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ABSOLUTE STABILITY OF TIME-DELAYED LURIE DIRECT CONTROL SYSTEMS WITH UNBOUNDED COEFFICIENTS

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Abstract. The absolute stability of a time-varying Lurie direct control system with time-varying delay is studied in this paper. Based on the Lyapunov stability theorem for non-autonomous delay differential systems, we tackle a class of stability analysis problems. Sufficient conditions to ensure the absolute stability of the time-delayed Lurie system with single nonlinearity are formulated by simple and easily verifiable inequalities. Subsequently, the conclusions are extended to the case of multiple nonlinearities. The results in this paper are not only effective for the Lurie systems with norm-unbounded coefficients but also applicable to this class of systems with bounded or constant coefficients. Finally, the stability results are illustrated by numerical simulations.

Keywords. Absolute stability; Lyapunov stability theorem; Lurie direct control system; Non-autonomous delay differential systems; Unbounded coefficients.

1. Introduction

Early in the 1940's, absolute stability concept was defined by Russian mathematicians Lurie and Postnikov in [1]. Since then, the absolute stability of Lurie systems has received considerable attention. For constant and uncertain Lurie systems, the corresponding stability theory was established; see, e.g., [2, 3, 4, 5, 6]. Recently, some new results have been developed for time-varying case; see, e.g., [7, 8, 9, 10, 11] and the references therein. By combining M-matrix property and Lyapunov theorems, some sufficient conditions to ensure absolute stability of time-varying Lurie direct control systems were derived in [7]. Moreover, some extensions to time-varying large-scale Lurie systems were developed in [8, 9].

It is known that time delay frequently appears in engineering systems. Also, it is always the significant cause of un-stability and poor performance. With respect to Lurie systems with time delay, fruitful absolute stability results have been achieved thus far. By the linear matrix inequality method, several stability criteria for time-delayed systems with sector-bounded

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nonlinearity were presented in [12]. By using a Lur'e-Postnikov function, some sufficient conditions for the absolute stability of time-delay Lurie direct control systems were presented in [13], and the relationship between these conditions and certain frequency-domain conditions was also provided. Furthermore, absolute stability for uncertain Lurie systems with time delay was analyzed in [14, 15, 16, 17, 18]. In addition, a class of more complicated neutral Lurie systems was considered in [19, 20].

It should be noted that all the existing results related to time-delayed Lurie systems mentioned above were established for the case that the system are constant or uncertain, while as for general time-varying Lurie systems, corresponding results are very few. In this paper, we investigate the absolute stability of time-varying Lurie direct control systems. In particular, the coefficients matrices of systems considered herein can be norm-unbounded. The aim of this paper is to give some simple and easily verifiable conditions to ensure the absolute stability of this class of Lurie systems based on the Lyapunov stability theorem. The selected Lyapunov-Krasovskii functional is composed of a quadratic term with respect to the state and an integral term related to time delay. Also, it is highlighted that the criteria proposed are not only effective for the Lurie systems with norm-unbounded coefficients but also applicable to such systems with bounded or constant coefficients.

Notations: $\lambda\left(A\right)$ stands for any eigenvalue of the square matrix A. Let $x=\left[\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array}\right]^T$. The notation $\|x\|$ stands for Euclidean norm of x , i.e., $\|x\|=\sqrt{\sum_{i=1}^{n}x_i^2}$. $\|A\|$ represents the matrix norm induced by Euclidean vector norm, namely $\|A\|=\max_{\|x\|=1}\|Ax\|$, and one can prove that $\|A\|=\sqrt{\lambda_{\max}\left(A^TA\right)}$. In the derivation of this paper, $\|A\|$ can be replaced by $\|A\|_F$, $\|A\|_F=\sqrt{\sum_{j=1}^{n}\sum_{k=1}^{m}\left|a_{kj}\right|^2}$, which is due to $\|A\|\leq\|A\|_F$ and the fact that the calculation of $\|A\|_F$ is not hard. $\overline{\lim\limits_{t\to\infty}}$ stands for the upper limit. For simplicity, let $\phi\left(s\right)=x\left(t+s\right), s\in\left[-\tau,0\right], t\geq 0$, $|||\phi|||=\sqrt{\int_{-\tau}^{0}\|\phi\left(s\right)\|^2ds}$.

To investigate the absolute stability of the Lurie direct control system with time-varying delay, we consider first the case of single nonlinearity. Next, the derived stability results are extended to the case of multiple nonlinearities. The Lyapunov theorem used in the proof was given in [21].

2. Absolute Stability of Time-Delayed Lurie Systems with Single Nonlinearity

Consider the following time-varying Lurie direct control system with single nonlinearity and time-varying delay

$$\left\{\begin{array}{l} \dot{x}\left(t\right)=A\left(t\right)x\left(t\right)+B\left(t\right)x\left(t-\tau\left(t\right)\right)+b\left(t\right)f\left(\sigma\left(t\right)\right), \\ \sigma\left(t\right)=c^T\left(t\right)x\left(t\right), \end{array}\right. \tag{2.1}$$

where $x\left(t\right)\in R^n$ is the state; $\sigma\left(t\right)\in R$ is the output; $A\left(t\right), B\left(t\right)$ are $n\times n$ time-varying matrices; $b\left(t\right), c\left(t\right)$ are n dimensional column vectors; $\tau\left(t\right)$ is the time delay, and $A\left(t\right), B\left(t\right), b\left(t\right), c\left(t\right)$ are continuous in $[0,+\infty)$. The nonlinearity $f\left(\cdot\right)$ is continuous and satisfies the sector condition:

$$F_{[0,k]}=\left\{f\left(\cdot\right)|f\left(0\right)=0; 0<\sigma\left(t\right)f\left(\sigma\left(t\right)\right)\leq k\sigma^2\left(t\right), \sigma\left(t\right)\neq 0\right\}, \tag{2.2}$$

where $k>0$ is a constant.

Definition 2.1. [22] System (2.1) is said to be absolutely stable if its zero solution is globally asymptotically stable for any nonlinearity $f \in F_{[0,k]}$.

The following assumptions are important for system (2.1).

A1: The time delay $\tau(t)$ is a continuous and piecewise differentiable function satisfying $0 \le \tau(t) \le \tau$ and $\dot{\tau}(t) \le w < 1$, where τ and w are constants. At the non-differential points of $\tau(t)$, $\tau(t)$ represents $\max[\dot{\tau}(t-0), \dot{\tau}(t+0)]$.

A2: For any $t \in [0,\infty)$, there exist symmetric positive-definite matrices P and G such that

$$\lambda\left(PA(t) + A^T(t)P + G\right) \le -\delta(t) \le -\delta,$$

where $\delta > 0$ is a constant.

A3: For any $t \in [0,\infty)$, we assume that

$$\frac{\|PB(t)\|}{\sqrt{\delta(t)(1-\alpha)\lambda_{\min}(G)}} \le \alpha,$$

where $\alpha > 0$ is a constant.

A4: For any $t \in [0,\infty)$, there exists $\varepsilon > 0$ such that

$$\frac{\left\|Pb(t) + \frac{1}{2}\varepsilon k C(t)\right\|}{\sqrt{\delta(t)\varepsilon}} \le \beta,$$

where $\beta > 0$ is a constant.

Theorem 2.2. *Under A1-A4, system (2.1) is absolutely stable if* $\alpha^2 + \beta^2 < 1$.

Proof. Using matrices P and G, we choose a Lyapunov-Krasovskii functional as

$$V(t,\phi) = x^T(t)Px(t) + \int_{t-\tau(t)}^{t} x^T(s)Gx(s)ds.$$

By using A1 and the property of the matrix norm, we see that $V(t,\phi)$ satisfies

$$\lambda_{\min}(P)\|x(t)\|^2 \le V(t,\phi) \le \lambda_{\max}(P)\|x(t)\|^2 + \lambda_{\max}(G)\int_{-\tau}^{0}\|x(t+s)\|^2 ds.$$

Thus

$$\lambda_{\min}(P)\|\phi(0)\|^2 \le V(t,\phi) \le \lambda_{\max}(P)\|\phi(0)\|^2 + \lambda_{\max}(G)|||\phi|||^2.$$

If $u(s) = \lambda_{\min}(P)s^2$, $v(s) = \lambda_{\max}(P)s^2$, and $w(s) = \lambda_{\max}(G)s^2$, then, for $t \ge 0$,

$$u(\|\phi(0)\|) \le V(t,\phi) \le v(\|\phi(0)\|) + w(|||\phi|||).$$

Hence, $V(t,\phi)$ satisfies the conditions of the Lyapunov theorem. Computing the time derivative of $V(t,\phi)$ along system (2.1) yields

$$\begin{aligned}\dot{V}(t,\phi)\big|_{(2.1)} &= 2x^T(t)P\dot{x}(t) + x^T(t)Gx(t) - (1-\dot{\tau}(t))x^T(t-\tau(t))Gx(t-\tau(t))\\ &= 2x^T(t)P(A(t)x(t) + B(t)x(t-\tau(t)) + b(t)f(\sigma(t))) + x^T(t)Gx(t)\\ &\quad - (1-\dot{\tau}(t))x^T(t-\tau(t))Gx(t-\tau(t))\\ &= x^T(t)\left(PA(t) + A^T(t)P + G\right)x(t) + 2x^T(t)PB(t)x(t-\tau(t))\\ &\quad + 2x^T(t)Pb(t)f(\sigma(t)) - (1-\dot{\tau}(t))x^T(t-\tau(t))Gx(t-\tau(t)).\end{aligned}$$

Observe that condition (2.2) is equivalent to $f^2(\sigma(t)) - kc^T(t)x(t)f(\sigma(t)) \le 0$. Taking A2 and the property of the matrix norm into account, for an arbitrary $\varepsilon > 0$, we obtain

$$
\begin{aligned}
&\dot{V}(t,\phi)\big|_{(2.1)} \\
&\le x^T(t)\left(PA(t)+A^T(t)P+G\right)x(t)+2x^T(t)PB(t)x(t-\tau(t))+2x^T(t)Pb(t)f(\sigma(t)) \\
&\quad -(1-\dot{\tau}(t))x^T(t-\tau(t))Gx(t-\tau(t))-\varepsilon f^2(\sigma(t))+\varepsilon kc^T(t)x(t)f(\sigma(t)) \\
&\le x^T(t)\left(PA(t)+A^T(t)P+G\right)x(t)+2x^T(t)PB(t)x(t-\tau(t)) \\
&\quad +2x^T(t)(Pb(t)+\frac{1}{2}\varepsilon kc(t))f(\sigma(t))-(1-\alpha)x^T(t-\tau(t))Gx(t-\tau(t))-\varepsilon f^2(\sigma(t)) \\
&\le -\delta(t)\|x(t)\|^2+2\|PB(t)\|\|x(t)\|\|x(t-\tau(t))\|+2\|Pb(t)+\frac{1}{2}\varepsilon kc(t)\|\|x(t)\||f(\sigma(t))| \\
&\quad -(1-\alpha)\lambda_{\min}(G)\|x(t-\tau(t))\|^2-\varepsilon f^2(\sigma(t)).
\end{aligned}
$$

In view of A3, A4, and the unbounded coefficient terms in system (2.1), we take $\sqrt{\delta(t)}\|x(t)\|$, $\sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau)\|$ and $\sqrt{\varepsilon}|f(\sigma(t))|$ as the following variables of the quadratic form. Notice that

$$
\begin{aligned}
\dot{V}(t,\phi)\big|_{(2.1)} &\le -\delta(t)\|x(t)\|^2+2\frac{\|PB(t)\|}{\sqrt{\delta(t)(1-\alpha)\lambda_{\min}(G)}}\left[\sqrt{\delta(t)}\|x(t)\|\right] \\
&\quad \cdot\left[\sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau(t))\|\right] \\
&\quad +2\frac{\left\|Pb(t)+\frac{1}{2}\varepsilon kc(t)\right\|}{\sqrt{\delta(t)\varepsilon}}\left[\sqrt{\delta(t)}\|x(t)\|\right] \\
&\quad \cdot\left[\sqrt{\varepsilon}|f(\sigma(t))|\right]-(1-\alpha)\lambda_{\min}(G)\|x(t-\tau(t))\|^2-\varepsilon f^2(\sigma(t)) \\
&\le -\delta(t)\|x(t)\|^2+2\alpha\sqrt{\delta(t)}\|x(t)\|\cdot\sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau(t))\| \\
&\quad +2\beta\left[\sqrt{\delta(t)}\|x(t)\|\right] \\
&\quad \cdot\left[\sqrt{\varepsilon}|f(\sigma(t))|\right]-(1-\alpha)\lambda_{\min}(G)\|x(t-\tau(t))\|^2-\varepsilon f^2(\sigma(t)).
\end{aligned}
$$

Then, the above inequality can be written as

$$
\begin{aligned}
&\dot{V}(t,\phi)\big|_{(2.1)} \\
&\le \begin{bmatrix} \sqrt{\delta(t)}\|x(t)\| \\ \sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau(t))\| \\ \sqrt{\varepsilon}|f(\sigma(t))| \end{bmatrix}^T D \begin{bmatrix} \sqrt{\delta(t)}\|x(t)\| \\ \sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau(t))\| \\ \sqrt{\varepsilon}|f(\sigma(t))| \end{bmatrix},
\end{aligned}
$$

where

$$
D=\begin{bmatrix} -1 & \alpha & \beta \\ \alpha & -1 & 0 \\ \beta & 0 & -1 \end{bmatrix}.
$$

It is clear that the matrix D is negative-definite if $\alpha^2+\beta^2<1$. Based on the Lyapunov theorem, system (2.1) is absolutely stable. This completes the proof. □

Corollary 2.3. *Under A1-A4, system (2.1) is absolutely stable if $\alpha^2+\beta^2<1$.*

To study the absolute stability of system (2.1), we only need to ensure that the above conditions are fulfilled as time tends to infinity. Therefore, the aforementioned $t \in [0,\infty)$ in A2-A4 can be written as $t \in [T,\infty), T \geq 0$. Furthermore, A3 and A4 can be rewritten as a new form of the upper limit, i.e., A5 and A6 below.

A5:

$$\overline{\lim_{t\to\infty}} \frac{\|PB(t)\|}{\sqrt{\delta(t)(1-\alpha)\lambda_{\min}(G)}} = \bar{\alpha},$$

where $\bar{\alpha}$ is a constant.

A6: There exists $\varepsilon > 0$ such that

$$\overline{\lim_{t\to\infty}} \frac{\left\|Pb(t) + \frac{1}{2}\varepsilon k C(t)\right\|}{\sqrt{\delta(t)\varepsilon}} = \bar{\beta},$$

where $\bar{\beta}$ is a constant.

Corollary 2.4. *Under A1, A2, A5, and A6, system (2.1) is absolutely stable if* $\bar{\alpha}^2 + \bar{\beta}^2 < 1$.

Corollary 2.5. *Under A1, A2, A5, and A6, system (2.1) is absolutely stable if* $\bar{\alpha} + \bar{\beta} < 1$.

It is worth pointing out that the above absolute stability criteria are also effective for time-delayed Lurie system with bounded or constant coefficients.

3. Absolute Stability of Time-Delayed Lurie Systems with Multiple Nonlinearities

Consider the following time-varying Lurie direct control system with multiple nonlinearities and time-varying delay

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)x(t-\tau(t)) + \sum_{j=1}^{m} b_j(t) f_j(\sigma_j(t)), \\ \sigma_i(t) = c_i^T(t)x(t)\ (i = 1,2,\cdots,m), \end{cases} \tag{3.1}$$

where $x(t) \in R^n$ is the state; $\sigma_i(t) \in R\,(i = 1,2,\cdots,m)$ are the outputs; $A(t), B(t)$ are $n \times n$ matrices; $b_i(t)$, $c_i(t)\,(i = 1,2,\cdots,m)$ are n dimensional column vectors; $\tau(t)$ is the time delay; $A(t), B(t), b_i(t), c_i(t)$ are continuous in $[0,\infty)$. The nonlinearities $f_i(\cdot)\,(i = 1,2,\cdots,m)$ are continuous and satisfy the sector condition

$$F_{[0,k_i]} = \left\{ f_i(\cdot) \,|\, f_i(0) = 0; 0 < \sigma_i(t) f_i(\sigma_i(t)) \leq k_i {\sigma_i}^2(t), \sigma_i(t) \neq 0 \right\}, \tag{3.2}$$

where $k_i > 0\,(i = 1,2,\cdots,m)$ are constants.

Definition 3.1. [22] System (3.1) is said to be absolutely stable if its zero solution is globally asymptotically stable for any nonlinearity $f_i(\cdot) \in F_{[0,k_i]}$.

In addition to the A1-A3 mentioned above, the following assumption is also crucial to system (3.1).

A7: For any $t \in [0,\infty)$, there exist $\varepsilon_i > 0\,(i = 1,2,\cdots,m)$ such that

$$\frac{\left\|Pb_i(t) + \frac{1}{2}\varepsilon_i k_i c_i(t)\right\|}{\sqrt{\delta(t)\varepsilon_i}} \leq \beta_i,$$

where $\beta_i > 0 (i = 1,2,\cdots,m)$ are constants.

Theorem 3.2. *Under A1-A3 and A7, system (3.1) is absolutely stable if* $\alpha^2+\sum_{i=1}^m \beta_i^2<1$.

Proof. Let us apply a Lyapunov-Krasovskii functional of the same form as in the proof of Theorem 2.2. Observe that $V(t,\phi)$ satisfies the conditions of the Lyapunov theorem.

By calculating the time derivative of $V(t,\phi)$ along system (3.1), we obtain

$$\begin{aligned}
\dot V(t,\phi)\big|_{(3.1)} &= 2x^T(t)P\dot x(t)+x^T(t)Gx(t)-(1-\dot\tau(t))x^T(t-\tau(t))Gx(t-\tau(t))\\
&= 2x^T(t)P(A(t)x(t)+B(t)x(t-\tau(t))+\sum_{j=1}^m b_j(t)f_j\left(\sigma_j(t)\right))+x^T(t)Gx(t)\\
&\quad -(1-\dot\tau(t))x^T(t-\tau(t))Gx(t-\tau(t))\\
&= x^T(t)\left(PA(t)+A^T(t)P+G\right)x(t)+2x^T(t)PB(t)x(t-\tau(t))\\
&\quad +2x^T(t)P\sum_{j=1}^m b_j(t)f_j\left(\sigma_j(t)\right)-(1-\dot\tau(t))x^T(t-\tau(t))Gx(t-\tau(t)).
\end{aligned}$$

By virtue of condition (3.2), we have $f_i^2(\sigma_i(t))-k_ic_i^T(t)x(t)f_i(\sigma_i(t))\le 0$. Accordingly, using A2 and the property of matrix norm and estimating $\dot V(t,\phi)\big|_{(3.1)}$ yield

$$\begin{aligned}
\dot V(t,\phi)\big|_{(3.1)} &\le x^T(t)\left(PA(t)+A^T(t)P+G\right)x(t)+2x^T(t)PB(t)x(t-\tau(t))\\
&\quad +2x^T(t)P\sum_{i=1}^m b_i(t)f_i(\sigma_i(t))-(1-\alpha)x^T(t-\tau(t))Gx(t-\tau(t))\\
&\quad -\sum_{i=1}^m \varepsilon_i f_i^2(\sigma_i(t))+\sum_{i=1}^m \varepsilon_i x^T(t)k_ic_i(t)f_i(\sigma_i(t))\\
&\le -\delta(t)\|x(t)\|^2+2\|PB(t)\|\,\|x(t)\|\,\|x(t-\tau(t))\|\\
&\quad +2\sum_{i=1}^m \|Pb_i(t)+\frac{1}{2}\varepsilon_ik_ic_i(t)\|\,\|x(t)\|\,|f_i(\sigma_i(t))|\\
&\quad -(1-\alpha)\lambda_{\min}(G)\|x(t-\tau(t))\|^2-\sum_{i=1}^m \varepsilon_i f_i^2(\sigma_i(t)).
\end{aligned}$$

Then, we regard $\sqrt{\delta(t)}\|x(t)\|$, $\sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau)\|$ and $\sqrt{\varepsilon_i}|f_i(\sigma_i(t))|\,(i=1,2,\cdots,m)$ as the variables of a quadratic form in the following. Further, from A3 and A7, one obtains

$$\begin{aligned}
\dot V(t,\phi)\big|_{(3.1)} &\le -\delta(t)\|x(t)\|^2+2\alpha\sqrt{\delta(t)}\|x(t)\|\cdot\sqrt{(1-\alpha)\lambda_{\min}(G)}\|x(t-\tau(t))\|\\
&\quad +2\sum_{i=1}^m \beta_i\left[\sqrt{\delta(t)}\|x(t)\|\right]\cdot\left[\sqrt{\varepsilon_i}|f_i(\sigma_i(t))|\right]-(1-\alpha)\lambda_{\min}(G)\|x(t-\tau(t))\|^2\\
&\quad -\sum_{i=1}^m \varepsilon_i f_i^2(\sigma_i(t)).
\end{aligned}$$

Observe that the inequality above can be rewritten as

$$\dot{V}(t,\phi)\big|_{(3.1)} \leq \begin{bmatrix} \sqrt{\delta(t)}\,\|x(t)\| \\ \sqrt{(1-\alpha)\lambda_{\min}(G)}\,\|x(t-\tau(t))\| \\ \sqrt{\varepsilon_1}\,|f_1(\sigma_1(t))| \\ \vdots \\ \sqrt{\varepsilon_m}\,|f_m(\sigma_m(t))| \end{bmatrix}^T D \begin{bmatrix} \sqrt{\delta(t)}\,\|x(t)\| \\ \sqrt{(1-\alpha)\lambda_{\min}(G)}\,\|x(t-\tau(t))\| \\ \sqrt{\varepsilon_1}\,|f_1(\sigma_1(t))| \\ \vdots \\ \sqrt{\varepsilon_m}\,|f_m(\sigma_m(t))| \end{bmatrix},$$

where

$$D = \begin{bmatrix} -1 & \alpha & \beta_1 & \cdots & \beta_m \\ \alpha & -1 & 0 & \cdots & 0 \\ \beta_1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_m & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

One can prove that the matrix D is negative-definite if $\alpha^2 + \sum\limits_{i=1}^{m} \beta_i^2 < 1$. By the Lyapunov theorem, system (3.1) is absolutely stable. The theorem is verified. □

Corollary 3.3. *Under A1-A3 and A7, system (3.1) is absolutely stable if* $\alpha + \sum_{i=1}^{m} \beta_i < 1$.

Similarly to the case of single nonlinearity, A7 can be replaced by another form, i.e., A8 below.

A8: There exist $\varepsilon_i > 0 (i = 1,2,\cdots,m)$ such that

$$\overline{\lim_{t\to\infty}} \frac{\left\|Pb_i(t) + \frac{1}{2}\varepsilon_i k_i c_i(t)\right\|}{\sqrt{\delta(t)\,\varepsilon_i}} = \bar{\beta}_i,$$

where $\bar{\beta}_i > 0 (i = 1,2,\cdots,m)$ are constants.

Corollary 3.4. *Under A1, A2, A5, and A8, system (3.1) is absolutely stable if* $\bar{\alpha}^2 + \sum_{i=1}^{m} \bar{\beta}_i^2 < 1$.

Corollary 3.5. *Under A1, A2, A5, and A8, system (3.1) is absolutely stable if* $\bar{\alpha} + \sum_{i=1}^{m} \bar{\beta}_i < 1$.

4. Numerical Simulations

To reinforce theoretical results, several numerical examples are presented in the following.

Example 4.1. Consider the following time-delayed Lurie direct control system with single nonlinearity

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2t-0.5 & 1 \\ -1 & -3t-0.5 \end{bmatrix} x(t) + \begin{bmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{\frac{t}{3}} \end{bmatrix} x(t-\tau(t)) + \begin{bmatrix} -2\sqrt{t} \\ \sqrt{t} \end{bmatrix} f(\sigma(t)), \\ \sigma(t) = \begin{bmatrix} \sqrt{t} & 1 \end{bmatrix} x(t), \end{cases} \tag{4.1}$$

where $\tau(t) = 3 + 0.5\sin t$ and $f(\cdot) \in F_{[0,2]}$.

This system is in the form of (2.1) with

$$A(t) = \begin{bmatrix} -2t-0.5 & 1 \\ -1 & -3t-0.5 \end{bmatrix}, B(t) = \begin{bmatrix} \sqrt{t} & 0 \\ 0 & \sqrt{\frac{t}{3}} \end{bmatrix}, b(t) = \begin{bmatrix} -2\sqrt{t} \\ \sqrt{t} \end{bmatrix}, c(t) = \begin{bmatrix} \sqrt{t} \\ 1 \end{bmatrix}.$$

For condition (2.2), $k=2$. Clearly, A1 is satisfied with $\tau=3.5$ and $w=0.5$. By letting $P=G=I$, we have

$$PA(t)+A^T(t)P+G=\begin{bmatrix} -4t & 0 \\ 0 & -6t \end{bmatrix}.$$

It is obvious that $\lambda\left(PA(t)+A^T(t)P+G\right)\leq -4t$. Hence A2 is fulfilled with $\delta(t)=4t$. Since $\lim\limits_{t\to\infty}\frac{\|PB(t)\|}{\sqrt{\delta(t)(1-\alpha)\lambda_{\min}(G)}}=\frac{1}{\sqrt{2}}$, then A3 is satisfied with $\bar{\alpha}=\frac{1}{\sqrt{2}}$. Then by taking $\varepsilon=2$, we obtain $\lim\limits_{t\to\infty}\frac{\left\|Pb(t)+\frac{1}{2}\varepsilon kc(t)\right\|}{\sqrt{\delta(t)\varepsilon}}=\frac{1}{\sqrt{8}}$. Thus A4 is satisfied with $\bar{\beta}=\frac{1}{\sqrt{8}}$. Finally, one can compute that $\bar{\alpha}^2+\bar{\beta}^2=\frac{1}{8}<1$, which means the conditions of Corollary 2.4 are satisfied. Hence, system (4.1) is absolutely stable. Letting $f(\sigma(t))=\sigma(t)+0.1\sin\sigma(t)$. It can be verify that $f(\cdot)\in F_{[0,2]}$. Additionally, the simulation result is obtained with initial condition $x(t)=\begin{bmatrix} 1 \\ -1 \end{bmatrix}, t\in[-3.5,0]$, as shown in Figure 1. It is clear that the zero solution of system (4.1) is asymptotically stable.

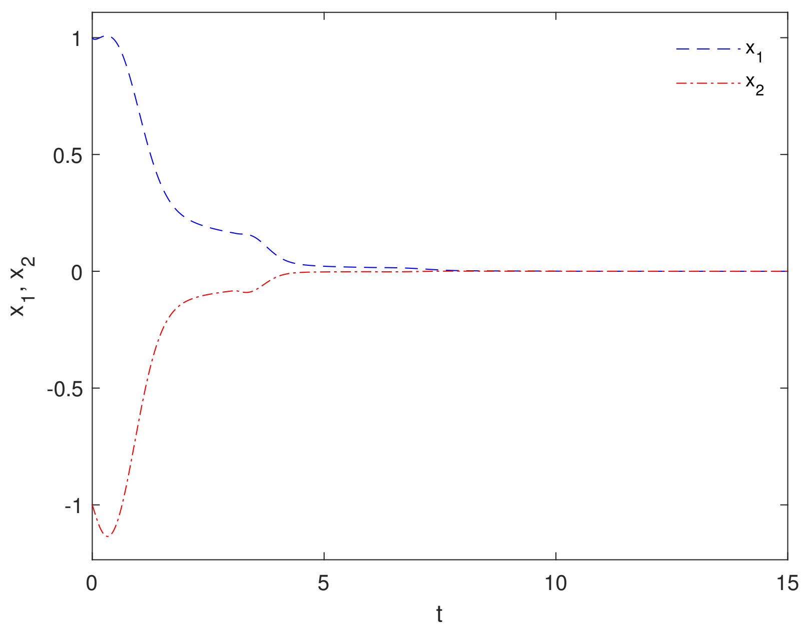

FIGURE 1. The state response of system (4.1) in Example 4.1.

Example 4.2. Let us continue with system (4.1), and the time delay is

$$\tau(t)=\begin{cases} 1, & t<2 \\ 0.5t, & 2\leq t\leq 4 \\ 2, & t>4 \end{cases}$$

while other parameters are unchanged. Here $\tau(t)\leq 2$, so $\tau=2$. We notice that $\tau(t)$ is not derivative at $t=2$ and $t=4$, but it has left and right derivative. By simple computation, A1 is fulfilled with $w=0.5$. Similarly to Example 4.1, this system is also absolutely stable. The numerical simulation result is shown in Figure 2.

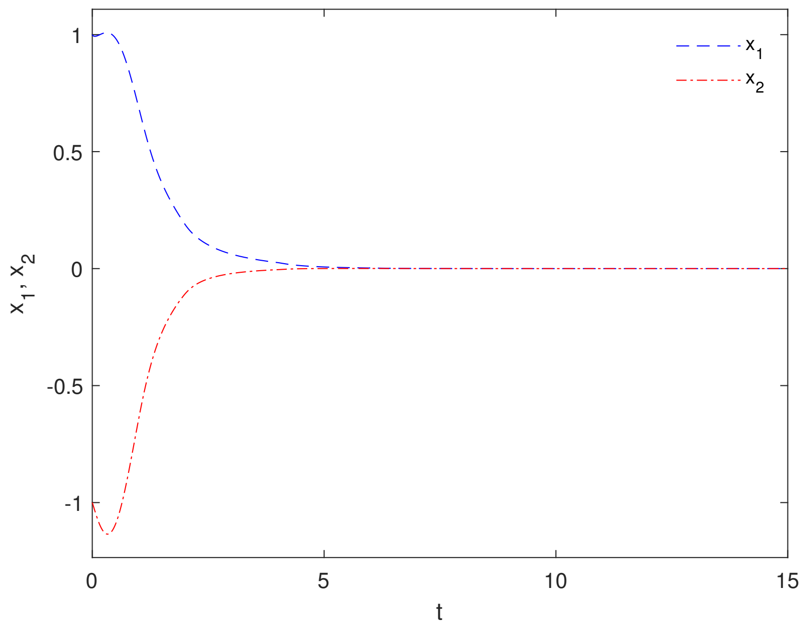

FIGURE 2. The state response of the system in Example 4.2.

Notice that the coefficients $A(t), B(t), b(t), c(t), \rho(t)$ are norm-unbounded in Example 4.1 and Example 4.2. This is the main highlight of this paper. All the conclusions are not only effective for Lurie direct control systems with unbounded coefficients but also valid for such systems with bounded or constant coefficients. Then, we present an example of Lurie system with constant coefficients.

Example 4.3. Consider the following constant Lurie direct control system with time-varying delay

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -4.5 & 1 \\ -1 & -4.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t-\tau(t)) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} f(\sigma(t)), \\ \sigma(t) = \begin{bmatrix} -1 & \sqrt{2} \end{bmatrix} x(t), \end{cases} \tag{4.2}$$

where $\tau(t) = 3 + 0.5\sin t$ and $f(\cdot) \in F_{[0,2]}$. This system is in the form of (2.1) with

$$A = \begin{bmatrix} -4.5 & 1 \\ -1 & -4.5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} -1 \\ \sqrt{2} \end{bmatrix}.$$

For condition (2.2), $k = 2$. Obviously, A1 is satisfied with $\tau = 3.5, w = 0.5$. By letting $P = G = I$, we have

$$PA + A^T P + G = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}.$$

Thus A2 is satisfied with $\delta(t) = \delta = 8$. Moreover, one has

$$\frac{\|PB\|}{\sqrt{\delta(t)(1-\alpha)\lambda_{\min}(G)}} = \frac{1}{2}.$$

Hence, A3 is fulfilled with $\alpha = \frac{1}{2}$. In addition, by letting $\varepsilon = 1$, it follows directly that

$$\frac{\left\|Pb + \frac{1}{2}\varepsilon kc\right\|}{\sqrt{\delta(t)\varepsilon}} = \frac{1}{2},$$

which implies that A4 is satisfied with $\beta = \frac{1}{2}$. It is straightforward to see $\alpha^2 + \beta^2 = \frac{1}{2} < 1$ holds. One can thus conclude from Theorem 2.2 that this system is absolutely stable. In order to carry out the numerical simulation, we select $f(\sigma(t)) = \sigma(t) + 0.1\sin\sigma(t)$ and assume that the initial condition is $x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, t \in [-3.5, 0]$. The simulation result is illustrated in Figure 3.

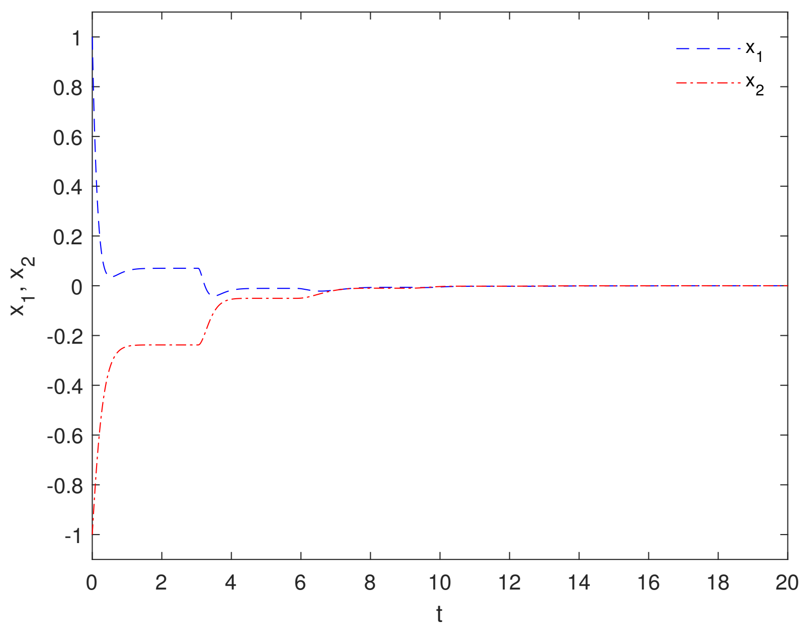

FIGURE 3. The state response of system (4.2) in Example 4.3.

As can be seen clearly in Figure 3, the states of system (4.2) converge to the origin asymptotically. This example illustrates that the stability conditions in this paper are effective for the constant Lurie direct control system with time-varying delay.

Example 4.4. Consider the following time-delayed Lurie direct control system with two nonlinearities

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -4t-0.5 & 1 \\ 0 & -6t-0.5 \end{bmatrix} x(t) + \begin{bmatrix} \sqrt{0.25t} & 0 \\ 0 & \sqrt{t} \end{bmatrix} x(t-\tau(t)) + \begin{bmatrix} \sqrt{3t} \\ \sqrt{t} \end{bmatrix} f_1(\sigma_1(t)) \\ \qquad + \begin{bmatrix} -t \\ \sqrt{t} \end{bmatrix} f_2(\sigma_2(t)), \\ \sigma_1(t) = \begin{bmatrix} -\sqrt{3t} & 1 \end{bmatrix} x(t), \\ \sigma_2(t) = \begin{bmatrix} t+1 & 1 \end{bmatrix} x(t), \end{cases} \tag{4.3}$$

where $\tau(t) = 3 + 0.5\sin t$, $f_1(\sigma) \in F_{[0,0.5]}$, and $f_2(\sigma) \in F_{[0,1]}$.

This system is in the form of (3.1) with

$$A(t)=\begin{bmatrix} -4t-0.5 & 1 \\ 0 & -6t-0.5 \end{bmatrix}, B(t)=\begin{bmatrix} \sqrt{0.25t} & 0 \\ 0 & \sqrt{t} \end{bmatrix}, b_1(t)=\begin{bmatrix} \sqrt{3t} \\ \sqrt{t} \end{bmatrix}, b_2(t)=\begin{bmatrix} -t \\ \sqrt{t} \end{bmatrix},$$

$$c_1(t)=\begin{bmatrix} -\sqrt{3t} \\ 1 \end{bmatrix}, c_2(t)=\begin{bmatrix} t+1 \\ 1 \end{bmatrix}.$$

For condition (3.2), here $k_1=0.5, k_2=1$. First, A1 is satisfied with $\tau=3.5, w=0.5$. By letting $P=G=I$, we have

$$PA(t)+A^T(t)P+G=\begin{bmatrix} -8t & 1 \\ 1 & -12t \end{bmatrix}.$$

By simple calculation, we obtain $\lambda\left(PA(t)+A^T(t)P+G\right)\leq -10t+\sqrt{4t^2+1}$. Let $T=\frac{2}{3}$. If $t>T$, then $\lambda\left(PA(t)+A^T(t)P+G\right)\leq -7.5t<-5$. Hence A2 is fulfilled with $\delta(t)=7.5t$ and $\delta=5$. Then,

$$\lim_{t\to\infty}\frac{\|PB(t)\|}{\sqrt{\delta(t)(1-\alpha)\lambda_{\min}(G)}}=\frac{1}{\sqrt{3.75}}.$$

Thus A5 is satisfied with $\bar{\alpha}=\frac{1}{\sqrt{3.75}}$. Furthermore, by letting $\varepsilon_1=4, \varepsilon_2=2$, we derive

$$\lim_{t\to\infty}\frac{\left\|Pb_1(t)+\frac{1}{2}\varepsilon_1 k_1 c_1(t)\right\|}{\sqrt{\delta(t)\varepsilon_1}}=\frac{1}{\sqrt{30}}$$

and

$$\lim_{t\to\infty}\frac{\left\|Pb_2(t)+\frac{1}{2}\varepsilon_2 k_2 c_2(t)\right\|}{\sqrt{\delta(t)\varepsilon_2}}=\frac{1}{\sqrt{15}},$$

which shows that A8 is satisfied with $\bar{\beta}_1=\frac{1}{\sqrt{30}}$ and $\bar{\beta}_2=\frac{1}{\sqrt{15}}$. In the end, since $\bar{\alpha}^2+\bar{\beta}_1^2+\bar{\beta}_2^2=\frac{11}{30}<1$, we arrive at the conclusion via Corollary 3.4 that this system is absolutely stable. The simulation is developed with the nonlinearities $f_1(\sigma(t))=0.2\sigma(t)+0.1\sin\sigma(t)$ and

$$f_2(\sigma(t))=\begin{cases} 0.125\sigma(t), & |\sigma(t)|<1, \\ 0.125\sigma^3(t), & 1\leq|\sigma(t)|\leq 2, \\ 0.5\sigma(t), & |\sigma(t)|>2, \end{cases}$$

and the initial condition $x(t)=\begin{bmatrix} 1 \\ -1 \end{bmatrix}, t\in[-3.5,0]$, as shown in Figure 4. Clearly, the zero solution of system (4.3) is asymptotically stable. It is observed that the simulation result agrees closely with the theoretical result.

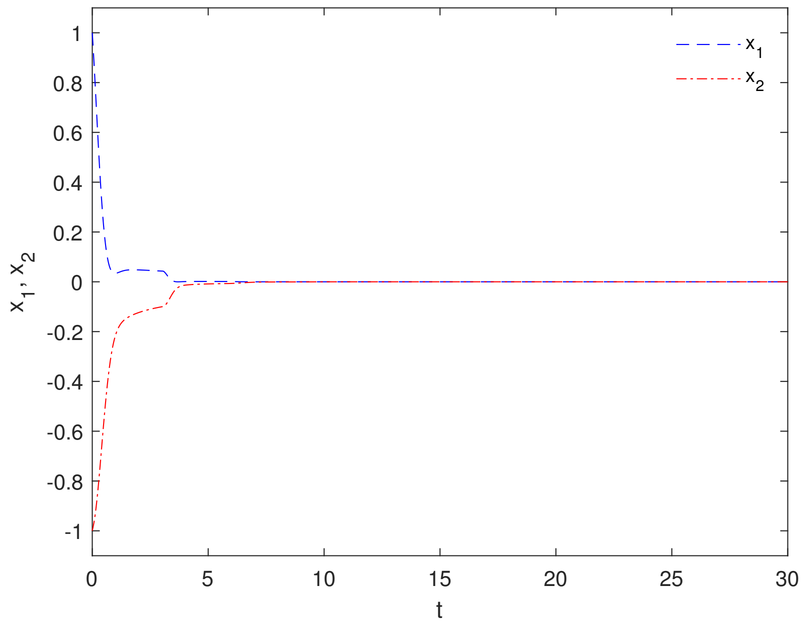

FIGURE 4. The state response of system (4.3) in Example 4.4.

5. CONCLUSIONS

The absolute stability of time-delayed Lurie direct control system was considered in this paper. From the Lyapunov theorem on time delay system, several simple and computable sufficient conditions were obtained by Lyapunov-Krasovskii approach. The criteria proposed in this paper are especially effective for time-delayed Lurie direct control system with unbounded coefficients and also valid for such system with bounded or constant coefficients. The stability results were illustrated by numerical simulations.

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