J. Nonlinear Funct. Anal. 2024 (2024) 4 https://doi.org/10.23952/jnfa.2024.4

MATHRES

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

MULTIVARIATE FIXED POINT AND MULTIVARIATE COMMON FIXED POINT THEOREMS FOR (ψ,φ)-WEAKLY CONTRACTIVE MAPPINGS WITH APPLICATIONS

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Abstract. In this paper, the concepts of multivariate $\star$-metric functions, multivariate coincidence points, multivariate common fixed points, and multivariate commutable mappings are introduced. In b-metric spaces, we propose multivariate fixed point and multivariate common fixed point theorems for (ψ,φ)-weakly contractions. As an application, the solvability of an integral equation is considered.

Keywords. b-metric space; Integral equation; Multivariate common fixed point; Weakly contraction mapping.

1. Introduction

In nonlinear analysis, the most famous result is the celebrated Banach contraction principle [1]. It plays a fundamental role in various nonlinear equations and has numerous real applications. In 1993, Czerwik [2] introduced a new concept, that is, the definition of b-metric. Indeed, this kind of spaces were earlier considered under various names; see [3]. Since then, existence results of fixed points of nonlinear operators were established in b-metric spaces; see, e.g., [4]-[7]. In [8], Mitrović et al. provided a new method to prove Czerwik's fixed point theorem in a b-metric space. By using of increased range of the Lipschitzian constants, Hussain et al. [9] presented a proof to the Fisher contraction theorem. Mustafa et al. [10] gave several fixed point theorems of some new types of the T-Chatterjea-contraction and the T-Kannan-contraction. Recently, Mitrović et al. [11] presented some new versions of these theorems in b-metric spaces. Savanović [12] also constructed some new results for multivalued quasi-contractions. Zoto et al. [13] and Huang et al. [14] considered some new F-contractions in b-metric-like spaces, respectively. In addition, in ordered b_2-metric spaces, some fixed point results of various contractive-type mappings were presented in [15] and Hussain et al. investigated topological and structural properties of generalized partial b-metric spaces in [16].

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Received September 21, 2023; Accepted January 26, 2024.

In 1997, Alber et al. [17] proposed a new type of contractive mappings which are named weakly contractive mappings. Rhoades [18] continued to study the concept of weakly contractions in a metric space. After that, in a partially ordered b-metric space, a coincidence point theorem of (ψ,φ)-weak contraction mappings was presented in [19]. At the same time, Choudhury et al. [20] extended the weakly contractive mapping by means of the concept of an altering distance function. Khan et al. [21] obtained some results involving the altering distance functions, and proposed relevant theorems. Guan [22] also generalized the weakly contraction mapping and obtained some common fixed point results.

In complete metric spaces, Su [23] proved a multivariate result for a contraction via a new metric function. In 2014, in a ordered S-metric space, Gupta [24] established several coupled common fixed point results. In 2009, Lakshmikantham [25] proposed the definition of a coupled coincidence point and used ingenious methods to transform the double-mapping problem into a single-mapping problem in a partially ordered metric space. With appropriate hypotheses, Petruşel et al. obtained some coupled fixed point results in the setting of b-metric spaces in [26], [27], and [28].

Inspired by [23], we aim to obtain several multivariate (common) fixed point theorems for (ψ,φ)-weakly contraction mappings in the framework of b-metric spaces.

2. Preliminaries

We first introduce some common theorems and concepts, which are important to establish our main results.

Definition 2.1. ([2]) Let $\mathscr{M}$ be a nonempty set and $\rho:\mathscr{M}\times\mathscr{M}\to[0,+\infty)$. ρ is called a b-metric if

(i) $\rho(\kappa,\iota)=0\Leftrightarrow\kappa=\iota,\forall\kappa,\iota\in\mathscr{M}$;

(ii) $\rho(\kappa,\iota)=\rho(\nu,\kappa),\forall\kappa,\iota\in\mathscr{M}$;

(iii) $\rho(\kappa,\iota)\leq s(\rho(\kappa,\zeta)+\rho(\iota,\zeta)),\forall\kappa,\iota,\zeta\in\mathscr{M}$,

where $s\geq1$ is a given real number.

It is usual that $(\mathscr{M},\rho)$ is called a b-metric space with coefficient $s\geq1$.

Example 2.2. ([29]) Let $\mathscr{M}$ be a nonempty set, and let λ be a metric on $\mathscr{M}$. For any $\kappa,\iota\in\mathscr{M}$, define $\rho(\kappa,\iota)=(\lambda(\kappa,\iota))^q$, where $q>1$ is a real number. It is clear that $\rho(\kappa,\iota)$ is a b-metric function with coefficient $s=2^{q-1}$.

Definition 2.3. ([30]) Let $(\mathscr{M},\rho)$ be a b-metric space with coefficient $s\geq1$. Let $\{\kappa_n\}$ in $\mathscr{M}$ be a sequence and $\kappa\in\mathscr{M}$. Then

(i) $\{\kappa_n\}$ is b-convergent to κ iff $\lim_{n\to+\infty}\rho(\kappa_n,\kappa)=0$;

(ii) $\{\kappa_n\}$ is Cauchy iff $\rho(\kappa_i,\kappa_j)\to0$ as $i,j\to+\infty$;

(iii) $(\mathscr{M},\rho)$ is complete iff each Cauchy sequence is a b-convergent sequence.

Definition 2.4. ([23]) Let $\Theta(\mu_1,\mu_2,\cdots,\mu_N)$ be a multivariate metric function on $\mathbb{R}^{+N}$. For any $\mu_i,\nu_i,\mu_i^n,\mu\in\mathbb{R}^+,i\in\{1,2,\cdots,N\}$, if

(1) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)$ is an increasing function for each variable μ_i;

(2) $\Theta(\mu_1+\nu_1,\mu_2+\nu_2,\cdots,\mu_N+\nu_N)\leq\Theta(\mu_1,\mu_2,\cdots,\mu_N)+\Theta(\nu_1,\nu_2,\cdots,\nu_N)$;

(3) $\Theta(\mu,\mu,\cdots,\mu)=\mu$;

(4) $\lim_{n\to+\infty}\Theta(\mu_1^n,\mu_2^n,\cdots,\mu_N^n)=0\Leftrightarrow\lim_{n\to+\infty}\mu_i^n=0,i\in\{1,2,\cdots,N\}$,

then Θ is said to be a multivariate semilinear non-negative real function.

The following functions are some basic examples.

Example 2.5. ([23])
(1) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)=\frac{1}{N}\sum_{i=1}^{N}\mu_i$.
(2) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)=\frac{1}{h}\sum_{i=1}^{N}l_i\mu_i$, where $l_i\in[0,1)$ and $0<h=\sum_{i=1}^{N}l_i<1$.
(3) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)=\sqrt{\frac{1}{N}\sum_{i=1}^{N}\mu_i^2}$.
(4) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)=\max\{\mu_1,\mu_2,\cdots,\mu_N\}$.

Definition 2.6. ([23]) Let $T:\mathscr{M}^N\to\mathscr{M}$ be an N-variable mapping. For an element $\vartheta\in\mathscr{M}$, ϑ is called a multivariate fixed point of T if $\vartheta=T(\vartheta,\vartheta,\cdots,\vartheta)$.

Definition 2.7. ([21]) A mapping $\sigma:\mathbb{R}^+\to\mathbb{R}^+$ is called an altering distance mapping if
(1) σ is nondecreasing and continuous;
(2) $\sigma(\upsilon)=0$ iff $\upsilon=0$.

Definition 2.8. ([31]) Let $T,S:\Pi\to\Pi$ be two self-mappings. If, for some $\vartheta\in\Pi$, $T\vartheta=S\vartheta$, then ϑ is called a coincidence point of T and S.

3. Multivariate Fixed Point Theorems

In this section, some new results for multivariate fixed point are provided.

Definition 3.1. Let $\Theta(\mu_1,\mu_2,\cdots,\mu_N)$ be a multivariate metric function on ${\mathbb{R}^+}^N$. For all positive real numbers μ, k, l, μ_i, and ν_i, where $i\in\{1,2,\cdots,N\}$, if
(i) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)$ is nondecreasing for each variable $\mu_i, i\in\{1,2,\cdots,N\}$;
(ii) $\Theta(k\mu_1+l\nu_1,k\mu_2+l\nu_2,\cdots,k\mu_N+l\nu_N)\le k\Theta(\mu_1,\mu_2,\cdots,\mu_N)+l\Theta(\nu_1,\nu_2,\cdots,\nu_N)$;
(iii) $\Theta(\mu,\mu,\cdots,\mu)=\mu$;
(iv) $\Theta(\mu_1,\mu_2,\cdots,\mu_N)=0\Rightarrow\mu_i=0, i\in\{1,2,\cdots,N\}$,
then the function Θ is called a multivariate $\star$-metric function.

Obviously, $\Theta(\mu_1,\mu_2,\cdots,\mu_N)=0\Rightarrow\mu_i=0, i\in\{1,2,\cdots,N\}$ if and only if

$$\lim_{n\to+\infty}\Theta(\mu_1^n,\mu_2^n,\cdots,\mu_N^n)=0\Rightarrow\lim_{n\to+\infty}\mu_i^n=0, i\in\{1,2,\cdots,N\}.$$

Theorem 3.2. *Let $(\mathscr{M},\rho)$ be a complete b-metric space with $s\ge1$. Let $F:\mathscr{M}^N\to\mathscr{M}$ be an N-variable mapping which satisfies*

$$\begin{aligned}\psi(s\rho(F\varepsilon,F\varsigma))\le&\psi(\Theta(\rho(\varepsilon_1,\varsigma_1),\rho(\varepsilon_2,\varsigma_2),\cdots,\rho(\varepsilon_N,\varsigma_N)))\\&-\varphi(\Theta(\rho(\varepsilon_1,\varsigma_1),\rho(\varepsilon_2,\varsigma_2),\cdots,\rho(\varepsilon_N,\varsigma_N))),\forall\varepsilon,\varsigma\in\mathscr{M}^N,\end{aligned}$$

where Θ is a multivariate $\star$-metric function, $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)\in\mathscr{M}^N,\varsigma=(\varsigma_1,\varsigma_2,\cdots,\varsigma_N)\in\mathscr{M}^N$, and ψ and φ are altering distance mappings. In that way, F possesses a unique multivariate fixed point $z\in\mathscr{M}$. Furthermore, for a $\mu_0\in\mathscr{M}^N$, the iteration $\{\mu_n\}\subset\mathscr{M}^N$ converges to $(z,z,\cdots,z)\in\mathscr{M}^N$, and the iterative sequence $\{F\mu_n\}\subset\mathscr{M}$ converges to $w\in\mathscr{M}$, where the *iterations* $\{\mu_n\}$ *and* $\{F\mu_n\}$ *are defined as follows:*

$$\begin{aligned}\mu_1 =&(F\mu_0, F\mu_0, \cdots, F\mu_0),\\ \mu_2 =&(F\mu_1, F\mu_1, \cdots, F\mu_1),\\ &\cdots\\ \mu_{n+1} =&(F\mu_n, F\mu_n, \cdots, F\mu_n).\end{aligned}$$

Proof. The proof is split into five steps.

Step 1. For all $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \in \mathscr{M}^N$ and $(\varsigma_1, \varsigma_2, \cdots, \varsigma_N) \in \mathscr{M}^N$, we define a two variable function D as follows:

$$D((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)) = \Theta(\rho(\varepsilon_1, \varsigma_1), \rho(\varepsilon_2, \varsigma_2), \cdots, \rho(\varepsilon_N, \varsigma_N)).$$

Then we obtain that D is a b-metric function on $\mathscr{M}^N$. In fact, we have

(1) $D((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)) = 0$ if and only if $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)$;

(2) $D((\varsigma_1, \varsigma_2, \cdots, \varsigma_N), (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)) = D((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), (\varsigma_1, \varsigma_2, \cdots, \varsigma_N))$.

Next, we prove the third condition of the b-metric. For all $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), (\varsigma_1, \varsigma_2, \cdots, \varsigma_N), (l_1, l_2, \cdots, l_N) \in \mathscr{M}^N$, one has

$$\begin{aligned}D((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)) =&\Theta(\rho(\varepsilon_1, \varsigma_1), \rho(\varepsilon_2, \varsigma_2), \cdots, \rho(\varepsilon_N, \varsigma_N))\\ \leq&\Theta(s\rho(\varepsilon_1, l_1) + s\rho(l_1, \varsigma_1), s\rho(\varepsilon_2, l_2) + s\rho(l_2, \varsigma_2),\\ &\cdots, s\rho(\varepsilon_N, l_N) + s\rho(l_N, \varsigma_N))\\ \leq&sD((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), (l_1, l_2, \cdots, l_N))\\ &+ sD((l_1, l_2, \cdots, l_N), (\varsigma_1, \varsigma_2, \cdots, \varsigma_N)).\end{aligned}$$

Thus D is a b-metric function on $\mathscr{M}^N$. Suppose that $\{\mu_n\} \subset \mathscr{M}^N$ is Cauchy, so one can obtain

$$\lim_{n,m\to+\infty} D(\mu_n, \mu_m) = \lim_{n,m\to+\infty} \Theta(\rho(\varepsilon_{1,n}, \varepsilon_{1,m}), \rho(\varepsilon_{2,n}, \varepsilon_{2,m}), \cdots, \rho(\varepsilon_{N,n}, \varepsilon_{N,m})) = 0$$

where $\mu_n = (\varepsilon_{1,n}, \varepsilon_{2,n}, \cdots, \varepsilon_{N,n})$ and $\mu_m = (\varepsilon_{1,m}, \varepsilon_{2,m}, \cdots, \varepsilon_{N,m})$. According to the definition of Θ, we have $\lim_{n,m\to+\infty} \rho(\varepsilon_{i,n}, \varepsilon_{i,m}) = 0$ for all $i \in \{1, 2, \cdots, N\}$, which means that each $\{\varepsilon_{i,n}\}$ is Cauchy. The completeness of $(\mathscr{M}, \rho)$ ensures that there exist $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N \in \mathscr{M}$ such that, for all $i \in \{1, 2, \cdots, N\}$, $\lim_{n\to+\infty} \rho(\varepsilon_{i,n}, \varepsilon_i) = 0$. Therefore $\lim_{n\to+\infty} D(\mu_n, \varepsilon) = 0$, where $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \in \mathscr{M}^N$, that is, $(\mathscr{M}^N, D)$ is complete.

Step 2. Let $F^* : \mathscr{M}^N \to \mathscr{M}^N$ be defined as follows:

$$F^*(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) = (F(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), F(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), \cdots, F(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N)),$$

for all $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \in \mathscr{M}^N$. Obviously, F^* is a mapping from $(\mathscr{M}^N, D)$ to $(\mathscr{M}^N, D)$. For each $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N), \varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N) \in \mathscr{M}^N$, we have

$$\begin{aligned}\psi(sD(F^*\varepsilon, F^*\varsigma)) =&\psi(sD((F\varepsilon, F\varepsilon, \cdots, F\varepsilon), (F\varsigma, F\varsigma, \cdots, F\varsigma)))\\ =&\psi(s\Theta(\rho(F\varepsilon, F\varsigma), \rho(F\varepsilon, F\varsigma), \cdots, \rho(F\varepsilon, F\varsigma)))\\ \leq&\psi(\Theta(\rho(\varepsilon_1, \varsigma_1), \rho(\varepsilon_2, \varsigma_2), \cdots, \rho(\varepsilon_N, \varsigma_N)))\\ &- \varphi(\Theta(\rho(\varepsilon_1, \varsigma_1), \rho(\varepsilon_2, \varsigma_2), \cdots, \rho(\varepsilon_N, \varsigma_N)))\\ =&\psi(D(\varepsilon, \varsigma)) - \varphi(D(\varepsilon, \varsigma)).\end{aligned} \tag{3.1}$$

Step 3. For all $\mu_0 \in \mathscr{M}^N$, define the sequence $\mu_{n+1} = F^*\mu_n$. If $\mu_{n_0} = \mu_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $\mu_n = \mu_{n+1} = F^*\mu_n$, that is, F^* has a fixed point μ_n. Without loss of generality, we suppose that $\mu_n \neq \mu_{n+1}$ for any nonnegative integer n. Then

$$\begin{aligned}\psi(D(\mu_n,\mu_{n+1})) \leq & \psi(sD(F^*\mu_{n-1},F^*\mu_n)) \\ \leq & \psi(D(\mu_{n-1},\mu_n)) - \varphi(D(\mu_{n-1},\mu_n)),\end{aligned}$$

which implies $D(\mu_n,\mu_{n+1}) \leq D(\mu_{n-1},\mu_n)$. It follows that $\{D(\mu_n,\mu_{n+1})\}$ is monotonically decreasing and there exists $\wp \geq 0$ satisfying $D(\mu_n,\mu_{n+1}) \to \wp$ as $n \to +\infty$. As a consequence, we obtain $\psi(\wp) \leq \psi(\wp) - \varphi(\wp)$, a contradiction except $\wp = 0$. Therefore, $\lim_{n\to+\infty} D(\mu_n,\mu_{n+1}) = 0$.

Step 4. We prove that $\{\mu_n\}$ is Cauchy. Suppose that there exists $\ell > 0$, and then we can obtain $\{\mu_{m_k}\}$ and $\{\mu_{n_k}\}$ of $\{\mu_n\}$ such that $n_k > m_k > k$, $D(\mu_{m_k},\mu_{n_k}) \geq \ell$, and $D(\mu_{m_k},\mu_{n_k-1}) < \ell$. Thus

$$\ell \leq D(\mu_{m_k},\mu_{n_k}) \leq sD(\mu_{m_k},\mu_{n_k-1}) + sD(\mu_{n_k-1},\mu_{n_k}) < s\ell + sD(\mu_{n_k-1},\mu_{n_k}).$$

It follows that $\ell \leq \limsup_{k\to+\infty} D(\mu_{m_k},\mu_{n_k}) \leq s\ell$ and $\frac{\ell}{s} \leq \limsup_{k\to+\infty} D(\mu_{m_k},\mu_{n_k-1}) \leq \ell$. Similarly, we have

$$D(\mu_{m_k},\mu_{n_k}) \leq sD(\mu_{m_k},\mu_{m_k-1}) + s^2D(\mu_{m_k-1},\mu_{n_k-1}) + s^2D(\mu_{n_k-1},\mu_{n_k})$$

and

$$D(\mu_{m_k-1},\mu_{n_k-1}) \leq sD(\mu_{m_k-1},\mu_{m_k}) + sD(\mu_{m_k},\mu_{n_k-1}),$$

so

$$\frac{\ell}{s^2} \leq \limsup_{k\to+\infty} D(\mu_{m_k-1},\mu_{n_k-1}) \leq s\ell.$$

In view of (3.1), one can deduce

$$\psi(sD(\mu_{m_k},\mu_{n_k})) \leq \psi(D(\mu_{m_k-1},\mu_{n_k-1})) - \varphi(D(\mu_{m_k-1},\mu_{n_k-1})).$$

In the same way, we obtain $\psi(s\ell) \leq \psi(s\ell) - \varphi(\frac{\ell}{s^2})$, which shows that $\frac{\ell}{s^2} = 0, i.e., \ell = 0$, a contradiction. Thus, $\{\mu_n\}$ is Cauchy.

Step 5. On the basis of completeness of $\mathscr{M}^N$, one can choose a $\mu \in \mathscr{M}^N$ satisfying $D(\mu_n,\mu) \to 0$ as $n \to +\infty$. In view of

$$\psi(D(F^*\mu_n,F^*\mu)) \leq \psi(D(\mu_n,\mu)) - \varphi(D(\mu_n,\mu)),$$

we have $\lim_{n\to+\infty} D(F^*\mu_n,F^*\mu) = 0$. Again, $D(\mu,F^*\mu) \leq sD(\mu,\mu_{n+1}) + sD(\mu_{n+1},F^*\mu)$. It follows that

$$0 \leq \lim_{n\to+\infty} D(\mu,F^*\mu) \leq s\lim_{n\to+\infty} D(\mu,\mu_{n+1}) + s\lim_{n\to+\infty} D(\mu_{n+1},F^*\mu) = 0.$$

Thus $\mu = F^*\mu$, i.e., F^* has a fixed point. Assume that ν is a fixed point of F^* with $\nu \neq \mu$, which yields

$$\psi(D(\mu,\nu)) \leq \psi(D(\mu,\nu)) - \varphi(D(\mu,\nu)),$$

which indicates $\mu = \nu$. Thus, F^* possesses the unique fixed point μ. For any $e_1,e_2,\cdots,e_N \in \mathscr{M}$, we see that $\mu_0 = (e_1,e_2,\cdots,e_N) \in \mathscr{M}^N$ and the iterative sequence $\mu_{n+1} = F^*\mu_n$. By contraction condition (3.1), one can deduce

$$\psi(D(\mu_n,\mu)) \leq \psi(D(\mu_{n-1},\mu)) - \varphi(D(\mu_{n-1},\mu)).$$

Taking the limit as $n \to +\infty$, one has $\lim_{n\to+\infty} D(\mu_n, \mu) = 0$. It follows that

$$\begin{aligned}\mu_1 =& F^*\mu_0 = (F\mu_0, F\mu_0, \cdots, F\mu_0),\\ \mu_2 =& F^*\mu_1 = (F\mu_1, F\mu_1, \cdots, F\mu_1),\\ &\cdots\\ \mu_{n+1} =& F^*\mu_n = (F\mu_n, F\mu_n, \cdots, F\mu_n),\end{aligned}$$

converges to $\mu \in \mathscr{M}^N$. Due to the special form of $\{\mu_n\}$, we can easily find a unique element $z \in \mathscr{M}$ satisfying $\mu = (z, z, \cdots, z)$. Therefore, $\{F\mu_n\}$ is convergent to $z \in \mathscr{M}$. By

$$F^*\mu = (F\mu, F\mu, \cdots, F\mu) = (z, z, \cdots, z), F\mu = F(z, z, \cdots, z),$$

we obtain $z = F(z, z, \cdots, z)$, that is, z is the unique multivariate fixed point of F. □

Example 3.3. Suppose $\mathscr{M} = [0, \frac{1}{2}]$ and $\rho(e,h) = (e-h)^2$. Define mappings $F : \mathscr{M}^2 \to \mathscr{M}$ and $\Theta : \mathbb{R}^{+2} \to \mathbb{R}^+$ by

$$F(e,h) = \frac{1}{12}(e+h)^2, \forall e,h \in \mathscr{M}, \quad \Theta(\varepsilon, \nu) = \frac{6}{5}(\frac{\varepsilon}{2} + \frac{\nu}{3}), \forall \varepsilon, \nu \in \mathbb{R}^+.$$

Define mappings $\psi(\iota) = \sqrt{\frac{\iota}{2}}$ and $\varphi(\iota) = \frac{5\iota}{18}$, for $\iota \in [0, +\infty)$.

Obviously, ρ is a b-metric function and $s = 2$. For any $e, h \in \mathscr{M}^2$ with $e = (\varepsilon_1, \varepsilon_2)$ and $h = (\varsigma_1, \varsigma_2)$, we have

$$\begin{aligned}\psi(s\rho(Fe,Fh)) &= \sqrt{\rho(\frac{1}{12}(\varepsilon_1+\varepsilon_2)^2, \frac{1}{12}(\varsigma_1+\varsigma_2)^2)}\\ &= \frac{1}{12}\sqrt{((\varepsilon_1+\varepsilon_2)^2 - (\varsigma_1+\varsigma_2)^2)^2}\\ &= \frac{1}{12}\cdot(\varepsilon_1+\varepsilon_2+\varsigma_1+\varsigma_2)\cdot(|\varepsilon_1-\varsigma_1| + |\varepsilon_2-\varsigma_2|)\\ &\le \frac{1}{6}\cdot(|\varepsilon_1-\varsigma_1| + |\varepsilon_2-\varsigma_2|)\\ &\le \frac{\sqrt{2}}{6}\cdot\sqrt{(\varepsilon_1-\varsigma_1)^2 + (\varepsilon_2-\varsigma_2)^2}\end{aligned}$$

and

$$\begin{aligned}&\psi(\Theta(\rho(\varepsilon_1,\varsigma_1), \rho(\varepsilon_2,\varsigma_2))) - \varphi(\Theta(\rho(\varepsilon_1,\varsigma_1), \rho(\varepsilon_2,\varsigma_2)))\\ =& \sqrt{\frac{1}{2}\cdot\frac{6}{5}\cdot(\frac{(\varepsilon_1-\varsigma_1)^2}{2} + \frac{(\varepsilon_2-\varsigma_2)^2}{3})} - \frac{5}{18}\cdot\frac{6}{5}\cdot(\frac{(\varepsilon_1-\varsigma_1)^2}{2} + \frac{(\varepsilon_2-\varsigma_2)^2}{3})\\ \ge& \frac{\sqrt{5}}{5}\cdot\sqrt{(\varepsilon_1-\varsigma_1)^2 + (\varepsilon_2-\varsigma_2)^2} - \frac{1}{6}\cdot\sqrt{(\varepsilon_1-\varsigma_1)^2 + (\varepsilon_2-\varsigma_2)^2}\\ \ge& \frac{\sqrt{2}}{6}\cdot\sqrt{(\varepsilon_1-\varsigma_1)^2 + (\varepsilon_2-\varsigma_2)^2}.\end{aligned}$$

Thus

$$\psi(s\rho(Fe,Fh) \le \psi(\Theta(\rho(\varepsilon_1,\varsigma_1), \rho(\varepsilon_2,\varsigma_2)) - \varphi(\Theta(\rho(\varepsilon_1,\varsigma_1), \rho(\varepsilon_2,\varsigma_2))), \forall e,h \in \mathscr{M}^2.$$

When $s = 2$ and $N = 2$, we can conclude that Theorem 3.2 is valid. It is also easy to see that F possesses the unique multivariate fixed point 0.

In Theorem 3.2, if $\psi(\iota)=2\iota,\varphi(\iota)=\iota,\Theta(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)=\frac{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_N}{N}$, then we conclude the following results.

Corollary 3.4. *Let $(\mathscr{M},\rho)$ be a complete b-metric space with $s\geq 1$. Let $F:\mathscr{M}^N\to\mathscr{M}$ be an N-variable mapping which satisfies:*

$$\rho(F\varepsilon,F\varsigma)\leq\frac{1}{2s}\cdot\frac{\rho(\varepsilon_1,\varsigma_1)+\rho(\varepsilon_2,\varsigma_2)+\cdots+\rho(\varepsilon_N,\varsigma_N)}{N},\forall\varepsilon,\varsigma\in\mathscr{M}^N,$$

where Θ is a multivariate $\star$-metric function, $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)\in\mathscr{M}^N$, and $\varsigma=(\varsigma_1,\varsigma_2,\cdots,\varsigma_N)\in\mathscr{M}^N$. In that way, F possesses a unique multivariate fixed point $w\in\mathscr{M}$ and for each $\mu_0\in\mathscr{M}^N$, $\{\mu_n\}\subset\mathscr{M}^N$ is convergent to $(w,w,\cdots,w)\in\mathscr{M}^N$, and $\{F\mu_n\}\subset\mathscr{M}$ is convergent to $w\in\mathscr{M}$, where the iterative sequences $\{\mu_n\}$ and $\{F\mu_n\}$ are defined as follows:

$$\begin{aligned}\mu_1=&(F\mu_0,F\mu_0,\cdots,F\mu_0),\\ \mu_2=&(F\mu_1,F\mu_1,\cdots,F\mu_1),\\ &\cdots\\ \mu_{n+1}=&(F\mu_n,F\mu_n,\cdots,F\mu_n).\end{aligned}$$

Corollary 3.5. *Let $(\mathscr{M},\rho)$ be a complete metric space. Let $F:\mathscr{M}^N\to\mathscr{M}$ be an N-variable mapping satisfies:*

$$\begin{aligned}\psi(\rho(F\varepsilon,F\varsigma)\leq&\psi(\Theta(\rho(\varepsilon_1,\varsigma_1),\rho(\varepsilon_2,\varsigma_2),\cdots,\rho(\varepsilon_N,\varsigma_N)))\\ &-\varphi(\Theta(\rho(\varepsilon_1,\varsigma_1),\rho(\varepsilon_2,\varsigma_2),\cdots,\rho(\varepsilon_N,\varsigma_N))),\forall\varepsilon,\varsigma\in\mathscr{M}^N,\end{aligned}$$

where Θ is a multivariate metric function, $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)\in\mathscr{M}^N,\varsigma=(\varsigma_1,\varsigma_2,\cdots,\varsigma_N)\in\mathscr{M}^N$ and ψ,φ are altering distance mappings. In that way, F posseses a unique multivariate fixed point $w\in\mathscr{M}$ and for each $\mu_0\in\mathscr{M}^N$, $\{\mu_n\}\subset\mathscr{M}^N$ is convergent to $(w,w,\cdots,w)\in\mathscr{M}^N$, and $\{F\mu_n\}\subset\mathscr{M}$ is convergent to $w\in\mathscr{M}$, where the sequences $\{\mu_n\}$ and $\{F\mu_n\}$ are defined as follows:

$$\begin{aligned}\mu_1=&(F\mu_0,F\mu_0,\cdots,F\mu_0),\\ \mu_2=&(F\mu_1,F\mu_1,\cdots,F\mu_1),\\ &\cdots\\ \mu_{n+1}=&(F\mu_n,F\mu_n,\cdots,F\mu_n).\end{aligned}$$

Remark 3.6. If $\psi(\upsilon)=(h+1)\upsilon$ and $\varphi(\upsilon)=\upsilon$, then Corollary 3.5 reduces to the Theorem 2.6 in [23].

4. An Application

In this section, we show that Corollary 3.4 is applicable to prove the solvability of the integral equation:

$$\varepsilon(\gamma)=\int_{\gamma_0}^{\gamma}\Phi(\varepsilon(r),\varepsilon(r),\cdots,\varepsilon(r),r)dr+\varepsilon_0,\gamma\in[\gamma_0-\kappa,\gamma_0+\kappa],\tag{4.1}$$

which is related to an initial-value problem

$$\begin{cases}\dfrac{d\varepsilon}{d\gamma}=\Phi(\varepsilon(\gamma),\varepsilon(\gamma),\cdots,\varepsilon(\gamma),\gamma),\gamma\in[\gamma_0-\kappa,\gamma_0+\kappa]\\ \varepsilon(\gamma_0)=\varepsilon_0,\end{cases}$$

where $\gamma_0, \kappa > 0$ are constants and $\Phi : \mathbb{R}^N \times [\gamma_0 - \kappa, \gamma_0 + \kappa] \to \mathbb{R}$ is a continuous mapping.

Let $\mathscr{M} = C([\gamma_0 - \kappa, \gamma_0 + \kappa])$ denote the set of all real continuous mappings on $[\gamma_0 - \kappa, \gamma_0 + \kappa]$. The standard metric on $\mathscr{M}$ is given by

$$d(\varepsilon, \varsigma) = \sup_{\gamma \in [\gamma_0 - \kappa, \gamma_0 + \kappa]} |\varepsilon(\gamma) - \varsigma(\gamma)|, \ \forall\ \varepsilon, \varsigma \in \mathscr{M}.$$

For $p \geq 2$, we define

$$\rho(\varepsilon, \varsigma) = (d(\varepsilon, \varsigma))^\beta = \sup_{\gamma \in [\gamma_0 - \kappa, \gamma_0 + \kappa]} |\varepsilon(\gamma) - \varsigma(\gamma)|^\beta \text{ for all } \varepsilon, \varsigma \in \mathscr{M}.$$

Then $(\mathscr{M}, \rho)$ is a complete b-metric space and $s = 2^{\beta - 1}$.

In the following, we define mapping $F : \mathscr{M}^N \to \mathscr{M}$ by

$$F\varepsilon(\gamma) = \int_{\gamma_0}^{\gamma} \Phi(\varepsilon_1(r), \varepsilon_2(r), \cdots, \varepsilon_N(r), r) dr + g(\gamma), \gamma \in [\gamma_0 - \kappa, \gamma_0 + \kappa],$$

where $\varepsilon(\gamma) = (\varepsilon_1(\gamma), \varepsilon_2(\gamma), \cdots, \varepsilon_N(\gamma)) \in \mathscr{M}^N$, $g \in \mathscr{M}$.

Theorem 4.1. *Assume that*

(i) $\Phi : \mathbb{R}^N \times [\gamma_0 - \kappa, \gamma_0 + \kappa] \to \mathbb{R}$ *is a continuous* $(N+1)$*-variables function,*

(ii) there exists $k(\gamma) \in L^1([\gamma_0 - \kappa, \gamma_0 + \kappa], \mathbb{R}^+)$ *such that*

$$\int_{\gamma_0 - \kappa}^{\gamma_0 + \kappa} k(\gamma) d\gamma \leq \frac{1}{2N},$$

and

$$|\Phi(\varepsilon_1(\gamma), \varepsilon_2(\gamma), \cdots, \varepsilon_N(\gamma), \gamma) - \Phi(\varsigma_1(\gamma), \varsigma_2(\gamma), \cdots, \varsigma_N(\gamma), \gamma)| \leq k(\gamma) \Sigma_{n=1}^N |\varepsilon_n(\gamma) - \varsigma_n(\gamma)|.$$

Then equation (4.1) has a unique solution $\varepsilon \in \mathscr{M}$.

Proof. Let $\varepsilon, \varsigma \in \mathscr{M}^N$ with $\varepsilon(\gamma) = (\varepsilon_1(\gamma), \varepsilon_2(\gamma), \cdots, \varepsilon_N(\gamma))$ and $\varsigma(\gamma) = (\varsigma_1(\gamma), \varsigma_2(\gamma), \cdots, \varsigma_N(\gamma))$. It follows from (i)-(ii) that

$$\begin{aligned}
|F\varepsilon(\gamma) - F\varsigma(\gamma)|^\beta &= |\int_{\gamma_0}^{\gamma} (\Phi(\varepsilon_1(r), \varepsilon_2(r), \cdots, \varepsilon_N(r), r) - \Phi(\varsigma_1(r), \varsigma_2(r), \cdots, \varsigma_N(r), r)) dr|^\beta \\
&\leq (\int_{\gamma_0}^{\gamma} |\Phi(\varepsilon_1(r), \varepsilon_2(r), \cdots, \varepsilon_N(r), r) - \Phi(\varsigma_1(r), \varsigma_2(r), \cdots, \varsigma_N(r), r)| dr)^\beta \\
&\leq (\int_{\gamma_0}^{\gamma} k(r) \Sigma_{n=1}^N |\varepsilon_n(r) - \varsigma_n(r)| dr)^\beta \\
&\leq (\int_{\gamma_0}^{\gamma} k(r) \Sigma_{n=1}^N \sup_{r \in [\gamma_0 - \kappa, \gamma_0 + \kappa]} |\varepsilon_n(r) - \varsigma_n(r)| dr)^\beta \\
&\leq (\Sigma_{n=1}^N \sup_{r \in [\gamma_0 - \kappa, \gamma_0 + \kappa]} |\varepsilon_n(r) - \varsigma_n(r)|)^\beta (\int_{\gamma_0}^{\gamma} k(r) dr)^\beta \\
&\leq N^{\beta - 1} (\Sigma_{n=1}^N \sup_{r \in [\gamma_0 - \kappa, \gamma_0 + \kappa]} |\varepsilon_n(r) - \varsigma_n(r)|^\beta) (\int_{\gamma_0}^{\gamma} k(r) dr)^\beta \\
&\leq N^\beta (\frac{1}{N} \Sigma_{n=1}^N \sup_{r \in [\gamma_0 - \kappa, \gamma_0 + \kappa]} |\varepsilon_n(r) - \varsigma_n(r)|^\beta) (\int_{\gamma_0}^{\gamma} k(r) dr)^\beta \\
&\leq \frac{1}{2s} (\frac{1}{N} \Sigma_{n=1}^N \rho(\varepsilon_n(r), \varsigma_n(r))),
\end{aligned}$$

which implies that

$$\rho(F\varepsilon(\gamma), F\varsigma(\gamma)) \leq \frac{1}{2s}\left(\frac{1}{N}\Sigma_{n=1}^{N}\rho(\varepsilon_n(r), \varsigma_n(r))\right).$$

Therefore, we obtain all the conditions of Corollary 3.4. In that way, mapping F possesses a unique multivariate fixed point $\varepsilon = (\varepsilon_0, \varepsilon_0, \cdots, \varepsilon_0) \in \mathscr{M}$. In this case, taking $g(\gamma_0) = \varepsilon_0(\gamma_0)$, we see that integral equation (4.1) possesses a unique solution $\varepsilon \in \mathscr{M}$. □

5. Multivariate Common Fixed Point Theorems

Definition 5.1. Let $\mathscr{M}$ be a nonempty set. Let $E : \mathscr{M}^N \to \mathscr{M}$ be an $N-$variable mapping, and let H be a self-mapping on $\mathscr{M}$. An element $e \in \mathscr{M}$ is called a multivariate coincidence point of E and H if $E(e, e, \cdots, e) = He$.

Definition 5.2. Let $\mathscr{M}$ be a nonempty set. Let $E : \mathscr{M}^N \to \mathscr{M}$ be an $N-$variable mapping, and let H be a self-mapping on $\mathscr{M}$. An element $e \in \mathscr{M}$ is called a multivariate common fixed point of E and H if $E(e, e, \cdots, e) = He = e$.

Definition 5.3. Let $\mathscr{M}$ be a nonempty set. Let $E : \mathscr{M}^N \to \mathscr{M}$ be an $N-$variable mapping, and let H be a self-mapping on $\mathscr{M}$. Then E and H are called multivariate commutable if they satisfy $E(He, He, \cdots, He) = HE(e, e, \cdots, e)$ for every $e \in \mathscr{M}$.

Definition 5.4. Let Ψ be the collection of mappings $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(ψ_1) for all $\varsigma > 0$, $\psi(\varsigma) < \varsigma$;

(ψ_2) ψ is strictly increasing and lower semi-continuous.

It is easy to obtain that $\lim_{n\to+\infty}\psi(\varsigma_n) = 0 \Rightarrow \lim_{n\to+\infty}\varsigma_n = 0$ for any bounded sequence $\{\varsigma_n\}$ with $\varsigma_n \in [0, +\infty)$. Also, for $\psi \in \Psi, \Phi_\psi$ denotes the collection $\varphi : [0, +\infty) \to [0, +\infty)$ which satisfies

(φ_1) if $\limsup_{n\to+\infty}\varsigma_n = r > 0$, then $\limsup_{n\to+\infty}\varphi(\varsigma_n) < \psi(r)$;

(φ_2) if $\lim_{n\to+\infty}\varsigma_n = 0$ for $\varsigma_n \in [0, +\infty)$, then $\lim_{n\to+\infty}\varphi(\varsigma_n) = 0$.

Theorem 5.5. *Let $(\mathscr{M}, \rho)$ be a complete b-metric space with $s \geq 1$. Let $E : \mathscr{M}^N \to \mathscr{M}$ be an $N-$variable mapping, and let $H : \mathscr{M} \to \mathscr{M}$ be continuous. Assume that there are $\psi \in \Psi$ and $\varphi \in \Phi_\psi$ satisfying*

$$\psi(s\rho(E\varepsilon, E\varsigma) \leq \varphi(\Theta(\rho(H\varepsilon_1, H\varsigma_1), \rho(H\varepsilon_2, H\varsigma_2), \cdots, \rho(H\varepsilon_N, H\varsigma_N))), \tag{5.1}$$

for all $\varepsilon, \varsigma \in \mathscr{M}^N$, where Θ is a multivariate $\star$-metric function, $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_N) \in \mathscr{M}^N$, and $\varsigma = (\varsigma_1, \varsigma_2, \cdots, \varsigma_N) \in \mathscr{M}^N$. If the following hypotheses are true:

(i) $E(\mathscr{M}^N) \subseteq H(\mathscr{M})$;

(ii) $H(\mathscr{M})$ is closed;

(iii) E and H are multivariate commutable,

then E and H possess a unique multivariate common fixed point $\varepsilon^* \in \mathscr{M}$.

Proof. **Step 1**. Fix $\varepsilon_0 \in \mathscr{M}$ such that $E(\varepsilon_0, \varepsilon_0, \cdots, \varepsilon_0) = \varsigma_1 \in \mathscr{M}$. Since $E(\mathscr{M}^N) \subseteq H(\mathscr{M})$, we can find $\varepsilon_1 \in \mathscr{M}$ with $H\varepsilon_1 = E(\varepsilon_0, \varepsilon_0, \cdots, \varepsilon_0) = \varsigma_1$. Since $E(\mathscr{M}^N) \subseteq H(\mathscr{M})$, we see that $\varepsilon_2 \in \mathscr{M}$ with $H\varepsilon_2 = E(\varepsilon_1, \varepsilon_1, \cdots, \varepsilon_1) = \varsigma_2$. Following this procedure, we can obtain sequences $\{\varepsilon_n\}$ and $\{\varsigma_n\}$ in $\mathscr{M}$ with $H\varepsilon_{n+1} = E(\varepsilon_n, \varepsilon_n, \cdots, \varepsilon_n) = \varsigma_{n+1}$. If $\varsigma_n = \varsigma_{n+1}$ for a $n \in \mathbb{N}$, one can obtain $H\varepsilon_n = E(\varepsilon_{n-1}, \varepsilon_{n-1}, \cdots, \varepsilon_{n-1}) = E(\varepsilon_n, \varepsilon_n, \cdots, \varepsilon_n) = H\varepsilon_{n+1}$, i.e., ε_n is a multivariate coincidence point of E and H.

Step 2. Assume that $\varsigma_n \neq \varsigma_{n+1}$ for all $n \in \mathbb{N}$. By (5.1), we have

$$\begin{aligned}\psi(\rho(H\varepsilon_n, H\varepsilon_{n+1})) \leq& \psi(s\rho(E(\varepsilon_{n-1}, \varepsilon_{n-1}, \cdots, \varepsilon_{n-1}), E(\varepsilon_n, \varepsilon_n, \cdots, \varepsilon_n)))\\ \leq& \varphi(\Theta(\rho(H\varepsilon_{n-1}, H\varepsilon_n), \rho(H\varepsilon_{n-1}, H\varepsilon_n), \cdots, \rho(H\varepsilon_{n-1}, H\varepsilon_n)))\\ =& \varphi(\rho(H\varepsilon_{n-1}, H\varepsilon_n)).\end{aligned}$$

Clearly, $\rho(H\varepsilon_n, H\varepsilon_{n+1}) > 0$ for all $n \in \mathbb{N}$. Then, for any n, we obtain

$$\psi(\rho(H\varepsilon_n, H\varepsilon_{n+1})) \leq \varphi(\rho(H\varepsilon_{n-1}, H\varepsilon_n)) < \psi(\rho(H\varepsilon_{n-1}, H\varepsilon_n)). \tag{5.2}$$

By the monotonicity of ψ, we can easily see that sequence $\{\rho(\varsigma_n, \varsigma_{n+1})\}$ is monotone decreasing and we can find $\upsilon \geq 0$ such that $\rho(\varsigma_n, \varsigma_{n+1}) \to \upsilon$ as $n \to +\infty$. Let us first assume that $\upsilon > 0$. Taking the limit as $n \to +\infty$ in (5.2), we find from the definitions of ψ and φ that

$$\psi(\upsilon) \leq \liminf_{n\to+\infty} \psi(\rho(\varsigma_n, \varsigma_{n+1})) \leq \limsup_{n\to+\infty} \psi(\rho(\varsigma_n, \varsigma_{n+1})) \leq \limsup_{n\to+\infty} \varphi(\rho(\varsigma_{n-1}, \varsigma_n)) < \psi(\upsilon),$$

which is a contradiction. Thus $\rho(\varsigma_n, \varsigma_{n+1}) \to 0$ as $n \to +\infty$.

Step 3. Prove that $\{\varsigma_n\}$ is Cauchy. Assume that there exists $\ell > 0$. One can find $\{\varsigma_{m_k}\}$ and $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ such that $n_k > m_k > k$ and

$$\rho(\varsigma_{m_k}, \varsigma_{n_k}) \geq \ell \text{ and } \rho(\varsigma_{m_k}, \varsigma_{n_k-1}) < \ell.$$

At the same time, n_k is the smallest index, which satisfies the above results. Thus

$$\ell \leq \rho(\varsigma_{m_k}, \varsigma_{n_k}) \leq s\rho(\varsigma_{m_k}, \varsigma_{n_k-1}) + s\rho(\varsigma_{n_k-1}, \varsigma_{n_k}) < s\ell + s\rho(\varsigma_{n_k-1}, \varsigma_{n_k}).$$

It follows that

$$\ell \leq \limsup_{k\to+\infty} \rho(\varsigma_{m_k}, \varsigma_{n_k}) \leq s\ell,$$
$$\frac{\ell}{s} \leq \limsup_{k\to+\infty} \rho(\varsigma_{m_k}, \varsigma_{n_k-1}) \leq \ell.$$

Again, according to the trigonometric inequality, we obtain

$$\rho(\varsigma_{m_k-1}, \varsigma_{n_k-1}) \leq s\rho(\varsigma_{m_k-1}, \varsigma_{m_k}) + s\rho(\varsigma_{m_k}, \varsigma_{n_k-1}),$$
$$\rho(\varsigma_{m_k}, \varsigma_{n_k}) \leq s\rho(\varsigma_{m_k}, \varsigma_{m_k-1}) + s^2\rho(\varsigma_{m_k-1}, \varsigma_{n_k-1}) + s^2\rho(\varsigma_{n_k-1}, \varsigma_{n_k}),$$

so

$$\frac{\ell}{s^2} \leq \limsup_{k\to+\infty} \rho(\varsigma_{m_k-1}, \varsigma_{n_k-1}) \leq s\ell.$$

Similarly, we can deduce

$$\ell \leq \liminf_{k\to+\infty} \rho(\varsigma_{m_k}, \varsigma_{n_k}) \leq s\ell.$$

In view of (5.1), we conclude that

$$\begin{aligned}\psi(s\rho(\varsigma_{m_k}, \varsigma_{n_k})) =& \psi(s\rho(E(\varepsilon_{m_k-1}, \varepsilon_{m_k-1}, \cdots, \varepsilon_{m_k-1}), E(\varepsilon_{n_k-1}, \varepsilon_{n_k-1}, \cdots, \varepsilon_{n_k-1})))\\ \leq& \varphi(\Theta(\rho(H\varepsilon_{m_k-1}, H\varepsilon_{n_k-1}), \cdots, \rho(H\varepsilon_{m_k-1}, H\varepsilon_{n_k-1})))\\ =& \varphi(\rho(H\varepsilon_{m_k-1}, H\varepsilon_{n_k-1})) = \varphi(\rho(\varsigma_{m_k-1}, \varsigma_{n_k-1})).\end{aligned}$$

Consequently,

$$\begin{aligned}\psi(s\ell) \leq & \psi(s \liminf_{k\to+\infty} \rho(\varsigma_{m_k}, \varsigma_{n_k}))\\ = & \psi(s \lim_{k\to+\infty} \rho(\varsigma_{m_{k_i}}, \varsigma_{n_{k_i}}))\\ \leq & \liminf_{k\to+\infty} \psi(s\rho(\varsigma_{m_{k_i}}, \varsigma_{n_{k_i}}))\\ \leq & \limsup_{k\to+\infty} \psi(s\rho(\varsigma_{m_{k_i}}, \varsigma_{n_{k_i}}))\\ \leq & \limsup_{k\to+\infty} \psi(s\rho(\varsigma_{m_k}, \varsigma_{n_k}))\\ \leq & \limsup_{k\to+\infty} \varphi(\rho(\varsigma_{m_k-1}, \varsigma_{n_k-1}))\\ < & \psi(s\ell).\end{aligned}$$

Obviously, this is a contradiction. Thus $\{\varsigma_n\}$ is Cauchy.

Step 4. According to the completeness of $\mathscr{M}$, we can find a $e^* \in \mathscr{M}$ with

$$\lim_{n\to+\infty} \rho(\varsigma_n, \varepsilon^*) = \lim_{n\to+\infty} \rho(H\varepsilon_n, \varepsilon^*).$$

Since $H(\mathscr{M})$ is closed, we deduce $\varepsilon^* \in H(\mathscr{M})$. In light of the continuity of H, we arrive at

$$\lim_{n\to+\infty} \rho(HH\varepsilon_n, H\varepsilon^*) = 0.$$

Use multivariate commutativity of E and H again, we conclude that

$$E(H\varepsilon_n, H\varepsilon_n, \cdots, H\varepsilon_n) = HE(\varepsilon_n, \varepsilon_n, \cdots, \varepsilon_n) = HH\varepsilon_{n+1}.$$

Hence,

$$\begin{aligned}\psi(\rho(HH\varepsilon_{n+1}, E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*))) \leq & \psi(s\rho(HH\varepsilon_{n+1}, E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)))\\ = & \psi(s\rho(E(H\varepsilon_n, H\varepsilon_n, \cdots, H\varepsilon_n), E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)))\\ \leq & \varphi(\Theta(\rho(HH\varepsilon_n, H\varepsilon^*), \cdots, \rho(HH\varepsilon_n, H\varepsilon^*)))\\ = & \varphi(\rho(HH\varepsilon_n, H\varepsilon^*)).\end{aligned}$$

Letting $n \to +\infty$, we have

$$\lim_{n\to+\infty} \rho(HH\varepsilon_{n+1}, E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)) = 0.$$

From the special form of triangular inequality, we conclude

$$\rho(H\varepsilon^*, E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)) \leq s\rho(H\varepsilon^*, HH\varepsilon_{n+1}) + s\rho(HH\varepsilon_{n+1}, E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)).$$

Letting $n \to +\infty$, we obtain $\rho(H\varepsilon^*, E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)) = 0$, that is, $H\varepsilon^* = E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)$. Therefore, E and H possess a multivariate coincidence point ε^*.

Step 5. Observe that

$$\begin{aligned}\psi(\rho(H\varepsilon_n, H\varepsilon^*)) \leq & \psi(s\rho(H\varepsilon_n, H\varepsilon^*))\\ = & \psi(s\rho(E(\varepsilon_{n-1}, \varepsilon_{n-1}, \cdots, \varepsilon_{n-1}), E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)))\\ \leq & \varphi(\Theta(\rho(H\varepsilon_{n-1}, H\varepsilon^*), \cdots, \rho(H\varepsilon_{n-1}, H\varepsilon^*)))\\ = & \varphi(\rho(H\varepsilon_{n-1}, H\varepsilon^*)).\end{aligned}$$

Clearly, $\rho(H\varepsilon_n, H\varepsilon^*) > 0$, and

$$\psi(\rho(H\varepsilon_n, H\varepsilon^*)) \leq \varphi(\rho(H\varepsilon_{n-1}, H\varepsilon^*)) < \psi(\rho(H\varepsilon_{n-1}, H\varepsilon^*)). \tag{5.3}$$

By the monotonicity of ψ, we can easily see that $\{\rho(H\varepsilon_n, H\varepsilon^*)\}$ is monotone decreasing. Thus there exists $\theta \geq 0$ satisfying $\rho(H\varepsilon_n, H\varepsilon^*) \to \theta$ as $n \to +\infty$. We can simply assume that $\theta > 0$. According to the properties of ψ and φ, taking the limit as $n \to +\infty$ in (5.3), we have

$$\psi(\theta) \leq \liminf_{n\to+\infty} \psi(\rho(H\varepsilon_n, H\varepsilon^*)) \leq \limsup_{n\to+\infty} \psi(\rho(H\varepsilon_n, H\varepsilon^*)) \leq \limsup_{n\to+\infty} \varphi(\rho(H\varepsilon_{n-1}, H\varepsilon^*)) < \psi(\theta),$$

which is a contradiction. Thus $\rho(H\varepsilon_n, H\varepsilon^*) \to 0$ as $n \to +\infty$. On other hand, we have

$$\rho(H\varepsilon^*, \varepsilon^*) \leq s\rho(H\varepsilon^*, H\varepsilon_n) + s\rho(H\varepsilon_n, \varepsilon^*).$$

Letting $n \to +\infty$, we see that $\rho(H\varepsilon^*, \varepsilon^*) = 0$. Thus $H\varepsilon^* = \varepsilon^* = E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*)$ and ε^* is a multivariate common fixed point of E and H. Letting ς^* be another multivariate common fixed point, we see that

$$\begin{aligned}
\psi(\rho(\varepsilon^*, \varsigma^*)) &\leq \psi(s\rho(E(\varepsilon^*, \varepsilon^*, \cdots, \varepsilon^*), E(\varsigma^*, \varsigma^*, \cdots, \varsigma^*)) \\
&\leq \varphi(\Theta(\rho(H\varepsilon^*, H\varsigma^*), \cdots, \rho(H\varepsilon^*, H\varsigma^*))) \\
&= \varphi(\rho(\varepsilon^*, \varsigma^*)) \\
&< \psi(\rho(\varepsilon^*, \varsigma^*)).
\end{aligned}$$

Therefore, ε^* is a unique multivariate common fixed point. □

Example 5.6. Let $\mathscr{M} = [0, \delta](0 < \delta < 1)$ and $\rho(\varepsilon, \varsigma) = (\varepsilon - \varsigma)^2$. It is obvious that $(\mathscr{M}, \rho)$ is a b-metric space with $s = 2$. Define mappings $E : \mathscr{M}^2 \to \mathscr{M}$, $H : \mathscr{M} \to \mathscr{M}$, and $\Theta : \mathbb{R}^{+2} \to \mathbb{R}^+$ by

$$\begin{aligned}
E(\varepsilon_1, \varepsilon_2) &= \frac{1}{16}(\varepsilon_1^2 + \varepsilon_2^2)^2, \forall \varepsilon_1, \varepsilon_2 \in \mathscr{M}, \\
H(\varepsilon) &= (\delta\varepsilon)^2, \forall \varepsilon \in \mathscr{M}, \\
\Theta(\kappa, \vartheta) &= \max\{\kappa, \vartheta\}, \forall \kappa, \vartheta \in \mathbb{R}^+.
\end{aligned}$$

Let $\psi(\upsilon) = \frac{\upsilon}{4}$ and $\varphi(\upsilon) = \frac{\upsilon}{8}$ for all $\upsilon \in [0, +\infty)$. For any $\varepsilon, \varsigma \in \mathscr{M}^2$ with $\varepsilon = (\varepsilon_1, \varepsilon_2), \varsigma = (\varsigma_1, \varsigma_2)$,

$$\begin{aligned}
\psi(s\rho(E\varepsilon, E\varsigma)) &= \frac{1}{4} \cdot 2 \cdot \rho(\frac{1}{16}(\varepsilon_1^2 + \varepsilon_2^2)^2, \frac{1}{16}(\varsigma_1^2 + \varsigma_2^2)^2) \\
&= \frac{1}{2} \cdot \frac{1}{256}((\varepsilon_1^2 + \varepsilon_2^2)^2 - (\varsigma_1^2 + \varsigma_2^2)^2)^2 \\
&= \frac{1}{2} \cdot \frac{1}{256}(\varepsilon_1^2 + \varepsilon_2^2 + \varsigma_1^2 + \varsigma_2^2)^2 \cdot ((\varepsilon_1^2 - \varsigma_1^2) + (\varepsilon_2^2 - \varsigma_2^2))^2 \\
&\leq \frac{\delta^4}{8} \max\{(\varepsilon_1^2 - \varsigma_1^2)^2, (\varepsilon_2^2 - \varsigma_2^2)^2\},
\end{aligned}$$

and

$$\begin{aligned}
&\varphi(\Theta(\rho(H\varepsilon_1, H\varsigma_1), \rho(H\varepsilon_2, H\varsigma_2)) \\
&= \frac{1}{8} \max\{(\delta^2\varepsilon_1^2 - \delta^2\varsigma_1^2)^2, (\delta^2\varepsilon_2^2 - \delta^2\varsigma_2^2)^2\} \\
&= \frac{\delta^4}{8} \max\{(\varepsilon_1^2 - \varsigma_1^2)^2, (\varepsilon_2^2 - \varsigma_2^2)^2\}.
\end{aligned}$$

It follows that

$$\psi(s\rho(E\varepsilon,E\varsigma)) \leq \varphi(\Theta(\rho(H\varepsilon_1,H\varsigma_1),\rho(H\varepsilon_2,H\varsigma_2))),\forall\varepsilon,\varsigma\in\mathscr{M}^2.$$

This proves that all hypotheses of Theorem 5.5 are fulfilled when $N=2$. Clearly, E and H possess a unique multivariate common fixed point 0.

If $\psi(\upsilon)=\frac{\upsilon}{2},\varphi(\upsilon)=\frac{\upsilon}{4}$ and $\Theta(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=\max\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n\}$ in Theorem 5.5, then we have the following result.

Corollary 5.7. *Let $(\mathscr{M},\rho)$ be a complete b-metric space with $s\geq 1$. Let $E:\mathscr{M}^N\to\mathscr{M}$ be an N-variable mapping, and let $H:\mathscr{M}\to\mathscr{M}$ be a continuous mapping satisfying the following contraction:*

$$\rho(E\varepsilon,E\varsigma)\leq\frac{1}{2s}\max\{\rho(H\varepsilon_1,H\varsigma_1),\rho(H\varepsilon_2,H\varsigma_2),\cdots,\rho(H\varepsilon_N,H\varsigma_N)\},\forall\varepsilon,\varsigma\in\mathscr{M}^N,$$

where Θ is a multivariate $\star-$metric function, $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)\in\mathscr{M}^N$, and $\varsigma=(\varsigma_1,\varsigma_2,\cdots,\varsigma_N)\in\mathscr{M}^N$. If the following hypotheses are true:
(i) $E(\mathscr{M}^N)\subseteq H(\mathscr{M})$;
(ii) $H(\mathscr{M})$ is closed;
(iii) E and H are multivariate commutable,
then E and H possess a unique multivariate common fixed point $\varepsilon^*\in\mathscr{M}$.

Corollary 5.8. *Let $(\mathscr{M},\rho)$ be a complete metric space. Let $E:\mathscr{M}^N\to\mathscr{M}$ be an $N-$variable mapping, and let $H:\mathscr{M}\to\mathscr{M}$ be a continuous mapping that satisfies*

$$\psi(\rho(E\varepsilon,E\varsigma) \leq\varphi(\Theta(\rho(H\varepsilon_1,H\varsigma_1),\rho(H\varepsilon_2,H\varsigma_2),\cdots,\rho(H\varepsilon_N,H\varsigma_N))),\forall\varepsilon,\varsigma\in\mathscr{M}^N,$$

where Θ is a multivariate metric function, $\varepsilon=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_N)\in\mathscr{M}^N$, and $\varsigma=(\varsigma_1,\varsigma_2,\cdots,\varsigma_N)\in\mathscr{M}^N$ and ψ,φ are both altering distance mappings. If the following hypotheses are true:
(i) $E(\mathscr{M}^N)\subseteq H(\mathscr{M})$;
(ii) $H(\mathscr{M})$ is closed;
(iii) E and H are multivariate commutable,
then the two mappings E and H possess a unique multivariate common fixed point $\varepsilon^*\in\mathscr{M}$.

6. Conclusions

In this paper, we introduced a multivariate $\star$-metric function and the concept of multivariate common fixed points and established multivariate fixed point and multivariate common fixed point results for (ψ,φ)-weakly contractions in a b-metric space. In addition, we provided examples and applications to support our main results. It is of interest to further consider whether our results can be deduced for the classes of spaces such as b-metric-like spaces, rectangular b-metric spaces, and S-metric spaces.

Acknowledgements
The first author was supported by the Science and Research Project Foundation of Liaoning Province Education Department No. JYTMS20231700.

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