z\rangle \\ &+2\frac{\lambda_{n_k}}{\theta_{n_k}}(1-\alpha_{n_k}(1-q))\langle f(z)-z, v_{2n_k}-z\rangle].\end{aligned} \tag{3.14}$$

According to the boundedness of $\{v_n\}$ and $\frac{\alpha_{n_k}}{\theta_{n_k}}$, $\frac{\lambda_{n_k}}{\theta_{n_k}} \in (0, \frac{1}{1-q})$, without loss of generality, we assume these two limits

$$\lim_{k\to\infty} 2\frac{\alpha_{n_k}}{\theta_{n_k}}\langle f(z)-z, v_{2n_k+1}-z\rangle,$$
$$\lim_{k\to\infty} 2\frac{\lambda_{n_k}}{\theta_{n_k}}(1-\alpha_{n_k}(1-q))\langle f(z)-z, v_{2n_k}-z\rangle$$

exit. By (3.14), we know that these two limits

$$\lim_{k\to\infty} \frac{\gamma_{n_k}(2-C_{n_k})}{\theta_{n_k}} \parallel J^A_{\beta_{n_k}} v_{2n_k} - v_{2n_k} \parallel^2,$$
$$\lim_{k\to\infty} \frac{\sigma_{n_k}(2-C'_{n_k})}{\theta_{n_k}} \parallel J^B_{\mu_{n_k}} v_{2n_k-1} - v_{2n_k-1} \parallel^2$$

exit, so there exit $K_1,\ K_2 > 0$ such that

$$\frac{\gamma_{n_k}(2-C_{n_k})}{\theta_{n_k}} \parallel J^A_{\beta_{n_k}} v_{2n_k} - v_{2n_k} \parallel^2 \le K_1,$$
$$\frac{\sigma_{n_k}(2-C'_{n_k})}{\theta_{n_k}} \parallel J^B_{\mu_{n_k}} v_{2n_k-1} - v_{2n_k-1} \parallel^2 \le K_2.$$

From the definition of $\theta_{n_k}$, we have $2-C_{n_k} > m$ $(\exists\, m>0)$ and $\frac{1}{\theta_{n_k}} > \frac{1}{(1-q)(\alpha_{n_k}+\lambda_{n_k})}$, from which we can immediately obtain that

$$\frac{m\gamma_{n_k}}{(1-q)(\alpha_{n_k}+\lambda_{n_k})}\|J^A_{\beta_{n_k}} v_{2n_k} - v_{2n_k}\|^2 \le K_1.$$

In view of condition (iv), we have

$$\lim_{k\to\infty} \| J^A_{\beta_{n_k}} v_{2n_k} - v_{2n_k}\| = 0.$$

Similarly,

$$\lim_{k\to\infty} \| J^B_{\mu_{n_k}} v_{2n_k-1} - v_{2n_k-1}\| = 0.$$

By Lemma 2.2 and condition (ii), we have

$$\|v_{2n_k} - J^A_\beta v_{2n_k}\| \le 2\|v_{2n_k} - J^A_{\beta_{n_k}} v_{2n_k}\| \to 0 \text{ (as } k\to\infty).$$

Similarly,

$$\|v_{2n_k-1} - J^B_\mu v_{2n_k-1}\| \le 2\|v_{2n_k-1} - J^B_{\mu_{n_k}} v_{2n_k-1}\| \to 0 \text{ (as } k\to\infty).$$

Lemma 2.3 implies that any weak cluster point of $\{v_{2n_k}\}$ belongs to $A^{-1}(0)$ and any weak cluster point of $\{v_{2n_k-1}\}$ belongs to $B^{-1}(0)$. Based on the definition of $\{v_{2n_k+1}\}$, it can be inferred that

$$\|v_{2n_k+1}-v_{2n_k}\| \le \alpha_{n_k}\left\|f(v_{2n_k})-v_{2n_k}\right\| + \gamma_{n_k}\|J^A_{\beta_{n_k}} v_{2n_k}-v_{2n_k}\| \to 0 \text{ (as } k\to\infty).$$

This implies that any weak cluster point of $\{v_{n_k}\}$ belongs to $S$. Without loss of generality, we assume that $\{v_{n_k}\}$ converges weakly to $v^* \in S$. In view of (3.14) and $z = P_S f(z)$, we have

$$\begin{aligned}\limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} \le \lim_{k\to\infty} [2\frac{\alpha_{n_k}}{\theta_{n_k}}\langle f(z)-z, v_{2n_k+1}-z\rangle \\ +2\frac{\lambda_{n_k}}{\theta_{n_k}}(1-\alpha_{n_k}(1-q))\langle f(z)-z, v_{2n_k}-z\rangle] \\ \le \frac{2}{1-q}\langle f(z)-z, v^*-z\rangle + \frac{2}{1-q}\langle f(z)-z, v^*-z\rangle \le 0.\end{aligned} \tag{3.15}$$

Consequently, we conclude from (3.10), (3.15) and Lemma 2.4 that $\|v_{2n+1}-z\| \to 0$ (as $n\to\infty$), which together with condition(i) and (3.8) yields $\|v_{2n}-z\| \to 0$ as $n\to\infty$. Thus $\|v_n - z\| \to 0$ as $n\to\infty$. □

Inspired by [2], we next consider the error-sequenced version of algorithm (3.1)-(3.2), i.e., algorithm (1.12)-(1.13). By the arguments of the proof of Theorem 3.1 above and the ideas in [2] and [21], we can obtain the following strong convergence result.

**Theorem 3.2.** *Let $\{x_n\}$ be the sequence generated by (1.12)-(1.13). Assume that*
(i) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \lambda_n = 0,\ \sum_{n=0}^{\infty} \lambda_n = \infty$;
(ii) $\beta_n \ge \beta > 0,\ \mu_n \ge \mu > 0,$ *for all* $n > 0$;
(iii) $\limsup_{n\to\infty} \gamma_n < 2$ *and* $\limsup_{n\to\infty} \sigma_n < 2$;
(iv) $\lim\_n \to \infty \frac{\alpha_n+\lambda_n}{\gamma_n} = 0$ *and* $\lim_{n\to\infty} \frac{\alpha_n+\lambda_n}{\sigma_n} = 0$.
*Then $\{x_n\}$ converges strongly to $P_S f(z)$, which is also the unique solution of the variational inequality: find $z \in S$ such that $\langle (I-f)z, x-z\rangle \ge 0$ for all $x\in S$ provided that any of the following error criteria is satisfied:*

(a) $\sum_{n=0}^{\infty} \|e_n\| \le \infty,\ \sum_{n=0}^{\infty} \|e'_n\| \le \infty$;
(b) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\|e'_n\|/\alpha_n \to 0$;
(c) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\|e'_n\|/\lambda_n \to 0$;
(d) $\|e_n\|/\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;
(e) $\|e_n\|/\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$;
(f) $\|e_n\|/\alpha_n \to 0$ and $\|e'_n\|/\alpha_n \to 0$;
(g) $\|e_n\|/\alpha_n \to 0$ and $\|e'_n\|/\lambda_n \to 0$;
(h) $\|e_n\|/\lambda_n \to 0$ and $\|e'_n\|/\alpha_n \to 0$;
(i) $\|e_n\|/\lambda_n \to 0$ and $\|e'_n\|/\lambda_n \to 0$;
(j) $\|e_n\|/\alpha_n \to 0$ and $\|e'_n\|/\alpha_{n-1} \to 0$;
(k) $\|e_{n-1}\|/\lambda_n \to 0$ and $\|e'_n\|/\alpha_{n-1} \to 0$;
(l) $\|e_{n-1}\|/\lambda_n \to 0$ and $\|e'_n\|/\lambda_n \to 0$;
(m) $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\|e'_n\|/\alpha_{n-1} \to 0$;
(n) $\|e_{n-1}\|/\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \|e'_n\| < \infty$.

*Proof.* Denote $C_n = \frac{\gamma_n}{1-\alpha_n}$ and $C_n' = \frac{\sigma_n}{1-\lambda_n}$. By the definition of $\{x_n\}$, $\{v_n\}$ and Theorem 3.1, we have $\limsup_{n\to\infty} C_n < 2$, $\limsup_{n\to\infty} C_n' < 2$, and

$$\begin{aligned}\|x_{2n+1}-v_{2n+1}\| \le &\|\alpha_n(f(x_{2n})-f(v_{2n}))+\delta_n(x_{2n}-v_{2n})+\gamma_n(J^A_{\beta_n}x_{2n}-J^A_{\beta_n}v_{2n})\|+\|e_n\| \\ \le &\alpha_n q\|x_{2n}-v_{2n}\|+(1-\alpha_n)\|(1-C_n)(x_{2n}-v_{2n}) \\ &+C_n(J^A_{\beta_n}x_{2n}-J^A_{\beta_n}v_{2n})\|+\|e_n\|.\end{aligned} \tag{3.16}$$

Since $J^A_{\beta_n}$ is $\frac{1}{2}-$averaged, we have $J^A_{\beta_n} = \frac{1}{2}I + \frac{1}{2}T_n$, where $T_n$ is nonexpansive for every $n \in N$, so (3.16) can be transformed into

$$\begin{aligned}\|x_{2n+1}-v_{2n+1}\| \le &\alpha_n q\|x_{2n}-v_{2n}\|+(1-\alpha_n)\|(1-C_n)(x_{2n}-v_{2n}) \\ &+C_n[\frac{1}{2}(x_{2n}-v_{2n})+\frac{1}{2}(Tx_{2n}-Tv_{2n})]\|+\|e_n\| \\ \le &\alpha_n q\|x_{2n}-v_{2n}\|+(1-\alpha_n)(1-\frac{C_n}{2})\|x_{2n}-v_{2n}\| \\ &+(1-\alpha_n)\frac{C_n}{2}\|Tx_{2n}-Tv_{2n}\|+\|e_n\| \\ \le &(1-\alpha_n(1-q))\|x_{2n}-v_{2n}\|+\|e_n\|.\end{aligned} \tag{3.17}$$

Similarly,

$$\|x_{2n}-v_{2n}\| \le (1-\lambda_n(1-q))\|x_{2n-1}-v_{2n-1}\|+\left\|e_n'\right\|. \tag{3.18}$$

Substituting (3.18) into (3.17) yields

$$\|x_{2n+1}-v_{2n+1}\| \le (1-\alpha_n(1-q))(1-\lambda_n(1-q))\,\|x_{2n-1}-v_{2n-1}\|+\|e_n\|+\left\|e_n'\right\|. \tag{3.19}$$

It follows from condition (i) and Lemma 2.4 that $\|x_{2n+1}-v_{2n+1}\| \to 0$ as $n\to\infty$. Similarly,

$$\begin{aligned}\|x_{2n}-v_{2n}\| \le &(1-\alpha_{n-1}(1-q))(1-\lambda_n(1-q))\,\|x_{2n-2}-v_{2n-2}\| \\ &+\|e_{n-1}\|+\left\|e_n'\right\| \to 0\ (as\quad n\to\infty).\end{aligned}$$

This completes the proof of the theorem. □

**Remark 3.3.** Theorem 3.2 gives the strong convergence of the ARA with viscosity and over relaxed factors. The convergence of the ARA is accelerated under the influence of the over-relaxed factors. This result can be seen as a generalization of the results in [2, 6].

If we replace the contraction map with $u \in H$ in (1.12)-(1.13) and the variation ranges of other parameters are kept unchanged, we have the following algorithm

$$x_{2n+1} = \alpha_n u+\delta_n x_{2n}+\gamma_n J^A_{\beta_n}x_{2n}+e_n, \quad n=0,1,..., \tag{3.20}$$

$$x_{2n} = \lambda_n u+\rho_n x_{2n-1}+\sigma_n J^B_{\mu_n}x_{2n-1}+e_n', \quad n=1,2,..., \tag{3.21}$$

where $u\in H$ is given, $A$ and $B$ are maximal montone operators, $\alpha_n,\ \lambda_n \in (0,1)$, $\delta_n,\ \rho_n \in (-2,1)$, $\gamma_n,\ \sigma_n \in (0,2)$ satisfying $\alpha_n+\delta_n+\gamma_n = 1$, $\lambda_n+\rho_n+\sigma_n = 1$, and $\{e_n\}$, and $\{e_n'\}$ are error sequences.

**Corollary 3.4.** *Let $\{x_n\}$ be the sequence generated by (3.20)-(3.21). Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$ *and* $\lim\limits_{n\to\infty}\lambda_n = 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$*;*

(ii) $\beta_n \ge \beta > 0$, $\mu_n \ge \mu > 0$, *for all* $n > 0$;

(iii) $\limsup_{n\to\infty}\gamma_n < 2$ *and* $\limsup_{n\to\infty}\sigma_n < 2$*;*

(iv) $\lim_{n\to\infty}\frac{\alpha_n+\lambda_n}{\gamma_n}=0$ *and* $\lim_{n\to\infty}\frac{\alpha_n+\lambda_n}{\sigma_n}=0$.

*Then* $\{x_n\}$ *converges strongly to* $P_S u$, *which is also the unique solution of the variational inequality:* $z\in S,\ \langle z-u, x-z\rangle \geq 0$ *for all* $x\in S$, *provided that any of the following error criteria is satisfied:*

(a) $\sum_{n=0}^{\infty}\|e_n\|\leq\infty,\ \sum_{n=0}^{\infty}\|e_n'\|\leq\infty$;
(b) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ and $\|e_n'\|/\alpha_n\to 0$;
(c) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ and $\|e_n'\|/\lambda_n\to 0$;
(d) $\|e_n\|/\alpha_n\to 0$ and $\sum_{n=1}^{\infty}\|e_n'\|<\infty$;
(e) $\|e_n\|/\lambda_n\to 0$ and $\sum_{n=1}^{\infty}\|e_n'\|<\infty$;
(f) $\|e_n\|/\alpha_n\to 0$ and $\|e_n'\|/\alpha_n\to 0$;
(g) $\|e_n\|/\alpha_n\to 0$ and $\|e_n'\|/\lambda_n\to 0$;
(h) $\|e_n\|/\lambda_n\to 0$ and $\|e_n'\|/\alpha_n\to 0$;
(i) $\|e_n\|/\lambda_n\to 0$ and $\|e_n'\|/\lambda_n\to 0$;
(j) $\|e_n\|/\alpha_n\to 0$ and $\|e_n'\|/\alpha_{n-1}\to 0$;
(k) $\|e_{n-1}\|/\lambda_n\to 0$ and $\|e_n'\|/\alpha_{n-1}\to 0$;
(l) $\|e_{n-1}\|/\lambda_n\to 0$ and $\|e_n'\|/\lambda_n\to 0$;
(m) $\sum_{n=0}^{\infty}\|e_n\|<\infty$ and $\|e_n'\|/\alpha_{n-1}\to 0$;
(n) $\|e_{n-1}\|/\lambda_n\to 0$ and $\sum_{n=1}^{\infty}\|e_n'\|<\infty$.

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