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A COMPARATIVE STUDY ON CHAOS CONTROL IN A FRACTIONAL-ORDER ROSSLER SYSTEM AND ITS TIME DELAYED FEEDBACK CONTROL SYSTEM

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Abstract. How to control chaos is a challenging problem in the Rossler system. A fractional-order Rossler system and its time delayed feedback control system are studied in this paper. A predictor-corrector algorithm for fraction-order Rossler systems and its time delayed feedback control system is given. The main aim of the current paper is to compare the control effects of our given methods, namely adjusting different fractional orders and time delayed feedback control items at different equations, to make the system reach a stable state. And in controlling chaos, the fractional-order Rossler system with time delayed feedback control is more effective. Numerical examples are conducted to confirm effectiveness of the proposed method.

Keywords. Fractional-order Rossler system; Time delayed feedback control; Predictor-corrector method; Chaos; Stable.

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1. Introduction

Fractional calculus, which is a generalization of integer order integration and differentiation, finds numerous real applications recently. Indeed, fractional calculus has time memory and long-range spatial correlation, and can more accurately describe physical phenomena and biochemical reaction processes of memory, heredity and path dependence than integer order integration; see, e.g., [1, 2, 3, 4, 5, 6] and the references therein.

Recently, various definitions of fractional calculus from various perspectives were presented. There are three common definitions, namely, Gronwald-Letnikov, Riemann-Liouville, and Caputo [7, 8, 9]. Based on the three definitions, some new definitions for fractional calculus were further introduced, such as Caputo-Fabrizio derivative (CFD) [10], memory dependent derivative [11], and so on. Here, we use the following expression

$$D_*^q y(x) = J^{m-q} y^{(m)}(x), q > 0,$$

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where $m = [q]$ is the value q rounded up to the nearest integer, $y^{(m)}$ is the convenient m-order derivative, and

$$J^{\beta} z(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} z(t) dt, \beta > 0.$$

is the β-order Riemann-Liouville integral operator. It is common that operator D_*^q is called q-order Caputo differential operator.

The Rossler system, a simple prototype with chaos, was introduced by Rossler in the 1970s [12]. In all equations, it only contains a nonlinear term. Furthermore, the Rossler system was abstracted from the Lorenz model [13]. Due to the simplicity of the Rossler system, it has become a criterion to detect the effectiveness of the chaos control method. The standard Rossler system is stated as

$$\begin{cases} \frac{dx}{dt} = -y - z, \\ \frac{dy}{dt} = x + ay, \\ \frac{dz}{dt} = b + z(x-c), \end{cases}$$

where x, y, z are control variables and a, b, c are parameters.

The fractional-order Rossler system is described by

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = -y - z, \\ \frac{d^{q_2}y}{dt^{q_2}} = x + ay, \\ \frac{d^{q_3}z}{dt^{q_3}} = b + z(x-c), \end{cases} \tag{1.1}$$

where its order is denoted by $q = (q_1, q_2, q_3)$, $0 < q_1, q_2, q_3 \leq 1$.

In order to obtain the desired stable region, some effective controllers need to be added. Recently, various chaos control methods have been proposed; see, e.g., [14, 15, 16]. An optimal control problem via Caputo fractional derivative was given by Vellappandi in [17]. For recent bifurcation control methods, we refer to [18, 19, 20, 21, 22, 23, 24, 25, 26]. The time delayed feedback control method was introduced by Pyragas [27]. For a dynamical system, the main idea of time delayed feedback control method is to obtain a continuous control by using a feedback term between the dynamical variable $M(t)$ and its delayed value. In all, a perturbation of the system is used as $E(t) = L(M(t) - M(t-\tau))$, where τ is the time delay and L is the feedback strength. The system can be stable by choosing the time delay and appropriate feedback term.

An Adams-type predictor-corrector method was given for the numerical solution of fractional differential equations in [28]. Bhalekar and Gejji extended the Adams-Bashforth-Moulton algorithm to solve the delay fractional-order differential equations and provided numerical illustrations to demonstrate utility of the method in [29].

The aim of this paper is to investigate the chaos control effect of fractional order model (1.1) and its time delayed feedback control system. By the time delayed feedback control method, different fractional orders and the control items at different equations are added to obtain the control effects. Predictor-corrector algorithm of fractional order model and its time delayed feedback control system are discussed. Numerical simulations verify our theoretical analysis, including waveform diagrams and phase portraits.

2. The Predictor-corrector Algorithm

2.1. **The fractional-order Rossler system.** The predictor-corrector scheme is used for fractional Rossler system (1.1), which is similar to the method in [28]. Consider the following differential equation

$$\begin{cases} D_*^q y(t) = f(t, y(t)), \\ y^{(k)}(0) = y_0^{(k)}, k = 0, 1, \dots, m-1, \end{cases} \tag{2.1}$$

System (2.1) is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds. \tag{2.2}$$

Let $h = \frac{t}{N}$, $t_n = nh$, $n = 0, 1, \dots, N$. Thus we can discretize equation (2.2) as

$$y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)), \tag{2.3}$$

where

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, j = 0, \\ (n-j-2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, 1 \le j \le n, \\ 1, j = n+1. \end{cases} \tag{2.4}$$

The predictor formula is

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t_{n+1}^k}{k!} + + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j)), \tag{2.5}$$

$$b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q). \tag{2.6}$$

The error estimate is $max_{j=0,1,\dots,N} \mid y(t_j) - y_h(t_j) \mid = O(h^p)$, where $p = min(2, 1+q)$.

Through the predictor-corrector method, system (1.1) can be discretized as follows

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-y_{n+1}^p - x_{n+1}^p) + \frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^n \alpha_{1,j,n+1}(-y_j - z_j), \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}[x_{n+1}^p + a y_{n+1}^p] + \frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^n \alpha_{2,j,n+1}[x_j + a y_j], \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}[b + z_{n+1}^p(x_{n+1}^p - c)] + \frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^n \alpha_{3,j,n+1}[b + z_j(x_j - c)], \end{cases} \tag{2.7}$$

$$\begin{cases} x_{n+1}^p = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^n \beta_{1,j,n+1}(-y_j - z_j), \\ y_{n+1}^p = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^n \beta_{2,j,n+1}[x_j + a y_j], \\ z_{n+1}^p = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^n \beta_{3,j,n+1}[b + z_j(x_j - c)], \end{cases} \tag{2.8}$$

$$\alpha_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, j = 0 \\ (n-j-2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, 1 \le j \le n \\ 1, j = n+1, \end{cases} \tag{2.9}$$

and

$$\beta_{i,j,n+1} = \frac{h^{q^i}}{q^i}[(n-j+1)^{q^i} - (n-j)^{q^i}], 0 \le j \le n, i = 1, 2, 3. \tag{2.10}$$

2.2. **The fractional-order Rossler system with time delayed feedback control.** In [30], Ding et al. investigated the integer-order Rossler chaotic system with time delayed feedback as follows

$$\begin{cases} \frac{dx}{dt} = -y - z + L(x(t-\tau) - x(t)), \\ \frac{dy}{dt} = x + ay, \\ \frac{dz}{dt} = b + z(x - c). \end{cases} \tag{2.11}$$

where τ is the time delay and L is the feedback strength. Equation (2.11) has two fixed points if $c^2 > 4ab$

$$(x_{1,2}, y_{1,2}, z_{1,2}) = (\frac{c \mp \sqrt{c^2 - 4ab}}{2}, \frac{-c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \mp \sqrt{c^2 - 4ab}}{2a}). \tag{2.12}$$

In [30], the stability of fixed point

$$(x_1, y_1, z_1) = (\frac{c - \sqrt{c^2 - 4ab}}{2}, \frac{-c + \sqrt{c^2 - 4ab}}{2a}, \frac{c - \sqrt{c^2 - 4ab}}{2a}). \tag{2.13}$$

is obtained. In the same way, the other one can be obtained immediately.

Using a time delayed feedback control method, we describe the fractional-order Rossler system with the time delayed feedback as follows

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = -y - z + L1(x(t-\tau) - x(t)), \\ \frac{d^{q_2}y}{dt^{q_2}} = x + ay, \\ \frac{d^{q_3}z}{dt^{q_3}} = b + z(x - c). \end{cases} \tag{2.14}$$

In (2.14), the time delayed feedback term is in the first equation, and $L1$ is the feedback strength.

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = -y - z, \\ \frac{d^{q_2}y}{dt^{q_2}} = x + ay + L2(y(t-\tau) - y(t)), \\ \frac{d^{q_3}z}{dt^{q_3}} = b + z(x - c). \end{cases} \tag{2.15}$$

In (2.15), the time delayed feedback term is in the second equation, and $L2$ is the feedback strength.

$$\begin{cases} \frac{d^{q_1}x}{dt^{q_1}} = -y - z, \\ \frac{d^{q_2}y}{dt^{q_2}} = x + ay, \\ \frac{d^{q_3}z}{dt^{q_3}} = b + z(x - c) + L3(z(t-\tau) - z(t)). \end{cases} \tag{2.16}$$

In (2.16), the time delayed feedback term is in the third equation, and $L3$ is the feedback strength. In (2.14), (2.15), and (2.16), $0 < q_1, q_2, q_3 \le 1$, and its order is denoted by $q = (q_1, q_2, q_3)$. Similar to [29], we study the predictor-corrector scheme for the delay differential equations of fractional order Rossler system (2.14). The predictor-corrector schemes of system (2.15) and (2.16) resemble system (2.14).

Consider the following delay fractional-order differential equations of [29]

$$\begin{cases} D_*^q y(t) = f(t, y(t), y(t-\tau)), t \in [0, T], 0 < q \le 1, \\ y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{2.17}$$

Use a uniform grid $t_n = nh : n = -k, -k+1, \cdots, -1, 0, 1, \cdots, N$, where k and N are intergers, that is, $h = \frac{t}{N}$ and $h = \frac{\tau}{k}$.

Setting

$$y_h(t_j) = \phi(t_j), j = -k, -k+1, \cdots, -1, 0, \tag{2.18}$$

we have

$$y_h(t_j - \tau) = y_h(jh - kh) = y_h(t_{j-k}), j = 0, 1, \cdots, N. \tag{2.19}$$

For given approximations

$$y_h(t_j) \approx y(t_j), j = -k, -k+1, \cdots, -1, 0, 1, \cdots, n, \tag{2.20}$$

we need to calculate $y_h(t_{n+1})$ by using

$$y(t_{n+1}) = \phi(0) + \frac{1}{\Gamma(q)} \int_0^{t_{n+1}} (t_{n+1} - \xi)^{q-1} f(\xi, y(\xi), y(\xi - \tau)) d\xi. \tag{2.21}$$

We use approximations $y_h(t_n)$ for $y(t_n)$ in (2.21). According to the product trapezoidal quadrature formula, the integral is evaluated in equation (2.21). The corrector formula is as follows

$$\begin{aligned} y_h(t_{n+1}) &= \phi(0) + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1}), y_h(t_{n+1} - \tau)) \\ &\quad + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)) \\ &= \phi(0) + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1}), y_h(t_{n+1-k})) \\ &\quad + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})), \end{aligned} \tag{2.22}$$

where $a_{j,n+1}$ are given by (2.4), and $y_h^p(t_{n+1})$ is called the predictor. In (2.22), we use the product rectangle rule to evaluate predictor term

$$\begin{aligned} y_h^p(t_{n+1}) &= \phi(0) + \frac{1}{\Gamma(q)} \sum_0^n b_{j,n+1} f(t_j, y_h(t_j), y_h(t_j - \tau)) \\ &= \phi(0) + \frac{1}{\Gamma(q)} \sum_0^n b_{j,n+1} f(t_j, y_h(t_j), y_h(t_{j-k})) \end{aligned} \tag{2.23}$$

where $b_{j,n+1}$ are given by (2.6).

According to the predictor-corrector method, we can discretize system (2.14) as follows

$$\begin{cases} x_{n+1} = \begin{cases} x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-y_{n+1}^p - x_{n+1}^p + L1(x_{n+1-k} - x_{n+1}^p)) \\ + \frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^n \alpha_{1,j,n+1}(-y_j - z_j + L1(x_{j-k} - x_j)), \end{cases} \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}[x_{n+1}^p + a y_{n+1}^p] + \frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^n \alpha_{2,j,n+1}[x_j + a y_j], \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}[b + z_{n+1}^p(x_{n+1}^p - c)] + \frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^n \alpha_{3,j,n+1}[b + z_j(x_j - c)] \end{cases} \tag{2.24}$$

and

$$\begin{cases} x_{n+1}^p = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^n \beta_{1,j,n+1}(-y_j - z_j + L1(x_{j-k} - x_j)), \\ y_{n+1}^p = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^n \beta_{2,j,n+1}[x_j + a y_j], \\ z_{n+1}^p = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^n \beta_{3,j,n+1}[b + z_j(x_j - c)]. \end{cases} \tag{2.25}$$

Furthermore, system (2.15) can be discretized as

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-y^p_{n+1} - x^p_{n+1}) + \frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^n \alpha_{1,j,n+1}(-y_j - z_j), \\ y_{n+1} = \begin{cases} y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}[x^p_{n+1} + ay^p_{n+1} + L2(y_{n+1-k} - y^p_{n+1})] \\ + \frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^n \alpha_{2,j,n+1}[x_j + ay_j + L2(y_{j-k} - y_j)], \end{cases} \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}[b + z^p_{n+1}(x^p_{n+1} - c)] + \frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^n \alpha_{3,j,n+1}[b + z_j(x_j - c)] \end{cases} \tag{2.26}$$

and

$$\begin{cases} x^p_{n+1} = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^n \beta_{1,j,n+1}(-y_j - z_j), \\ y^p_{n+1} = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^n \beta_{2,j,n+1}[x_j + ay_j + L2(y_{j-k} - y_j)], \\ z^p_{n+1} = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^n \beta_{3,j,n+1}[b + z_j(x_j - c)]. \end{cases} \tag{2.27}$$

System (2.16) can be discretized as

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)}(-y^p_{n+1} - x^p_{n+1}) + \frac{h^{q_1}}{\Gamma(q_1+2)} \sum_{j=0}^n \alpha_{1,j,n+1}(-y_j - z_j), \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)}[x^p_{n+1} + ay^p_{n+1}] + \frac{h^{q_2}}{\Gamma(q_2+2)} \sum_{j=0}^n \alpha_{2,j,n+1}[x_j + ay_j], \\ z_{n+1} = \begin{cases} z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)}[b + z^p_{n+1}(x^p_{n+1} - c) + L3(z_{n+1-k} - z^p_{n+1})] \\ + \frac{h^{q_3}}{\Gamma(q_3+2)} \sum_{j=0}^n \alpha_{3,j,n+1}[b + z_j(x_j - c) + L3(z_{j-k} - z_j)] \end{cases} \end{cases} \tag{2.28}$$

and

$$\begin{cases} x^p_{n+1} = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^n \beta_{1,j,n+1}(-y_j - z_j), \\ y^p_{n+1} = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^n \beta_{2,j,n+1}[x_j + ay_j], \\ z^p_{n+1} = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^n \beta_{3,j,n+1}[b + z_j(x_j - c) + L3(z_{j-k} - z_j)]. \end{cases} \tag{2.29}$$

where $\alpha_{i,j,n+1}$ and $\beta_{i,j,n+1}$ are given by (2.9) and (2.10).

3. Comparison and Discussion by Numerical Simulations

The Rossler system becomes a criterion to detect the effectiveness of the chaos control method because of its simplicity. It is well known that the integer-order Rossler model is a chaos system. Next, according to the derived discrete schemes in the section above, we give a comparative study on chaos control for the fractional-order Rossler system and its time delayed feedback control system by waveform plots and phases.

Table 1. The fractional Rossler system (1.1)

q=(1,1,1)	chaos
q=(0.96,1,1)	chaos
q=(0.84,1,1)	chaos
q=(0.72,1,1)	chaos
q=(0.66,1,1)	chaos
q=(1,0.96,1)	chaos
q=(1,0.84,1)	chaos
q=(1,0.72,1)	chaos
q=(1,0.6,1)	stable
q=(1,1,0.96)	chaos
q=(1,1,0.84)	chaos
q=(1,1,0.74)	chaos
q=(1,1,0.72)	stable

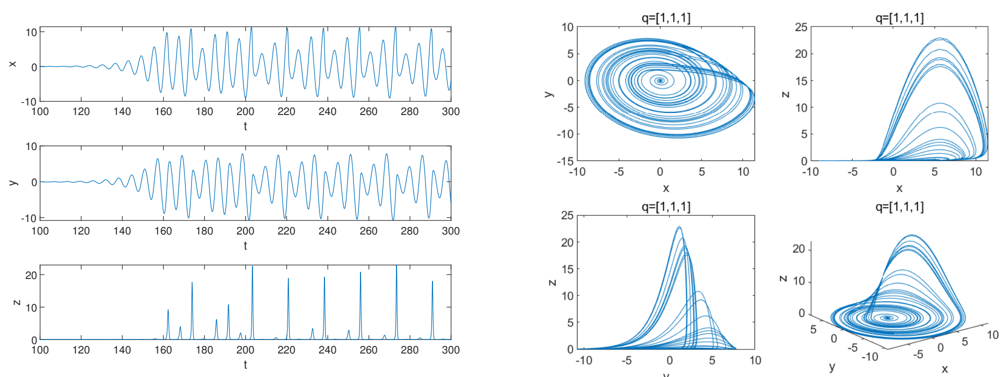

FIGURE 1. The waveformplot and phase of the predictor-corrector algorithm for Rossler system (1.1) (integer order): $q = (1, 1, 1)$

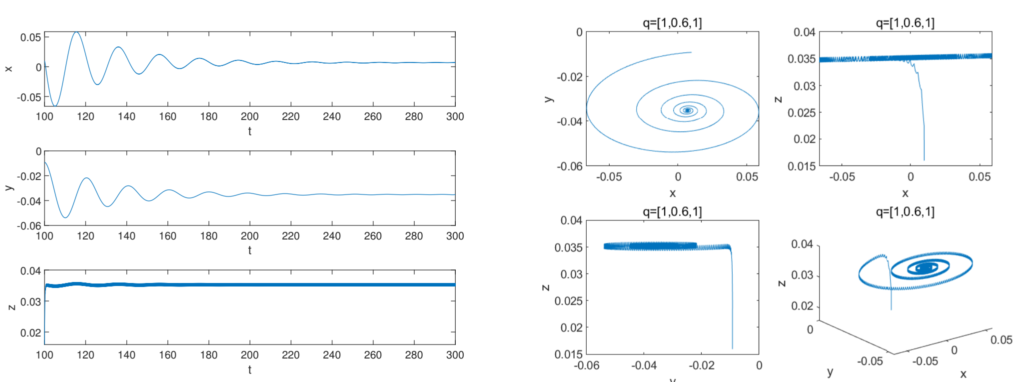

FIGURE 2. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (1.1): $q = (1, 0.6, 1)$

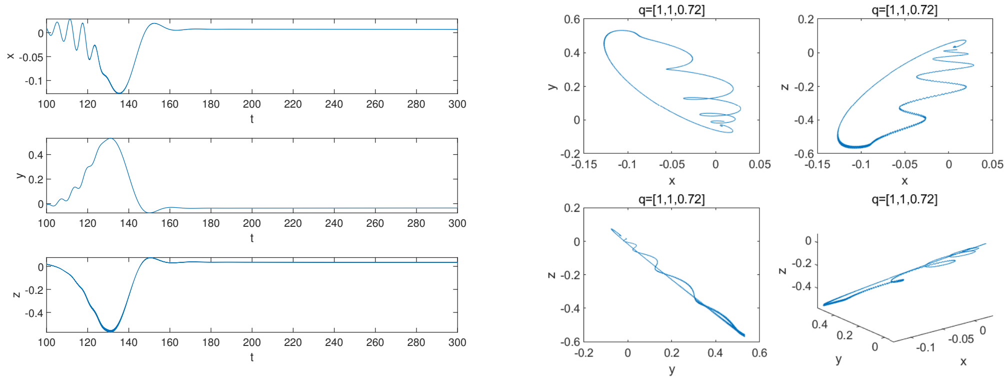

FIGURE 3. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (1.1): $q = (1, 1, 0.72)$

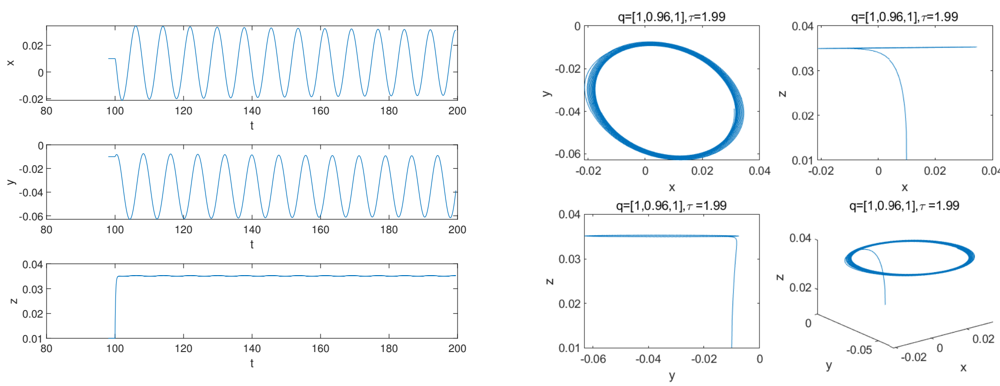

FIGURE 4. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (2.14): $q = (1, 0.96, 1), \tau = 1.99$

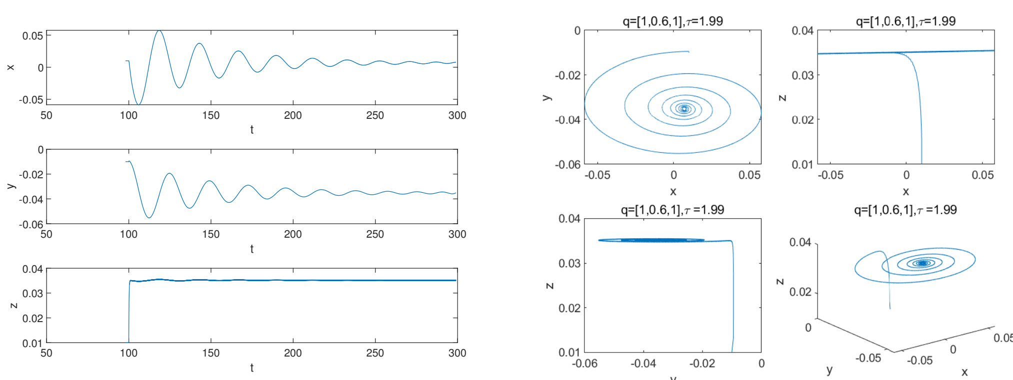

FIGURE 5. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (2.14): $q = (1, 0.6, 1), \tau = 1.99$

Table 2. The fractional Rossler system (2.14)with delay for $\tau = 1.99$

(q_1, q_2, q_3)	$\tau = 1.99$
(0.96,1,1)	chaos
(0.84,1,1)	chaos
(0.72,1,1)	chaos
(0.66,1,1)	chaos
(1,0.96,1)	periodic solution
(1,0.84,1)	stable
(1,0.72,1)	stable
(1,0.6,1)	stable
(1,1,0.96)	periodic solution
(1,1,0.84)	periodic solution
(1,1,0.74)	periodic solution
(1,1,0.72)	stable

Table 3. The fractional Rossler system (2.14) with delay for different τ

(q_1, q_2, q_3)	$\tau = 1.99$	$\tau = 3$	$\tau = 4.6$
(1,1,1)	periodic solution	stable	periodic solution
(0.96,1,1)	chaos	stable	stable
(0.84,1,1)	chaos	stable	stable
(1,0.96,1)	periodic solution	stable	stable
(1,0.84,1)	stable	stable	stable
(1,1,0.96)	periodic solution	stable	periodic solution
(1,1,0.84)	periodic solution	stable	periodic solution
(1,1,0.72)	stable	stable	periodic solution

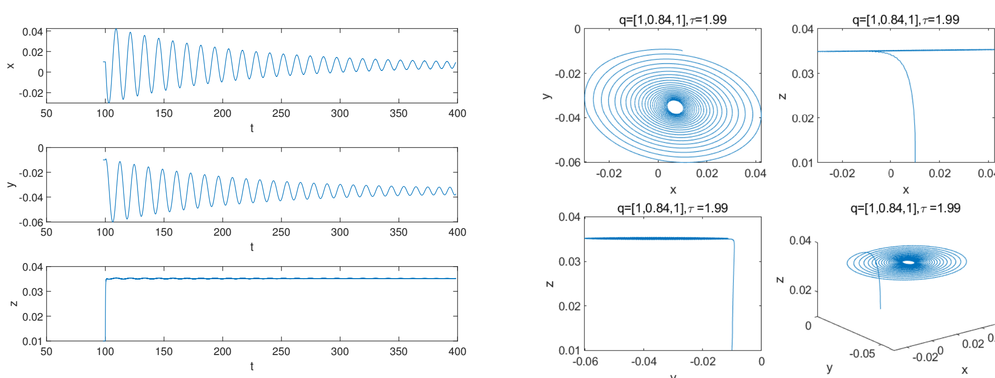

FIGURE 6. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (2.14): $q = (1, 0.84, 1), \tau = 1.99$

Let $a = 0.2, b = 0.2$, and $c = 0.7$ for the fractional-order Rossler system (1.1), system (2.11) and the fractional-order Rossler system with time delayed feedback control (2.14)-(2.15)-(2.16).

In [30], Ding et al. gave numerical simulations for the integer-order Rossler system (2.11) with different delay τ. Simulation results indicate that the Rossler chaotic system can be controlled to be stable by choosing appropriate controlled parameters. In this paper, through Tables 1-3

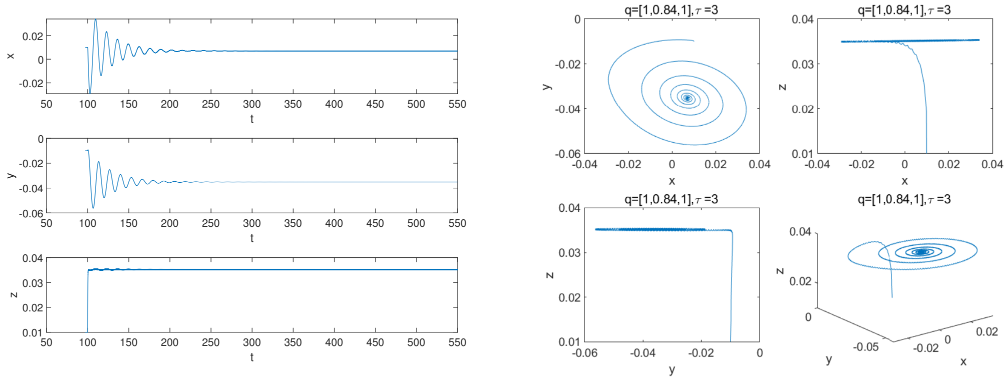

FIGURE 7. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (2.14): $q = (1, 0.84, 1), \tau = 3$

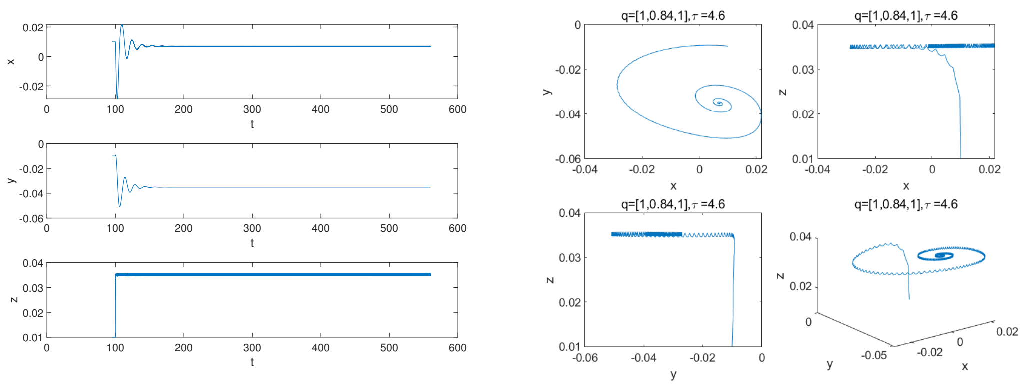

FIGURE 8. The waveformplot and phase of the predictor-corrector algorithm for fractional-order Rossler system (2.14): $q = (1, 0.84, 1), \tau = 4.6$

and Figures 1-8, we can obtain the following results of the fractional-order Rossler system (1.1) and the fractional-order Rossler time delayed feedback control system (2.14)-(2.15)-(2.16). The main experimental conclusions are summarized as follows.

From Table 1, we can obtain two states (chaos and stable solution) by choosing different $q = (q_1, q_2, q_3)$. From Figure 1, we give waveform plots and phase diagrams of the integer order Rossler chaos system. From Figures 2-3, we plot the waveform plots and phase diagrams of fractional-order Rossler system (1.1) at different $q = (q_1, q_2, q_3)$ (Figure 2: $q = (1, 0.6, 1)$ and Figure 3: $q = (1, 1, 0.72)$). The fractional-order Rossler system (1.1) is stable at $q = (1, 0.6, 1)$ (or $q = (1, 1, 0.72)$).

From Table 2, we can obtain three results (chaos, period solution, and stable solution) by choosing different $q = (q_1, q_2, q_3)$. From Figures 4-5, we plot waveform plots and phase diagrams about time delayed feedback control analysis of a fractional-order Rossler system (2.14) at different $q = (q_1, q_2, q_3)$ (Figure 4: $q = (1, 0.96, 1)$ period solution, Figure 5: $q = (1, 0.6, 1)$ stable solution).

In Table 3, for $\tau = 1.99, \tau = 3$, and $\tau = 4.6$, we receive different results (chaos, periodic solution, and stable solution) when $q = (q_1, q_2, q_3)$ takes different values. From Figures 6-8, we can show that a fractional-order Rossler system (2.14) with time delayed feedback control is stable at $q = (1, 0.84, 1)$ and different $\tau = 1.99, \tau = 3, \tau = 4.6$. The fractional-order Rossler system (2.14) with time delayed feedback control is very effective for the control of chaotic system.

Through above analysis, we find that the fractional-order Rossler system (1.1) and its time delayed feedback control system (2.14) can make the chaotic system become stable. And the fractional-order Rossler system with time delayed feedback control is very effective in controlling chaos.

4. Conclusions

According to Adams-type predictor-corrector method, we investigated the fractional-order Rossler system. We deduced that system (1.1) has period or stable solutions at different $q = (q_1, q_2, q_3)$. The predictor-corrector algorithm of the fractional delay differential equation was used to make the fractional Rossler system reach a stable state from the chaotic state. Therefore, through our discussion and comparison, the integer-order Rossler model is a chaos system. The fractional-order Rossler system and its time delayed feedback control system are useful for the control of chaotic system. In addition, the fractional-order Rossler system with time delayed feedback control is more effective than the fractional-order Rossler system in controlling chaos.

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