J. Nonlinear Funct. Anal. 2024 (2024) 3 https://doi.org/10.23952/jnfa.2024.3

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

NEW STRONGLY CONVERGENT ITERATIVE METHODS FOR NONEXPANSIVE MAPPINGS

WANRONG ZHAN, HAI YU*, FENGHUI WANG

Department of Mathematics, Luoyang Normal University, Luoyang 471022, China

Abstract. In this paper, we introduce two new iterative algorithms, Halpern-Krasnosel'skiĭ-Mann iteration (HKM) and the Krasnosel'skiĭ-Mann-Halpern iteration (KMH) for fixed points of nonexpansive mappings. Under mild conditions, we proved the strong convergence theorems of the algorithms for fixed points of nonexpansive mappings in Hilbert spaces. Finally, we give a numerical example for illustrating the efficiency of the given algorithms in comparison with existing algorithms in the literatures.

Keywords. Convex feasibility problem; Halpern iteration; Krasnosel'skiĭ-Mann iteration; Nonexpansive mapping; Strong convergence.

1. Introduction

Let C be a nonempty, convex and closed set in a real Hilbert apace H with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. A mapping $T: C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$. The set of fixed points of a mapping $T: C \to C$ is defined by $F(T) := \{x \in C \mid Tx = x\}$.

Various efficient iterative methods for finding fixed points of nonexpansive mappings have been investigated recently; see, e.g., [1, 3, 10, 14, 16] and the references therein. One of the celebrated iterations is the Krasnosel'skiĭ-Mann iteration [8, 12], which generates, with an initial point $x_0 \in C$ arbitrarily chosen, a sequence $\{x_n\}$ by the iteration process:

$$x_{n+1} = (1-\lambda_n)x_n + \lambda_n T x_n, \tag{1.1}$$

where $\{\lambda_n\}$ is real sequence in $[0,1]$. It was proved in [15] that $\{x_n\}$ generated by iteration (1.1) converges weakly to a fixed point of T provided that $F(T)$ is nonempty and λ_n satisfies $\sum_{n=0}^{\infty} \lambda_n(1-\lambda_n) = \infty$.

However, the Krasnosel'skiĭ-Mann iteration has only weak convergence, as noted by a counterexample in [5]. In order to obtain strong convergence, in 1967, Halpern [6] proposed the

*Corresponding author.

E-mail address: yuhai2000@126.com (H. Yu)

Received September 13, 2023; Accepted January 25, 2024.

Halpern iteration: $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \tag{1.2}$$

where $u \in C$ is an arbitrary (but fixed) point in C, and $\alpha_n = n^{-a}, a \in (0,1)$.

In 1977, Lions [9] proved the strong convergence of $\{x_n\}$ to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0$.

But both Halpern's and Lion's conditions imposed on sequence $\{\alpha_n\}$ excluded the natural and important choice $\alpha_n = \frac{1}{n+1}$. To overcome this, Wittmann [21] in 1992 proved the strong convergence of $\{x_n\}$ by using the control conditions (C1) and the following condition (C3):

(C3) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

In 2002, Xu [22, 23] proved the strong convergence of $\{x_n\}$ by replacing condition (C2) or (C3) with the following condition (C4):

(C4) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

In order to further study the control condition on parameter α_n, many authors modified the Halpern iteration for nonexpansive mappings. C.E. Chidume and C.O. Chidume [2] and Suzuki [18] gave the simpler modification of Halpern iteration: $x_0 \in C$,

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\big((1-\delta)x_n + \delta Tx_n\big), \tag{1.3}$$

where $\delta \in (0,1)$ is a constant, $\{\alpha_n\}$ satisfies condition (C1) only. Kim and Xu [7] proposed the following iteration method: $x_0 \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. Assume that $\{\alpha_n\}$ satisfies conditions (C1) and (C3), and $\{\beta_n\}$ satisfies the following condition (C5):

(C5) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=0}^{\infty} \beta_n = \infty$ and $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T. Yao [25] also proved the same conclusion if $\{\alpha_n\}$ satisfies condition (C1), and $\{\beta_n\}$ satisfies the following condition (C6):

(C6) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Song et al. [17] presented two new iteration methods: $x_0 \in C$,

$$x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)x_n) + (1-\beta_n)Tx_n, \tag{1.5}$$

and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T(\alpha_n u + (1-\alpha_n)x_n). \tag{1.6}$$

They proved that, under conditions (C1) and (C6), the sequences generated by (1.5) and (1.6) converge strongly to a fixed point of T, respectively.

In this paper, inspired by the above research, we propose two new iteration processes for finding a fixed point of T. Our method is different from (1.3)-(1.6). More precisely, the first iteration, called the Halpern-Krasnosel'skiĭ-Mann iteration (HKM iteration), is as follows: $x_0 \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Ty_n, \end{cases} \tag{1.7}$$

where $u \in C$ denotes a fixed vector and $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0,1)$. The second iteration, called the Krasnosel'skiĭ-Mann-Halpern iteration (KMH iteration), is defined as follows: $x_0 \in C$,

$$\begin{cases} y_n = \alpha_n u + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n, \end{cases} \tag{1.8}$$

where $u \in C$ denotes a fixed vector and $\{\alpha_n\}$, $\{\beta_n\}$ are in $(0,1)$. Under conditions (C1) and (C6), we prove the strong convergence theorems of the iterations for finding the fixed point of the nonexpansive mappings in Hilbert spaces.

2. Preliminaries

Throughout this paper, we always assume that H is a Hilbert space and I is its identity operator. Let $\{x_n\} \subseteq H$ be a sequence. $\omega_w(x_n)$ (resp., $\omega(x_n)$) stands for the set of cluster points in the weak (resp., strong) topology. '$x_n \rightharpoonup x$' (resp.,'$x_n \to x$') means the weak (resp., strong) convergence of $\{x_n\}$ to x.

Lemma 2.1. [1] *Let $C \subseteq H$ be a nonempty, convex, and closed subset, and let $P_C : H \to C$ be the metric projection from H on C. Then, for all $x \in H$ and $y \in C$, $\langle x - P_C x, y - P_C x\rangle \le 0$.*

Lemma 2.2. *Let X be a real inner product space. Then:*

(i) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in X.$

(ii) $\|tx+sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2, \quad \forall x,y \in X, \forall s,t \in \mathbb{R}.$

Lemma 2.3. [1] *Let D be a nonempty, convex, and closed subset of H and let $T : D \to D$ be a nonexpansive mapping with $F(T) \ne \emptyset$. Let $\{x_n\}$ be a sequence in D and let x be a vector in H. If $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

Lemma 2.4. [23] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n,\ n \ge 1,$$

where (a) $\{a_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\alpha_n = \infty$; (b) $\limsup \sigma_n \le 0$; and (c) $\gamma_n \ge 0 (n \ge 1)$, $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then $a_n \to 0$ as $n \to \infty$.

Lemma 2.5. [11] *Let $\{\Gamma_n\}$ be a real sequence, which does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\ge 0}$ of $\{\Gamma_n\}$ such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \text{for all} \quad j \ge 0.$$

Let the integer sequence $\{\tau(n)\}_{n\ge n_0}$ be defined by

$$\tau(n) = \max\{k \le n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\tau(n)\}_{n\ge n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$, and, for all $n \ge n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \quad \Gamma_n \le \Gamma_{\tau(n)+1}.$$

3. Convergence Analysis

Firstly, we give the strong convergence proof of the HKM iteration.

Theorem 3.1. *Let $\{x_n\}$ be generated by the HKM iteration (1.7). Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ and conditions (C1) and (C6) hold. Then $\{x_n\}$ strongly converges to z, where $z := P_{F(T)}u$.*

Proof. We first demonstrate that $\{x_n\}$ is a bounded sequence. Fix $\bar{x} \in F(T)$. It follows from (1.7) and the nonexpansiveness of mapping T that

$$\begin{aligned}\|y_n - \bar{x}\| &\leq \beta_n \|x_n - \bar{x}\| + (1-\beta_n)\|Tx_n - \bar{x}\| \\ &\leq \|x_n - \bar{x}\|,\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq \alpha_n \|u - \bar{x}\| + (1-\alpha_n)\|Ty_n - \bar{x}\| \\ &\leq \alpha_n \|u - \bar{x}\| + (1-\alpha_n)\|y_n - \bar{x}\| \\ &\leq \alpha_n \|u - \bar{x}\| + (1-\alpha_n)\|x_n - \bar{x}\| \\ &\leq \max\{\|u - \bar{x}\|, \|x_n - \bar{x}\|\},\end{aligned}$$

which indicates that

$$\|x_{n+1} - \bar{x}\| \leq \max\{\|u - \bar{x}\|, \|x_0 - \bar{x}\|\}.$$

Hence $\{x_n\}$ is bounded. Let $\Gamma_n = \|x_n - z\|^2$. We now demonstrate that $\{x_n\}$ converges strongly to z by considering two possible cases on $\{\Gamma_n\}$.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}_{n=n_0}^{\infty}$ is non-increasing. Thus $\{\|x_n - z\|\}_{n=1}^{\infty}$ converges, and we therefore obtain

$$\|x_n - z\| - \|x_{n+1} - z\| \to 0, n \to \infty. \tag{3.1}$$

From Lemma 2.2 (i) and (ii), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2 \|Ty_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|y_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)\|\beta_n(x_n - z) + (1-\beta_n)(Tx_n - z)\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &= (1-\alpha_n)\big(\beta_n\|x_n - z\|^2 + (1-\beta_n)\|Tx_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2\big) \\ &\quad + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + 2\alpha_n \langle u - z, x_{n+1} - z\rangle.\end{aligned} \tag{3.2}$$

Using the boundedness of $\{x_n\}$, one sees that

$$\begin{aligned}&(1-\alpha_n)\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M_1\end{aligned} \tag{3.3}$$

for some $M_1 > 0$. By conditions (C1) and (C6), we can assume without loss of generality that there exists $\varepsilon_1 > 0$ such that $(1-\alpha_n)\beta_n(1-\beta_n) \geq \varepsilon_1$ for all $n \geq 0$. Hence, we obtain from (3.3) and (3.1) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.4}$$

By the boundedness of $\{x_n\}$, we can take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u-z, x_n - z\rangle = \lim_{i\to\infty} \langle u-z, x_{n_i} - z\rangle,$$

and $x_{n_i} \rightharpoonup x^*$. From Lemma 2.3 and (3.4), we obtain $x^* \in F(T)$. This along with Lemma 2.1 implies that

$$\begin{aligned}
\limsup_{n\to\infty} \langle u-z, x_{n+1} - z\rangle &= \limsup_{n\to\infty} \langle u-z, x_n - z\rangle \\
&= \lim_{i\to\infty} \langle u-z, x_{n_i} - z\rangle \\
&= \langle u-z, x^* - z\rangle \leq 0.
\end{aligned}$$

It follows from (3.2) that

$$\|x_{n+1} - z\|^2 \leq (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n \langle u-z, x_{n+1} - z\rangle. \tag{3.5}$$

Applying Lemma 2.4 to (3.5), we obtain that $\lim_{n\to\infty} \|x_n - z\| = 0$. Thus, $x_n \to z = P_{F(T)}u$.

Case 2. Assume that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then, according to Lemma 2.5, $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

It follows from (3.3) that

$$\begin{aligned}
&(1-\alpha_{\tau(n)})\beta_{\tau(n)}(1-\beta_{\tau(n)})\|x_{\tau(n)} - Tx_{\tau(n)}\|^2 \\
&\leq \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1} - z\rangle \\
&\leq 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1} - z\rangle,
\end{aligned}$$

which implies that

$$\|x_{\tau(n)} - Tx_{\tau(n)}\| \to 0. \tag{3.6}$$

From the boundedness of $\{x_n\}$ and condition (C1), we obtain

$$\begin{aligned}
\|x_{\tau(n)+1} - x_{\tau(n)}\| &\leq \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\alpha_{\tau(n)})\|Ty_{\tau(n)} - x_{\tau(n)}\| \\
&\leq \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + \|Ty_{\tau(n)} - Tx_{\tau(n)}\| + \|Tx_{\tau(n)} - x_{\tau(n)}\| \\
&\leq \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + \|y_{\tau(n)} - x_{\tau(n)}\| + \|Tx_{\tau(n)} - x_{\tau(n)}\| \\
&\leq \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\beta_{\tau(n)})\|Tx_{\tau(n)} - x_{\tau(n)}\| + \|Tx_{\tau(n)} - x_{\tau(n)}\| \\
&\leq \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + 2\|Tx_{\tau(n)} - x_{\tau(n)}\| \to 0, \quad n \to \infty.
\end{aligned} \tag{3.7}$$

Since $\{x_{\tau(n)}\}$ is bounded, one sees that there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to some x^*. From Lemma 2.3 and (3.6), we have $x^* \in F(T)$. This along with Lemma 2.1 implies that

$$\limsup_{n\to\infty}\langle u-z, x_{\tau(n)}-z\rangle = \langle u-z, x^*-z\rangle \le 0.$$

It follows from (3.7) that $\limsup_{n\to\infty}\langle u-z, x_{\tau(n)+1}-z\rangle \le 0$. From (3.5), we obtain

$$\Gamma_{\tau(n)+1} \le (1-\alpha_{\tau(n)})\Gamma_{\tau(n)} + 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1}-z\rangle.$$

Thus, we have

$$\begin{aligned}\alpha_{\tau(n)}\Gamma_{\tau(n)} &\le \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1}-z\rangle\\ &\le 2\alpha_{\tau(n)}\langle u-z, x_{\tau(n)+1}-z\rangle,\end{aligned}$$

which implies that $\Gamma_{\tau(n)} \le 2\langle u-z, x_{\tau(n)+1}-z\rangle$, and

$$\limsup_{n\to\infty}\Gamma_{\tau(n)} \le 2\limsup_{n\to\infty}\langle u-z, x_{\tau(n)+1}-z\rangle \le 0.$$

Therefore $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. It follows from (3.7) that

$$\begin{aligned}\sqrt{\Gamma_{\tau(n)+1}} &= \|x_{\tau(n)+1}-z\|\\ &\le \|x_{\tau(n)+1}-x_{\tau(n)}+x_{\tau(n)}-z\|\\ &\le \sqrt{\Gamma_{\tau(n)}} + \|x_{\tau(n)+1}-x_{\tau(n)}\| \to 0.\end{aligned}$$

By Lemma 2.5, we obtain for all sufficiently large n that $0 \le \Gamma_n \le \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, $\{x_n\}$ converges strongly to z. □

Remark 3.2. The HKM iteration is actually composed of one-step Krasnosel'skiĭ-Mann iteration and one-step Halpern iteration.

Secondly, the strong convergence theorem of the KMH iteration is given as follows.

Theorem 3.3. *Let $\{x_n\}$ be generated by the KMH iteration (1.8). Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ and conditions $(C1)$ and $(C6)$ hold. Then $\{x_n\}$ strongly converges to z, where $z := P_{F(T)}u$.*

Proof. We first demonstrate that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Fix $\bar{x} \in F(T)$. We obtain from (1.8) and the nonexpansiveness of mapping T that

$$\begin{aligned}\|y_n-\bar{x}\| &\le \alpha_n\|u-\bar{x}\| + (1-\alpha_n)\|Tx_n-\bar{x}\|\\ &\le \alpha_n\|u-\bar{x}\| + (1-\alpha_n)\|x_n-\bar{x}\|\\ &\le \max\{\|u-\bar{x}\|, \|x_n-\bar{x}\|\}.\end{aligned} \tag{3.8}$$

Thus, we have

$$\begin{aligned}\|x_{n+1}-\bar{x}\| &\le \beta_n\|y_n-\bar{x}\| + (1-\beta_n)\|Ty_n-\bar{x}\|\\ &\le \|y_n-\bar{x}\|\\ &\le \max\{\|u-\bar{x}\|, \|x_n-\bar{x}\|\}.\end{aligned}$$

Using induction, this implies that $\|x_{n+1}-\bar{x}\| \le \max\{\|u-\bar{x}\|, \|x_0-\bar{x}\|\}$. Therefore, $\{x_n\}$ is bounded. From (3.8), one sees that $\{y_n\}$ is also bounded.

Let $\Gamma_n = \|x_n - z\|^2$. We now divide the proof into two possible cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\|x_n - z\|\}_{n=1}^{\infty}$ converges, and we therefore obtain

$$\|x_n - z\| - \|x_{n+1} - z\| \to 0, n \to \infty. \tag{3.9}$$

From Lemma 2.2, we have

$$\begin{aligned}\|y_n - z\|^2 &\le (1-\alpha_n)^2\|Tx_n - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle \\ &\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \beta_n\|y_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\ &\le \|y_n - z\|^2 - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2 \\ &\le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle - \beta_n(1-\beta_n)\|y_n - Ty_n\|^2. \end{aligned} \tag{3.10}$$

It follows from the boundedness of $\{y_n\}$ that

$$\begin{aligned}\beta_n(1-\beta_n)\|y_n - Ty_n\|^2 &\le (1-\alpha_n)\|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle \\ &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle \\ &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M_2 \end{aligned} \tag{3.11}$$

for some $M_2 > 0$. From condition (C6), without loss of generality, we can assume that there exists $\varepsilon_2 > 0$ such that $\beta_n(1-\beta_n) \ge \varepsilon_2$ for all $n \ge 0$. Hence, it follows from (3.11), (3.9), and condition (C1) that

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0. \tag{3.12}$$

By the boundedness of $\{y_n\}$, we can take a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z, y_n - z\rangle = \lim_{i\to\infty}\langle u - z, y_{n_i} - z\rangle,$$

and $y_{n_i} \rightharpoonup y^*$. From Lemma 2.3 and (3.12), we have $y^* \in F(T)$. By Lemma 2.1, we obtain

$$\limsup_{n\to\infty}\langle u - z, y_n - z\rangle = \lim_{i\to\infty}\langle u - z, y_{n_i} - z\rangle = \langle u - z, y^* - z\rangle \le 0.$$

It follows from (3.10) that

$$\|x_{n+1} - z\|^2 \le (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, y_n - z\rangle. \tag{3.13}$$

Applying Lemma 2.4 to (3.13) and using condition (C1), we have $\lim_{n\to\infty}\|x_n - z\| = 0$.

Case 2. Assume that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \ge 0$. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \le n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then according to Lemma 2.5, $\{\tau(n)\}$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ for all $n \ge n_0$. It follows from (3.11) that

$$\begin{aligned}\beta_{\tau(n)}(1-\beta_{\tau(n)})\|y_{\tau(n)} - Ty_{\tau(n)}\|^2 &\le \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + 2\alpha_{\tau(n)}\langle u - z, y_{\tau(n)} - z\rangle \\ &\le 2\alpha_{\tau(n)}\langle u - z, y_{\tau(n)} - z\rangle.\end{aligned}$$

This implies that

$$\|y_{\tau(n)} - Ty_{\tau(n)}\| \to 0. \tag{3.14}$$

By the boundedness of $\{y_{\tau(n)}\}$, we can take a subsequence of $\{y_{\tau(n)}\}$, without loss of generality, still denoted by $\{y_{\tau(n)}\}$, such that $y_{\tau(n)} \rightharpoonup y^*$. This along with Lemma 2.3 and (3.14) implies that $y^* \in F(T)$. It follows from Lemma 2.1 that

$$\limsup_{n\to\infty}\langle u-z, y_{\tau(n)}-z\rangle = \langle u-z, y^*-z\rangle \le 0.$$

From (3.13), we obtain

$$\Gamma_{\tau(n)+1} \le (1-\alpha_{\tau(n)})\Gamma_{\tau(n)} + 2\alpha_{\tau(n)}\langle u-z, y_{\tau(n)}-z\rangle. \tag{3.15}$$

Thus, we have

$$\begin{aligned}\alpha_{\tau(n)}\Gamma_{\tau(n)} &\le \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + 2\alpha_{\tau(n)}\langle u-z, y_{\tau(n)}-z\rangle \\ &\le 2\alpha_{\tau(n)}\langle u-z, y_{\tau(n)}-z\rangle.\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}\Gamma_{\tau(n)} \le 2\limsup_{n\to\infty}\langle u-z, y_{\tau(n)}-z\rangle \le 0,$$

and hence $\lim_{n\to\infty}\Gamma_{\tau(n)} = 0$. This together with (3.15) implies that $\lim_{n\to\infty}\Gamma_{\tau(n)+1} = 0$. By Lemma 2.5, we obtain for all sufficiently large n that $0 \le \Gamma_n \le \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, $\{x_n\}$ converges strongly to z. □

Remark 3.4. The KMH iteration is actually composed of one-step Halpern iteration and one-step Krasnosel'skiĭ-Mann iteration.

Remark 3.5. The two results in this paper remain true if one replaces the iteration with the so-called viscosity process defined as follows: $x_0 \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)Ty_n, \end{cases} \tag{3.16}$$

or

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \\ x_{n+1} = \beta_n y_n + (1-\beta_n)Ty_n, \end{cases} \tag{3.17}$$

where f is a contraction with coefficient $\rho(0<\rho<1)$. Let $\{x_n\}$ be generated by (3.16) or (3.17). Then, under conditions (C1) and (C6), $\{x_n\}$ strongly converges to z, where z is the unique fixed point of $P_{F(T)}f$. For the proof technique, we refer to [13, 19, 24].

4. Numerical Experiment

In this section, we present a numerical experiment to illustrate the performance of the proposed algorithms. All algorithms are performed in MATLAB R2016a on an Intel(R) Core(TM) i7-8565U laptop with 16 GB RAM.

Example 4.1. In this example, we apply our Algorithms to solve the classic two-sets convex feasibility problem in finite-dimensional Euclidean space [4]. This problem is formally stated as follows:

$$\text{Find} \quad x^* \in A \cap B,$$

where the feasible sets $A, B \subseteq \mathbb{R}^m$ are nonempty, convex, and closed sets. Let $T = P_A P_B$. It is easy to verify that T is a nonexpansive mapping and $F(T) = A \cap B$.

We consider a two-sets convex feasibility problem where the two nonempty, convex, and closed sets are $A = \{x \in \mathbb{R}^m \mid \|x - O_1\| \le r_1\}$ and $B = \{x \in \mathbb{R}^m \mid \|x - O_2\| \le r_2\}$, which are both balls, where $O_i \in \mathbb{R}^m$ and $r_i \in \mathbb{R}^+, i = 1,2$.

We apply algorithms (1.3)-(1.8) to solve Example 4.1. In this example, we choose $O_1 = (0,0,\cdots,0)^T$, $O_2 = (1,1,0,0,\cdots,0)^T$ and $r_1 = r_2 = 1$. In all these algorithms, we take $u = (0,0,\cdots,0)^T$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{n+1} + \frac{1}{6}$, and $\delta = \frac{1}{6}$. We choose different values of m and x_0, and plot the graphs of $Error = \|x_{n+1} - x_n\|$ against number of iterations n. The stopping criterion is $Error = \|x_{n+1} - x_n\| < \varepsilon = 10^{-6}$. The results are shown in Figure 1. From Figure 1, our algorithms demonstrates better performance than the algorithms (1.3)-(1.6).

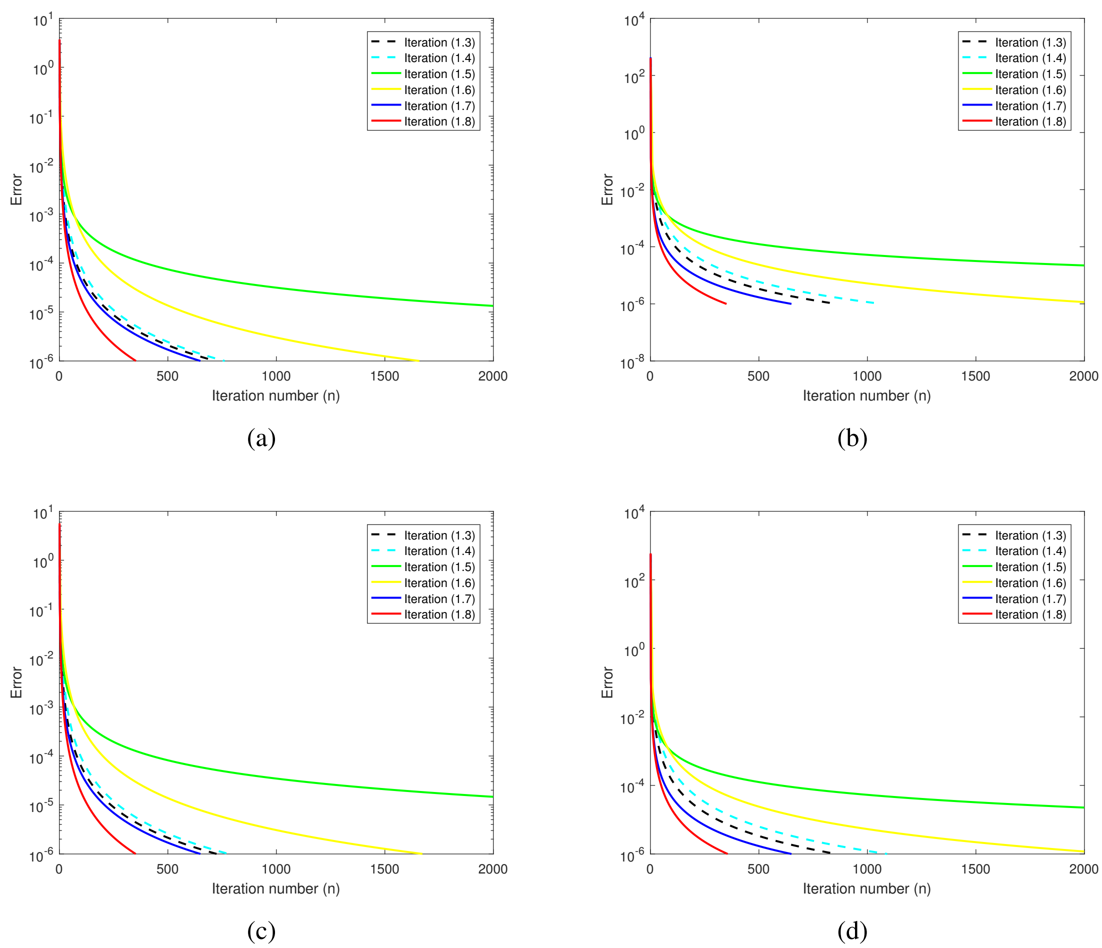

FIGURE 1. Example 4.1: (a): $m = 50$, $x_0 \in \mathbb{R}^m$ is generated from uniform distribution in the interval $[0,1]$; (b): $m = 50$, $x_0 \in \mathbb{R}^m$ is generated from uniform distribution in the interval $[0,100]$; (c): $m = 100$, $x_0 \in \mathbb{R}^m$ is generated from uniform distribution in the interval $[0,1]$; (d): $m = 100$, $x_0 \in \mathbb{R}^m$ is generated from uniform distribution in the interval $[0,100]$.

Funding

This work was supported by the National Natural Science Foundation of China (Nos. 11971216 and 62072222).

REFERENCES

[1] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Space, Springer-Verlag, New York, 2011.
[2] C.E. Chidume, C.O. Chidume, Iterative approximation of fixed points of nonexpansive mappings, J. Math. Anal. Appl. 318 (2006) 288-295.
[3] S.Y. Cho, Weak convergence of a splitting algorithm in Hilbert spaces, J. Appl. Anal. Comput. 7 (2017) 427-438.
[4] P.L. Combettes, The convex feasibility problem in image recovery, Advances in imaging and electron physics, In: P. Hawkes, ed., Vol. 95, pp. 155-270, Academic Press, New York, 1996.
[5] A. Genel, J. Lindenstrass, An example concerning fixed points, Israel J. Math. 22 (1975) 81-86.
[6] B. Halpern, Fixed points of nonexpanding maps, Bull. Amer. Math. Soc. 73 (1967) 957-961.
[7] T.H. Kim, H.K. Xu, Strong convergence of modified Mann iterations, Nonlinear Anal. 61 (2005) 51-60.
[8] M.A. Krasnoselskii, Two remarks on the method of successive approximations, Uspekhi Mat. Nauk, 10 (1955) 123-127.
[9] P.L. Lions, Approximation de points fixes de contractions, C. R. Acad. Sci. Paris Ser. A-B 284 (1977) 1357-1359.
[10] L. Liu, B. Tan, S.Y. Cho, On the resolution of variational inequality problems with a double-hierarchical structure, J. Nonlinear Convex Anal. 21 (2020) 377-386.
[11] P.E. Mainge, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, Set-Valued Anal. 16 (2008) 899-912.
[12] W.R. Mann, Mean value methods in iteration, Bull. Amer. Math. Soc. 4 (1953) 506-510.
[13] A. Moudafi, Viscosity approximation methods for fixed-points, J. Math. Anal. Appl. 241 (2000) 46-55.
[14] X. Qin, A weakly convergent method for splitting problems with nonexnpansive mappings, J. Nonlinear Convex Anal. 24 (2023) 1033-1043.
[15] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. 67 (1979) 274-276.
[16] Y. Shehu, J.C. Yao, Weak convergence of two-step inertial iteration for countable family of quasi-nonexpansive mappings, Ann. Math. Sci. Appl. 7 (2022) 259-279.
[17] Y. Song, H. Li, Strong convergence of iterative sequences for nonexpansive mappings, Appl. Math. Lett. 22 (2009) 1500-1507.
[18] T. Suzuki, A sufficient and necessary condition for Halpern-type strong convergence to fixed points of non-expansive mappings, Proc. Amer. Math. Soc. 135 (2007) 99-106.
[19] T. Suzuki, Moudafi's viscosity approximations with Meir-Keeler contractions, J. Math. Anal. Appl. 325 (2007) 342-352.
[20] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005) 227-239.
[21] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. (Basel) 58 (1992) 486-491.
[22] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, Bull. Aust. Math. Soc. 65 (2002) 109-113.
[23] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 2 (2002) 240-256.
[24] H.K. Xu, Viscosity approximation methods for nonexpansive mappings, J. Math. Anal. Appl. 298 (2004) 279-291.
[25] Y. Yao, R. Chen, J.C. Yao, Strong convergence and certain control conditions for modified Mann iteration, Nonlinear Anal. 68 (2008) 1687-1693.