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BOUNDARY VALUE PROBLEMS OF FRACTIONAL ORDER: ANALYZING ITERATIVE SYSTEMS WITH POSITIVE SOLUTIONS

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Abstract. This study addresses a fractional-order iterative system

$$\mathsf{D}_{0^+}^{\mathfrak{i}_1}\left[\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(\mathfrak{q})\right]+\sigma(\mathfrak{q})\hbar_{\mathfrak{t}}(\mathbf{r}_{\mathfrak{t}+1}(\mathfrak{q}))=0, 0<\mathfrak{q}<1, 1<\mathfrak{i}_1,\mathfrak{i}_2<2,$$

satisfying two-point integral boundary conditions

$$\mathbf{r}_{\mathfrak{t}}(0)=\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(0)=0, \mathbf{r}_{\mathfrak{t}}(1)=\int_0^1\Lambda(\mathfrak{z})\mathbf{r}_{\mathfrak{t}}(\mathfrak{z})d\mathfrak{z}$$

and

$$\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(1)=\int_0^1\Lambda(\mathfrak{z})\big(\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(\mathfrak{z})\big)d\mathfrak{z}.$$

Here, we consider the system where $\mathfrak{t}$ is an integer from 1 to $\mathfrak{w}$, and $\mathbf{r}_{\mathfrak{w}+1}$ equals $\mathbf{r}_1$ and $\sigma(\mathfrak{q})$ belongs to the Lebesgue space $L^{\mathfrak{p}}[0,1]$ for $1\leq\mathfrak{p}\leq\infty$ and may exhibit singularities within $(0,1/2)$. The study employs Riemann-Liouville fractional derivatives of orders $\mathfrak{i}_1$ and $\mathfrak{i}_2$, denoted as $\mathsf{D}_{0^+}^{\mathfrak{i}}$. By leveraging fixed-point index theory within a Banach space cone, the existence of infinitely many positive solutions to this problem is established. Furthermore, in the specific case where $\mathfrak{w}$ equals 1, sufficient conditions for a unique solution are derived by using Rus's theorem within a complete metric space.

Keywords. Fractional boundary value problems; Iterative equation; Positive solutions; Rus's theorem; two fractional derivatives.

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1. INTRODUCTION

Fractional calculus has emerged as a valuable tool across diverse fields, including economics, life sciences, and engineering. Its ability to model systems with memory and hereditary properties has led to widespread applications in areas, such as control theory, materials science,

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and biophysics; see, e.g., [3, 7, 8, 15, 16]. Fractional order differential equations(FDEs) have become instrumental in describing complex phenomena that exhibit non-local behavior [4]. A substantial body of research has focused on establishing the existence and uniqueness of solutions to fractional differential equations, particularly in the context of boundary value problems (BVPs). Various theoretical frameworks have been employed to address these challenges. For example, Abbas et al. [1] discussed Hadamard fractional integral equations with random effects. This work is relevant as it lays the foundation for understanding stability in fractional systems, especially those involving stochastic elements, which are crucial when tackling iterative boundary value problems. Ahmad et al. [2] contributed by presenting results on fractional differential equations of arbitrary order with nonlocal integral boundary conditions. Their work strengthens the framework for addressing non-local behaviors in fractional BVPs, which is central to the discussion in this paper. Additionally, Alzabut et al. [6] explored fractional order nonlocal thermistor boundary value problems on time scales, providing a connection between fractional models and their applications in physical systems, thereby supporting the iterative system approach employed here. Ferreira's results [9] on the existence and uniqueness of solutions for two-point fractional boundary value problems directly underpin the theoretical analysis in this paper. Meanwhile, Guswanto and Suzuki's work [10] on mild solutions for fractional semi-linear differential equations complements the present discussion by outlining conditions pertinent to semi-linear systems, which are also considered in this paper. In 2017, Mahmudov and Matar [12] investigated hybrid differential equations with arbitrary fractional orders. Their approach aligns closely with the extension of differential equations to hybrid forms, a theme explored here. In a similar vein, Rao et al. [13] addressed positive solutions for nonlinear iterative systems of fractional boundary value problems, which forms a key part of the theoretical exploration in this work. In 2019, Vivek et al. [19] introduced hybrid fractional differential equations with complex orders, which is crucial for the advanced discussion on fractional models and their applications in this manuscript. Furthermore, Toyoda and Watanabe [18] made significant strides by establishing the existence and uniqueness theorem for a fractional order BVP, which is given as:

$$\begin{gathered} \mathsf{D}_{0^+}^{\mathfrak{i}_1}\left(\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}(\mathfrak{q})\right) = \hbar(\mathfrak{q},\mathbf{r}(\mathfrak{q})), 0<\mathfrak{q}<1, \\ \mathbf{r}(0)=\mathbf{r}(1)=\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}(0)=\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}(1)=0, \end{gathered}$$

where $1<\mathfrak{i}_1,\mathfrak{i}_2\le 2$, $3<\mathfrak{i}_1+\mathfrak{i}_2<4$ and $\mathsf{D}_{0^+}^{\times}$ is the Riemann-Liouville(RL) fractional derivative of order $\times\in\{\mathfrak{i}_1,\mathfrak{i}_2\}$. The two fractional order derivatives with boundary conditions are a rich and evolving area of research with significant potential for advancing our understanding of complex systems. Their study involves tackling mathematical challenges and leveraging new techniques to address real-world problems more effectively. Drawing on previous research, this paper establishes the existence of infinite positive solutions for an iterative system of fractional-order BVPs described by

$$\left.\begin{gathered} \mathsf{D}_{0^+}^{\mathfrak{i}_1}\left(\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(\mathfrak{q})\right)+\sigma(\mathfrak{q})\hbar_{\mathfrak{t}}(\mathbf{r}_{\mathfrak{t}+1}(\mathfrak{q}))=0, 0<\mathfrak{q}<1, \\ \mathbf{r}_{\mathfrak{w}+1}(\mathfrak{q})=\mathbf{r}_1(\mathfrak{q}), 0<\mathfrak{q}<1, \end{gathered}\right\} \tag{1.1}$$

$$\left.\begin{aligned} \mathbf{r}_{\mathfrak{t}}(0)=0,\, \mathbf{r}_{\mathfrak{t}}(1)=\int_0^1 \Lambda(\mathfrak{z})\mathbf{r}_{\mathfrak{t}}(\mathfrak{z})d\mathfrak{z}, \\ \mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(0)=0,\, \mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(1)=\int_0^1 \Lambda(\mathfrak{z})\big(\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_{\mathfrak{t}}(\mathfrak{z})\big)d\mathfrak{z}, \end{aligned}\right\} \tag{1.2}$$

where $\mathfrak{t}\in\{1,2,\cdots,\mathfrak{w}\}$, $3<\mathfrak{i}_1+\mathfrak{i}_2<4$ and $\mathsf{D}_{0^+}^{\times}$ is a Riemann-Liouville fractional derivative of order $\times\in\{\mathfrak{i}_1,\mathfrak{i}_2\}$, $\sigma(\mathfrak{q})\in L^{\mathrm{p}}[0,1](1\le \mathrm{p}\le\infty)$ has singularities in $(0,1/2)$, by an application of fixed point index theory (in cone) in a Banach space. Also, we derive sufficient conditions for the existence of unique solution of the problem by Rus's theorem in a complete metric space. Studying iterative fractional-order BVPs offers significant advantages compared to traditional non-iterative differential equations, particularly in their application to complex models. For instance, iterative fractional BVPs are more suitable for accurately modeling phenomena such as infectious disease dynamics and the motion of charged particles with delayed interactions. These types of problems are often inadequately described by standard non-iterative differential equations, highlighting the need for fractional-order models to capture their unique behaviors and interactions. Through the article, we presume that the conditions below hold:

(H_1) $\hbar_{\mathfrak{t}}:[0,+\infty)\to[0,+\infty)$ and $\Lambda:[0,1]\to[0,+\infty)$ are continuous,

(H_2) there exist a sequence $\{\mathfrak{q}_{\mathfrak{s}}\}_{\mathfrak{s}=1}^{\infty}$, with $0<\mathfrak{q}_{\mathfrak{s}+1}<\mathfrak{q}_{\mathfrak{s}}<\frac{1}{2}$, $\lim_{\mathfrak{s}\to\infty}\mathfrak{q}_{\mathfrak{s}}=\mathfrak{q}^*<\frac{1}{2}$, and $\lim_{\mathfrak{q}\to\mathfrak{q}_r}\sigma(\mathfrak{q})=+\infty$, $\mathfrak{s}\in\mathbb{N}$, There also exists $\sigma^*>0$ with $\sigma^*<\sigma(\mathfrak{q})<\infty$ a.e on $[0,1]$.

(H_3) for $\text{ß}\in\{\mathfrak{i}_1,\mathfrak{i}_2\}$, $0<\Lambda_{\text{ß}}<1$, where $\Lambda_{\text{ß}}=\int_0^1\mathfrak{z}^{\text{ß}-1}\Lambda(\mathfrak{z})d\mathfrak{z}$.

2. Preliminaries

Let $y\in\mathscr{C}[0,1]$. To analyze the BVP (1.1)-(1.2), we begin by examining the following linear BVP

$$\mathsf{D}_{0^+}^{\mathfrak{i}_2}\mathbf{r}_1(\mathfrak{q})+y(\mathfrak{q})=0,\ 0<\mathfrak{q}<1, \tag{2.1}$$

$$\mathbf{r}_1(0)=0,\ \mathbf{r}_1(1)=\int_0^1\Lambda(\mathfrak{p})\mathbf{r}_1(\mathfrak{p})d\mathfrak{p}, \tag{2.2}$$

Lemma 2.1. *Linear BVP (2.1)-(2.2) possesses a unique solution* $\mathbf{r}_1(\mathfrak{q})=\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})y(\mathfrak{p})d\mathfrak{p}$, *where*

$$\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})=\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{q}^{\mathfrak{i}_2-1}}{1-\Lambda_{\mathfrak{i}_2}}\mathsf{Q}_{\mathfrak{i}_2}(\mathfrak{p}),$$

$$\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})=\frac{1}{\Gamma(\mathfrak{i}_2)}\begin{cases}\mathfrak{q}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}, & \mathfrak{q}\le\mathfrak{p},\\ \mathfrak{q}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}-(\mathfrak{q}-\mathfrak{p})^{\mathfrak{i}_2-1}, & \mathfrak{p}\le\mathfrak{q},\end{cases}$$

and

$$\mathsf{Q}_{\mathfrak{i}_2}(\mathfrak{p})=\int_0^1\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p}_1)\Lambda(\mathfrak{p})d\mathfrak{p}_1.$$

Proof. The proof is trivial, so we omit the details here. □

Lemma 2.2. *The function* $\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$ *exhibits the below characteristics:*

(i) $\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})\ge 0$ *and continuous on* $[0,1]\times[0,1]$,

(ii) $\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})\le\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})$ *for* $\mathfrak{q},\mathfrak{p}\in[0,1]$,

(iii) $\eta^{\mathfrak{i}_2-1}\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})\le\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$ *for* $\mathfrak{q}\in[\eta,1-\eta]$, $\mathfrak{p}\in[0,1]$ *and for some* $\eta\in(0,\frac{1}{2})$.

Proof. The proof of (i) is straightforward. Observing the behavior of $\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$, it decreases as $\mathfrak{q}$ increases for $\mathfrak{p}\le\mathfrak{q}$ and it is increases as $\mathfrak{q}$ increases for $\mathfrak{q}\le\mathfrak{p}$. This confirms (ii). We now proceed to prove (iii). For the case $\mathfrak{p}\le\mathfrak{q}$, we have

$$\frac{\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})}{\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})}=\frac{\mathfrak{q}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}-(\mathfrak{q}-\mathfrak{p})^{\mathfrak{i}_2-1}}{\mathfrak{p}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}}\ge\frac{\mathfrak{q}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}}{\mathfrak{p}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}}\ge 1,$$

and

$$\frac{\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})}{\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})}=\frac{\mathfrak{q}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}}{\mathfrak{p}^{\mathfrak{i}_2-1}(1-\mathfrak{p})^{\mathfrak{i}_2-1}}\ge\frac{\mathfrak{q}^{\mathfrak{i}_2-1}}{\mathfrak{p}^{\mathfrak{i}_2-1}}\ge\eta^{\mathfrak{i}_2-1}\text{ for }\mathfrak{q}\in[\eta,1-\eta],\mathfrak{p}\in[0,1].$$

This completes the proof. □

Lemma 2.3. *Define* $\mathsf{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})=\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})+\frac{1}{1-\Lambda_{\mathfrak{i}_2}}\mathsf{Q}_{\mathfrak{i}_2}(\mathfrak{p})$. *Then the kernel function* $\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$ *exhibits the following characteristics:*

(i) $\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$ *is nonnegative and continuous on* $[0,1]\times[0,1]$,
(ii) $\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})\le\mathsf{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})$ *for* $\mathfrak{q},\mathfrak{p}\in[0,1]$,
(iii) $\eta^{\mathfrak{i}_2-1}\mathsf{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})\le\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$ *for* $\mathfrak{q}\in[\eta,1-\eta],\mathfrak{p}\in[0,1]$ *and for some* $\eta\in(0,\frac{1}{2})$.

Proof. Points (i) and (ii) are straightforward. To establish (iii), consider $\eta\in(0,\frac{1}{2})$ and $\mathfrak{p}\in[0,1]$. According to Lemma 2.2, we obtain the following inequality

$$\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})=\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{q}^{\mathfrak{i}_2-1}}{1-\Lambda_{\mathfrak{i}_2}}\mathsf{Q}_{\mathfrak{i}_2}(\mathfrak{p})\ge\eta^{\mathfrak{i}_2-1}\mathscr{H}_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})+\frac{\eta^{\mathfrak{i}_2-1}}{1-\Lambda_{\mathfrak{i}_2}}\mathsf{Q}_{\mathfrak{i}_2}(\mathfrak{p})=\eta^{\mathfrak{i}_2-1}\mathsf{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}).$$

□

Lemma 2.4. *Consider* $\mathrm{x}\in\mathscr{C}[0,1]$. *The BVP given by*

$$\mathsf{D}^{\mathfrak{i}_1}_{0^+}\big(\mathsf{D}^{\mathfrak{i}_2}_{0^+}\mathbf{r}_1(\mathfrak{q})\big)+\mathrm{x}(\mathfrak{q})=0,\ 0<\mathfrak{q}<1,\tag{2.3}$$

$$\left.\begin{aligned}\mathbf{r}_1(0)=0,\mathbf{r}_1(1)&=\int_0^1\mathbf{r}_1(\mathfrak{z})d\mathfrak{z},\\ \mathsf{D}^{\mathfrak{i}_2}_{0^+}\mathbf{r}_1(0)=0,\mathsf{D}^{\mathfrak{i}_2}_{0^+}\mathbf{r}_1(1)&=\int_0^1\Lambda(\mathfrak{z})\big(\mathsf{D}^{\mathfrak{i}_2}_{0^+}\mathbf{r}_1(\mathfrak{z})\big)d\mathfrak{z}\end{aligned}\right\}\tag{2.4}$$

has a unique solution

$$\mathbf{r}_1(\mathfrak{q})=\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})\left[\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{z})\mathrm{x}(\mathfrak{z})d\mathfrak{z}\right]d\mathfrak{p},\tag{2.5}$$

where $\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$ *is defined in Lemma 2.1 and*

$$\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})=\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})+\frac{\mathfrak{q}^{\mathfrak{i}_1-1}}{1-\Lambda_{\mathfrak{i}_1}}\mathsf{Q}_{\mathfrak{i}_1}(\mathfrak{p}),$$

$$\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})=\frac{1}{\Gamma(\mathfrak{i}_1)}\begin{cases}\mathfrak{q}^{\mathfrak{i}_1-1}(1-\mathfrak{p})^{\mathfrak{i}_1-1}-(\mathfrak{q}-\mathfrak{p})^{\mathfrak{i}_1-1}, & \mathfrak{p}\le\mathfrak{q},\\ \mathfrak{q}^{\mathfrak{i}_1-1}(1-\mathfrak{p})^{\mathfrak{i}_1-1}, & \mathfrak{q}\le\mathfrak{p},\end{cases}$$

and

$$\mathsf{Q}_{\mathfrak{i}_1}(\mathfrak{p})=\int_0^1\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{p}_1)\Lambda(\mathfrak{p})d\mathfrak{p}_1.$$

Proof. Let $y_1(\mathfrak{q}) = \mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}_1(\mathfrak{q})$ for $0<\mathfrak{q}<1$. Then the boundary value problem

$$\mathsf{D}_{0+}^{\mathfrak{i}_1}\big(\mathsf{D}^{\mathfrak{i}_2}\mathbf{r}_1(\mathfrak{q})\big)+\mathtt{x}(\mathfrak{q})=0,\ 0<\mathfrak{q}<1,$$
$$\mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}_1(0)=0,\ \mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}_1(1)=\int_0^1\Lambda(\mathfrak{z})\big(\mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}_1(\mathfrak{z})\big)d\mathfrak{z}$$

is equivalent to the problem

$$\left.\begin{aligned}\mathsf{D}_{0+}^{\mathfrak{i}_1}y_1(\mathfrak{q})+\mathtt{x}(\mathfrak{q})=0,\ 0<\mathfrak{q}<1,\\ y_1(0)=0,\ y_1(1)=\int_0^1\Lambda(\mathfrak{z})y_1(\mathfrak{z})d\mathfrak{z}.\end{aligned}\right\}\tag{2.6}$$

Equation (2.6) has a unique solution $y_1(\mathfrak{q})=\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})\mathtt{x}(\mathfrak{p})d\mathfrak{p}$ according to Lemma 2.1. That is,

$$\mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}_1(\mathfrak{q})=\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})\mathtt{x}(\mathfrak{p})d\mathfrak{p}.\tag{2.7}$$

Also,

$$\mathbf{r}_1(\mathfrak{q})=\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})\left[\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{z})\mathtt{x}(\mathfrak{z})d\mathfrak{z}\right]d\mathfrak{p}.$$

is the only solution to the differential equation (2.7) with boundary conditions

$$\mathbf{r}_1(0)=0\ \text{ and }\ \mathbf{r}_1(1)=\int_0^1\Lambda(\mathfrak{z})\mathbf{r}_1(\mathfrak{z})d\mathfrak{z},$$

according to Lemma 2.1. That brings the proof to a close. □

Lemma 2.5. *The properties of the function* $\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ *are as follows:*

(i) $\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ *is non-negative and continuous on* $[0,1]\times[0,1]$,
(ii) $\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})\le\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{p})$ *for* $\mathfrak{q},\mathfrak{p}\in[0,1]$,
(iii) $\eta^{\mathfrak{i}_2-1}\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{p})\le\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ *for* $\mathfrak{q}\in[\eta,1-\eta],\mathfrak{p}\in[0,1]$ *and for some* $\eta\in(0,\frac{1}{2})$.

Lemma 2.6. *Define* $\mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p})=\mathscr{H}_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{p})+\frac{1}{1-\Lambda_{\mathfrak{i}_1}}\mathtt{Q}_{\mathfrak{i}_1}(\mathfrak{p})$. *Then kernel function* $\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ *exhibits the following characteristics:*

(i) $\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ *is non-negative and continuous on* $[0,1]\times[0,1]$,
(ii) $\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})\le\mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p})$ *for* $t,\mathfrak{p}\in[0,1]$,
(iii) $\eta^{\mathfrak{i}_2-1}\mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p})\le\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ *for* $\mathfrak{q}\in[\eta,1-\eta],\mathfrak{p}\in[0,1]$ *and for some* $\eta\in(0,\frac{1}{2})$.

A solution to system (1.1)–(1.2) is equivalent to an $\mathfrak{w}$-tuple $(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_{\mathfrak{w}})$ satisfying

$$\begin{aligned}\mathbf{r}_1(\mathfrak{q})=&\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p}_1)\Big[\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{p}_3,\mathfrak{p}_4)\sigma(\mathfrak{p}_4)\hbar_2\cdots\\ &\hbar_{\mathfrak{w}-1}\Big[\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1\end{aligned}$$

and

$$\begin{aligned}\mathbf{r}_{\mathtt{t}}(\mathfrak{q})&=\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})\left[\int_0^1\mho_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{z})\sigma(\mathfrak{z})\hbar_{\mathfrak{w}}(\mathbf{r}_{\mathfrak{w}+1}(\mathfrak{q}))d\mathfrak{q}\right]d\mathfrak{p},\ \mathtt{t}=2,3,\cdots,\mathfrak{w},\\ \mathbf{r}_{\mathfrak{w}+1}(\mathfrak{q})&=\mathbf{r}_1(\mathfrak{q}),\ 0<\mathfrak{q}<1.\end{aligned}$$

Let $\mathscr{B}$ represent the Banach space $\mathscr{C}([0,1],\mathbb{R})$ with the norm $\|\mathbf{r}\| = \max_{\mathfrak{q}\in[0,1]} |\mathbf{r}(\mathfrak{q})|$. For $\eta \in (0,1/2)$, the cone $\mathscr{P}_\eta$ within $\mathscr{B}$ is defined by

$$\mathscr{P}_\eta = \Big\{\mathbf{r}\in\mathscr{B} : \mathbf{r}(\mathfrak{q}) \geq 0, \min_{\mathfrak{q}\in[\eta,1-\eta]} \mathbf{r}(\mathfrak{q}) \geq \eta^{\mathfrak{i}_2-1}\|\mathbf{r}\|\Big\},$$

For any $\mathbf{r}_1 \in \mathscr{P}_\eta$, define an operator $\aleph : \mathscr{P}_\eta \to \mathscr{B}$ by

$$(\aleph\mathbf{r}_1)(\mathfrak{q}) = \int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p}_1)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_3,\mathfrak{p}_4)\sigma(\mathfrak{p}_4)\hbar_2\cdots$$
$$\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1.$$

Lemma 2.7. *For every $\eta\in(0,1/2)$, the operator $\aleph$ maps $\mathscr{P}_\eta$ into itself and is completely continuous on $\mathscr{P}_\eta$.*

Proof. Let $\eta\in(0,1/2)$. Since $\hbar_{\mathfrak{t}}(\mathbf{r}_{\mathfrak{t}+1}(\mathfrak{p}))$ is nonnegative for $\mathfrak{p}\in[0,1]$, it follows that $\mathbf{r}_1\in\mathscr{P}_\eta$. Given that $\mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p})$, $\mho_{\mathfrak{i}_1}(\mathfrak{q},\mathfrak{p})$ are nonnegative for all $t,\mathfrak{p}\in[0,1]$, we have $\aleph(\mathbf{r}_1(\mathfrak{q}))\geq 0$ for all $\mathfrak{q}\in[0,1]$ with $\mathbf{r}_1\in\mathscr{P}_\eta$. Consequently, by applying Lemma 2.3 and Lemma 2.6, we obtain

$$\begin{aligned}
&\min_{\mathfrak{q}\in[\eta,1-\eta]}(\aleph\mathbf{r}_1)(\mathfrak{q})\\
&= \min_{\mathfrak{q}\in[\eta,1-\eta]}\Bigg\{\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p}_1)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_3,\mathfrak{p}_4)\sigma(\mathfrak{p}_4)\hbar_2\cdots\\
&\quad\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1\Bigg\}\\
&\geq \eta^{\mathfrak{i}_2-1}\int_0^1 \mathfrak{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_1)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_3,\mathfrak{p}_4)\sigma(\mathfrak{p}_4)\hbar_2\cdots\\
&\quad\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1\\
&\geq \eta^{\mathfrak{i}_2-1}\max_{\mathfrak{q}\in[0,1]}\Bigg\{\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p}_1)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_3,\mathfrak{p}_4)\sigma(\mathfrak{p}_4)\hbar_2\cdots\\
&\quad\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1\Bigg\}\\
&\geq \eta^{\mathfrak{i}_2-1}\max_{\mathfrak{q}\in[0,1]}|\aleph\mathbf{r}_1(\mathfrak{q})|.
\end{aligned}$$

Therefore, $\aleph$ maps $\mathscr{P}_\eta$ into itself. As a result, $\aleph$ is a completely continuous operator, which can be established using classical methods and the Arzelà-Ascoli theorem. □

3. Infinite Positive Solutions

The specific theorems below are used to support the existence of infinite positive solutions for the BVP (1.1)–(1.2).

Theorem 3.1 ([11])**.** *Let $\mathscr{B}$ be a Banach space and $\mathscr{P}$ be a cone within $\mathscr{B}$. For $r>0$, define the set $\mathrm{E}_r=\{\mathbf{r}\in\mathscr{P}:\|\mathbf{r}\|<r\}$. Suppose that $\aleph:\mathscr{P}\cap\overline{\mathrm{E}}_r\to\mathscr{P}$ is a completely continuous operator with the property that $\aleph\mathbf{r}\neq\mathbf{r}$ for all $\mathbf{r}\in\partial\mathrm{E}_r$.*

(a) *If $\|\aleph\mathbf{r}\|<\|\mathbf{r}\|$ for all $\mathbf{r}\in\partial\mathrm{E}_r$, then $i(\aleph,\mathrm{E}_r,\mathscr{P})=1$.*

(b) *If* $\|\aleph\mathbf{r}\| \geq \|\mathbf{r}\|$ *for all* $\mathbf{r} \in \partial\mathtt{E}_r$, *then* $i(\aleph, \mathtt{E}_r, \mathscr{P}) = 0$.

Theorem 3.2 ([11]). *Let $\mathscr{P}$ be a retract of a real Banach space $\mathscr{B}$. For any bounded, relatively open subset $\mathtt{E}$ of $\mathscr{P}$ and any completely continuous operator $\aleph : \overline{\mathtt{E}} \to \mathscr{P}$ that has no fixed points on $\partial\mathtt{E}$ (relative to $\mathscr{P}$), there exists an integer $i(\aleph, \mathtt{E}, \mathscr{P})$ that satisfies the following properties:*

($\hbar_1$) ***Normalization**: If $\aleph(\mathbf{r})$ is constantly equal to some $\mathbf{r}_0 \in \mathtt{E}$ for every $\mathbf{r} \in \overline{\mathtt{E}}$, then $i(\aleph, \mathtt{E}, \mathscr{P}) = 1$.*

($\hbar_2$) ***Additivity**: If $\mathtt{E}_1$ and $\mathtt{E}_2$ are disjoint open subsets of $\mathtt{E}$ such that $\aleph$ has no fixed points on $\overline{\mathtt{E}} \setminus (\mathtt{E}_1 \cup \mathtt{E}_2)$, then $i(\aleph, \mathtt{E}, \mathscr{P}) = i(\aleph, \mathtt{E}_1, \mathscr{P}) + i(\aleph, \mathtt{E}_2, \mathscr{P})$.*

($\hbar_3$) ***Homotopy Invariance**: If $\mathfrak{F} : [0,1] \times \overline{\mathtt{E}} \to \mathscr{P}$ is a completely continuous operator such that $\mathfrak{F}(\mathfrak{q}, \mathbf{r}) \neq \mathbf{r}$ for any $(\mathfrak{q}, \mathbf{r}) \in [0,1] \times \partial\mathtt{E}$, then $i(\mathfrak{F}(\mathfrak{q}, \cdot), \mathtt{E}, \mathscr{P})$ is independent of $\mathfrak{q} \in [0,1]$.*

($\hbar_4$) ***Permanence**: If $\mathtt{F}$ is a retract of $\mathscr{P}$ and $\aleph(\overline{\mathtt{E}}) \subset \mathtt{F}$, then $i(\aleph, \mathtt{E}, \mathscr{P}) = i(\aleph, \mathtt{E} \cap \mathtt{F}, \mathtt{F})$.*

($\hbar_5$) ***Excision**: If $\mathtt{E}_0$ is an open subset of $\mathtt{E}$ and $\aleph$ has no fixed points in $\overline{\mathtt{E}} \setminus \mathtt{E}_0$, then $i(\aleph, \mathtt{E}, \mathscr{P}) = i(\aleph, \mathtt{E}_0, \mathscr{P})$.*

($\hbar_6$) ***Existence of Solutions**: If $i(\aleph, \mathtt{E}, \mathscr{P}) \neq 0$, then $\aleph$ has at least one fixed point in $\mathtt{E}$.*

Moreover, $i(\aleph, \mathtt{E}, \mathscr{P})$ is uniquely defined.

Theorem 3.3. *(Hölder's Inequality) Let $h \in L^{\mathtt{p}}[0,1]$ and $g \in L^{\mathtt{q}}[0,1]$, where $\mathtt{p} > 1$ and $\mathtt{q} > 1$, with $\frac{1}{\mathtt{p}} + \frac{1}{\mathtt{q}} = 1$. Then, $hg \in L^1[0,1]$ and the inequality $\|hg\|_1 \leq \|h\|_{\mathtt{p}} \|g\|_{\mathtt{q}}$ holds. Additionally, if $h \in L^1[0,1]$ and $g \in L^\infty[0,1]$, then $hg \in L^1[0,1]$ and $\|hg\|_1 \leq \|h\|_1 \|g\|_\infty$.*

In this study, we prove the existence of infinitely many positive solutions for three scenarios involving $\sigma \in L^{\mathtt{p}}[0,1]$: specifically when $0 < \mathtt{p} < 1$, $\mathtt{p} = 1$, and $\mathtt{p} = \infty$.

We begin by examining the case where $\mathtt{p} < 1$.

Theorem 3.4. *Assume that conditions (H_1) through (H_3) are satisfied. Let $\{\eta_{\mathtt{s}}\}_{\mathtt{s}=1}^{\infty}$ be a sequence such that $\mathfrak{q}_{\mathtt{s}+1} < \eta_{\mathtt{s}} < \mathfrak{q}_{\mathtt{s}}$ for $\mathtt{s} = 1, 2, 3, \ldots$. Define sequences $\{\mathtt{B}_{\mathtt{s}}\}_{\mathtt{s}=1}^{\infty}$ and $\{\mathtt{A}_{\mathtt{s}}\}_{\mathtt{s}=1}^{\infty}$ such that*

$$\mathtt{B}_{\mathtt{s}+1} < \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\mathtt{A}_{\mathtt{s}} < \mathtt{A}_{\mathtt{s}} < \mathtt{L}\mathtt{A}_{\mathtt{s}} < \mathtt{B}_{\mathtt{s}}, \text{ and } \mathtt{L}\mathtt{A}_{\mathtt{s}} < \mathscr{O}_1\mathtt{B}_{\mathtt{s}},\ k \in \mathbb{N},$$

where

$$\mathtt{L} = \max\left\{\left[\eta_1^{\mathfrak{i}_2+\mathfrak{i}_1-2}\sigma^* \int_{\eta_1}^{1-\eta_1} \mathtt{Q}_{\mathfrak{i}_2}^*(\mathfrak{p})d\mathfrak{p}\left[\int_{\eta_1}^{1-\eta_1} \mathtt{Q}_{\mathfrak{i}_1}^*(\mathfrak{p})d\mathfrak{p}\right]\right]^{-1},\ 1\right\}.$$

Further, assume that $\hbar_{\mathtt{t}}$ satisfies

($\mathscr{J}_1$) $\hbar_{\mathtt{t}}(\mathbf{r}(\mathfrak{q})) \leq \mathscr{O}_1\mathtt{B}_{\mathtt{s}}$ *for all* $0 \leq \mathbf{r}(\mathfrak{q}) \leq \mathtt{B}_{\mathtt{s}}$ *and* $\mathfrak{q} \in [0,1]$, *where* $\mathscr{O}_1 < \left[\|\mathtt{Q}_{\mathfrak{i}_1}^*\|_{\mathtt{q}} \|\sigma\|_{\mathtt{p}} \int_0^1 \mathtt{Q}_{\mathfrak{i}_2}^*(\mathfrak{p})d\mathfrak{p}\right]^{-1}$.

($\mathscr{J}_2$) $\hbar_{\mathtt{t}}(\mathbf{r}(\mathfrak{q})) \geq \mathtt{L}\mathtt{A}_{\mathtt{s}}$ *for all* $\eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\mathtt{A}_{\mathtt{s}} \leq \mathbf{r}(\mathfrak{q}) \leq \mathtt{A}_{\mathtt{s}}$ *and* $\mathfrak{q} \in [\eta_{\mathtt{s}}, 1-\eta_{\mathtt{s}}]$.

Then the iterative system described by (1.1)–(1.2) has infinite solutions $\{(\mathbf{r}_1^{[k]}, \mathbf{r}_2^{[k]}, \ldots, \mathbf{r}_{\mathfrak{w}}^{[k]})\}_{\mathtt{s}=1}^{\infty}$ and satisfies $\mathbf{r}_{\mathtt{t}}^{[k]}(\mathfrak{q}) \geq 0$ on $(0,1)$ for all $\mathtt{t} = 1, 2, \ldots, \mathfrak{w}$ and $k \in \mathbb{N}$.

Proof. Let $\{\mathtt{E}_{\mathtt{s}}\}_{\mathtt{s}=1}^{\infty}$ and $\{\mathtt{F}_{\mathtt{s}}\}_{\mathtt{s}=1}^{\infty}$ be sequences of open subsets within the space $\mathscr{B}$, defined by $\mathtt{E}_{\mathtt{s}} = \{\mathbf{r} \in \mathscr{B} : \|\mathbf{r}\| < \mathtt{B}_{\mathtt{s}}\}$ and $\mathtt{F}_{\mathtt{s}} = \{\mathbf{r} \in \mathscr{B} : \|\mathbf{r}\| < \mathtt{A}_{\mathtt{s}}\}$. Consider the sequence $\{\eta_{\mathtt{s}}\}_{\mathtt{s}=1}^{\infty}$ as stated in the assumptions, noting that $\mathfrak{q}^* < \mathfrak{q}_{\mathtt{s}+1} < \eta_{\mathtt{s}} < \mathfrak{q}_{\mathtt{s}} < \frac{1}{2}$ holds for every k in $\mathbb{N}$. Next, we define the cone $\mathscr{P}_{\eta_{\mathtt{s}}}$ as

$$\mathscr{P}_{\eta_{\mathtt{s}}} = \left\{\mathbf{r} \in \mathscr{B} : \mathbf{r}(\mathfrak{q}) \geq 0 \text{ and } \min_{\mathfrak{q} \in [\eta_{\mathtt{s}}, 1-\eta_{\mathtt{s}}]} \mathbf{r}(\mathfrak{q}) \geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\|\mathbf{r}\|\right\}.$$

For a fixed k and $\mathbf{r}_1 \in \partial\mathtt{E}$, one has

$$\mathtt{A}_{\mathtt{s}} = \|\mathbf{r}_1\| \geq \mathbf{r}_1(\mathfrak{q}) \geq \min_{\mathfrak{q}\in[\eta_{\mathtt{s}},1-\eta_{\mathtt{s}}]} \mathbf{r}_1(\mathfrak{q}) \geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\|\mathbf{r}_1\| \geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\mathtt{A}_{\mathtt{s}}.$$

By $(\mathscr{J}2)$, for $\mathfrak{p}_{2\mathfrak{w}-2} \in [\eta_{\mathtt{s}}, 1-\eta_{\mathtt{s}}]$, we have

$$\begin{aligned}
&\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\\
&\quad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\\
&\quad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})\Big[\eta_{\mathtt{s}}^{\mathfrak{i}_1-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\\
&\quad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\eta_{\mathtt{s}}^{\mathfrak{i}_1-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})\Big[\mathtt{L}\mathtt{A}_{\mathtt{s}}\sigma^*\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}})d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\\
&\quad\geq \mathtt{L}\mathtt{A}_{\mathtt{s}}\eta_1^{\mathfrak{i}_2+\mathfrak{i}_1-2}\sigma^*\int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})d\mathfrak{p}_{2\mathfrak{w}-1}\Big[\int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}})d\mathfrak{p}_{2\mathfrak{w}}\Big]\\
&\quad\geq \mathtt{A}_{\mathtt{s}}.
\end{aligned}$$

A corresponding argument applies for $\mathfrak{p}_{2\mathfrak{w}-4} \in [\eta_{\mathtt{s}}, 1-\eta_{\mathtt{s}}]$ leading to the conclusion that

$$\begin{aligned}
&\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-4},\mathfrak{p}_{2\mathfrak{w}-3})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-3},\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2})\\
&\qquad\times\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\\
&\qquad\qquad\times\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\Big]d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3}\\
&\quad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-3})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-3},\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2})\\
&\qquad\times\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\\
&\qquad\qquad\times\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\Big]d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3}\\
&\quad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-3})\Big[\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-3},\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2})\\
&\qquad\times\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\\
&\qquad\qquad\times\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\Big]d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3}\\
&\quad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-3})\Big[\eta_{\mathtt{s}}^{\mathfrak{i}_1-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2})\times\hbar_{\mathfrak{w}-1}(\mathtt{A}_{\mathtt{s}})d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3}\\
&\qquad\geq \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\eta_{\mathtt{s}}^{\mathfrak{i}_1-1}\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-3})\Big[\mathtt{L}\mathtt{A}_{\mathtt{s}}\sigma^*\int_{\eta_{\mathtt{s}}}^{1-\eta_{\mathtt{s}}} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-2})d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3}\\
&\qquad\geq \mathtt{L}\mathtt{A}_{\mathtt{s}}\eta_1^{\mathfrak{i}_2+\mathfrak{i}_1-2}\sigma^*\int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-3})d\mathfrak{p}_{2\mathfrak{w}-3}\Big[\int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-2})d\mathfrak{p}_{2\mathfrak{w}-2}\Big]\\
&\qquad\geq \mathtt{A}_{\mathtt{s}}.
\end{aligned}$$

Proceeding with a step-by-step refinement process, we arrive at

$$(\aleph\vartheta_1)(\mathfrak{q}) = \int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{q},\mathfrak{p}_1)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_3,\mathfrak{p}_4)\cdots \\ \hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}}) \\ \times \hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1 \geq \mathtt{A}_\mathtt{s}.$$

Therefore, $\|\aleph\mathbf{r}_1\| \geq \|\mathbf{r}_1\|$. Consequently, by invoking Theorem 3.1, we obtain

$$i(\aleph,\mathtt{E}_\mathtt{s},\mathscr{P}_{\eta_\mathtt{s}}) = 0. \tag{3.1}$$

On the other hand, let $\mathbf{r}_1 \in \partial\mathtt{F}_\mathtt{s}$. Then $\mathbf{r}_1(\mathfrak{p}) \leq \mathtt{B}_\mathtt{s} = \|\mathbf{r}_1\|$ for all $\mathfrak{p}\in[0,1]$. By $(\mathscr{J}_1)$ and $0<\mathfrak{p}_{2\mathfrak{w}-2}<1$, we have

$$\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1} \\ \leq \int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1} \\ \leq \mathscr{O}_1\mathtt{B}_\mathtt{s}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}.$$

Let $\mathtt{q}>1$ be such that $\frac{1}{\mathtt{q}}+\frac{1}{\mathtt{p}}=1$. Using Hölder's inequality, we derive

$$\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1} \\ \leq \mathscr{O}_1\mathtt{B}_\mathtt{s}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})d\mathfrak{p}_{2\mathfrak{w}-1}\Big[\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})d\mathfrak{p}_{2\mathfrak{w}}\Big] \\ \leq \mathscr{O}_1\mathtt{B}_\mathtt{s}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})d\mathfrak{p}_{2\mathfrak{w}-1}\|\mathtt{Q}^*_{\mathfrak{i}_1}\|_\mathtt{q}\|\sigma\|_\mathtt{p} \\ \leq \mathtt{B}_\mathtt{s}.$$

A corresponding argument applies for $\mathfrak{p}_{2\mathfrak{w}-4}\in(0,1)$, leading to the conclusion that

$$\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-4},\mathfrak{p}_{2\mathfrak{w}-3})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-3},\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2}) \\ \times\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}}) \\ \times\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\Big]d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3} \\ \leq \int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-4},\mathfrak{p}_{2\mathfrak{w}-3})\Big[\int_0^1 \mho_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-3},\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2})\hbar_{\mathfrak{w}-1}(\mathtt{B}_\mathtt{s})d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3} \\ \leq \mathscr{O}_1\mathtt{B}_\mathtt{s}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-3})\Big[\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}_{2\mathfrak{w}-2})\sigma(\mathfrak{p}_{2\mathfrak{w}-2})d\mathfrak{p}_{2\mathfrak{w}-2}\Big]d\mathfrak{p}_{2\mathfrak{w}-3} \\ \leq \mathscr{O}_1\mathtt{B}_\mathtt{s}\int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}_{2\mathfrak{w}-1})d\mathfrak{p}_{2\mathfrak{w}-1}\|\mathtt{Q}^*_{\mathfrak{i}_1}\|_\mathtt{q}\|\sigma\|_\mathtt{p} \\ \leq \mathtt{B}_\mathtt{s}.$$

Proceeding with a step-by-step refinement process, we arrive at

$$(\aleph\mathbf{r}_1)(\mathfrak{q}) = \int_0^1 \mho_{i_2}(\mathfrak{q},\mathfrak{p}_1)\Big[\int_0^1 \mho_{i_1}(\mathfrak{p}_1,\mathfrak{p}_2)\sigma(\mathfrak{p}_2)\hbar_1\Big[\int_0^1 \mho_{i_2}(\mathfrak{p}_2,\mathfrak{p}_3)\Big[\int_0^1 \mho_{i_1}(\mathfrak{p}_3,\mathfrak{p}_4)\sigma(\mathfrak{p}_4)\hbar_2\cdots$$
$$\hbar_{\mathfrak{w}-1}\Big[\int_0^1 \mho_{i_2}(\mathfrak{p}_{2\mathfrak{w}-2},\mathfrak{p}_{2\mathfrak{w}-1})\Big[\int_0^1 \mho_{i_1}(\mathfrak{p}_{2\mathfrak{w}-1},\mathfrak{p}_{2\mathfrak{w}})\sigma(\mathfrak{p}_{2\mathfrak{w}})$$
$$\times\hbar_{\mathfrak{w}}(\mathbf{r}_1(\mathfrak{p}_{2\mathfrak{w}}))d\mathfrak{p}_{2\mathfrak{w}}\Big]d\mathfrak{p}_{2\mathfrak{w}-1}\cdots d\mathfrak{p}_2\Big]d\mathfrak{p}_1 \le \mathtt{B}_\mathtt{s}.$$

Since $\mathtt{B}_\mathtt{s} = \|\mathbf{r}_1\|$ for $\mathbf{r}_1 \in \partial\mathtt{F}_\mathtt{s}$, we obtain $\|\aleph\mathbf{r}_1\| \le \|\mathbf{r}_1\|$. Thus Theorem 3.1 implies that

$$i(\aleph,\mathtt{F}_\mathtt{s},\mathscr{P}_{\eta_\mathtt{s}}) = 1. \tag{3.2}$$

Given that $\mathtt{A}_\mathtt{s} < \mathtt{B}_\mathtt{s}$ for all $k \in \mathbb{N}$, utilizing the results from equations (3.2) and (3.1), we can apply the additivity property of the fixed-point index, $(\hbar_2)$ to conclude that $i(\aleph,\mathtt{F}_\mathtt{s}\setminus\overline{\mathtt{E}}_\mathtt{s}) = 1$ for all $k \in \mathbb{N}$. Consequently, by the fixed-point property $(\hbar_6)$, $\aleph$ possesses a fixed point $\mathbf{r}_1^{[k]}$ within $\mathtt{F}_\mathtt{s}\setminus\overline{\mathtt{E}}_\mathtt{s}$, where $\mathbf{r}_1^{[k]}(\mathfrak{q}) \ge 0$ over $(0,1)$ for $k \in \mathbb{N}$. Setting $\mathbf{r}_{\mathfrak{w}+1} = \mathbf{r}_1$, we derive an infinite sequence of positive solutions $\{(\mathbf{r}_1^{[k]},\mathbf{r}_2^{[k]},\ldots,\mathbf{r}_{\mathfrak{w}}^{[k]})\}_{\mathtt{s}=1}^{\infty}$ for system (1.1)–(1.2), which are iteratively defined by

$$\mathbf{r}_\mathtt{t}(\mathfrak{q}) = \int_0^1 \mho_{i_2}(\mathfrak{q},\mathfrak{p})\left[\int_0^1 \mho_{i_1}(\mathfrak{p},\mathfrak{z})\hbar_\mathtt{t}(\mathbf{r}_{\mathtt{t}+1}(\mathfrak{z}))\,d\mathfrak{z}\right]d\mathfrak{p}, \quad \text{where } \mathtt{t} = \mathfrak{w},\mathfrak{w}-1,\ldots,2,1.$$

Thus the proof is concluded. □

For the cases where $\mathtt{p} = 1$ and $\mathtt{p} = \infty$, theorems are established as follows.

Theorem 3.5. *Consider the conditions* (H_1) *through* (H_3) *and suppose that* $\{\eta_\mathtt{s}\}_{\mathtt{s}=1}^{\infty}$ *is a sequence where* $\mathfrak{q}_{\mathtt{s}+1} < \eta_\mathtt{s} < \mathfrak{q}_\mathtt{s}$ *for* $\mathtt{s} = 1,2,3,\ldots$. *Assume further that* $\{\mathtt{B}_\mathtt{s}\}_{\mathtt{s}=1}^{\infty}$ *and* $\{\mathtt{A}_\mathtt{s}\}_{\mathtt{s}=1}^{\infty}$ *satisfy the inequalities:* $\mathtt{B}_{\mathtt{s}+1} < \eta_\mathtt{s}^{i_2-1}\mathtt{A}_\mathtt{s} < \mathtt{A}_\mathtt{s} < \mathtt{L}\mathtt{A}_\mathtt{s} < \mathtt{B}_\mathtt{s}$ *and* $\mathtt{L}\mathtt{A}_\mathtt{s} < \mathscr{O}_2\mathtt{B}_\mathtt{s}$, *for* $k \in \mathbb{N}$. *Assume also that* $\hbar_\mathtt{t}$ *fulfills condition* $(\mathscr{J}_2)$ *and*

$(\mathscr{J}_3)$ $\hbar_\mathtt{t}(\mathbf{r}(\mathfrak{q})) \le \mathscr{O}_2\mathtt{B}_\mathtt{s}$ *for all* $0 \le \mathbf{r}(\mathfrak{q}) \le \mathtt{B}_\mathtt{s}$ *and* $\mathfrak{q} \in [0,1]$, *where*

$$\mathscr{O}_2 < \min\left\{\left[\|\mathtt{Q}_{i_1}^*\|_\infty\|\sigma\|_1\int_0^1 \mathtt{Q}_{i_2}^*(\mathfrak{p})\,d\mathfrak{p}\right]^{-1},\mathtt{L}\right\}.$$

Then, system (1.1)–(1.2) *has an infinite number of solutions* $\{(\mathbf{r}_1^{[k]},\mathbf{r}_2^{[k]},\ldots,\mathbf{r}_{\mathfrak{w}}^{[k]})\}_{\mathtt{s}=1}^{\infty}$, *where* $\mathbf{r}_\mathtt{t}^{[k]}(\mathfrak{q}) \ge 0$ *for* $\mathtt{t} = 1,2,\ldots,\mathfrak{w}$ *and* $k \in \mathbb{N}$.

Theorem 3.6. *Suppose that conditions* (H_1)-(H_3) *hold. Let the sequence* $\{\eta_\mathtt{s}\}_{\mathtt{s}=1}^{\infty}$ *be such that* $\mathfrak{q}_{\mathtt{s}+1} < \eta_\mathtt{s} < \mathfrak{q}_\mathtt{s}$ *for* $\mathtt{s} = 1,2,3,\ldots$. *Consider sequences* $\{\mathtt{B}_\mathtt{s}\}_{\mathtt{s}=1}^{\infty}$ *and* $\{\mathtt{A}_\mathtt{s}\}_{\mathtt{s}=1}^{\infty}$ *that satisfy the following conditions:* $\mathtt{B}_{\mathtt{s}+1} < \eta_\mathtt{s}^{i_2-1}\mathtt{A}_\mathtt{s} < \mathtt{L}\mathtt{A}_\mathtt{s} < \mathtt{B}_\mathtt{s}$ *and* $\mathtt{L}\mathtt{A}_\mathtt{s} < \mathscr{O}_3\mathtt{B}_\mathtt{s}$, *where* $k \in \mathbb{N}$. *Assume additionally that* $\hbar_\mathtt{t}$ *satisfies condition* $(\mathscr{J}_2)$, *along with the following condition:*

$(\mathscr{J}_4)$ *The inequality* $\hbar_\mathtt{t}(\mathbf{r}(\mathfrak{q})) \le \mathscr{O}_3\mathtt{B}_\mathtt{s}$ *holds for all* $0 \le \mathbf{r}(\mathfrak{q}) \le \mathtt{B}_\mathtt{s}$ *and* $\mathfrak{q} \in [0,1]$, *where*

$$\mathscr{O}_3 < \min\left\{\left[\|\mathtt{Q}_{i_1}^*\|_1\|\sigma\|_\infty\int_0^1 \mathtt{Q}_{i_2}^*(\mathfrak{p})\,d\mathfrak{p}\right]^{-1},\mathtt{L}\right\}.$$

Under these conditions, the iterative system defined by (1.1)–(1.2) *possesses an infinite number of solutions* $\{(\mathbf{r}_1^{[k]},\mathbf{r}_2^{[k]},\ldots,\mathbf{r}_{\mathfrak{w}}^{[k]})\}_{\mathtt{s}=1}^{\infty}$ *such that* $\mathbf{r}_\mathtt{t}^{[k]}(\mathfrak{q}) \ge 0$ *for* $\mathtt{t} = 1,2,\ldots,\mathfrak{w}$ *and* $k \in \mathbb{N}$.

4. Existence of The Unique Solution

This section establishes sufficient conditions for the unique solvability of the BVP

$$\left.\begin{aligned} \mathsf{D}_{0+}^{\mathfrak{i}_1}\left[\mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}(\mathfrak{q})\right]+\mathtt{f}(\mathfrak{q},\mathbf{r}(\mathfrak{q}))=0,\, 0<\mathfrak{q}<1,\, 1<\mathfrak{i}_1,\mathfrak{i}_2<2,\\ \mathbf{r}(0)=\mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}(0)=0,\, \mathbf{r}(1)=\int_0^1\Lambda(\mathfrak{z})\mathbf{r}(\mathfrak{z})d\mathfrak{z}\\ \mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}(1)=\int_0^1\Lambda(\mathfrak{z})\left(\mathsf{D}_{0+}^{\mathfrak{i}_2}\mathbf{r}(\mathfrak{z})\right)d\mathfrak{z}, \end{aligned}\right\}. \tag{4.1}$$

We employ Rus's fixed point theorem, utilizing two distinct metrics, to achieve this. For a detailed exposition of Rus's theorem, we refer to [5, 17]. The following assumption will underpin our subsequent analysis:

(H_4) $\mathtt{f}:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and there exists a $\mathtt{M}>0$ such that

$$|\mathtt{f}(\mathfrak{p},\mathtt{y}_1)-\mathtt{f}(\mathfrak{p},\mathtt{y}_2)|\le\mathtt{M}|\mathtt{y}_1-\mathtt{y}_2| \text{ for all } \mathtt{y}_1,\mathtt{y}_2\in[0,+\infty).$$

Let $\mathscr{X}=C([0,1])$ denote the space of all continuous real-valued functions on the interval $[0,1]$. For functions $\mathtt{x}_1,\mathtt{x}_2\in\mathscr{X}$, we introduce two distinct metrics on $\mathscr{X}$:

$$\mathtt{d}(\mathtt{x}_1,\mathtt{x}_2)=\max_{t\in[0,1]}|\mathtt{x}_1(t)-\mathtt{x}_2(t)|, \tag{4.2}$$

$$\rho(\mathtt{x}_1,\mathtt{x}_2)=\left[\int_0^1|\mathtt{x}_1(t)-\mathtt{x}_2(t)|^{\mathtt{p}}dt\right]^{\frac{1}{\mathtt{p}}},\ \mathtt{p}>1. \tag{4.3}$$

The pair $(C([0,1]),\mathtt{d})$, where $\mathtt{d}$ is defined as in (4.2), is known to be a complete metric space. Meanwhile, the pair $(C([0,1]),\rho)$, with ρ given in (4.3), also forms a metric space. The connection between these two metrics within the space $\mathscr{X}$ can be described by

$$\rho(\mathtt{x}_1,\mathtt{x}_2)\le\mathtt{d}(\mathtt{x}_1,\mathtt{x}_2) \text{ for all } \mathtt{x}_1,\mathtt{x}_2\in\mathscr{X}. \tag{4.4}$$

Theorem 4.1 (Rus [14])**.** *Let $\mathscr{X}$ be a nonempty set equipped with two metrics, $\mathtt{d}$ and ρ. Assume that $(\mathscr{X},\mathtt{d})$ is a complete metric space. Suppose there is a mapping $\aleph:\mathscr{X}\to\mathscr{X}$ that is continuous with respect to the metric $\mathtt{d}$ and satisfies the following conditions for all $\mathtt{x}_1,\mathtt{x}_2\in\mathscr{X}$:*

$$\mathtt{d}(\aleph(\mathtt{x}_1),\aleph(\mathtt{x}_2))\le\alpha\rho(\mathtt{x}_1,\mathtt{x}_2), \tag{4.5}$$

for some $\alpha>0$, and

$$\rho(\aleph(\mathtt{x}_1),\aleph(\mathtt{x}_2))\le\beta\rho(\mathtt{x}_1,\mathtt{x}_2), \tag{4.6}$$

where $0<\beta<1$. Under these conditions, there exists a unique element $\mathtt{x}^*\in\mathscr{X}$ such that $\aleph(\mathtt{x}^*)=\mathtt{x}^*$.

Theorem 4.2. *Define $\mho_{\mathfrak{i}_2}^*$ as $\mho_{\mathfrak{i}_2}^*=\int_0^1\mho_{\mathfrak{i}_2}(\mathfrak{p},\mathfrak{p})d\mathfrak{p}$. If (H_1) and (H_4) are satisfied, and $\mathtt{p}>1$, $\mathtt{q}>1$ are fixed constants with $\frac{1}{\mathtt{p}}+\frac{1}{\mathtt{q}}=1$, and*

$$\mho_{\mathfrak{i}_2}^*\mathtt{M}_1\left[\int_0^1|\mho_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}}\,d\mathfrak{p}\right]^{\frac{1}{\mathtt{q}}}<1, \tag{4.7}$$

then the BVP (4.1) has a unique nontrivial solution in $\mathscr{X}$.

Proof. Let $\mathbf{r} \in \mathscr{X}$, and define $\aleph : \mathscr{X} \to \mathscr{B}$ by

$$(\aleph \mathbf{r})(\mathfrak{q}) = \int_0^1 \mho_{\mathtt{i}_2}(\mathfrak{q},\mathfrak{p}) \left[\int_0^1 \mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{z}) \mathtt{f}(\mathfrak{z},\mathbf{r}(\mathfrak{z})) d\mathfrak{z} \right] d\mathfrak{p}.$$

Then, by Lemma 2.4, it can be seen that the fixed point of $\aleph$ is a solution to problem (4.1). Let $\mathbf{r}_1, \mathbf{r}_2 \in C([0,1])$ and $\mathfrak{q} \in [0,1]$. Applying Hölder's inequality, we obtain

$$\begin{aligned}
&|(\aleph \mathbf{r}_1)(t) - (\aleph \mathbf{r}_2)(t)| \\
&\quad = \left| \int_0^1 \mho_{\mathtt{i}_2}(\mathfrak{q},\mathfrak{p}) \left[\int_0^1 \mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{z}) \left[\mathtt{f}(\mathfrak{z},\mathbf{r}_1(\mathfrak{z})) - \mathtt{f}(\mathfrak{z},\mathbf{r}_2(\mathfrak{z})) \right] d\mathfrak{z} \right] d\mathfrak{p} \right| \\
&\quad \le \int_0^1 |\mho_{\mathtt{i}_2}(\mathfrak{p},\mathfrak{p})| \left[\int_0^1 \mho_{\mathtt{i}_1}(\mathfrak{z},\mathfrak{z}) \mathtt{M}_1 |\mathbf{r}_1(\mathfrak{z}) - \mathbf{r}_2(\mathfrak{z})| d\mathfrak{z} \right] d\mathfrak{p} \\
&\quad \le \mho_{\mathtt{i}_2}^* \mathtt{M}_1 \left[\int_0^1 |\mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}} d\mathfrak{p} \right]^{\frac{1}{\mathtt{q}}} \left[\int_0^1 |\mathbf{r}_1(\mathfrak{p}) - \mathbf{r}_2(\mathfrak{p})|^{\mathtt{p}} d\mathfrak{p} \right]^{\frac{1}{\mathtt{p}}} \\
&\quad \le \mho_{\mathtt{i}_2}^* \mathtt{M}_1 \left[\int_0^1 |\mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}} d\mathfrak{p} \right]^{\frac{1}{\mathtt{q}}} \rho(\mathbf{r}_1,\mathbf{r}_2).
\end{aligned} \tag{4.8}$$

Define

$$\alpha = \mho_{\mathtt{i}_2}^* \mathtt{M}_1 \left[\int_0^1 |\mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}} \, d\mathfrak{p} \right]^{\frac{1}{\mathtt{q}}}.$$

Then we have

$$\mathtt{d}(\aleph \mathbf{r}_1, \aleph \mathbf{r}_2) \le \alpha \rho(\mathbf{r}_1,\mathbf{r}_2), \tag{4.9}$$

for some $\alpha > 0$ and for all $\mathbf{r}_1, \mathbf{r}_2 \in \mathscr{X}$. Thus, inequality (4.5) from Theorem 4.1 is satisfied. Applying (4.4) to (4.9), we obtain

$$\mathtt{d}(\aleph \mathbf{r}_1, \aleph \mathbf{r}_2) \le \alpha \rho(\mathbf{r}_1,\mathbf{r}_2) \le \alpha \mathtt{d}(\mathbf{r}_1,\mathbf{r}_2).$$

Given any $\varepsilon > 0$, choosing $\delta = \varepsilon/\alpha$ ensures that $\mathtt{d}(\aleph \mathbf{r}_1, \aleph \mathbf{r}_2) < \varepsilon$ whenever $\mathtt{d}(\mathbf{r}_1,\mathbf{r}_2) < \delta$. Hence, $\aleph$ is continuous on $\mathscr{X}$ with respect to the metric $\mathtt{d}$.

Finally, to show that $\aleph$ is a contraction mapping on $\mathscr{X}$ with respect to ρ, we refer to (4.8). For each pair $\mathbf{r}_1, \mathbf{r}_2 \in \mathscr{X}$, we consider

$$\begin{aligned}
&\left[\int_0^1 |(\aleph \mathbf{r}_1)(t) - (\aleph \mathbf{r}_2)(t)|^{\mathtt{p}} dt \right]^{\frac{1}{\mathtt{p}}} \\
&\qquad \le \left[\int_0^1 \left| \mho_{\mathtt{i}_2}^* \mathtt{M}_1 \left[\int_0^1 |\mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}} d\mathfrak{p} \right]^{\frac{1}{\mathtt{q}}} \rho(\mathbf{r}_1,\mathbf{r}_2) \right|^{\mathtt{p}} dt \right]^{\frac{1}{\mathtt{p}}} \\
&\qquad \le \mho_{\mathtt{i}_2}^* \mathtt{M}_1 \left[\int_0^1 |\mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}} d\mathfrak{p} \right]^{\frac{1}{\mathtt{q}}} \rho(\mathbf{r}_1,\mathbf{r}_2).
\end{aligned}$$

That is

$$\rho(\aleph \mathbf{r}_1, \aleph \mathbf{r}_2) \le \mho_{\mathtt{i}_2}^* \mathtt{M}_1 \left[\int_0^1 |\mho_{\mathtt{i}_1}(\mathfrak{p},\mathfrak{p})|^{\mathtt{q}} d\mathfrak{p} \right]^{\frac{1}{\mathtt{q}}} \rho(\mathbf{r}_1,\mathbf{r}_2).$$

Given the condition (4.7), we obtain $\rho(\aleph \mathbf{r}_1, \aleph \mathbf{r}_2) \leq \beta \rho(\mathbf{r}_1, \mathbf{r}_2)$, where $\beta < 1$ and for all $\mathbf{r}_1, \mathbf{r}_2 \in \mathscr{X}$. According to Theorem 4.1, this implies that the operator $\aleph$ has a unique fixed point in $\mathscr{X}$. Consequently, the BVP (4.1) has a unique nontrivial solution. □

In the special case where $\mathtt{p} = 2$ and $\mathtt{q} = 2$, Theorem 4.2 simplifies to the following result.

Corollary 4.3. *Assume that conditions* (H_1) *and* (H_4) *are satisfied, and* $\mho^*_{\mathfrak{i}_2}\mathtt{M}_1\sqrt{\int_0^1 |\mho_{\mathfrak{i}_1}(\mathfrak{p},\mathfrak{p})|^2\, d\mathfrak{p}} < 1$. *Then BVP* (4.1) *has a unique nontrivial solution in* $\mathscr{X}$.

5. Examples

We present two illustrative examples to showcase the practical implications of our primary findings.

Example 4.1. Examine the fractional order BVP defined as

$$\left.\begin{array}{c} \mathsf{D}^{\frac{8}{5}}\left(\mathsf{D}^{\frac{3}{2}}\mathbf{r}_{\mathtt{t}}(\mathfrak{q})\right) + \sigma(\mathfrak{q})\hbar_{\mathtt{t}}(\mathbf{r}_{\mathtt{t}+1}(\mathfrak{q})) = 0, 0 < \mathfrak{q} < 1, \mathtt{t} = 1,2, \\ \mathbf{r}_3(\mathfrak{q}) = \mathbf{r}_1(\mathfrak{q}), 0 < \mathfrak{q} < 1, \end{array}\right\} \tag{5.1}$$

$$\left.\begin{array}{c} \mathbf{r}_{\mathtt{t}}(0) = 0, \mathbf{r}_{\mathtt{t}}(1) = \displaystyle\int_0^1 \Lambda(\mathfrak{z})\mathbf{r}_{\mathtt{t}}(\mathfrak{z})d\mathfrak{z}, \mathtt{t} = 1,2, \\ \mathsf{D}^{\frac{3}{2}}\mathbf{r}_{\mathtt{t}}(0) = 0, \mathsf{D}^{\frac{3}{2}}\mathbf{r}_{\mathtt{t}}(1) = \displaystyle\int_0^1 \Lambda(\mathfrak{z})\left(\mathsf{D}^{\frac{3}{2}}\mathbf{r}_{\mathtt{t}}(\mathfrak{z})\right)d\mathfrak{z}, \mathtt{t} = 1,2, \end{array}\right\} \tag{5.2}$$

where $\Lambda(\mathfrak{z}) = 1$, $\sigma(\mathfrak{q}) = \frac{1}{|\mathfrak{q}-1/3|}$. For $\mathtt{j} = 1,2$, consider

$$\hbar(\mathbf{r}) = \hbar_{\mathtt{j}}(\mathbf{r}) = \begin{cases} 11\times 10^{-4}, & \mathbf{r} \in (10^{-4}, +\infty), \\ \frac{280\times 10^{-(k+2)} - 11\times 10^{-4k}}{10^{-(4k+2)} - 10^{-4k}}(x - 10^{-4k}) + 11\times 10^{-4k}, & \\ & \mathbf{r} \in \left[10^{-(4k+2)}, 10^{-4k}\right], \\ 280\times 10^{-(4k+2)}, & \mathbf{r} \in \left(\frac{1}{5^{1/2}}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right), \\ \frac{280\times 10^{-(4k+2)} - 11\times 10^{-(4k+4)}}{\frac{1}{5^{1/2}}\times 10^{-(4k+2)} - 10^{-(4k+4)}}(x - 10^{-(4k+4)}) + 11\times 10^{-(4k+4)}, & \\ & \mathbf{r} \in \left(10^{-(4k+4)}, \frac{1}{5^{1/2}}\times 10^{-(4k+2)}\right], \\ 0, & \mathbf{r} = 0, \end{cases}$$

Let

$$\mathfrak{q}_{\mathtt{s}} = \frac{31}{64} - \sum_{r=1}^{\mathtt{s}} \frac{1}{4(r+1)^4},\ \eta_{\mathtt{s}} = \frac{1}{2}(\mathfrak{q}_{\mathtt{s}} + \mathfrak{q}_{\mathtt{s}+1}),\ \mathtt{s} = 1,2,3,\cdots.$$

Then

$$\eta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$\mathfrak{q}_{\mathtt{s}+1} < \eta_{\mathtt{s}} < \mathfrak{q}_{\mathtt{s}}, \eta_{\mathtt{s}} > \frac{1}{5}.$$

Therefore,

$$\eta_{\mathtt{s}}^{\mathfrak{i}_2-1} > \frac{1}{5^{1/2}}, \mathtt{s} = 1,2,3,\cdots.$$

It is easy to see that

$$\mathfrak{q}_1 = \frac{15}{32} < \frac{1}{2}, \ \mathfrak{q}_{\mathtt{s}} - \mathfrak{q}_{\mathtt{s}+1} = \frac{1}{4(j+2)^4}, \ \mathtt{s} = 1,2,3,\cdots.$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^4} = \frac{\pi^4}{90}$ and $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, it follows that

$$\mathfrak{q}^* = \lim_{r\to\infty} \mathfrak{q}_r = \frac{31}{64} - \sum_{\mathtt{t}=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5}.$$

Also, $\sigma^* = \frac{3}{2}$, $\mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p}) = \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p}) = \frac{4}{3}(1-\mathfrak{p})^{1/2}$. So,

$$\eta_1^{\mathfrak{i}_2+\mathfrak{i}_1-2}\sigma^* \int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})d\mathfrak{p}\Big[\int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p})d\mathfrak{p}\Big] \approx 0.003571565744,$$

$$\begin{aligned}
\mathtt{L} &= \max\left\{\left[\eta_1^{\mathfrak{i}_2+\mathfrak{i}_1-2}\sigma^* \int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})d\mathfrak{p}\Big[\int_{\eta_1}^{1-\eta_1} \mathtt{Q}^*_{\mathfrak{i}_1}(\mathfrak{p})d\mathfrak{p}\Big]\right]^{-1}, \ 1\right\} \\
&= \{279.9892461, 1\} = 279.9892461,
\end{aligned}$$

$$\|\mathtt{Q}^*_{\mathfrak{i}_1}\|_2 \approx 0.9428090414, \ \int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})d\mathfrak{p} \approx 0.9,$$

and

$$\left[\|\mathtt{Q}^*_{\mathfrak{i}_1}\|_{\mathtt{q}}\|\sigma\|_{\mathtt{p}} \int_0^1 \mathtt{Q}^*_{\mathfrak{i}_2}(\mathfrak{p})d\mathfrak{p}\right]^{-1} \approx 12.50034791 \text{ for } \mathtt{p} = \mathtt{q} = 2.$$

Take $\mathscr{O}_1 = 12$. By taking $\mathtt{B}_{\mathtt{s}} = 10^{-4k}$ and $\mathtt{A}_{\mathtt{s}} = 10^{-(4k+2)}$, we have

$$\begin{aligned}
\mathtt{B}_{\mathtt{s}+1} = 10^{-(4k+4)} &< \frac{1}{5^{1/2}} \times 10^{-(4k+2)} < \eta_{\mathtt{s}}^{\mathfrak{i}_2-1}\mathtt{A}_{\mathtt{s}} \\
&< \mathtt{A}_{\mathtt{s}} = 10^{-(4k+2)} < \mathtt{B}_{\mathtt{s}} = 10^{-4k},
\end{aligned}$$

$\mathtt{L}\mathtt{A}_{\mathtt{s}} = 279.9892461 \times 10^{-(4k+2)} < 12 \times 10^{-4k} = \mathscr{O}_1\mathtt{B}_{\mathtt{s}}$, $\mathtt{s} = 1,2,3,\cdots$. Furthermore, $\hbar_1$ and $\hbar_2$ meet the defined growth criteria:

$$\hbar_1(\mathbf{r}) = \hbar_2(\mathbf{r}) \leq \mathscr{O}_1\mathtt{B}_{\mathtt{s}} = 12 \times 10^{-4k}, \ \mathbf{r} \in \left[0, 10^{-4k}\right]$$

$$\hbar_1(\mathbf{r}) = \hbar_2(\mathbf{r}) \geq \mathtt{L}\mathtt{A}_{\mathtt{s}} = 279.9892461 \times 10^{-(4k+2)}, \ \mathbf{r} \in \left[\frac{1}{5^{1/2}} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

Since all the conditions of Theorem 3.4 are met, Theorem 3.4 guarantees that the BVP (5.1)–(5.2) possesses infinitely many positive solutions ${\mathbf{r}^{[k]}}_{\mathtt{s}=1}^{\infty}$. Each solution $\mathbf{r}^{[k]}$ satisfies the inequality $10^{-(4k+2)} \leq \|\mathbf{r}^{[k]}\| \leq 10^{-4k}$ for every $\mathtt{s} = 1,2,3,\ldots$.

The plots were generated using Python, specifically utilizing the `numpy`, `matplotlib`, and `scipy` packages to compute and visualize the solutions of (5.1)-(5.2). The behavior of the

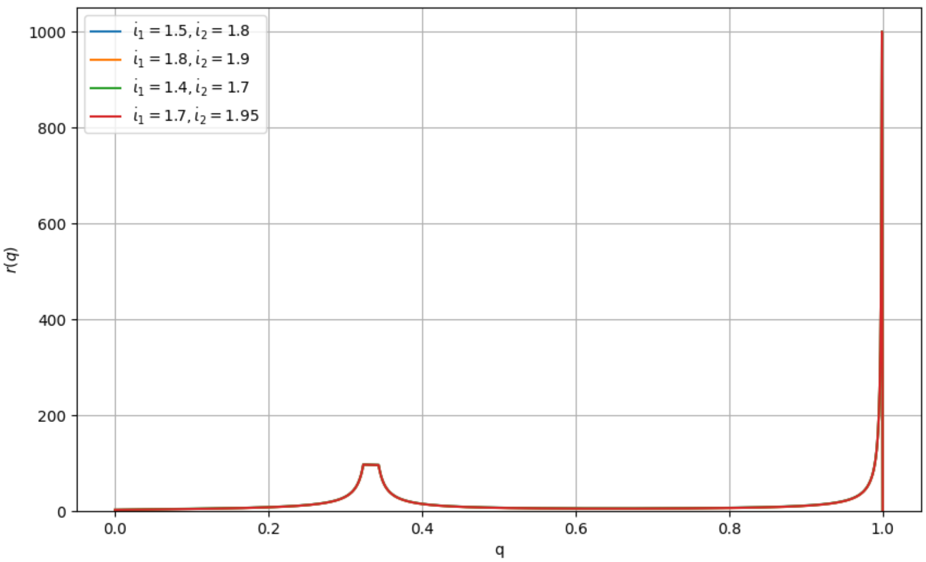

FIGURE 1. Solution curves for different fractional orders, showing how the solutions vary with changes in i_1 and i_2 within the range of 0 to 1000

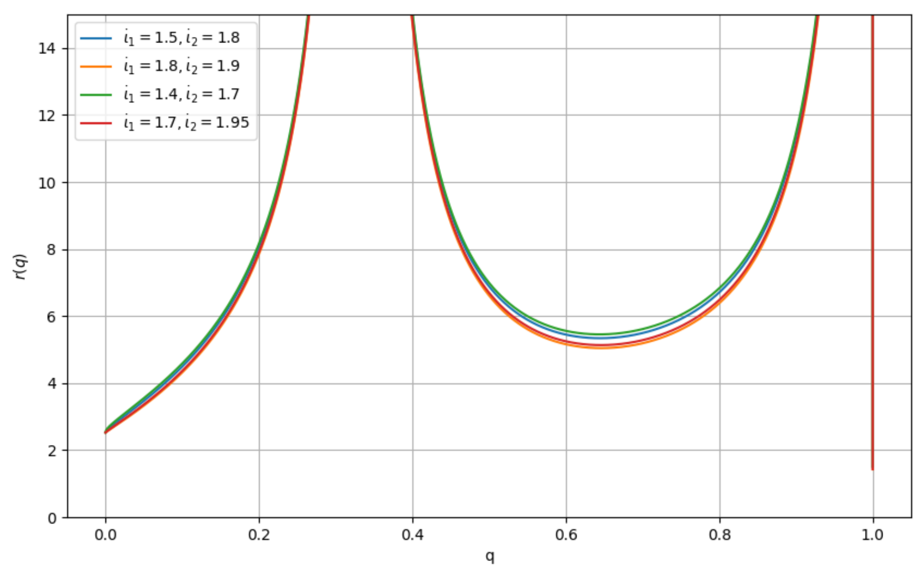

FIGURE 2. Solution curves for different fractional orders, showing how the solutions vary with changes in i_1 and i_2 within the range of 0 to 15

solution curves varies significantly depending on the fractional orders (i_1, i_2) used. Higher fractional orders result in smoother and more stable solutions, reflecting the influence of memory in fractional calculus, while lower orders introduce more oscillatory behavior. The $\hbar$ function effectively smooths the solutions, especially when $\mathbf{r}(\mathfrak{q})$ is small, ensuring that the boundary conditions are met and preventing large discontinuities. Near $\mathfrak{q} = 1/3$, the $\sigma(\mathfrak{q})$ function introduces noticeable peaks due to its singularity, but the solutions remain bounded due to the clamping mechanism applied to $\sigma(\mathfrak{q})$. Overall, the solutions start from zero at $\mathfrak{q} = 0$ and exhibit varying degrees of oscillations before converging near $\mathfrak{q} = 1$, with higher fractional orders producing more gradual changes and lower orders leading to more pronounced oscillations.

Example 4.2. We examine the fractional order BVP given below:

$$\mathsf{D}^{\frac{9}{5}}\left(\mathsf{D}^{\frac{3}{2}}\mathbf{r}(\mathfrak{q})\right) + \mathtt{f}(\mathfrak{q}, \mathbf{r}(\mathfrak{q})) = 0,\ 0 < \mathfrak{q} < 1, \tag{5.3}$$

$$\left.\begin{aligned} \mathbf{r}(0) = 0,\ \mathbf{r}(1) = \int_0^1 \Lambda(\mathfrak{z})\mathbf{r}(\mathfrak{z})d\mathfrak{z}, \\ \mathsf{D}^{\frac{3}{2}}\mathbf{r}(0) = 0,\ \mathsf{D}^{\frac{3}{2}}\mathbf{r}(1) = \int_0^1 \Lambda(\mathfrak{z})\left(\mathsf{D}^{\frac{3}{2}}(\mathbf{r}(\mathfrak{z}))\right)d\mathfrak{z}, \end{aligned}\right\} \tag{5.4}$$

where $\mathtt{f}(\mathfrak{q}, \mathbf{r}) = \mathfrak{q} + c\cos\mathbf{r}$, $\Lambda(\mathfrak{p}) = \mathfrak{p}$. Then

$$\mho^*_{\mathfrak{i}_2} = \int_0^1 \mho_{\mathfrak{i}_2}(\mathfrak{p}, \mathfrak{p})d\mathfrak{p} \approx 1.355165229$$

and

$$\int_0^1 |\mho_{\mathfrak{i}_1}(\mathfrak{p}, \mathfrak{p})|^2 d\mathfrak{p} \approx 3.223245968.$$

By Mean value theorem, there exists a $\theta \in (\mathbf{r}_1, \mathbf{r}_2)$ such that

$$|\mathtt{f}(\mathfrak{q}, \mathbf{r}_1) - \mathtt{f}(\mathfrak{q}, \mathbf{r}_2)| = |c||\sin\theta||\mathbf{r}_1 - \mathbf{r}_2| \leq |c||\mathbf{r}_1 - \mathbf{r}_2|.$$

So, we have, for $\mathtt{p} = \mathtt{q} = 2$,

$$\mho^*_{\mathfrak{i}_2}\mathtt{M}_1\sqrt{\left[\int_0^1 |\mho_{\mathfrak{i}_1}(\mathfrak{p}, \mathfrak{p})|^2 d\mathfrak{p}\right]} = |c| \times 2.432982437$$

By Theorem 4.3, there exists a unique solution for (5.3)-(5.4), for all values of c such that $|c| < 0.4110181746$.

Conclusion

In this study, we established the existence of infinite positive solutions for an iterative system of singular fractional BVPs. By employing the fixed point index theory within a cone in a Banach space, we demonstrated the conditions under which these solutions exist. Our results extend the existing body of research on fractional differential equations by addressing systems characterized by singularities and iterative boundary conditions. This work not only broadens the theoretical understanding of fractional boundary value problems but also provides a framework for future investigations into more complex systems involving non-local behaviors and hereditary properties.

Future Work

Future research could explore several avenues to build upon our findings. One potential direction is to investigate the uniqueness and stability of the positive solutions for different classes of singular fractional boundary value problems. Additionally, applying the established theoretical framework to practical scenarios in engineering, biology, and physics could yield valuable insights into real-world systems with fractional dynamics. Another area of interest is the numerical approximation of the solutions, which would facilitate the application of our theoretical results to computational models. Finally, extending the study to multi-dimensional fractional boundary value problems could open new pathways for understanding complex systems in higher dimensions.

Novelty and Applications

The novelty of our findings lies in the innovative use of fixed point index theory in cone for the iterative system of singular fractional boundary value problems, which, to our knowledge, has not been previously explored in this context. This approach allows for the handling of singularities within the system, providing a robust method for establishing the existence of solutions. The applications of our work are diverse and impactful, potentially influencing fields such as materials science, where understanding the behavior of materials with memory and hereditary properties is crucial, and in control theory, where fractional dynamics play a significant role. Our results offer a new perspective and tools for researchers and practitioners dealing with complex systems governed by fractional differential equations.

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