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TWO SELF-ADAPTIVE CQ ALGORITHMS FOR THE SPLIT FEASIBILITY PROBLEM WITH MULTIPLE OUTPUT SETS

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Abstract. The split feasibility problem with multiple output sets is to find a point $x^* \in \bigcap_{i=1}^{t} C_i$ such that $A_j x^* \in Q_j,\ j = 1,2,\ldots,r$ where $C_i \subset H$ and $Q_j \subset H_j$ are nonempty, convex, and closed subsets, H and H_j are Hilbert spaces, and $A_j : H \to H_j$ are linear and bounded operators. In this paper, we present two self-adaptive ball-relaxed CQ algorithms. Under mild conditions, we establish strong convergence and provide numerical experiments to illustrate the effectiveness of the proposed algorithms.

Keywords. Ball-relaxation; CQ Algorithm; Inverse problem; Split feasibility problem with multiple output sets; Self-adaptive step-size.

1. INTRODUCTION

Let C and Q be nonempty, convex, and closed subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded and linear operator with its adjoint A^*. The split feasibility problem (SFP) is to find a vector x^* such that

$$x^* \in C, \qquad Ax^* \in Q. \tag{1.1}$$

The SFP was first introduced by Censor and Elfving [5] for modeling certain inverse problem, which plays an important role in medical image reconstruction and in signal processing [3, 4]. Various algorithms for solving (1.1) have been presented and analyzed recently. Among them, a classical method for solving the SFP is Bryne's CQ algorithm [3, 4] which does not involve inverse matrix. The CQ algorithm is as follows:

$$x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n), \tag{1.2}$$

where P_C denotes the metric projection onto set C, and the step-size τ is in $(0, 2/\|A\|)$.

It is known that the projections onto a general convex and closed subsets might be hard to be implemented. If the convex sets have some particular structures, such as hyperplanes,

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half-spaces, balls, and so on, then they can be calculated explicitly. From the viewpoint of computation, Yang [25] defined two sequences of half-spaces $\{C_n\}$ and $\{Q_n\}$ by

$$C_n = \{x \in H_1 | \ c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\}, \tag{1.3}$$

and

$$Q_n = \{y \in H_2 | \ q(Ax_n) + \langle \eta_n, y - Ax_n \rangle \leq 0\}, \tag{1.4}$$

where $\xi_n \in \partial c(x_n)$, $\eta_n \in \partial q(Ax_n)$, $c : H_1 \to (-\infty, +\infty]$, and $q : H_2 \to (-\infty, +\infty]$ are convex and subdifferentiable functions such that

$$C = \{x \in H_1 : c(x) \leq 0\}, \quad Q = \{y \in H_2 : q(y) \leq 0\}.$$

Yang proved that $C \subset C_n$ and $Q \subset Q_n$ and proposed the half-space relaxed CQ algorithm below:

$$x_{n+1} = P_{C_n}(x_n - \tau A^*(I - P_{Q_n})Ax_n), \tag{1.5}$$

where $\tau \in (0, 2/\|A\|)$.

Yu et al. [26] introduced another ball-relaxed CQ method for solving the SFP under the condition that functions c and q are ν-strongly convex lower semi-continuous and γ-strongly convex lower semi-continuous, respectively. They defined two sequences of closed balls by

$$C_n^b = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n \rangle + \frac{\nu}{2}\|x - x_n\|^2 \ \leq 0\},$$

and

$$Q_n^b = \{y \in H_2 : q(Ax_n) + \langle \eta_n, y - Ax_n \rangle + \frac{\gamma}{2}\|y - Ax_n\|^2 \ \leq 0\},$$

where $\xi_n \in \partial c(x_n)$ and $\eta_n \in \partial q(Ax_n)$. The ball-relaxed CQ algorithm is formulated as follows:

$$x_{n+1} = P_{C_n^b}(x_n - \tau_n^b A^*(I - P_{Q_n^b})Ax_n),$$

where τ_n^b is the step-size.

Note that the step-size τ in (1.2) and (1.5) depends on $\|A\|$, the operator norm, which is hard to compute or estimate in practice. Hence, authors introduced variable step-sizes that does not require the calculation of the operator norm. In particular, López et al. [10] introduced the following variable step-size in (1.2):

$$\tau_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2},$$

where $0 < \rho_n < 4$ and $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$. They proved the weak convergence of their algorithm.

Recently, Ma et al. [15] introduced another kind of step-size which is bounded away from zero as follows:

$$\tau_{n+1} = \begin{cases} \min\{\frac{2\delta f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \Phi_n \tau_n + \Psi_n\}, & \nabla f_n(x_n) \neq 0, \\ \Phi_n \tau_n + \Psi_n \ , & \text{otherwise;} \end{cases}$$

where $\delta \in (0,1)$, $f_n(x_n) = \frac{1}{2}\|(I - P_{Q_n})Ax_n\|^2$, $\{\Phi_n\}$ and $\{\Psi_n\}$ are sequences of nonnegative numbers such that $\{\Phi_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(\Phi_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\Psi_n < \infty$.

In order to improve the convergence rate of the algorithms, the inertial acceleration was widely applied. It was firstly proposed by Polyak in 1964 [17] for solving smooth convex minimization problems. Inertial algorithms are a two-step iterative method and the next iterative is defined by making use of the previous two iterates.

In [1], Alvarez and Attouch employed the inertial extrapolation technique for improving the performance of the celebrated proximal point algorithm. In [7], Sun et al. proposed an inertial relaxed CQ algorithm by applying the inertial extrapolation technique in (1.5):

$$\begin{cases} \omega_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = P_{C_n}(\omega_n - \tau A^*(I - P_{Q_n})A\omega_n), \end{cases} \tag{1.6}$$

where $\tau \in (0, 2/\|A\|)$, $\theta_n \in [0, \overline{\theta_n}]$, $\overline{\theta_n} = \min\{\theta, (\max\{n^2\|x_n - x_{n-1}\|^2, n^2\|x_n - x_{n-1}\|\})^{-1}\}, \theta \in (0,1)$, C_n and Q_n are the half-space relaxations defined by (1.3) and (1.4). It was proved that the iterative sequence generated by (1.6) is weakly convergent to a solution of the SFP. There are many inertial algorithms that greatly improved the performance of their non-inertial versions; see, e.g., [8, 11, 14, 21]

The multiple-sets split feasibility problem (MSSFP), introduced by Censor et al. [6], is to find a vector $x^* \in C_i$ such that

$$x^* \in C_i, \quad Ax^* \in Q_j, \quad i = 1, 2, \ldots, t, \ j = 1, 2, \ldots, r, \tag{1.7}$$

where $C_i, \ i = 1, 2, \ldots, t$ and $Q_j, \ j = 1, 2, \ldots, r$ are nonempty, convex and closed subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be bounded and linear operator. Note that x^* solves the MSSFP if and only if the distance from x^* to C_i is zero and the distance from Ax^* to Q_j is also zero. Based on this idea, Censor et al. [6] defined the following proximity function $g(x)$ to measure the distance of a point to all sets:

$$g(x) := \frac{1}{2}\sum_{i=1}^{t} \rho_i \|(I - P_{C_i})x\|^2 + \frac{1}{2}\sum_{j=1}^{r} \pi_j \|(I - P_{Q_j})Ax\|^2,$$

where $\rho_i, \ i = 1, 2, \ldots, t$ and $\pi_j, \ j = 1, 2, \ldots, r$ are all positive constants with $\sum_{i=1}^{t} \rho_i + \sum_{j=1}^{r} \pi_j = 1$. Then the gradient descent method can be applied to the algorithms; see, e.g., [12, 23, 24] for pertinent results.

In 2020, Reich and Tuyen [20] proposed and studied the following split problem: let X and $X_j, \ j = 1, 2, \ldots, N$ be Banach or Hilbert spaces, and let $A_j : X \to X_j, \ j = 1, 2, \ldots, N$ be mappings from X to X_j. Suppose that (P) and $(P_j), \ j = 1, 2, \ldots, N$ are $N+1$ problems on X and X_j, respectively. Find a vector $x^* \in X$ such that x^* is a solution to (P) and $A_j x^*$ is a solution to (P_j) for all $j = 1, 2, \ldots, N$. As a special case of the split problem above, Reich et al. [18, 19, 20] proposed and studied the following split feasibility problem with multiple output sets in Hilbert spaces: let C and $Q_j, \ j = 1, 2, \ldots, N$ be nonempty, convex and closed subsets of real Hilbert spaces H and $H_j, \ j = 1, 2, \ldots, N$, respectively, and let $A_j : H \to H_j, \ j = 1, 2, \ldots, N$ be bounded and linear operators. Find a vector x^* such that

$$x^* \in C, \quad A_j x^* \in Q_j, \quad j = 1, 2, \ldots, N. \tag{1.8}$$

In this paper, we investigate the following problem: let $C_i, \ i = 1, 2, \ldots, t$ and $Q_j, \ j = 1, 2, \ldots, r$ be nonempty, convex and closed subsets of real Hilbert spaces H and $H_j, \ j = 1, 2, \ldots, r$, respectively, and let $A_j : H \to H_j, \ j = 1, 2, \ldots, r$ be bounded and linear operators with their adjoint A_j^*. Find a vector $x^* \in H$ with the property

$$x^* \in C_i, \quad A_j x^* \in Q_j, \quad i = 1, 2, \ldots, t, \ j = 1, 2, \ldots, r. \tag{1.9}$$

If $i = j = 1$, then problem (1.9) is reduced to problem (1.1). If $A_j \equiv A$, then problem (1.9) is reduced to problem (1.7). If $i = 1$, then problem (1.9) is reduced to problem (1.8). Let Ω denote the set of solutions of problem (1.9). Throughout this paper, one always assumes that $\Omega \neq \emptyset$. Note that x^* is a solution to problem (1.9) if and only if the distance from x^* to C_i is zero and the distance from $A_j x^*$ to Q_j is also zero. Similar with MSSFP (1.7), we can also define a proximity function $h(x)$ to measure the distance of a point to all sets:

$$h(x) := \frac{1}{2}\sum_{i=1}^{t} \rho_i \|(I - P_{C_i^b})x\|^2 + \frac{1}{2}\sum_{j=1}^{r} \beta_j \|(I - P_{Q_j^b})A_j x\|^2,$$

where $\rho_i,\ i = 1,2,\dots,t$ and $\beta_j,\ j = 1,2,\dots,r$ are all positive constants with $\sum_{i=1}^{t} \rho_i = 1$, $\sum_{j=1}^{r} \beta_j = 1$, C_i^b and Q_j^b here are the ball relaxations of C_i and Q_j defined as in (1.10) and (1.11) below. This proximity function is convex and differentiable with gradient

$$\nabla h(x) = \sum_{i=1}^{t} \rho_i (I - P_{C_i^b})x + \sum_{j=1}^{r} \beta_j A_j^* (I - P_{Q_j^b})A_j x.$$

In this paper, we assume that the nonempty, convex and closed subsets C_i and Q_j are defined by:

$$C_i = \{x \in H : c_i(x) \leq 0\} \quad \text{and} \quad Q_j = \{y \in H_j : q_j(y) \leq 0\},$$

where $c_i : H \to (-\infty, +\infty],\ i = 1,2,\dots,t,$ and $q_j : H_j \to (-\infty, +\infty],\ j = 1,2,\dots,r$ are ν_i- and γ_j-strongly convex lower semi-continuous functions, respectively, and each c_i and q_j are subdifferentiable on H and H_j, respectively.

Motivated by the algorithm proposed in [22] and [26], we introduce two ball-relaxed CQ algorithms for solving the problem (1.9) in which the metric projections were computed onto the closed balls C_i^b and Q_j^b instead of the closed set C_i and Q_j, respectively. For all $n \in \mathbb{N}$, the balls $C_{i,n}^b$ are defined by

$$C_{i,n}^b = \{x \in H_1 : c_i(x_n) + \langle \xi_{i,n}, x - x_n \rangle + \frac{\nu_i}{2}\|x - x_n\|^2 \ \leq 0\}, \tag{1.10}$$

where $\xi_{i,n} \in \partial c_i(x_n)$ is the subgradient of c_i at x_n, and

$$Q_{j,n}^b = \{y \in H_2 : q_j(A_j x_n) + \langle \eta_{j,n}, y - A_j x_n \rangle + \frac{\gamma_j}{2}\|y - A_j x_n\|^2 \ \leq 0\}, \tag{1.11}$$

where $\eta_{j,n} \in \partial q_j(A_j x_n)$ is the subgradient of q_j at $A_j x_n$. In our algorithms, the step-size is motivated by Ma et al. [15], which is bounded from zero. Under some mild conditions, we establish strong convergence theorems of the proposed algorithms.

The paper is arranged as follows. In Section 2, some basic concepts and lemmas are proposed. The main results are presented in Section 3. Numerical experiments are provided in Section 4, the last section.

2. Preliminaries

In this section, we recall some definitions and basic results that are used in this paper. Throughout this paper, we always assume that H is a real Hilbert space. We borrow the symbols $\rightharpoonup$ and $\to$ to represent the weak and strong convergence, respectively. For any sequence $\{x_n\}$, let $\omega_n(x_n)$ be the set of the weak cluster points of $\{x_n\}$, that is, $\omega_n(x_n) = \{x \mid \exists \{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup x,\ n_i \to \infty\}$.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H, and $T : C \to H$ be a mapping and recall the following definitions.

(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(2) T is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle, \quad \forall x, y \in C,$$

where I denotes the identity mapping.

For each $x \in H$, there exists a unique point $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u \in C} \|x - u\|. \tag{2.1}$$

The mapping $P_C : H \to C$ defined by (2.1) is called the metric projection of H onto C.

We denote the set of fixed points of operator T by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in C \mid Tx = x\}$. Clearly, $\mathrm{Fix}(P_C) = C$. Moreover, the metric projection P_C has the following well-known properties.

Lemma 2.1. [2] *Let C be a nonempty, convex and closed subset of a real Hilbert space H, and let P_C be the metric projection from H onto C. Then, for all $x, y \in H$ and $z \in C$,*
(1) $\langle x - P_C x, z - P_C x \rangle \leq 0$;
(2) $\|P_C x - P_C y\| \leq \|x - y\|$;
(3) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle$;
(4) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|x - P_C x\|^2$;
(5) $\|(I - P_C)x - (I - P_C)y\|^2 \leq \langle x - y, (I - P_C)x - (I - P_C)y \rangle$.

Definition 2.2. [2] Let $f : H \to (-\infty, +\infty]$ be a proper function.

(1) f is lower semi-continuous at x if $x_n \to x$ implies $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

(2) f is weakly lower semi-continuous at x if $x_n \rightharpoonup x$ implies $f(x) \leq \liminf_{n \to \infty} f(x_n)$.

(3) f is lower semi-continuous on H if it is lower semi-continuous at every point $x \in H$ and f is weakly lower semi-continuous on H if it is weakly lower semi-continuous at every point $x \in H$.

Lemma 2.3. [2] *Let $f : H \to (-\infty, +\infty]$ be a proper and convex function. Then f is lower semi-continuous if and only if it is weakly lower semi-continuous.*

Definition 2.4. Let $f : H \to (-\infty, +\infty]$ be a proper function. A vector $u \in H$ is a subgradient of f at point x if $f(y) \geq f(x) + \langle u, y - x \rangle$ for all $y \in H$. The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the subdifferential of f at x. If $\partial f(x) \neq \emptyset$, then f is said to be subdifferentiable at x.

If the function f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$.

Lemma 2.5. *Let $g : H \to (-\infty, +\infty]$ be a strongly convex function with constant β. Then, for all $x, y \in H$,*

$$g(y) \geq g(x) + \langle \xi, y - x \rangle + \frac{\beta}{2} \|y - x\|^2, \ \xi \in \partial g(x).$$

Lemma 2.6. *Let H be a real Hilbert spaces. Then, for all $x, y \in H$,*
(1) $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\langle x, y\rangle$;
(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(3) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2,\ \forall \alpha \in \mathbb{R}$;
(4) $\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{i<j} \alpha_i \alpha_j \|x_i - x_j\|^2,\ \forall \alpha_i \in \mathbb{R},\ \sum_{i=1}^m \alpha_i = 1,\ x_i \in H$.

Lemma 2.7. [16] *Let $\{\varphi_n\}$ be a sequence of nonnegative numbers fulfilling:*

$$\varphi_{n+1} \leq \Phi_n \varphi_n + \Psi_n,\ \forall n \in \mathbb{N},$$

where $\{\Phi_n\}$ and $\{\Psi_n\}$ are sequences of nonnegative numbers with $\{\Phi_n\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(\Phi_n - 1) < \infty$, and $\sum_{n=1}^{\infty} \Psi_n < \infty$. Then $\lim_{n\to\infty} \varphi_n$ exists.

Lemma 2.8. [9] *Assume that $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n b_n, \quad n \geq 0,$$

$$s_{n+1} \leq s_n - \delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\delta_n\}$ is a sequence of nonnegative real numbers, $\{\alpha_n\}, \{b_n\}$ and $\{\gamma_n\}$ are three sequences in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \gamma_n = 0$;
(iii) $\lim_{i\to\infty} \delta_{n_i} = 0$ *yields* $\limsup_{i\to\infty} b_{n_i} \leq 0$ *for any subsequence $\{n_i\}$ of $\{n\}$.*
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.9. [13] *Let $\{s_n\}$ be a sequence of nonnegative real numbers such that $s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n b_n$, $n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{b_n\}$ is a sequence of nonnegative real numbers such that*
(i) $\sum_{n=0}^{\infty} |\alpha_n b_n| < \infty$, *or* $\limsup_{n\to\infty} b_n \leq 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then $\lim_{n\to\infty} s_n = 0$.

3. Algorithms and Their Convergence

In this section, we introduce two inertial Halpern-type ball-relaxed CQ algorithms for solving the split feasibility problem with multiple output sets in Hilbert spaces, and prove their strong convergence under some mild conditions.

Algorithm 3.1. *Step 0. Take $\tau_1 > 0$, $\lambda_1 > 0$, $\delta_1, \delta_2, \delta_3 \in (0,1)$, $\{\theta_n\} \subset [0, \overline{\theta}) \subset [0,1)$ and $\{\alpha_n\} \in (0,1)$. Choose the sequence $\{\Phi_n\}, \{\overline{\Phi_n}\}$ and $\{\Psi_n\}, \{\overline{\Psi_n}\}$ satisfying Lemma 2.7. Give x_0, x_1, and $u \in H$ arbitrarily. Let the integer $n \geq 1$.*
Step 1. Compute $\omega_n = x_n + \theta_n(x_n - x_{n-1})$.
Step 2. Compute $u_{i,n} = P_{C_{i,n}^b} \omega_n$ for all $i = 1, 2, \ldots, t$ and set

$$d_{1,n} = \max_{i=1,\ldots,t} \{\|u_{i,n} - \omega_n\|\}, \tag{3.1}$$

$$L_{1,n} = \{i \in \{1, \ldots, t\} : \|u_{i,n} - \omega_n\| = d_{1,n}\}.$$

Step 3. Compute $v_{j,n} = P_{Q_{j,n}^b} A_j \omega_n$ for all $j = 1, 2, \ldots, r$ and set

$$d_{2,n} = \max_{j=1,\ldots,r} \{\|v_{j,n} - A_j \omega_n\|\}, \tag{3.2}$$

$$L_{2,n} = \{j \in \{1,...,r\} : \|v_{j,n} - A_j\omega_n\| = d_{2,n}\}.$$

Step 4. Let $\Gamma_n := \max\{d_{1,n}, d_{2,n}\}$. *If* $\Gamma_n = d_{1,n}$, *compute*

$$z_n = \omega_n - \tau_n \sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n, \quad \rho_{i,n} > \delta_3, \sum_{i\in L_{1,n}} \rho_{i,n} = 1,$$

where $\tau_1 = \tau_0$, *and for* $n \geq 2$,

$$\tau_{n+1} = \begin{cases} \min\{\frac{2\delta_1(d_{1,n})^2}{\|\sum_{i\in L_{1,n}}\rho_{i,n}(I-P_{C^b_{i,n}})\omega_n\|^2}, \Phi_n\tau_n + \Psi_n\}, & \|\sum_{i\in L_{1,n}}\rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2 \neq 0, \\ \Phi_n\tau_n + \Psi_n \ , & \text{otherwise.} \end{cases} \tag{3.3}$$

If $\Gamma_n = d_{2,n}$, *compute*

$$z_n = \omega_n - \lambda_n \sum_{j\in L_{2,n}} \beta_{j,n}A^*_j(I - P_{Q^b_{j,n}})A_j\omega_n, \quad \beta_{j,n} > \delta_3, \sum_{j\in L_{2,n}} \beta_{j,n} = 1,$$

where $\lambda_1 = \lambda_0$, *and for* $n \geq 2$,

$$\lambda_{n+1} = \begin{cases} \min\{\frac{2\delta_2(d_{2,n})^2}{\|\sum_{j\in L_{2,n}}\beta_{j,n}A^*_j(I-P_{Q^b_{j,n}})A_j\omega_n\|^2}, \overline{\Phi_n}\lambda_n + \overline{\Psi_n}\}, & \|\sum_{j\in L_{2,n}}\beta_{j,n}A^*_j(I - P_{Q^b_{j,n}})A_j\omega_n\|^2 \neq 0, \\ \overline{\Phi_n}\lambda_n + \overline{\Psi_n} \ , & \text{otherwise.} \end{cases}$$

Step 5. Compute

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \sum_{i\in L_{1,n}} \rho_{i,n}P_{C^b_{i,n}} z_n,$$

set $n \leftarrow n + 1$, *and go to Step 1.*

The following proposition shows the property of sequences $\{\tau_n\}$ and $\{\lambda_n\}$, which is useful to the proof of our convergence theorems.

Proposition 3.2. *Let* $\{\tau_n\}$ *and* $\{\lambda_n\}$ *be the sequences generated by Algorithm 3.1. Then* $\lim_{n\to\infty}\tau_n = \tau$, $\lim_{n\to\infty}\lambda_n = \lambda$ *with* $\tau \geq \min\{2\delta_1, \tau_1\} > 0$, $\lambda \geq \min\{\frac{2\delta_2}{M^2}, \lambda_1\} > 0$, *where* $M = \max_{1\leq j\leq n}\|A_j\|$.

Proof. The definition of $d_{1,n}$ yields that

$$d_{1,n} = \sum_{i\in L_{1,n}} \rho_{i,n}\|(I - P_{C^b_{i,n}})\omega_n\| \geq \|\sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|.$$

In the case of $\|\sum_{i\in L_{1,n}}\rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2 \neq 0$, we obtain that

$$\frac{2\delta_1(d_{1,n})^2}{\|\sum_{i\in L_{1,n}}\rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2} \geq 2\delta_1 > 0.$$

This further yields that

$$\begin{aligned} \tau_{n+1} &= \min\{\frac{2\delta_1(d_{1,n})^2}{\|\sum_{i\in L_{1,n}}\rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2}, \Phi_n\tau_n + \Psi_n\} \\ &\geq \min\{2\delta_1, \tau_n\}, \end{aligned}$$

where we used the assumptions $\Phi_n \geq 1$ and $\Psi_n \geq 0$.

Next, we prove that sequence $\{\tau_n\}$ has a lower bound $\min\{2\delta_1, \tau_1\}$. In fact, if $n = 1$, then $\tau_1 \geq \min\{2\delta_1, \tau_1\}$. Suppose that the inequality $\tau_k \geq \min\{2\delta_1, \tau_1\}$ holds for $n = k \geq 1$. When $n = k+1$, one has

$$\tau_{k+1} \geq \min\{2\delta_1, \tau_k\} \geq \min\{2\delta_1, \tau_1\}.$$

By induction, one sees that sequence $\{\tau_n\}$ has a lower bound $\min\{2\delta_1, \tau_1\} > 0$. From (3.3), one has $\tau_{n+1} \leq \Phi_n \tau_n + \Psi_n$. From Lemma 2.7, it follows that $\lim_{n\to\infty} \tau_n$ exists. Setting $\lim_{n\to\infty} \tau_n = \tau$, one has $\tau \geq \min\{2\delta_1, \tau_1\} > 0$.

Similarly, the definition of $d_{2,n}$ yields that

$$d_{2,n} = \sum_{j\in L_{2,n}} \beta_{j,n} \|(I - P_{Q^b_{j,n}}) A_j \omega_n\| \geq \| \sum_{j\in L_{2,n}} \beta_{j,n} (I - P_{Q^b_{j,n}}) A_j \omega_n\|.$$

Thus

$$\begin{aligned} Md_{2,n} &\geq \sum_{j\in L_{2,n}} \beta_{j,n} \|A_j^*\| \|(I - P_{Q^b_{j,n}}) A_j \omega_n\| \\ &\geq \| \sum_{j\in L_{2,n}} \beta_{j,n} A_j^* (I - P_{Q^b_{j,n}}) A_j \omega_n\|, \end{aligned}$$

where $M = \max_{1\leq j\leq n} \|A_j\|$. In the case of $\|\sum_{j\in L_{2,n}} \beta_{j,n} A_j^*(I - P_{Q^b_{j,n}}) A_j \omega_n\|^2 \neq 0$, one concludes that

$$\begin{aligned} \lambda_{n+1} &= \min\{\frac{2\delta_2 (d_{2,n})^2}{\|\sum_{j\in L_{2,n}} \beta_{j,n} A_j^*(I - P_{Q^b_{j,n}}) A_j \omega_n\|^2}, \overline{\Phi_n} \lambda_n + \overline{\Psi_n}\} \\ &\geq \min\{\frac{2\delta_2}{M^2}, \lambda_n\}. \end{aligned}$$

By induction, one can similarly obtains that $\{\lambda_n\}$ has a lower bound $\min\{\frac{2\delta_2}{M^2}, \lambda_1\} > 0$. Hence, $\lim_{n\to\infty} \lambda_n$ exists and

$$\lambda = \lim_{n\to\infty} \lambda_n \geq \min\{\frac{2\delta_2}{M^2}, \lambda_1\} > 0.$$

□

Theorem 3.3. *Assume that the solution set of (1.9) is nonempty, i.e., $\Omega \neq \emptyset$. Suppose that the following conditions hold*
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\theta_n \|x_n - x_{n-1}\|}{\alpha_n} = 0$.
Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $P_\Omega u$.

Proof. Let $z = P_\Omega u \in \Omega$. Since $C_i \subset C^b_{i,n}$ for all $i = 1, \ldots, t$ and $Q_j \subset Q^b_{j,n}$ for all $j = 1, \ldots, r$ we have $z = P_{C_i} z = P_{C^b_{i,n}} z$ and $A_j z = P_{Q_j} A_j z = P_{Q^b_{j,n}} A_j z$. We consider the following two cases. *Case 1.* $\Gamma_n = d_{1,n}$. From Lemma 2.1 (5), we have

$$\begin{aligned} \langle \omega_n - z, \sum_{i\in L_{1,n}} \rho_{i,n} (I - P_{C^b_{i,n}}) \omega_n \rangle &= \langle \omega_n - z, \sum_{i\in L_{1,n}} \rho_{i,n} (I - P_{C^b_{i,n}}) \omega_n - \sum_{i\in L_{1,n}} \rho_{i,n} (I - P_{C^b_{i,n}}) z \rangle \\ &= \sum_{i\in L_{1,n}} \rho_{i,n} \langle \omega_n - z, (I - P_{C^b_{i,n}}) \omega_n - (I - P_{C^b_{i,n}}) z \rangle \\ &\geq \sum_{i\in L_{1,n}} \rho_{i,n} \|(I - P_{C^b_{i,n}}) \omega_n\|^2 \\ &= (d_{1,n})^2, \end{aligned}$$

which together with the definition of τ_{n+1} yields that

$$\begin{aligned}\|z_n - z\|^2 &= \|\omega_n - z - \tau_n \sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2 \\ &= \|\omega_n - z\|^2 + \tau_n^2 \|\sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2 - 2\tau_n\langle \omega_n - z, \sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\rangle \\ &\le \|\omega_n - z\|^2 + \tau_n^2 \|\sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2 - 2\tau_n(d_{1,n})^2 \\ &\le \|\omega_n - z\|^2 + 2\tau_n(d_{1,n})^2(\frac{\delta_1\tau_n}{\tau_{n+1}} - 1).\end{aligned}$$

Note that Proposition 3.2 indicates that

$$\lim_{n\to\infty}(1 - \frac{\delta_1\tau_n}{\tau_{n+1}}) = 1 - \delta_1,$$

with $\delta_1 \in (0,1)$. There exist a positive integer N and $\rho \in (\delta_1, 1)$ such that, for all $n \ge N$, $1 - \frac{\delta_1\tau_n}{\tau_{n+1}} > 1 - \rho > 0$. Thus,

$$\|z_n - z\|^2 \le \|\omega_n - z\|^2 + 2\tau_n(d_{1,n})^2(\rho - 1),\ n \ge N. \tag{3.4}$$

Let $y_n = \sum_{i\in L_{1,n}} \rho_{i,n} P_{C^b_{i,n}} z_n$. From Lemma 2.6 (4), Lemma 2.1 (4), and (3.4), we deduce that, for all $n \ge N$,

$$\begin{aligned}\|y_n - z\|^2 &= \|\sum_{i\in L_{1,n}} \rho_{i,n}(P_{C^b_{i,n}} z_n - P_{C^b_{i,n}} z)\|^2 \\ &\le \sum_{i\in L_{1,n}} \rho_{i,n}\|P_{C^b_{i,n}} z_n - P_{C^b_{i,n}} z\|^2 \\ &\le \sum_{i\in L_{1,n}} \rho_{i,n}(\|z_n - z\|^2 - \|(I - P_{C^b_{i,n}})z_n\|^2) \\ &= \|z_n - z\|^2 - \sum_{i\in L_{1,n}} \rho_{i,n}\|P_{C^b_{i,n}} z_n - \omega_n + \tau_n \sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2 \\ &\le \|\omega_n - z\|^2 + 2\tau_n(d_{1,n})^2(\rho - 1) - \sum_{i\in L_{1,n}} \rho_{i,n}\|P_{C^b_{i,n}} z_n - \omega_n + \tau_n \sum_{i\in L_{1,n}} \rho_{i,n}(I - P_{C^b_{i,n}})\omega_n\|^2.\end{aligned} \tag{3.5}$$

From the definition of ω_n, it follows that

$$\begin{aligned}\|\omega_n - z\| &= \|x_n + \theta_n(x_n - x_{n-1}) - z\| \\ &\le \|x_n - z\| + \theta_n\|x_n - x_{n-1}\| \\ &= \|x_n - z\| + \alpha_n\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|.\end{aligned} \tag{3.6}$$

The condition (ii) reads that there is a constant $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le M_1. \tag{3.7}$$

Rearranging (3.5)-(3.7), we derive that, for all $n \ge N$,

$$\|y_n - z\| \le \|x_n - z\| + \alpha_n M_1,$$

and hence

$$\begin{aligned}\|x_{n+1}-z\| &\le \alpha_n\|u-z\|+(1-\alpha_n)\|y_n-z\|\\ &\le \alpha_n\|u-z\|+(1-\alpha_n)(\|x_n-z\|+\alpha_n M_1)\\ &\le \alpha_n(\|u-z\|+M_1)+(1-\alpha_n)\|x_n-z\|\\ &\le \max\{\|x_n-z\|,\|u-z\|+M_1\}\\ &\le \cdots \le \max\{\|x_N-z\|,\|u-z\|+M_1\}.\end{aligned}$$

This means that sequence $\{x_n\}$ is bounded, so $\{\omega_n\}$ is bounded, too. By Lemma 2.6 (2) and (3.5), we derive that, for all $n \ge N$,

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n)\|y_n-z\|^2+2\alpha_n\langle u-z,x_{n+1}-z\rangle\\ &\le (1-\alpha_n)(\|\omega_n-z\|^2+2\tau_n(d_{1,n})^2(\rho-1)\\ &\quad -\sum_{i\in L_{1,n}}\rho_{i,n}\|P_{C_{i,n}^b}z_n-\omega_n+\tau_n\sum_{i\in L_{1,n}}\rho_{i,n}(I-P_{C_{i,n}^b})\omega_n\|^2)\\ &\quad +2\alpha_n\langle u-z,x_{n+1}-z\rangle.\end{aligned} \tag{3.8}$$

By the definition of ω_n, we have

$$\begin{aligned}\|\omega_n-z\|^2 &= \|x_n+\theta_n(x_n-x_{n-1})-z\|^2\\ &= \|x_n-z\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n\langle x_n-z,x_n-x_{n-1}\rangle\\ &\le \|x_n-z\|^2+\theta_n^2\|x_n-x_{n-1}\|^2+2\theta_n\|x_n-z\|\|x_n-x_{n-1}\|.\end{aligned}$$

Let $M_2=\sup_{n\ge N}\{\beta\|x_n-x_{n-1}\|,\|x_n-z\|\}$. For all $n \ge N$, we have

$$\|\omega_n-z\|^2 \le \|x_n-z\|^2+3M_2\theta_n\|x_n-x_{n-1}\|. \tag{3.9}$$

Substituting (3.9) into (3.8) yields that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n)\|x_n-z\|^2-2(1-\alpha_n)\tau_n(d_{1,n})^2(1-\rho)\\ &\quad +2\alpha_n\langle u-z,x_{n+1}-z\rangle+3M_2\theta_n\|x_n-x_{n-1}\|\\ &\quad -(1-\alpha_n)\sum_{i\in L_{1,n}}\rho_{i,n}\|P_{C_{i,n}^b}z_n-\omega_n+\tau_n\sum_{i\in L_{1,n}}\rho_{i,n}(I-P_{C_{i,n}^b})\omega_n\|^2.\end{aligned} \tag{3.10}$$

Let

$$\begin{aligned}s_n &= \|x_n-z\|^2,\\ \gamma_n &= 3M_2\theta_n\|x_n-x_{n-1}\|+2\alpha_n\langle u-z,x_{n+1}-z\rangle,\\ b_n &= 3M_2\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|+2\langle u-z,x_{n+1}-z\rangle,\end{aligned}$$

and

$$\delta_n = 2(1-\alpha_n)\tau_n(d_{1,n})^2(1-\rho)+(1-\alpha_n)\sum_{i\in L_{1,n}}\rho_{i,n}\|P_{C_{i,n}^b}z_n-\omega_n+\tau_n\sum_{i\in L_{1,n}}\rho_{i,n}(I-P_{C_{i,n}^b})\omega_n\|^2.$$

Then (3.10) can be transformed to

$$s_{n+1}\le (1-\alpha_n)s_n+\alpha_n b_n,\ n\ge 0,$$

and

$$s_{n+1}\le s_n-\delta_n+\gamma_n,\ n\ge 0.$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\delta_{n_k}\leq 0$, which yields that

$$\lim_{k\to\infty} 2(1-\alpha_{n_k})\tau_{n_k}(d_{1,n_k})^2(1-\rho)=0,$$

and

$$\lim_{k\to\infty}(1-\alpha_{n_k})\sum_{i\in L_{1,n_k}}\rho_{i,n}\|P_{C^b_{i,n_k}}z_{n_k}-\omega_{n_k}+\tau_{n_k}\sum_{i\in L_{1,n_k}}\rho_{i,n_k}(I-P_{C^b_{i,n_k}})\omega_{n_k}\|^2=0.$$

By Proposition 3.2 and condition (i), we deduce that

$$\lim_{k\to\infty} d_{1,n_k}=0,$$

and

$$\lim_{k\to\infty}\|P_{C^b_{i,n_k}}z_{n_k}-\omega_{n_k}+\tau_{n_k}\sum_{i\in L_{1,n_k}}\rho_{i,n}(I-P_{C^b_{i,n_k}})\omega_{n_k}\|=0.$$

Thus

$$\lim_{k\to\infty}\|(I-P_{C^b_{i,n_k}})\omega_{n_k}\|=0,\ i=1,2,...,t,$$

and

$$\lim_{k\to\infty}\|(I-P_{Q^b_{j,n_k}})A_j\omega_{n_k}\|=0,\ j=1,2,...,r. \tag{3.11}$$

Now, we prove that $\omega_\omega(x_{n_k})\subset\Omega$. Since $\{x_{n_k}\}$ is bounded, then $\omega_\omega(x_{n_k})\neq\emptyset$. Take $z^*\in\omega_\omega(x_{n_k})$ arbitrarily, there exists a subsequence $\{x_{n_{k_m}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_m}}\rightharpoonup z^*$ and

$$\limsup_{k\to\infty}\langle u-z,x_{n_k}-z\rangle=\lim_{k\to\infty}\langle u-z,x_{n_{k_m}}-z\rangle.$$

The definition of ω_{n_k} and condition (ii) imply that

$$\|x_{n_k}-\omega_{n_k}\|=\theta_{n_k}\|x_{n_k}-x_{n_{k-1}}\|=\alpha_{n_k}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_{k-1}}\|\to 0,\ k\to\infty. \tag{3.12}$$

Thus, $\omega_{n_{k_m}}\rightharpoonup z^*$ and hence $A_j\omega_{n_{k_m}}\rightharpoonup A_jz^*$ for $j=1,2,...,r$. Since $P_{Q^b_{j,n_{k_m}}}A_j\omega_{n_{k_m}}\in Q^b_{j,n_{k_m}}$, it follows from the definition of $Q^b_{j,n_{k_m}}$ that, for each $j=1,...,r$,

$$q_j(A_j\omega_{n_{k_m}})\leq\langle\eta_{j,n_{k_m}},A_j\omega_{n_{k_m}}-P_{Q^b_{j,n_{k_m}}}A_j\omega_{n_{k_m}}\rangle-\frac{\gamma_j}{2}\|A_j\omega_{n_{k_m}}-P_{Q^b_{j,n_{k_m}}}A_j\omega_{n_{k_m}}\|^2, \tag{3.13}$$

where $\eta_{j,n_{k_m}}\in\partial q_j(A_j\omega_{n_{k_m}})$. Since ∂q_j is bounded on bounded sets, there exists a constant $M_3>0$ such that $\|\eta_{j,n_{k_m}}\|\leq M_3$. By (3.11) and (3.13), we obtain

$$q_j(A_j\omega_{n_{k_m}})\leq\|\eta_{j,n_{k_m}}\|\|(I-P_{Q^b_{j,n_{k_m}}})A_j\omega_{n_{k_m}}\|\leq M_3\|(I-P_{Q^b_{j,n_{k_m}}})A_j\omega_{n_{k_m}}\|\to 0,\ m\to\infty.$$

Thus q_j being lower semi-continuous yields that

$$q_j(A_jz^*)\leq\liminf_{m\to\infty}q(A_j\omega_{n_{k_m}})=0.$$

Therefore, $A_jz^*\in Q_j,\ j=1,2,...,r.$

On the other hand, in view of $P_{C^b_{i,n_{k_m}}}\omega_{n_{k_m}}\in C^b_{i,n_{k_m}}$, we obtain that

$$\begin{aligned}
c_i(\omega_{n_{k_m}}) &\leq\langle\xi_{i,n_{k_m}},\omega_{n_{k_m}}-P_{C^b_{i,n_{k_m}}}\omega_{n_{k_m}}\rangle-\frac{\nu_i}{2}\|\omega_{n_{k_m}}-P_{C^b_{i,n_{k_m}}}\omega_{n_{k_m}}\|^2\\
&\leq\|\xi_{i,n_{k_m}}\|\|(I-P_{C^b_{i,n_{k_m}}})\omega_{n_{k_m}}\|\\
&\leq M_4\|(I-P_{C^b_{i,n_{k_m}}})\omega_{n_{k_m}}\|\to 0,\ m\to\infty,
\end{aligned}$$

where $M_4 > 0$ is a constant such that $\|\xi_{i,n_{k_m}}\| \leq M_4$, since ∂c_i is bounded on bounded sets. This together with c_i being weakly lower semi-continuous yields

$$c_i(z^*) \leq \liminf_{m\to\infty} c(\omega_{n_{k_m}}) = 0,\ i = 1,2,...,t.$$

Hence $z^* \in C_i,\ i = 1,2,...,t$. Consequently, $z^* \in \Omega$, and thus $\omega_\omega(x_{n_k}) \subset \Omega$. Therefore, we obtain by Lemma 2.1 (1) that

$$\limsup_{k\to\infty}\langle u-z, x_{n_k}-z\rangle = \lim_{m\to\infty}\langle u-z, x_{n_{k_m}}-z\rangle = \langle u-z, z^*-z\rangle \leq 0. \tag{3.14}$$

Meanwhile, we also have

$$\|y_{n_k}-\omega_{n_k}\| \leq \|y_{n_k}-\omega_{n_k}+\tau_{n_k}\sum_{i\in L_{1,n_k}}\rho_{i,n_k}(I-P_{C^b_{i,n_k}})\omega_{n_k}\| + \tau_{n_k}\sum_{i\in L_{1,n_k}}\rho_{i,n_k}\|(I-P_{C^b_{i,n_k}})\omega_{n_k}\| \to 0 \tag{3.15}$$

as $k\to\infty$. Thus condition (i), (3.12) and, (3.15) read that

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &\leq \|x_{n_k+1}-\omega_{n_k}\| + \|\omega_{n_k}-x_{n_k}\|\\ &= \|\alpha_{n_k}u+(1-\alpha_{n_k})y_{n_k}-\omega_{n_k}\| + \|x_{n_k}-\omega_{n_k}\|\\ &\leq \alpha_{n_k}\|u-\omega_{n_k}\| + (1-\alpha_{n_k})\|y_{n_k}-\omega_{n_k}\| + \|x_{n_k}-\omega_{n_k}\| \to 0,\ k\to\infty.\end{aligned} \tag{3.16}$$

Combining (3.14), (3.16), and condition (i), we obtain

$$\begin{aligned}\limsup_{k\to\infty} b_{n_k} &= \limsup_{k\to\infty}(2\langle u-z, x_{n_k+1}-z\rangle + 3M_2\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|)\\ &= \lim_{k\to\infty} 2\langle u-z, x_{n_k+1}-x_{n_k}\rangle + \limsup_{k\to\infty} 2\langle u-z, x_{n_k}-z\rangle\\ &\quad + 3M_2\lim_{k\to\infty}\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\| \leq 0.\end{aligned}$$

Therefore, it follows from Lemma 2.8 that $\{x_n\}$ converges strongly to the solution $z = P_\Omega u$.

Case 2. $\Gamma_n = d_{2,n}$. In this case, $z_n = \omega_n - \lambda_n\sum_{j\in L_{2,n}}\beta_{j,n}A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n$. From Lemma 2.1 (5), we have

$$\begin{aligned}&\langle\omega_n - z, \sum_{j\in L_{2,n}}\beta_{j,n}A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\rangle\\ &= \sum_{j\in L_{2,n}}\beta_{j,n}\langle A_j\omega_n - A_jz, (I-P_{Q^b_{j,n}})A_j\omega_n - (I-P_{Q^b_{j,n}})A_jz\rangle\\ &\geq \sum_{j\in L_{2,n}}\beta_{j,n}\|(I-P_{Q^b_{j,n}})A_j\omega_n\|^2 = (d_{2,n})^2,\end{aligned}$$

which together with the definition of λ_{n+1} yields that

$$\begin{aligned}&\|z_n - z\|^2\\ &= \|\omega_n - z - \lambda_n\sum_{j\in L_{2,n}}\beta_{j,n}A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\|^2\\ &= \|\omega_n - z\|^2 + \lambda_n^2\|\sum_{j\in L_{2,n}}\beta_{j,n}A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\|^2 - 2\lambda_n\langle\omega_n - z, \sum_{j\in L_{2,n}}\beta_{j,n}A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\rangle\\ &\leq \|\omega_n - z\|^2 + \lambda_n^2\|\sum_{j\in L_{2,n}}\beta_{j,n}A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\|^2 - 2\lambda_n(d_{2,n})^2\\ &\leq \|\omega_n - z\|^2 + 2\lambda_n(d_{2,n})^2(\frac{\delta_2\lambda_n}{\lambda_{n+1}} - 1).\end{aligned}$$

Since $\lim_{n\to\infty}(1-\frac{\delta_2\lambda_n}{\lambda_{n+1}}) = 1-\delta_2$, and $\delta_2 \in (0,1)$, there exist a positive integer $\tilde{N}$ and $\tilde{\rho} \in (\delta_2, 1)$ such that $\forall n \geq \tilde{N}, 1-\frac{\delta_2\lambda_n}{\lambda_{n+1}} > 1-\tilde{\rho} > 0$. Thus,

$$\|z_n - z\|^2 \leq \|\omega_n - z\|^2 + 2\lambda_n(d_{2,n})^2(\tilde{\rho}-1), \; n \geq \tilde{N}. \tag{3.17}$$

Let $y_n = \sum_{i\in L_{1,n}} \rho_{i,n} P_{C^b_{i,n}} z_n$. From (3.17), we deduce that, for all $n \geq \tilde{N}$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|\sum_{i\in L_{1,n}} \rho_{i,n}(P_{C^b_{i,n}} z_n - P_{C^b_{i,n}} z)\|^2 \\
&\leq \sum_{i\in L_{1,n}} \rho_{i,n}(\|z_n - z\|^2 - \|(I-P_{C^b_{i,n}})z_n\|^2) \\
&= \|z_n - z\|^2 - \sum_{i\in L_{1,n}} \rho_{i,n}\|(I-P_{C^b_{i,n}})z_n\|^2 \\
&\leq \|\omega_n - z\|^2 + 2\lambda_n(d_{2,n})^2(\tilde{\rho}-1) - \sum_{i\in L_{1,n}} \rho_{i,n}\|P_{C^b_{i,n}} z_n - \omega_n \\
&\quad + \lambda_n \sum_{j\in L_{2,n}} \beta_{j,n} A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\|^2.
\end{aligned}$$

Similar as in Case 1, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 \leq &(1-\alpha_n)\|x_n - z\|^2 - 2(1-\alpha_n)\lambda_n(d_{2,n})^2(1-\tilde{\rho}) \\
&+ 2\alpha_n\langle u - z, x_{n+1} - z\rangle + 3M_2\theta_n\|x_n - x_{n-1}\| \\
&- (1-\alpha_n)\sum_{i\in L_{1,n}} \rho_{i,n}\|P_{C^b_{i,n}} z_n - \omega_n + \lambda_n \sum_{j\in L_{2,n}} \beta_{j,n} A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\|^2.
\end{aligned}$$

Employing Lemma 2.8 again, we can also conclude that $\{x_n\}$ converges strongly to the solution $z^* = P_\Omega u$. □

Remark 3.4. If, in Case1, $\|\sum_{i\in L_{1,n}} \rho_{i,n}(I-P_{C^b_{i,n}})\omega_n\| = 0$, which means

$$0 = \langle \sum_{i\in L_{1,n}} \rho_{i,n}(I-P_{C^b_{i,n}})\omega_n, \omega_n - z\rangle \geq \sum_{i\in L_{1,n}} \rho_{i,n}\|(I-P_{C^b_{i,n}})\omega_n\|^2 = (d_{1,n})^2 \geq 0,$$

then we have by the definition of $d_{1,n}$ that

$$\|(I-P_{C^b_{i,n}})\omega_n\| = 0, \; i = 1,\dots,t, \tag{3.18a}$$

$$\|(I-P_{Q^b_{j,n}})A_j\omega_n\| = 0, \; j = 1,\dots,r. \tag{3.18b}$$

If, in Case2, $\|\sum_{j\in L_{2,n}} \beta_{j,n} A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\| = 0$, (3.18) also holds. Note that (3.18) implies that $\omega_n \in \Omega$. Thus the algorithm stops after a finite number of iterates. So in the proof of the theorem and proposition, we assume that $\|\sum_{i\in L_{1,n}} \rho_{i,n}(I-P_{C^b_{i,n}})\omega_n\| \neq 0$ and $\|\sum_{j\in L_{2,n}} \beta_{j,n} A_j^*(I-P_{Q^b_{j,n}})A_j\omega_n\| \neq 0$, i.e., we assume that the sequence $\{x_n\}$ generated by Algorithm 3.1 contains infinity number of iterates.

Next, we present the second algorithm of this paper. In the following algorithm, the inertial parameter θ_n is chosen as the same as in Algorithm 3.1.

Algorithm 3.5. *Step 0. Take* $\tau_1 > 0$, $\lambda_1 > 0$, $\delta_1, \delta_2 \subset (0,1)$, $\{\theta_n\} \subset [0, \overline{\theta}) \subset [0,1)$ *and* $\{\alpha_n\} \in (0,1)$. *Choose the sequence* $\{\Phi_n\}, \{\overline{\Phi_n}\}$ *and* $\{\Psi_n\}, \{\overline{\Psi_n}\}$ *satisfying the Lemma 2.7. Give* x_0, x_1,

and $u \in H$ arbitrarily. Let the integer $n \geq 1$.
Step 1. Compute $\omega_n = x_n + \theta_n(x_n - x_{n-1})$.
Step 2. Compute $u_{i,n} = P_{C^b_{i,n}} \omega_n$ for all $i = 1, 2, \dots, t$ and set

$$d_{1,n} = \max_{i=1,\dots,t} \{\|u_{i,n} - \omega_n\|\},$$

$$L_{1,n} = \{i \in \{1, \dots, t\} : \|u_{i,n} - \omega_n\| = d_{1,n}\}.$$

Sstep 3. Compute $v_{j,n} = P_{Q^b_{j,n}} A_j \omega_n$ for all $j = 1, 2, \dots, r$ and set

$$d_{2,n} = \max_{j=1,\dots,r} \{\|v_{j,n} - A_j \omega_n\|\},$$

$$L_{2,n} = \{j \in \{1, \dots, r\} : \|v_{j,n} - A_j \omega_n\| = d_{2,n}\}.$$

Step 4. Choose $m \in L_{1,n}$ and $s \in L_{2,n}$, and compute

$$z_n = \omega_n - \tau_n (I - P_{C^b_{m,n}}) \omega_n - \lambda_n A_s^* (I - P_{Q^b_{s,n}}) A_s \omega_n,$$

where $\tau_1 = \tau_0$, $\lambda_1 = \lambda_0$ and for $n \geq 2$,

$$\tau_{n+1} = \begin{cases} \min\{\frac{\delta_1 (d_{1,n})^2}{\|(I - P_{C^b_{m,n}})\omega_n\|^2}, \Phi_n \tau_n + \Psi_n\}, & \|(I - P_{C^b_{m,n}})\omega_n\|^2 \neq 0, \\ \Phi_n \tau_n + \Psi_n, & \text{otherwise}; \end{cases} \tag{3.19}$$

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\delta_2 (d_{2,n})^2}{\|A_s^*(I - P_{Q^b_{s,n}}) A_s \omega_n\|^2}, \overline{\Phi_n} \lambda_n + \overline{\Psi_n}, & \|A_s^*(I - P_{Q^b_{s,n}}) A_s \omega_n\|^2 \neq 0, \\ \overline{\Phi_n} \lambda_n + \overline{\Psi_n}, & \text{otherwise}. \end{cases} \tag{3.20}$$

Step 5. Compute

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) \sum_{i \in L_{1,n}} \rho_{i,n} P_{C^b_{i,n}} z_n,$$

set $n \leftarrow n + 1$, and go to Step 1.

Proposition 3.6. *Let $\{\tau_n\}$, $\{\lambda_n\}$ be the sequences generated by Algorithm 3.5. Then $\lim_{n\to\infty} \tau_n = \tau$, $\lim_{n\to\infty} \lambda_n = \lambda$ with $\tau \geq \min\{\delta_1, \tau_1\} > 0$, $\lambda \geq \min\{\frac{\delta_2}{\|M\|^2}, \lambda_1\} > 0$, where $M = \max_{1 \leq j \leq n} \|A_j\|$.*

Proof. In the case of $\|(I - P_{C^b_{m,n}})\omega_n\|^2 \neq 0$, one has $\|(I - P_{C^b_{m,n}})\omega_n\| = d_{1,n}$, so

$$\tau_{n+1} = \min\{\frac{\delta_1 (d_{1,n})^2}{\|(I - P_{C^b_{m,n}})\omega_n\|^2}, \Phi_n \tau_n + \Psi_n\} \geq \min\{\delta_1, \tau_n\},$$

where $\Phi_n \geq 1$ and $\Psi_n \geq 0$. By induction, sequence $\{\tau_n\}$ has a positive lower bound $\min\{\delta_1, \tau_1\}$. From (3.19), we have $\tau_{n+1} \leq \Phi_n \tau_n + \Psi_n$. Lemma 2.7 then reads that $\lim_{n\to\infty} \tau_n$ exists and we denote $\lim_{n\to\infty} \tau_n = \tau$. It is obvious that $\tau > 0$. In the case of $\|A_s^*(I - P_{Q^b_{s,n}}) A_s \omega_n\|^2 \neq 0$, similarly, we have

$$M d_{2,n} \geq \|A_s^*\| \|(I - P_{Q^b_{s,n}}) A_s \omega_n\| \geq \|A_s^*(I - P_{Q^b_{s,n}}) A_s \omega_n\|,$$

where $M = \max_{1 \leq j \leq r} \|A_j\|$. This further presents that

$$\lambda_{n+1} = \min\{\frac{\delta_2 (d_{2,n})^2}{\|A_s^*(I - P_{Q^b_{s,n}}) A_s \omega_n\|^2}, \overline{\Phi_n} \lambda_n + \overline{\Psi_n}\} \geq \min\{\frac{\delta_2}{M^2}, \lambda_n\}.$$

By induction, we can also obtain that sequence $\{\lambda_n\}$ has a lower bound $\min\{\frac{\delta_2}{M^2}, \lambda_1\}$. Hence, $\lim_{n\to\infty}\lambda_n$=$\lambda > 0$. □

Theorem 3.7. *Assume that the solution set of (1.9) is nonempty, i.e., $\Omega \neq \emptyset$. Suppose that the conditions (i) and (ii) in Theorem 3.3 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to $P_\Omega u$.*

Proof. Let $z = P_\Omega u$. From Lemma 2.1 (5), we have

$$\begin{aligned}
&\langle \omega_n - z, \tau_n(I-P_{C^b_{m,n}})\omega_n + \lambda_n A_s^*(I-P_{Q^b_{s,n}})A_s\omega_n\rangle \\
&= \tau_n\langle \omega_n - z, (I-P_{C^b_{m,n}})\omega_n\rangle + \lambda_n\langle \omega_n - z, A_s^*(I-P_{Q^b_{s,n}})A_s\omega_n\rangle \\
&= \tau_n\langle \omega_n - z, (I-P_{C^b_{m,n}})\omega_n - (I-P_{C^b_{m,n}})z\rangle + \lambda_n\langle A_s\omega_n - A_s z, (I-P_{Q^b_{s,n}})A_s\omega_n - (I-P_{Q^b_{s,n}})z\rangle \\
&\geq \tau_n(d_{1,n})^2 + \lambda_n(d_{2,n})^2,
\end{aligned}$$

which together with (3.19) and (3.20) yields that

$$\begin{aligned}
\|z_n - z\|^2 &= \|\omega_n - z\|^2 + \|\tau_n(I-P_{C^b_{m,n}})\omega_n + \lambda_n A_s^*(I-P_{Q^b_{s,n}})A_s\omega_n\|^2 \\
&\quad - 2\langle \omega_n - z, \tau_n(I-P_{C^b_{m,n}})\omega_n + \lambda_n A_s^*(I-P_{Q^b_{s,n}})A_s\omega_n\rangle \\
&\leq \|\omega_n - z\|^2 + 2\tau_n^2\|(I-P_{C^b_{m,n}})\omega_n\|^2 + 2\lambda_n^2\|A_s^*(I-P_{Q^b_{s,n}})A_s\omega_n\|^2 \\
&\quad - 2\tau_n(d_{1,n})^2 - 2\lambda_n(d_{2,n})^2 \\
&\leq \|\omega_n - z\|^2 + 2\tau_n(d_{1,n})^2(\frac{\delta_1\tau_n}{\tau_{n+1}} - 1) + 2\lambda_n(d_{2,n})^2(\frac{\delta_2\lambda_n}{\lambda_{n+1}} - 1).
\end{aligned}$$

Proposition 3.6 indicates that

$$\lim_{n\to\infty}(1 - \frac{\delta_1\tau_n}{\tau_{n+1}}) = 1 - \delta_1, \qquad \lim_{n\to\infty}(1 - \frac{\delta_2\lambda_n}{\lambda_{n+1}}) = 1 - \delta_2,$$

with $\delta_1,\ \delta_2 \in (0,1)$. There exist a positive integer $\overline{N}$ and $\overline{\rho} \in (\max\{\delta_1,\ \delta_2\}, 1)$ such that, for all $n \geq \overline{N}$, $1 - \frac{\delta_1\tau_n}{\tau_{n+1}} > 1 - \overline{\rho} > 0$ and $1 - \frac{\delta_2\lambda_n}{\lambda_{n+1}} > 1 - \overline{\rho} > 0$. Thus,

$$\|z_n - z\|^2 \leq \|\omega_n - z\|^2 + 2\tau_n(d_{1,n})^2(\overline{\rho} - 1) + 2\lambda_n(d_{2,n})^2(\overline{\rho} - 1),\ n \geq \overline{N}. \tag{3.21}$$

Let $y_n = \sum_{i\in L_{1,n}}\rho_{i,n}P_{C^b_{i,n}}z_n$. From (3.21), we deduce that, for all $n \geq \overline{N}$,

$$\begin{aligned}
\|y_n - z\|^2 &\leq \sum_{i\in L_{1,n}}\rho_{i,n}(\|z_n - z\|^2 - \|(I-P_{C^b_{i,n}})z_n\|^2) \\
&\leq \|\omega_n - z\|^2 + 2\tau_n(d_{1,n})^2(\overline{\rho} - 1) + 2\lambda_n(d_{2,n})^2(\overline{\rho} - 1) \\
&\quad - \sum_{i\in L_{1,n}}\rho_{i,n}\|P_{C^b_{i,n}}z_n - \omega_n + \tau_n(I-P_{C^b_{m,n}})\omega_n + \lambda_n A_s^*(I-P_{Q^b_{s,n}})A_s\omega_n\|^2.
\end{aligned} \tag{3.22}$$

Similar as in Algorithm 3.1, we have for all $n \geq \overline{N}$ that $\|y_n - z\| \leq \|x_n - z\| + \alpha_n M_1$. It follows that

$$\begin{aligned}\|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|y_n - z\| \\ &\leq \alpha_n \|u - z\| + (1-\alpha_n)(\|x_n - z\| + \alpha_n M_1) \\ &\leq \alpha_n(\|u - z\| + M_1) + (1-\alpha_n)\|x_n - z\| \\ &\leq \max\{\|x_n - z\|, \|u - z\| + M_1\} \\ &\leq \cdots \leq \max\{\|x_N - z\|, \|u - z\| + M_1\}.\end{aligned}$$

This means that the sequence $\{x_n\}$ is bounded. Thus $\{\omega_n\}$ is bounded, too. By (3.22) and Lemma 2.6 (2), we derive that for all $n \geq \overline{N}$

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|y_n - z\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)(\|\omega_n - z\|^2 + 2\tau_n (d_{1,n})^2(\overline{\rho} - 1) + 2\lambda_n (d_{2,n})^2(\overline{\rho} - 1) \\ &\quad - \sum_{i \in L_{1,n}} \rho_{i,n} \|P_{C^b_{i,n}} z_n - \omega_n + \tau_n (I - P_{C^b_{m,n}})\omega_n + \lambda_n A_s^* (I - P_{Q^b_{s,n}}) A_s \omega_n\|^2) \\ &\quad + 2\alpha_n \langle u - z, x_{n+1} - z\rangle.\end{aligned}$$

Observe that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 - 2(1-\alpha_n)\tau_n (d_{1,n})^2(1-\overline{\rho}) - 2(1-\alpha_n)\lambda_n (d_{2,n})^2(1-\overline{\rho}) \\ &\quad + 3M_2\theta_n \|x_n - x_{n-1}\| - (1-\alpha_n) \sum_{i \in L_{1,n}} \rho_{i,n} \|P_{C^b_{i,n}} z_n - \omega_n + \tau_n (I - P_{C^b_{m,n}})\omega_n \\ &\quad + \lambda_n A_s^* (I - P_{Q^b_{s,n}}) A_s \omega_n\|^2 + 2\alpha_n \langle u - z, x_{n+1} - z\rangle.\end{aligned} \tag{3.23}$$

where $M_2 = \sup_{n \geq N}\{\beta \|x_n - x_{n-1}\|, \|x_n - z\|\}$. Let

$$\begin{aligned}s_n &= \|x_n - z\|^2, \\ \gamma_n &= 3M_2\theta_n \|x_n - x_{n-1}\| + 2\alpha_n \langle u - z, x_{n+1} - z\rangle, \\ b_n &= 2\langle u - z, x_{n+1} - z\rangle + 3M_2 \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|,\end{aligned}$$

and

$$\begin{aligned}\delta_n &= 2(1-\alpha_n)(d_{1,n})^2(1-\overline{\rho})\tau_n + 2(1-\alpha_n)(d_{2,n})^2(1-\overline{\rho})\lambda_n \\ &\quad + (1-\alpha_n)\sum_{i \in L_{1,n}} \rho_{i,n} \|P_{C^b_{i,n}} z_n - \omega_n + \tau_n (I - P_{C^b_{m,n}})\omega_n + \lambda_n A_s^* (I - P_{Q^b_{s,n}}) A_s \omega_n\|^2.\end{aligned}$$

Then (3.23) can be transformed to the following inequlities:

$$\begin{aligned}s_{n+1} &\leq (1-\alpha_n)s_n + \alpha_n b_n, \ n \geq 0, \\ s_{n+1} &\leq s_n - \delta_n + \gamma_n, \ n \geq 0.\end{aligned}$$

From Lemma 2.8, we see that the sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to $z^* = P_\Omega u$. □

Remark 3.8. In Algorithm 3.1 and Algorithm 3.5, one way to determine the inertial parameter θ_n is

$$\theta_n = \begin{cases} \min\{\frac{\varepsilon_n}{\|x_n - x_{n-1}\|}, \overline{\theta}\}, & \text{if } x_n \neq x_{n-1}, \\ \overline{\theta}, & \text{otherwise;} \end{cases} \tag{3.24}$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = o(\alpha_n)$, $\overline{\theta} \in (0,1)$.

4. Numerical Experiments

In this section, we provide numerical experiments for the split feasibility problem with multiple output sets in Hilbert spaces. Firstly, we compare our algorithms with the algorithm in [18] and the algorithm in [19], which are denoted by MSP1 and MSP2, respectively. Secondly, we both apply the ball-relaxed projection method and the half-space relaxation projection method to our algorithm, which are denoted by BRM and HRM, respectively. All codes were written in MATLAB R2018b and performed on a PC Desktop Intel(R) Core(TM) i7-8700 CPU @ 3.20GHz 3.19 GHz, RAM 8.00 GB.

Example 4.1. Consider the following split feasibility problem with multiple output sets. Let $H_1 = \mathbb{R}^N$ and $H_2 = \mathbb{R}^{N(j+1)}$ for each $j = 1,...,r$. Let $A_j : \mathbb{R}^N \to \mathbb{R}^{N(j+1)}$ be given by $A_j = (a_{pq})_{N\times N(j+1)}$ with randomly generated $a_{pq} \in [-1,1]$. Let $N = 50$ and $r = 50$. Find $x^* \in \mathbb{R}^N$ with the property

$$x^* \in \cap_{i=1}^2 C_i \text{ such that } A_j x^* \in Q_j,\ j = 1,2,...,r,$$

where

$$C_1 = \{x \in \mathbb{R}^N : \sum_{k=1}^N 10^{\frac{k-1}{N-1}} x_k^2 - 1 \le 0\},$$

$$C_2 = \{x \in \mathbb{R}^N : \sum_{k=1}^N 10^{\frac{N-k}{N-1}} x_k^2 - 1 \le 0\},$$

and

$$Q_j = \{y \in \mathbb{R}^{N(j+1)} : \sum_{k=j}^{N(j+1)} 10^{\frac{k-1}{N(j+1)-1}} y_k^2 - 1 \le 0\}.$$

Now, we investigate the numerical behavior of our proposed algorithms for different choices of methods. For the convenience of comparison, we randomly generate an initial value x_0, which is used in the three algorithms simultaneously and set $x_{-1} = x_0$. The inertial parameter θ_n is defined in (3.24) with $\varepsilon_n = \frac{1}{n^{2.1}}$ and $\overline{\theta} = 0.01$. We use $E_n = \frac{1}{2}(d_{1,n} + d_{2,n})$ to measure the error of the n-th iterate, where $d_{1,n}$ and $d_{2,n}$ are defined by (3.1) and (3.2), respectively. If $E_n < 10^{-8}$, then the iteration progress stops.

In BRM and HRM, we set $\tau_1 = \lambda_1 = 0.1$, $\Phi_n = \overline{\Phi}_n = 1 + \frac{10^4}{n^{1.1}}$, $\Psi_n = \overline{\Psi}_n = 0$, $\delta_1 = \delta_2 = \delta \in (0,1)$, and $\alpha_n = \frac{1}{n}$.

In MSP1, the sequence $\{x_n\}$ is generated by

$$\begin{cases} \omega_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \omega_n - \gamma_n A_{\bar{j}}^*(I - P_{Q_{\bar{j},n}})A_{\bar{j}}\omega_n, \\ x_{n+1} = \frac{1}{2}(P_{C_{1,n}} z_n + P_{C_{2,n}} z_n), \end{cases}$$

where $\bar{j}$ denotes the index such that $\|A_{\bar{j}}x_n - P_{Q_{\bar{j},n}}A_{\bar{j}}x_n\| = \max_{j=1,...,r}\{\|A_j x_n - P_{Q_{j,n}}A_j x_n\|\}$, $\gamma_1 = 0.1$, and γ_n is defined by

$$\gamma_n = \begin{cases} \frac{2\delta\|A_{\bar{j}}x_n - P_{Q_{\bar{j},n}}A_{\bar{j}}x_n\|^2}{\|A_{\bar{j}}^*(I-P_{Q_{\bar{j},n}})A_{\bar{j}}x_n\|^2}, & \text{if } \|A_{\bar{j}}^*(I - P_{Q_{\bar{j},n}})A_{\bar{j}}x_n\| > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\delta \in (0,1)$, $C_{i,n}$ and $Q_{j,n}$ denote the half-space relaxation of C_i and Q_j, $i = 1,2,\ j = 1,...,r$.

In MSP2, $\{x_n\}$ is generated by

$$\begin{cases} \omega_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \omega_n - \gamma_n \sum_{j=1}^{r} A_j^*(I - P_{Q_{j,n}})A_j\omega_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{2}(P_{C_{1,n}}z_n + P_{C_{2,n}}z_n), \end{cases}$$

where $\gamma_1 = 0.1$, $\gamma_n = \frac{1.9}{r\max_{j=1,...,r}\{\|A_j\|^2\}}$, and $\alpha_n = \frac{1}{n+2}$.

We consider two cases whether u is in the solution set or not. In Table 1, we take $u = (0.01, 0.01, 0, ..., 0)$, which belongs to the solution set; in Table 2, we take $u = (0.01, 0.018, 0, ..., 0)$, which does not belong to the solution set.

It is observed from Table 1 and Table 2 that as δ increases, BRM, HRM, and MSP1 take fewer steps and less time to reach the stopping criterion. For each fixed δ, our algorithm outperforms MSP1 and MSP2 in terms of the number of iterations and CPU time, which supports the superiority of the step-size selection of our algorithms. BRM is better than HRM in terms of CPU time even though they have the same number of iterations, which shows the advantage of the ball-relaxed projection method over the half-space relaxation projection method for this example.

TABLE 1. Computational Results with BRM, HRM, MSP1, and MSP2

δ	Iteration				CPU time			
	BRM	HRM	MSP1	MSP2	BRM	HRM	MSP1	MSP2
0.1	**10**	**10**	126		**0.4294**	0.8151	5.3266	
0.3	**6**	**6**	44		**0.3139**	0.6451	1.9208	
0.5	**5**	**5**	22	34	**0.3792**	0.4487	0.9215	3.3603
0.7	**4**	**4**	13		**0.2039**	0.3479	0.7243	
0.9	**3**	**3**	9		**0.1624**	0.4964	0.4990	

TABLE 2. Computational Results with BRM, HRM, MSP1, and MSP2

δ	Iteration				CPU time			
	BRM	HRM	MSP1	MSP2	BRM	HRM	MSP1	MSP2
0.2	**9**	**9**	65		**0.3930**	0.7792	2.7077	
0.4	**6**	**6**	32	35	**0.2999**	0.5801	1.4353	3.4197
0.6	**4**	5	16		**0.2458**	0.4137	0.8247	
0.8	**4**	**4**	11		**0.2064**	0.3045	0.5066	

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