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CONVERGENCE TOPOLOGIES AND f-STATISTICALLY CAUCHY SEQUENCES

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Abstract. Modulus statistical convergence has already been transported to the general scope of uniform spaces. Here, the novel notion of f^*-statistically Cauchy sequence is introduced and studied in uniform spaces. Certain properties are verified, in particular, the relationship between f^*-statistically Cauchy sequences and f-statistically Cauchy sequences is unveiled. Finally, f-statistical limit points and f-statistical cluster points of a sequence are considered and studied, again in the general context of uniform spaces.

Keywords. Convergence topologies; Cauchy sequences; Cluster points; Limit points; Modulus function; Statistical convergence.

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1. Introduction

The concept of statistical convergence was firstly introduced by Steinhaus [34] and Fast [12] and later revisited by Schoenberg [33]. Thereafter, this notion has been studied by many authors in various spaces. Fridy [13] proved that a number sequence is statistically convergent if and only if it is statistically Cauchy. The statistical convergence in Banach spaces was studied by Kolk [20]. Maddox [26] extended the concept of statistical convergence to sequences with values in arbitrary locally convex Hausdorff topological vector spaces. Connor et al. [9] provided relevant results that relate statistical convergence to classical properties of Banach spaces. For other works concerning statistical convergence, we refer to [4, 5, 7, 17, 19, 27, 32].

The notion of a modulus function was introduced by Nakano [29]. In [31], Ruckle investigated the sequence space defined by a modulus function. In [25], Maddox introduced and discussed some properties of sequence spaces defined by means of a modulus function. Recently

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[1, 2, 23], it was firstly introduced the density of a subset of natural numbers and statistical convergence in normed spaces by means of a modulus, and some basic and important properties were studied. Afterwards, this concept has been deeply studied and investigated by many authors. In [8], Bhardwaj and Dhawan obtained Korovkin-type approximation theorems by using the notion of lacunary statistical convergence via a modulus function. In [18], Kama and Altay introduced some multiplier sequence spaces relying on statistical convergence by a modulus function, achieving characterizations of completeness and barrelledness of a normed space and an Orlicz-Pettis theorem. In [22], León-Saavedra et al. gave relations between the statistical convergence and the strong Cesàro convergence defined by a modulus function. In [6], Altay et al. introduced and investigated the concepts of statistical continuity, statistical derivative and strongly Cesàro derivative for real-valued functions by means of a modulus function. Afterwards, García-Pacheco and Kama [15] extended the concept of statistical convergence by a modulus function to the most general setting so far: uniform spaces. Finally, statistical convergence and related notions has been transported to other ambiences such as matrix sequences [35], double sequences [36], and uniform convergence [24].

In Summability Theory, uniform convergence is often present. Although it is intuitively clear what is meant by uniform convergence, many times there exists a lack of a clear understanding of uniform convergence in formal topological terms. Uniformities come into play to bridge this gap. There exists a closed relationship between topological spaces and uniform spaces [16]. At a first glance, topological spaces are more general than uniform spaces. However, certain notions related to convergence and continuity (such us uniform continuity, uniform convergence, Cauchy sequences, and completeness) are more natural to be defined in uniform spaces rather than in topological spaces. As we will see in Subsection 3.3, the notion of uniform convergence in general topological spaces might lead to strange topologies that not model properly the necessities and the situations that appear in Summability Theory.

Uniform spaces provide the right setting to define Cauchy structures (filter bases, nets, and sequences) as well as completeness. Therefore, uniform spaces are also the appropriate setting to study f-statistical Cauchy sequences. In this sense, it is important to obtain a version of [15, Proposition 3.3] (Proposition 2.10) for f-statistical Cauchy sequences. Our Proposition 3.1 only provides one implication. Our future work aims at achieving the converse implication. The field of Summability and Convergence is constantly getting enriched with extensions of statistical convergence by moduli to very different ambiences. This manuscript takes one leap further on this trend by transporting statistical convergence by moduli to uniform spaces. The objective of this manuscript is to keep exploring convergence topologies as well as statistical convergence by a modulus function in uniform spaces. The novel notion of f^*-statistically Cauchy sequence is introduced and studied, as well as f-statistical limit points and f-statistical cluster points of a sequence.

2. PRELIMINARIES

This section is aimed at presenting some definitions and results on which we will strongly rely in this work.

2.1. **Uniform spaces.** The most general category that we will work with is the one of uniform spaces, where the objects are the uniform spaces and the morphisms are the uniformly continuous functions. Although, some times we jump up to general topological spaces and general continuous functions.

Definition 2.1 (Uniform space)**.** Let X be a set. A uniformity on X is a filter $\mathscr{U} \subseteq \mathscr{P}(X \times X)$ satisfying, for every $U \in \mathscr{U}$, the following conditions:

(1) $U \subseteq X \times X$ is a reflexive internal binary relation on X, that is, $\Delta_X \subseteq U$, where $\Delta_X := \{(x,x) : x \in X\}$ is the diagonal of X.
(2) There exists $V \in \mathscr{U}$ such that $V \circ V \subseteq U$, where $V \circ V := \{(v,w) \in X \times X : \exists u \in X \ \ (v,u),(u,w) \in V\}$.
(3) $U^{-1} \in \mathscr{U}$, where $U^{-1} := \{(v,u) \in X \times X : (u,v) \in V\}$.

The pair $(X,\mathscr{U})$ is called a uniform space. The elements of $\mathscr{U}$ are called entourages or vicinities.

Every filter base of $\mathscr{U}$ is called a base of entourages or vicinities. For every $x \in X$ and every $U \in \mathscr{U}$, $U[x] := \{y \in X : (x,y) \in U\}$. An entourage U is said to be symmetric provided that $U = U^{-1}$. If U is an entourage, then $V := U \cap U^{-1}$ is a symmetric entourage. If $\mathscr{B}$ is a base of entourages, then $\mathscr{B}_1 := \left\{U \cap U^{-1} : U \in \mathscr{B}\right\}$ is a base of (symmetric) entourages. Given two uniformities $\mathscr{U}_1, \mathscr{U}_2$ on X, we say that $\mathscr{U}_1$ is coarser than $\mathscr{U}_2$, or $\mathscr{U}_2$ is finer than $\mathscr{U}_1$, if $\mathscr{U}_1 \subseteq \mathscr{U}_2$. Every subset Y of X inherits its uniform structure by means of the uniformity $\mathscr{U}_Y := \{U \cap (Y \times Y) : U \in \mathscr{U}\}$.

Every uniform space becomes a topological space by defining the topology by means of the entourages. Let X be a uniform space. Then

$$\tau := \{A \subseteq X : \forall a \in A\ \exists U \subseteq X \times X \text{ entourage such that } U[a] \subseteq A\} \cup \{\varnothing\}$$

is a topology on X that turns it into a regular topological space. Also, for every $A \subseteq X \setminus \{\varnothing\}$, $\operatorname{int}(A) = \{a \in A : \exists U \subseteq X \times X \text{ entourage such that } U[a] \subseteq A\}$. As a consequence, $\{U[x] : U \text{ entourage}\}$ is base of neighborhoods of every $x \in X$. This topology is known as uniform topology.

Uniformities provide the right structure to define notions such as uniform continuity or Cauchy sequences or nets. For instance, a function $f : X \to Y$ between uniform spaces X,Y is said to be uniformly continuous provided that, for every entourage V of Y, there exists an entourage U of X such that $f(U) \subseteq V$, where $f(U) := \{(f(x_1), f(x_2)) \in Y \times Y : (x_1,x_2) \in U\}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in a uniform space X is said to be a Cauchy sequence provided that, for every entourage U in X, there exists $n_U \in \mathbb{N}$ in such a way that $(x_p,x_q) \in U$ for all $p,q \in \mathbb{N}$ with $p,q \geq n_U$. The notion of Cauchy sequence can be easily generalized to that of Cauchy net or Cauchy prefilter. A net $(f_\lambda)_{\lambda\in\Lambda}$ of functions valued on a uniform space X and defined on an arbitrary set I, endowed with an upward directed collection $\mathscr{G}$ of subsets of I, converges to a function $f_0 : I \to X$ if and only if, for every $G \in \mathscr{G}$ and every entourage $V \in X$, there exists $\lambda_{G,U} \in \Lambda$ such that $(f_\lambda(i), f_0(i)) \in V$ for all $i \in G$ and all $\lambda \in \Lambda$ with $\lambda \geq \lambda_{G,U}$. In fact, the set of all functions from I to X, X^I, can be naturally endowed with a uniform structure, called convergence uniformity, via the base of entourages given by $\{\mathscr{V}(G,V) : G \in \mathscr{G} \text{ and } V \text{ is an entourage of } X\}$, where for every entourage V of X and every $G \in \mathscr{G}$, $\mathscr{V}(G,V) := \left\{(f,g) \in X^I \times X^I : \forall i \in G\ \ (f(i),g(i)) \in V\right\}$.

Special cases of uniform spaces are the pseudometric spaces and the topological groups.

Example 2.2. Let X be a pseudometric space. The sets of the form

$$\mathsf{U}_\delta := \{(x,y) \in X \times X : d(x,y) < \delta\},$$

for every $\delta > 0$, form a base of entourages whose generated filter is called the pseudometric uniformity. For every $x \in X$ and every $\delta > 0$, $\mathsf{U}_\delta[x] = \mathsf{U}(x,\delta)$, that is, the open ball of center x and radius δ.

Example 2.3. Let G be a topological group. The sets of the form

$$U_V := \left\{(g,h) \in G \times G : gh^{-1} \in V\right\},$$

for every $V \subseteq G$ neighborhood of 1, constitute a base of entourages whose generated filter is called the group uniformity. For every $g \in G$ and every $V \subseteq G$ neighborhood of 1, $\mathsf{U}_V[g] = V^{-1}g$.

2.2. **Modulus functions and f-density.** The following notion, which is crucial to the development of the ambience of statistical convergence and is the one of modulus function, which was originally introduced in [29].

Definition 2.4 (Modulus functions)**.** A modulus is a function f from $[0,\infty)$ to $[0,\infty)$ that satisfies the following conditions:

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x+y) \leq f(x) + f(y)$ for $x, y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0.

Observe that from the previous definition it can be inferred that f must be continuous everywhere on $[0,\infty)$, and $f\left(\frac{x}{r}\right) \geq \frac{1}{r} f(x)$ for all $x \in \mathbb{R}^+$ and all $r \in \mathbb{N}$. A modulus f may be bounded or unbounded. For example, $f(x) = \frac{x}{x+1}$ is bounded, whereas $f(x) = x^p$, for $0 < p < 1$, is unbounded.

A modulus function f is said to be compatible [22] if, for any $\varepsilon > 0$, there exists $\tilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \geq n_0$. Examples [22] of compatible modulus functions are $f(x) = x + \log(x+1)$ and $f(x) = x + \frac{x}{x+1}$. Examples of non-compatible modulus functions are $f(x) = \log(x+1)$ and $f(x) = W(x)$, where W is the W-Lambert function restricted to $\mathbb{R}^+$. By relying on modulus functions, the notion of f-density was coined in [2].

Definition 2.5 (f-Density)**.** Let f be a modulus function. The f-density of a subset $A \subseteq \mathbb{N}$ is defined as

$$d_f(A) := \lim_{n\to\infty} \frac{f(\mathrm{card}(A \cap [1,n]))}{f(n)}$$

if the limit exists.

If $f(x) = x$ for all $x \in [0,\infty)$, then the concept of f-density of a subset $A \subseteq \mathbb{N}$ reduces to natural density, denoted by $d(A)$. As usual, when working with f-densities, whenever we write the expression $d_f(A)$, we will implicitly assume that $d_f(A)$ exists. The following remark completes [15, Remark 2.9].

Remark 2.6. Let f be a modulus function. For every $A, B \subseteq \mathbb{N}$ such that $d_f(A), d_f(B)$ exist:

(1) d_f is increasing, that is, $d_f(A) \leq d_f(B)$ whenever $A \subseteq B$. Thus $0 = d_f(\varnothing) \leq d_f(A) \leq d_f(\mathbb{N}) = 1$.
(2) d_f is subadditive, that is, $d_f(A \cup B) \leq d_f(A) + d_f(B)$ provided that $d_f(A \cup B)$ exists. If $d_f(A) = d_f(B) = 0$, then $d_f(A \cup B)$ exists. Hence it is 0.
(3) An example displayed in [3] shows that d_f is not additive even for disjoint pairs of subsets of $\mathbb{N}$.
(4) If $d_f(A) = 0$, then $d_f(\mathbb{N} \setminus A) = 1$.
(5) In [3, Example 2.1], it is shown that the converse to the previous proposition does not hold, that is, $d_f(A) = 1$ does not necessarily mean $d_f(\mathbb{N} \setminus A) = 0$.
(6) $d_f(A) = 0$ implies $d(A) = 0$.
(7) $d_f(A) = 0$ implies $d_f(C)$ exists and is 0 for all $C \subseteq A$.
(8) If $A \subseteq \mathbb{N}$ is finite and f is unbounded, then $d_f(A) = 0$.

From the above properties, it is not hard to infer that the collection of all subsets of $\mathbb{N}$ with f-density 0 is an ideal of $\mathscr{P}(\mathbb{N})$. Even more, if f is unbounded, then all finite subsets of $\mathbb{N}$ have null f-density, meaning that the union of all sets with null f-density is the whole of $\mathbb{N}$. Thus, under the assumption that f is unbounded, the collection of all subsets of $\mathbb{N}$ with f-density 0 is a bornology of $\mathscr{P}(\mathbb{N})$.

The following lemmas, which can be found in [23, Lemma 2.8] and [2, Lemma 3.4], respectively, play an important role in the upcoming sections.

Lemma 2.7. *If f is an unbounded modulus function and $(A_i)_{i\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of naturals with f-density 0, then there exists another sequence of subsets of naturals $(B_i)_{i\in\mathbb{N}}$ such that $B_i \subseteq A_i$ and $A_i \setminus B_i$ is finite for every $i \in \mathbb{N}$, and $B := \bigcup_{i\in\mathbb{N}} B_i$ has f-density 0.*

Lemma 2.8. *If H is a infinite subset of $\mathbb{N}$, then there exists an unbounded modulus function f such that $d_f(H) = 1$.*

2.3. **f-Statistical convergence.** Next, f-density leads to f-statistical convergence, which in uniform spaces was originally defined in [15].

Definition 2.9 (f-Statistical convergence)**.** Let X be a uniform space. Let f be a modulus function. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be f-statistically convergent to $x_0 \in X$ if the set $\{n \in \mathbb{N} : (x_n, x_0) \notin U\}$ has f-density 0 for every entourage $U \subseteq X$. We denote by $f\text{-st}\lim(x_n)$ the set of all f-statistical limits of $(x_n)_{n\in\mathbb{N}}$.

Previous definition unveils that f-statistical convergence is a particular case of ideal convergence. The following propositions [15] summarize the relationships between f-statistical convergence, statistical convergence, and usual convergence in uniform spaces.

Proposition 2.10. *Let X be a uniform space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is convergent to $x_0 \in X$ if and only if $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0 for every unbounded modulus f. In short,*

$$\lim(x_n) = \bigcap \{f\text{-st}\lim(x_n) : f \textit{ unbounded modulus function}\}.$$

Proposition 2.11. *Let X be a uniform space. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x_0 \in X$. The following statements hold:*

(1) If there exists a modulus f such that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0, then $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to x_0. In short,

$$\bigcup\{f\text{-st}\lim(x_n) : f \textit{ modulus function}\} \subseteq \text{st}\lim(x_n).$$

(2) Conversely, if $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to x_0, then $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0 for every compatible modulus function f. In short,

$$\text{st}\lim(x_n) \subseteq \bigcap\{f\text{-st}\lim(x_n) : f \textit{ compatible modulus function}\}.$$

As previously mentioned, according to Remark 2.6, the collection of all subsets of $\mathbb{N}$ with f-density 0 is a bornology of $\mathscr{P}(\mathbb{N})$ (for f unbounded). Therefore, f-statistical convergence is a particular case of ideal convergence. In [28, 30], the notion of ideal Cauchy sequence was coined and studied in linear structures. Therefore, it makes sense to define and consider f-statistically Cauchy sequences in uniform spaces, since metric linear structures are particular cases of uniform spaces.

The notion of f-statistically Cauchy sequence was also coined in [15]. Even though topological spaces are more general than uniform spaces, as far as we are concerned and for the reasons previously exposed above, uniform spaces provide the most general context where f-statistically Cauchy sequences can be naturally defined. We would like to make the reader notice that f-statistical convergence has already been studied in previous works, however, as highlighted in the Introduction, uniform spaces provide the natural ambience to define f-statistically Cauchy sequences and related notions.

Definition 2.12 (f-Statistical Cauchy)**.** Let X be a uniform space. Let f be a modulus function. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be f-statistically Cauchy if for every entourage $U \subseteq X \times X$ there exists some $m \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : (x_n, x_m) \notin U\}$ has f-density 0.

Also, in [15] a necessary condition was obtained for a f-statistically Cauchy sequence to be f-statistically convergent.

Proposition 2.13. *Let X be a uniform space with a countable base of entourages. Let f be an unbounded modulus function. If X is complete, then every f-statistically Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is f-statistically convergent.*

3. Results

The goal of this section is to continue the study of modulus statistical convergence on uniform spaces initiated in [15].

3.1. **f-Statistically Cauchy sequences.** The first result in this subsection is a partial version of [15, Proposition 3.3] (see also Proposition 2.10) for f-Cauchy sequences in uniform spaces.

Proposition 3.1. *Let X be a uniform space. If a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is Cauchy, then $(x_n)_{n\in\mathbb{N}}$ is f-statistically Cauchy for every unbounded modulus f.*

Proof. Fix an arbitrary unbounded modulus f. For every symmetric entourage $U \subseteq X \times X$, we see that a natural n_U can be found satisfying that $(x_p, x_q) \in U$ for all $p, q \geq n_U$. This means that $\{n \in \mathbb{N} : (x_n, x_U) \notin U\}$ has f-density 0 because it is finite. This assures that $(x_n)_{n\in\mathbb{N}}$ is f-statistically Cauchy. $\square$

Next theorem relates f-statistical convergence and f-statistical Cauchy in uniform spaces.

Theorem 3.2. *Let X be a uniform space and f an unbounded modulus function. If a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is f-statistically convergent, then it is a f-statistically Cauchy sequence.*

Proof. Take $x_0 \in f\text{-st}\lim(x_n)$ and consider any symmetric entourage U in X. There exists another entourage V such that $V \circ V \subseteq U$. Define $A := \{n \in \mathbb{N} : (x_n, x_0) \notin V\}$. Since $x_0 \in f\text{-st}\lim(x_n)$, we have that $d_f(A) = 0$. Choose $m \in \mathbb{N}$ such that $x_m \in V[x_0]$. Note that $(x_n, x_0) \in V$ for all $n \in \mathbb{N} \setminus A$. Since $(x_0, x_m) \in V$ and $V \circ V \subseteq U$, we have that $(x_n, x_m) \in U$ for all $n \in \mathbb{N} \setminus A$. Therefore, $\{n \in \mathbb{N} : (x_n, x_m) \notin U\} \subseteq A$. Since $d_f(A) = 0$, we have $d_f(\{n \in \mathbb{N} : (x_n, x_m) \notin U\}) = 0$, that is, $(x_n)_{n\in\mathbb{N}}$ is a f-statistically Cauchy sequence. □

Next theorem establishes a sufficient condition for a uniform space to be sequentially complete in terms of f-statistical convergence and f-statistical Cauchy.

Theorem 3.3. *Let X be a uniform space. For every unbounded modulus f, if every f-statistically Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X is f-statistically convergent in X, then X is sequentially complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is f-statistically Cauchy by the observation made at the beginning of this section. Hence, by hypothesis, it is f-statistically convergent in X, say to some x_0 in X. By relying on Proposition 2.10, $(x_n)_{n\in\mathbb{N}}$ is convergent to x_0 in X. Therefore, the proof is finished. □

The novel concepts of f^*-statistical convergence and f^*-statistical Cauchy will be presented in the upcoming definitions, which are strongly motivated by [15, Theorem 3.6].

Definition 3.4 (f^*-Statistical convergence)**.** Let X be a uniform space. Let f be a modulus function. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be f^*-statistically convergent to $x_0 \in X$ if there exists a subset $A \subseteq \mathbb{N}$ satisfying that $d_f(A) = 0$ and $(x_n)_{n\in\mathbb{N}\setminus A}$ is convergent to x_0. In this case, we denote it by $f^*\text{-st}\lim(x_n) = x_0$.

Definition 3.5 (f^*-Statistical Cauchy)**.** Let X be a uniform space. Let f be a modulus function. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be f^*-statistically Cauchy if there exists a subset $A \subseteq \mathbb{N}$ satisfying that $d_f(A) = 0$ and $(x_n)_{n\in\mathbb{N}\setminus A}$ is a Cauchy sequence.

In [15, Theorem 3.6], it was proved that if X is a uniform space endowed with a countable base of entourages, then $f\text{-st}\lim(x_n) = f^*\text{-st}\lim(x_n)$ for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and every unbounded modulus f. We seek a similar result for f^*-statistical Cauchy.

Proposition 3.6. *Let X be a uniform space and f an unbounded modulus function. If a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is f^*-statistically Cauchy, then it is f-statistically Cauchy as well.*

Proof. Let U be any symmetric entourage in X. Since $(x_n)_{n\in\mathbb{N}}$ is f^*-statistically Cauchy, there exists $A \subseteq \mathbb{N}$ such that $d_f(A) = 0$ and $(x_n)_{n\in\mathbb{N}\setminus A}$ is Cauchy. That is, there exists $n_U \in \mathbb{N} \setminus A$ such that $n, m \in \mathbb{N} \setminus A$ and $n, m \geq n_U$ imply $(x_n, x_m) \in U$. Next, we consider $B := \{n \in \mathbb{N} : (x_n, x_{n_U}) \notin U\}$. The following inclusion holds $B \subseteq A \cup \{1, \dots, n_U - 1\}$. Since $d_f(A) = 0$ and $d_f(\{1, \dots, n_U - 1\}) = 0$, we have $d_f(A \cup \{1, \dots, n_U - 1\}) = 0$. Therefore, considering $B \subseteq A \cup \{1, \dots, n_U - 1\}$, we obtain that $d_f(B) = 0$. As a consequence, $(x_n)_{n\in\mathbb{N}}$ is f-statistically Cauchy. □

To obtain the converse of Proposition 3.6, the following result is necessary.

Proposition 3.7. *Let X be a uniform space and f be an unbounded modulus function. If $(x_n)_{n\in\mathbb{N}} \subseteq X$ is f-statistically Cauchy, then for every entourage $U \subseteq X\times X$ there exists $M\subseteq \mathbb{N}$ with $d_f(M)=0$ such that $(x_n,x_k)\in U$ whenever $n,k\notin M$.*

Proof. Fix an arbitrary symmetric entourage U of X. We see that there exists another entourage $V\subseteq X\times X$ such that $V\circ V\subseteq U$. By hypothesis, there can be found $m\in\mathbb{N}$ such that $d_f(\{n\in\mathbb{N} : (x_n,x_m)\notin V\})=0$. Next, we define $M:=\{n\in\mathbb{N} : (x_n,x_m)\notin V\}$. If we choose $n,k\notin M$, then $(x_n,x_m)\in V$ and $(x_k,x_m)\in V$, which means that $(x_n,x_k)\in V\circ V\subseteq U$. □

We are finally in the right position to provide the converse to Proposition 3.6.

Theorem 3.8. *Let X be a uniform space with a countable base of entourages and f an unbounded modulus function. If a sequence $(x_n)_{n\in\mathbb{N}}\subseteq X$ is f-statistically Cauchy, then it is f^*-statistically Cauchy as well.*

Proof. Take $\mathscr{B}=\{U_i : i\in\mathbb{N}\}$ as a countable base of symmetric entourages in X with $U_1\supseteq U_2\supseteq U_3\supseteq\cdots$. By Proposition 3.7, there exists $C_i\subseteq\mathbb{N}$ with $d_f(C_i)=0$ such that $m,n\notin C_i$ implies $(x_n,x_m)\in U_i$. Now, let us define the sets $A_1:=C_1$, $A_2:=C_2\setminus C_1$, $A_3:=C_3\setminus(C_1\cup C_2)$, and so on. Since $d_f(C_i)=0$ for each $i\in\mathbb{N}$, $(A_i)_{i\in\mathbb{N}}$ is a countable family of mutually disjoint sets with $d_f(A_i)=0$ for each $i\in\mathbb{N}$. By Lemma 2.7, there exists a sequence $(B_i)_{i\in\mathbb{N}}$ such that $B_i\subseteq A_i$ and $A_i\setminus B_i$ is finite for every $i\in\mathbb{N}$, and $B:=\bigcup_{i\in\mathbb{N}}B_i$ has f-density 0. We will prove that $(x_n)_{n\in\mathbb{N}\setminus B}$ is a Cauchy sequence. Indeed, fix an arbitrary entourage $U\subseteq X\times X$. Since $\mathscr{B}$ is base of entourages, there exists $k\in\mathbb{N}$ with $U_k\subseteq U$. Notice that

$$C_k\setminus B\subseteq\bigcup_{i=1}^{k}A_i\setminus B_i.$$

Thus, $C_k\setminus B$ is finite because, so is $A_i\setminus B_i$ for every $i\in\mathbb{N}$. Finally, if $n,m\in\mathbb{N}\setminus B$ are such that $n,m>\max(C_k\setminus B)$, then $n,m\notin C_k$, meaning that $(x_n,x_m)\in U_k\subseteq U$. This means that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for $n\in\mathbb{N}\setminus B$. Consequently, $(x_n)_{n\in\mathbb{N}}$ is a f^*-statistically Cauchy sequence. □

This subsection will be concluded with the following representative example of Theorem 3.8.

Example 3.9. Consider the compatible unbounded modulus $f(x)=x+\dfrac{x}{1+x}$ [22]. Notice that $A:=\{n^3 : n\in\mathbb{N}\}$ satisfies that $d_f(A)=0$. Indeed, $\operatorname{card}(A\cap[1,n])=\lfloor\sqrt[3]{n}\rfloor$ for all $n\in\mathbb{N}$. Then

$$\begin{aligned} 0 \ &\leq\ d_f(A)=\lim_{n\to\infty}\frac{f(\operatorname{card}(A\cap[1,n]))}{f(n)}=\lim_{n\to\infty}\frac{f\left(\lfloor\sqrt[3]{n}\rfloor\right)}{f(n)}\\ &\leq\ \lim_{n\to\infty}\frac{f\left(\sqrt[3]{n}\right)}{f(n)}=\lim_{n\to\infty}\frac{\sqrt[3]{n}+\frac{\sqrt[3]{n}}{1+\sqrt[3]{n}}}{n+\frac{n}{1+n}}=0.\end{aligned}$$

Consider the uniform space X given by the absolutely valued field $\mathbb{Q}\left[\sqrt{3}\right]$ (endowed with the Euclidean absolute value). Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$x_n:=\begin{cases}\sum_{k=0}^{n}\frac{(-1)^k}{2k+1}, & n\in\mathbb{N}\setminus A\\ \sqrt[3]{n}+\sqrt{3}, & n\in A.\end{cases}$$

Notice that $(x_n)_{n\in\mathbb{N}}$ is an f-statistically Cauchy sequence and an f^*-statistically Cauchy sequence in $\mathbb{Q}\left[\sqrt{3}\right]$.

3.2. **f-Statistical limit points and f-statistical cluster points.** In [14], Fridy and Orhan introduced the notions of statistical limit point and statistical cluster point of a sequence of real numbers. In [11], Di Maio and Koćinac studied the statistical limit points and statistical cluster points of a sequence on any topological space. Later, Lahiri and Das [10, 21] considered limit points and cluster points for ideal convergence in topological spaces. Recently, Listán-García [23] provided the definitions of statistical limit point and statistical cluster point of a sequence on Banach space. Here, we will redefine the previous two concepts in the scope of uniform spaces via a modulus function.

Definition 3.10 (f-Statistical limit points)**.** Let X be a uniform space and let f be an unbounded modulus function. A point $x\in X$ is said to be an f-statistical limit point of a sequence $(x_n)_{n\in\mathbb{N}}\subseteq X$ if there exists an strictly increasing sequence $K=(n_k)_{k\in\mathbb{N}}$ of naturals with $d_f(K)\neq 0$ such that $\lim_{k\in\mathbb{N}} x_{n_k}=x$. We denote the set of all f-statistical limit points of $(x_n)_{n\in\mathbb{N}}$ as $\mathscr{F}_f(x_n)$.

Definition 3.11 (f-Statistical cluster points)**.** Let X be a uniform space and let f be a unbounded modulus function. A point $x\in X$ is said to be an f-statistical cluster point of a sequence $(x_n)_{n\in\mathbb{N}}\subseteq X$ if, for each symmetric entourage $U\subseteq X\times X$, the f-density of the set $\{n\in\mathbb{N}:(x_n,x)\in U\}$ is not zero. We denote the set of all f-statistical cluster points of $(x_n)_{n\in\mathbb{N}}$ as $\mathscr{C}_f(x_n)$.

For a sequence $(x_n)_{n\in\mathbb{N}}$ in a uniform space X, let $\mathscr{F}_{st}(x_n)$, $\mathscr{C}_{st}(x_n)$, and $\mathscr{L}(x_n)$ be the sets of statistical limit points of $(x_n)_{n\in\mathbb{N}}$, of all statistical cluster points of $(x_n)_{n\in\mathbb{N}}$, and of limit points of $(x_n)_{n\in\mathbb{N}}$, respectively. Since for any unbounded modulus function f and any $A\subseteq X$,

$$d_f(A)=0\Rightarrow d(A)=0,$$

the following inclusions hold:

$$\mathscr{F}_{st}(x_n)\subseteq\mathscr{F}_f(x_n) \text{ and } \mathscr{C}_{st}(x_n)\subseteq\mathscr{C}_f(x_n).$$

The following proposition generalizes [11, Lemma 2.1 and Theorem 2.2].

Proposition 3.12. *Let X be a uniform space and f an unbounded modulus function. For any sequence $(x_n)_{n\in\mathbb{N}}\subseteq X$, $\mathscr{F}_f(x_n)\subseteq\mathscr{C}_f(x_n)$. If, in addition, X is endowed with a countable base of entourages, then $\mathscr{C}_f(x_n)\subseteq\mathscr{L}(x_n)$.*

Proof. Fix an arbitrary $y\in\mathscr{F}_f(x_n)$. There exists a infinite subset $K\subseteq\mathbb{N}$ with $d_f(K)\neq 0$ such that for each symmetric entourage $U\subseteq X\times X$ there exists $k_0\in\mathbb{N}$ with $x_k\in U[y]$ for every $k\geq k_0$. Then $\{k\in K:(x_k,y)\notin U\}\subseteq K\cap[1,k_0]$ is finite, so

$$d_f\left(\{k\in K:(x_k,y)\notin U\}\right)=0 \tag{3.1}$$

in view of the fact that f is unbounded.

On the other hand,

$$K\setminus\{k\in K:(x_k,y)\notin U\}\subseteq\{n\in\mathbb{N}:(x_n,y)\in U\}.$$

Hence, since d_f is increasing, we obtain that

$$d_f\left(K\setminus\{k\in K:(x_k,y)\notin U\}\right)\leq d_f\left(\{n\in\mathbb{N}:(x_n,y)\in U\}\right). \tag{3.2}$$

From $d_f(K) \neq 0$, (3.1), and the subadditivity of d_f, we conclude that the f-density on the left-hand side of (3.2) is not zero, meaning that $d_f(\{n \in \mathbb{N} : (x_n, y) \in U\}) \neq 0$. In other words, $y \in \mathscr{C}_f(x_n)$. Now, we show that $\mathscr{C}_f(x_n) \subseteq \mathscr{L}(x_n)$ by assuming that $\mathscr{B} = \{U_i : i \in \mathbb{N}\}$ is a countable base of symmetric entourages in X with $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$. Fix an arbitrary $y \in \mathscr{C}_f(x_n)$. For every $i \in \mathbb{N}$, $d_f(K_i) \neq 0$, where $K_i := \{n \in \mathbb{N} : (x_n, y) \in U_i\}$. Let n_i be fixed in K_i for every $i \in \mathbb{N}$ in such a way that $n_1 < n_2 < \cdots$ in view of the fact that each K_i is infinite. We show that $y \in \lim(x_{n_k})$. Take any symmetric entourage U. We can choose $i \in \mathbb{N}$ such that $U_i \subseteq U$. If $n_k \geq n_i$, then $x_{n_k} \in U_k \subseteq U_i \subseteq U$. As a consequence, $y \in \lim(x_{n_k})$. This means that $y \in \mathscr{L}(x_n)$. □

Next proposition is a refinement of [10, Theorem 1(1)] via a more direct proof.

Proposition 3.13. *Let X be a uniform space and f an unbounded modulus function. For any sequence $(x_n)_{n \in \mathbb{N}}$ in X, the set $\mathscr{C}_f(x_n)$ is closed.*

Proof. Take an arbitrary element $x \in \overline{\mathscr{C}_f(x_n)}$. Fix a symmetric entourage $U \in \mathscr{U}$. We prove that the f-density of $\{n \in \mathbb{N} : (x_n, x) \in U\}$ is not zero. Consider a symmetric entourage V such that $V \circ V \subseteq U$. By hypothesis, $V[x] \cap \mathscr{C}_f(x_n) \neq \emptyset$, so a point $y \in V[x] \cap \mathscr{C}_f(x_n)$ can be chosen. Next, the f-density of the set $\{n \in \mathbb{N} : (x_n, y) \in V\}$ is positive. Observe that

$$\{n \in \mathbb{N} : (x_n, y) \in V\} \subseteq \{n \in \mathbb{N} : (x_n, x) \in U\}.$$

The increasing condition of d_f allows that

$$0 \neq d_f(\{n \in \mathbb{N} : (x_n, y) \in V\}) \leq d_f(\{n \in \mathbb{N} : (x_n, x) \in U\}).$$

As a consequence, $x \in \mathscr{C}_f(x_n)$. □

Next Theorem generalizes [11, Theorem 2.12] for the case of uniform spaces since in the latter reference statistical convergence is considered not through a modulus function.

Theorem 3.14. *Let X be a uniform space and f an unbounded modulus function. If $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are sequences in X such that $d_f(\{n \in \mathbb{N} : x_n \neq y_n\}) = 0$, then $\mathscr{F}_f(x_n) = \mathscr{F}_f(y_n)$ and $\mathscr{C}_f(x_n) = \mathscr{C}_f(y_n)$.*

Proof. Fix an arbitrary $y \in \mathscr{F}_f(x_n)$. There exists an infinite subset $K \subseteq \mathbb{N}$ with $d_f(K) \neq 0$ such that $\lim_{k \in K} x_k = y$. Subadditivity of d_f leads to

$$d_f(K \setminus \{n \in \mathbb{N} : x_n \neq y_n\} \geq d_f(K) - d_f(\{n \in \mathbb{N} : x_n \neq y_n\}) = d_f(K) \neq 0.$$

By taking $H := K \setminus \{n \in \mathbb{N} : x_n \neq y_n\}$, then $\lim_{h \in H} y_h = y$ and so $y \in \mathscr{F}_f(y_n)$. A symmetric argument shows that $\mathscr{F}_f(y_n) \subseteq \mathscr{F}_f(x_n)$. Let us prove now that $\mathscr{C}_f(x_n) = \mathscr{C}_f(y_n)$. Take any $y \in \mathscr{C}_f(x_n)$ and fix an arbitrary symmetric entourage $U \subseteq X \times X$. Thus, $d_f(\{n \in \mathbb{N} : (x_n, y) \in U\}) > 0$. Since the following inclusion holds

$$\{n \in \mathbb{N} : (x_n, y) \in U\} \setminus \{n \in \mathbb{N} : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : (y_n, y) \in U\},$$

the increasing condition of d_f, together with its subadditivity, allows that

$$\begin{aligned} d_f(\{n \in \mathbb{N} : (y_n, y) \in U\}) &\geq d_f(\{n \in \mathbb{N} : (x_n, y) \in U\} \setminus \{n \in \mathbb{N} : x_n \neq y_n\}) \\ &\geq d_f(\{n \in \mathbb{N} : (x_n, y) \in U\}) - d_f(\{n \in \mathbb{N} : x_n \neq y_n\}) \\ &= d_f(\{n \in \mathbb{N} : (x_n, y) \in U\}) > 0. \end{aligned}$$

As a consequence, $y \in \mathscr{C}_f(y_n)$. A symmetric reasoning shows that $\mathscr{C}_f(y_n) \subseteq \mathscr{C}_f(x_n)$. □

The following is a representative example of Theorem 3.14.

Example 3.15. Consider the compatible unbounded modulus $f(x) = x + \log(1+x)$ [22]. Notice that $A := \{n^2 : n \in \mathbb{N}\}$ satisfies that $d_f(A) = 0$. Indeed, $\operatorname{card}(A \cap [1,n]) = \lfloor \sqrt{n} \rfloor$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} 0 &\leq d_f(A) = \lim_{n\to\infty} \frac{f(\operatorname{card}(A\cap[1,n]))}{f(n)} = \lim_{n\to\infty} \frac{f(\lfloor\sqrt{n}\rfloor)}{f(n)} \\ &\leq \lim_{n\to\infty} \frac{f(\sqrt{n})}{f(n)} = \lim_{n\to\infty} \frac{\sqrt{n}+\log(1+\sqrt{n})}{n+\log(1+n)} = 0. \end{aligned}$$

Consider the uniform space X given by the topological group of surjective linear isometries on an infinite-dimensional separable complex Hilbert spaces H (recall that H can be identified by ℓ_2). Consider the sequences $(T_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ of surjective linear isometries on H defined by

$$T_n\left(\sum_{i=1}^{\infty} t_i e_i\right) = \sum_{i=1}^{\infty} t_{\alpha_n(i)} e_i \quad \text{and} \quad S_n\left(\sum_{i=1}^{\infty} t_i e_i\right) = \sum_{i=1}^{\infty} t_{\beta_n(i)} e_i,$$

where $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of H and α_n, β_n are permutations of $\mathbb{N}$ defined as follows for every $n \in \mathbb{N}$:

$$\alpha_n := \begin{cases} \sigma(n,n+1), & n \in \mathbb{N}\setminus A \\ \sigma\left(n,(\sqrt{n}+1)^2\right), & n \in A \end{cases} \quad \text{and} \quad \beta_n := \begin{cases} \sigma(n,n+1), & n \in \mathbb{N}\setminus A \\ \sigma\left(n,(\sqrt{n}+2)^2\right), & n \in A \end{cases}$$

where $\sigma(i,j)$ is the cycle (permutation) of $\mathbb{N}$ that fixes all elements of $\mathbb{N}\setminus\{i,j\}$ and switches i,j. Notice that $\alpha_n = \beta_n$ if and only if $n \in \mathbb{N}\setminus A$, meaning that $T_n = S_n$ if and only if $n \in \mathbb{N}\setminus A$. In accordance with Theorem 3.14, $\mathscr{F}_f(T_n) = \mathscr{F}_f(S_n)$ and $\mathscr{C}_f(T_n) = \mathscr{C}_f(S_n)$.

Theorem 3.16. *Let X be a uniform space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then*

$$\mathscr{L}(x_n) \subseteq \bigcup \left\{\mathscr{C}_f(x_n) : f \text{ is an unbounded modulus function}\right\}.$$

Proof. Let us fix an arbitrary $y \in \mathscr{L}(x_n)$. There exists an infinite subset $K \subseteq \mathbb{N}$ such that $\lim_{k\in K} x_k = y$. Since K is a infinite set, [2, Lemma 3.4] allows the existence of an unbounded modulus function g such that $d_g(K) = 1$. Suppose on the contrary that $y \notin \mathscr{C}_g(x_n)$. There exists a symmetric entourage $U_g \subseteq X \times X$ in such a way that the g-density of the set $\{n \in \mathbb{N} : (x_n, y) \in U_g\}$ is zero. In particular, the increasing condition of d_g implies that $d_g(\{k \in K : (x_k,y) \in U_g\}) = 0$. On the other hand, since $(x_k)_{k\in K}$ converges to y in the uniform topology of X, we conclude that $\{k \in K : (x_k,y) \notin U_g\}$ is finite, meaning that $d_g(\{k \in K : (x_k,y) \notin U_g\}) = 0$. Finally, the subadditivity of d_g forces a contradiction with the fact that $d_g(K) = 1$. □

By gluing together Proposition 3.12 and Theorem 3.16, the following final corollary is obtained.

Corollary 3.17. *Let X be a uniform space with a countable base of entourages. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then*

$$\mathscr{L}(x_n) = \bigcup \left\{\mathscr{C}_f(x_n) : f \text{ is an unbounded modulus function}\right\}.$$

3.3. **Finest convergence topologies vs. uniform product topologies.** In what follows, $\mathscr{P}(I)$ stands for the power set of I and X^I denotes the family of mappings from I to X, for X,I arbitrary sets. Recall that a subset $\mathscr{G} \subseteq \mathscr{P}(I)$ is said to be upward directed provided that, for any $G_1, G_2 \in \mathscr{G}$, there exists $G_3 \in \mathscr{G}$ such that $G_1 \cup G_2 \subseteq G_3$. In a similar way, downward directed is defined.

Remark 3.18. Let I be a nonempty set and $\mathscr{G} \subseteq \mathscr{P}(I)$ upward directed. Let X be a nonempty set and $\mathscr{T} \subseteq \mathscr{P}(X)$ downward directed. Consider the family $\{\mathscr{U}(G,T) : G \in \mathscr{G}, T \in \mathscr{T}\}$, where $\mathscr{U}(G,T) := \{f \in X^I : f(G) \subseteq T\}$. Then:

(1) If $\mathscr{U}(G,T) \neq \varnothing$ for all $G \in \mathscr{G}$ and all $T \in \mathscr{T}$, then $\{\mathscr{U}(G,T) : G \in \mathscr{G}, T \in \mathscr{T}\}$ is a filter base in X^I.
(2) If $\bigcup\{\mathscr{U}(G,T) : G \in \mathscr{G}, T \in \mathscr{T}\} = X^I$, then $\{\mathscr{U}(G,T) : G \in \mathscr{G}, T \in \mathscr{T}\}$ is a subbase for a topology on X^I.

Under the settings of Proposition 3.18, notice that $\mathscr{U}(G,T) = \prod_{i\in I} T_i$, where

$$T_i := \begin{cases} X & i \in I \setminus G, \\ T & i \in G, \end{cases}$$

and $\mathscr{U}(G_1,T_1) \cap \cdots \cap \mathscr{U}(G_k,T_k) = \prod_{i\in I} S_i$, where

$$S_i := \begin{cases} X & i \in I \setminus (G_1 \cup \cdots \cup G_k), \\ \bigcap_{i\in G_j} T_j & i \in G_1 \cup \cdots \cup G_k, \end{cases}$$

when X^I is seen as a product. Previous remark motivates the following definition.

Definition 3.19 (Convergence topology)**.** Let I be a nonempty set and $\mathscr{G} \subseteq \mathscr{P}(I)$ upward directed. Let X be a topological space. The convergence topology of X^I generated by $\mathscr{G}$ is defined as the topology generated by the subbase $\{\mathscr{U}(G,U) : G \in \mathscr{G}, U \subseteq X \text{ open}\}$.

A trivial fact is that, if $\mathscr{G} \subseteq \mathscr{F} \subseteq \mathscr{P}(I)$ are upward directed, then the convergence topology generated by $\mathscr{G}$ is clearly coarser than the convergence topology generated by $\mathscr{F}$. Also, if $F \subseteq X^I$, then the inherited convergence topology of F is generated by the subbase $\{\mathscr{U}(G,U) \cap F : G \in \mathscr{G}, U \subseteq X \text{ open}\}$. Another easy fact is that if $\mathscr{G} \subseteq \mathscr{P}(I)$ is a bornology and X is Hausdorff, then the convergence topology on X^I induced by $\mathscr{G}$ is Hausdorff as well.

Example 3.20 (Compact-open topology)**.** Let Y,X be topological spaces. If $\mathscr{K}$ stands for the family of compact subsets of Y, then convergence topology of X^Y generated by $\mathscr{K}$ is called the compact-open topology.

The compact-open topology is a classical example of convergence topology in the literature of General Topology. The problem with the convergence topology is that it is hard to guess what this topology looks like, unless $\mathscr{G} := \mathscr{P}_{\mathtt{f}}(I)$ (the set of finite subsets of I). In this situation, the convergence topology generated by $\mathscr{G}$ is the pointwise convergence topology, that is, the product topology of X^I. Nevertheless, if $\mathscr{G}$ is not the collection of finite subsets of I, then the convergence topology generated by $\mathscr{G}$ could result into a strange topology.

Example 3.21 (Finest convergence topology)**.** Let I be a nonempty set and X a topological space. Consider the finest convergence topology on X^I, that is, the one generated by $\mathscr{G} := \mathscr{P}(I)$, and look at X^I as a direct product. For every subset $G \subseteq I$ and every open subset $U \subseteq X$, we have

$$\mathscr{U}(G,U) = \left\{ f \in X^I : f(G) \subseteq U \right\} = \prod_{i \in I} U_i,$$

where

$$U_i = \begin{cases} U & i \in G, \\ X & i \notin G, \end{cases}$$

for all $i \in I$.

Under the settings of the previous example, observe that the convergence topology of X^I generated by $\mathscr{G} := \mathscr{P}(I)$ is coarser than the uniform product topology of X^I (the topology on X^I where the product of open sets is always open). Our novel result in this final subsection serves to confirm that the convergence topology generated by the whole power set results into a very strange (and very strong) topology except for the case of finite index sets.

Theorem 3.22. *Let I be a nonempty set and X a topological space. The convergence topology on X^I generated by $\mathscr{G} := \mathscr{P}(I)$ coincides with the uniform product topology if and only if for every family $(U_i)_{i \in I}$ of nonempty open subsets of X and for every $(x_i)_{i \in I} \in \prod_{i \in I} U_i$ there exists a finite number of nonempty open subsets $O_1, \ldots, O_l$ of X such that for each $i \in I$ there exists $j_i \in \{1, \ldots, l\}$ with $x_i \in O_{j_i} \subseteq U_i$.*

Proof. Suppose first that the uniform convergence topology on X^I coincides with the uniform product topology. Let $(U_i)_{i \in I}$ be a family of nonempty open subsets of X and fix an arbitrary $(x_i)_{i \in I} \in \prod_{i \in I} U_i$. By hypothesis, there exist $G_1, \ldots, G_k \in \mathscr{G}$ and nonempty open subsets $V_1, \ldots, V_k$ of X such that that $(x_i)_{i \in I} \in \mathscr{U}(G_1, V_1) \cap \cdots \cap \mathscr{U}(G_k, V_k) \subseteq \prod_{i \in I} U_i$. Notice that $\mathscr{U}(G_1, V_1) \cap \cdots \cap \mathscr{U}(G_k, V_k) = \prod_{i \in I} W_i$, where

$$W_i := \begin{cases} X & i \in I \setminus (G_1 \cup \cdots \cup G_k), \\ \bigcap_{i \in G_j} V_j & i \in G_1 \cup \cdots \cup G_k. \end{cases}$$

Obviously, for every $i \in I$, $x_i \in W_i \subseteq U_i$. Next, let us denote by $O_1, \ldots, O_l$ to all the nonempty finite intersections of $V_1, \ldots, V_k$. Since $\mathscr{U}(G, X) = X^I$ for any $G \in \mathscr{G}$, we may assume without any loss of generality that $V_k = X$. Fix an arbitrary $i_0 \in I$. If $i_0 \in I \setminus (G_1 \cup \cdots \cup G_k)$, then $U_{i_0} = W_{i_0} = X$, so $x_{i_0} \in V_k \subseteq U_{i_0}$. If $i_0 \in G_1 \cup \cdots \cup G_k$, then $x_{i_0} \in W_{i_0} \subseteq U_{i_0}$ and $W_{i_0} = \bigcap_{i_0 \in G_j} V_j$, meaning that there exists $j_{i_0} \in \{1, \ldots, l\}$ with $O_{j_{i_0}} = W_{i_0}$. Conversely, fix an arbitrary open set $\prod_{i \in I} U_i$ of the base of the uniform product topology and take any $(x_i)_{i \in I} \in \prod_{i \in I} U_i$. By assumption, there exists a finite number of nonempty open subsets $O_1, \ldots, O_l$ of X such that for each $i \in I$ there exists $j_i \in \{1, \ldots, l\}$ with $x_i \in O_{j_i} \subseteq U_i$. Then it only suffices to notice that

$$(x_i)_{i \in I} \in \mathscr{U}(G_1, O_1) \cap \cdots \cap \mathscr{U}(G_l, O_l) \subseteq \prod_{i \in I} U_i,$$

where $G_h := \{i \in I : j_i = h\}$ for every $h \in \{1, \ldots, l\}$. $\square$

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REFERENCES

[1] A. Aizpuru, M. C. Listán-García, d F. Rambla-Barreno, Double density by moduli and statistical convergence, Bull. Belg. Math. Soc. Simon Stevin 19 (2012) 663-673.

[2] A. Aizpuru, M. C. Listán-García, d F. Rambla-Barreno, Density by moduli and statistical convergence, Quaest. Math. 37 (2014) 525-530.

[3] A. Aizpuru, M. C. Listán-García, d F. Rambla-Barreno, Density by moduli and statistical convergence, Quaest. Math. 37 (2014) 525-530.

[4] A. Aizpuru, M. Nicasio-Llach, F. Rambla-Barreno, A remark about the Orlicz-Pettis theorem and the statistical convergence, Acta Math. Sin. (Engl. Ser.) 26 (2010) 305-310.

[5] A. Aizpuru, M. Nicasio-Llach, About the statistical uniform convergence, Bull. Braz. Math. Soc. (N.S.) 39 (2008), 173-182.

[6] B. Altay, F.J. García-Pacheco, R. Kama, On f-strongly Cesàro and f-statistical derivable functions, AIMS Math. 7 (2022) 11276-11291.

[7] P. Baliarsingh, U. Kadak, M. Mursaleen, On statistical convergence of difference sequences of fractional order and related Korovkin type approximation theorems, Quaest. Math. 41 (2018) 1117-1133.

[8] V.K. Bhardwaj, S. Dhawan, Application of f-lacunary statistical convergence to approximation theorems, J. Inequal. Appl. 2018 (2018) 281.

[9] J. Connor, M. Ganichev, V. Kadets, A characterization of Banach spaces with separable duals via weak statistical convergence, J. Math. Anal. Appl. 244 (2000) 251-261.

[10] P. Das, Some further results on ideal convergence in topological spaces, Topology Appl. 159 (2012) 10-11, 2621-2626.

[11] G. Di Maio, L. D. R. Kočinac, Statistical convergence in topology, Topology Appl. 156 (2008) 28-45.

[12] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241-244.

[13] J. A. Fridy, On statistical convergence, Analysis, 5 (1985) 301-313.

[14] J. A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific J. Math. 160 (1993) 43-51.

[15] F.J. García-Pacheco, R. Kama, f-statistical convergence on topological modules, Electron. Res. Arch. 30 (2022) 2183-2195.

[16] I. James, Topologies and uniformities. Exp. and rev. version of topological and uniform spaces, 1987, exp. and rev. version of toplogical and uniform spaces, 1987 ed., Springer Undergrad. Math. Ser., Springer, London, 1999.

[17] R. Kama, On some vector valued multiplier spaces with statistical Cesáro summability, Filomat 33 (2019) 5135-5147.

[18] R. Kama, B. Altay, Multiplier sequence spaces defined by f-statistical summability and Orlicz-Pettis theorem, Numer. Funct. Anal. Optim. 42 (2021) 1410-1422.

[19] A. Karakas, Y. Altin, H. Altinok, Almost statistical convergence of order β of sequences of fuzzy numbers, Soft Computing 20 (2016) 3611-3616.

[20] E. Kolk, The statistical convergence in Banach spaces, Tartu Ül. Toimetised (1991), 41-52.

[21] B. K. Lahiri, P. Das, I and I^*-convergence in topological spaces, Math. Bohem. 130 (2005) 153-160.

[22] F. León-Saavedra, M. del Carmen Listán-García, F.J. Pérez Fernández, M.P. Romero de la Rosa, On statistical convergence and strong Cesàro convergence by moduli, J. Inequal. Appl. 2019 (2019) 298.

[23] M. C. Listán-García, f-statistical convergence, completeness and f-cluster points, Bull. Belg. Math. Soc. Simon Stevin 23 (2016) 235-245.

[24] M. C. Listán-García, On uniform f-statistical convergence of sequences of functions, Quaest. Math. 46 (2023) 1643-1651.

[25] I. J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc. 100 (1986) 161-166.

[26] I. J. Maddox, Statistical convergence in a locally convex space, Math. Proc. Cambridge Philos. Soc. 104 (1988) 141-145.

[27] M. Mursaleen, S. A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comput. Appl. Math. 233 (2009) 142-149.

[28] A. Nabiev, S. Pehlivan, M. Gürdal, On $\mathscr{I}$-Cauchy sequences, Taiwanese J. Math. 11 (2007) 569-576.

[29] H. Nakano, Concave modulars, J. Math. Soc. Japan 5 (1953) 29-49.
[30] A. Or, G. Karabacak, Ideal convergence and ideal Cauchy sequences in intuitionistic fuzzy metric spaces, Math. Morav. 27 (2023) 113-128.
[31] W.H. Ruckle, FK spaces in which the sequence of coordinate vectors is bounded, Canadian J. Math. 25 (1973), 973-978.
[32] E. Savaş, R.F. Patterson, Lacunary statistical convergence of multiple sequences, Appl. Math. Lett. 19 (2006) 527-534.
[33] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361-375.
[34] H. Steinhaus, Sur la convergence ordinarie et la convergence asymptotique, Colloq. Math. 2 (1951) 73-74.
[35] E.N. Yıldı rım, Statistical convergence of matrix sequences, Konuralp J. Math. 12 (2024) 74-79.
[36] Y. Zhu, A. Shen, Z. Wang, W. Peng, On deferred f-statistical convergence for double sequences, Open Math. 22 (2024) 20230174.