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MANN VISCOSITY TSENG TYPE METHOD FOR INCLUSION AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce a modification of the forward-backward splitting algorithm with different step-size rules for a monotone inclusion problem and a fixed point problem of a nonexpansive mapping. Our algorithm is based on the Mann viscosity and Tseng type methods. Under some assumptions, such as Lipschitz continuous and monotone, we obtain a strong convergence theorem of common solutions in Hilbert spaces. Finally, we also apply our theorem to some problems and provide numerical examples to support the effectiveness of our algorithm.

Keywords. Forward-backward splitting method; Monotone inclusion; Nonexpansive mapping; Tseng method.

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1. INTRODUCTION

Let H be a Hilbert space. Recall that a mapping S is said to be nonexpansive iff, for all $x,y \in H$, $\|Sx-Sy\| \leq \|x-y\|$. The set of fixed points of mapping S is denoted by $F(S)$, that is $F(S)=\{x\in H : Sx=x\}$, in this paper. It is known that the fixed point set of nonexpansive mappings are convex and closed. Let $B : H \to 2^H$ be a set-valued operator. The variational inclusion problem (VIP), which was considered by Martinet [10], is to find a point $x^* \in H$ such that $0 \in Bx^*$. This problem is now considered as a unified framework for problems in various fields, such as, economics, network, transportation, and so on. To solve this problem, the following proximal point method was investigated, for a given $x_1 \in H$, $x_{n+1} = J^B_{\lambda_n} x_n$, $n \geq 1$, where $\{\lambda_n\} \subset (0,\infty)$ and $J^B_{\lambda_n} = (I+\lambda_n B)^{-1}$ is the resolvent of operator B, which was assumed to be maximally monotone; see [1, 13, 14, 21] for some related works.

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Furthermore, if the considered set-valued operator can be split, one has the problem of finding x^* such that

$$0 \in Ax^* + Bx^*, \tag{1.1}$$

where $A : H \to H$ is a single-valued operator and $B : H \to 2^H$ is a set-valued operator. We refer [2, 7, 11, 18, 22] for some efficient solution methods for the split inclusion problem. Note that several popular problems, such as variational inequality problems, saddle problems, equilibrium problems, all can be remodeled as the split inclusion problem. Numerous efficient methods were introduced and investigated for these problem, see, e.g., [6, 8, 16, 19, 20] and the references therein.

The forward-backward splitting method is one of popular iterative methods used to solve problem (1.1). It defines a sequence $\{x_n\}$ via, for any $x_1 \in H$, $x_{n+1} = J^B_{\lambda_n}(I - \lambda_n A)x_n$, where $\{\lambda_n\}$ is a sequence of positive real numbers and A and B are maximal monotone operators; see, e.g., [3, 4, 17, 23, 29] and the references therein.

Furthermore, Tseng [23], in 2000, proposed a modified forward-backward splitting method to solve problem (1.1), which is known as Tseng's splitting algorithm, for any $x_1 \in H$,

$$\begin{aligned} y_n &= J^B_{\lambda_n}(I - \lambda_n A)x_n, \\ x_{n+1} &= P_C\big(y_n - \lambda_n(Ay_n - Ax_n)\big), \qquad \forall n \in \mathbb{N}, \end{aligned}$$

where P_C denote the metric (nearest point) projection onto C, λ_n is chosen to be the largest $\lambda \in \{\delta, \delta l, \delta l^2, \dots\}$ satisfying $\lambda\|Ay_n - Ax_n\| \le \mu\|y_n - x_n\|$ when $\delta > 0$ and $l, \mu \in (0,1)$. Tseng proved that the iterative sequences converges weakly to a zero of $A + B$.

As a convex feasibility problem, the problem of finding a point $x^* \in H$ such that

$$x^* \in F(S) \cap (A+B)^{-1}0 =: \Omega, \tag{1.2}$$

where $F(S)$ denotes the fixed point set of S, which is nonexpansive, is now under the spotlight.

In 2019, Suparatulatorn and Khemphet [20] studied problem (1.2) and introduced the following modified Tseng type algorithm, for any $x_1 \in H$,

$$\begin{aligned} z_n &= J^B_{\lambda_n}(I - \lambda_n A)x_n, \\ y_n &= z_n - \lambda_n(Az_n - Ax_n), \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n - \beta_n)x_n + \beta_n S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying some suitable conditions and $\{\lambda_n\}$ is defined by

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\theta\|x_n - z_n\|}{\|Ax_n - Az_n\|}, \lambda_n\right\}, & \text{if } Ax_n - Az_n \neq 0; \\ \lambda_n, & \text{otherwise}, \end{cases}$$

where $\lambda_1 > 0$ and $\theta \in (0,1)$. They proved that $\{x_n\}$ converges strongly to a solution point of problem (1.2).

On the other hand, Tan et al. [24], in 2020, studied the modified Mann viscosity method with the inertial algorithm to solve fixed point problems. For arbitrary x_0, $x_1 \in H$, their algorithm reads

$$\begin{aligned} z_n &= x_n + \mu_n(x_n - x_{n-1}), \\ y_n &= \beta_n z_n + (1-\beta_n) S z_n, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) y_n, \qquad \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, and $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ and some suitable conditions. They demonstrated that $\{x_n\}$ converges strongly to $z = P_{F(S)} f(z)$.

In this paper, motivated and inspired by the results mentioned above, we further investigate problem (1.2). We present an iteratie algorithms to find a common solution to problem (1.2) and provide some suitable conditions to guarantee that the constructed sequence $\{x_n\}$ converges strongly to a point in Ω.

2. Preliminaries

In this section, we present some auxiliary tools, lemmas and definitions, to prove our main theorem. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H. The strong convergence and weak convergence of $\{x_n\}$ to x are written by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Recall that a mapping S is said to be Lipschitz if there exists $L \geq 0$ such that, for all $x, y \in H$. $\|Sx - Sy\| \leq L\|x - y\|$. The number L, associated with S, is called a Lipschitz constant. Moreover, if $L \in [0,1)$, we say that S is a contraction. If $L = 1$, we say that S is nonexpansive. S is said to be firmly nonexpansive if, for all $x, y \in H$, $\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy \rangle$. S is said to be β-inverse strongly monotone (β-ism) if, for a positive real number β and for all $x, y \in H$, $\langle Sx - Sy, x - y \rangle \geq \beta \|Sx - Sy\|^2, .$

Now, we collect some important properties for our proof.

Lemma 2.1. [30] *The following statements are true:*
(i) If $A : H \to H$ is β-ism, then A is $\frac{1}{\beta}$-Lipschitz continous and monotone mapping.
(ii) If $A : H \to H$ is β-ism and $\lambda \in (0, \beta]$, then $S := I - \lambda A$ is firmly nonexpansive.

Let B be a set-valued mapping on space H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. Recall that B is said to be monotone if, $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone mapping B is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. To a maximal monotone operator B and a constant $\lambda > 0$, its resolvent J_λ^B is defined by $J_\lambda^B := (I + \lambda B)^{-1} : H \to D(B)$. Notice that J_λ^B is a single-valued and firmly nonexpansive mapping, and $F(J_\lambda^B) = B^{-1}0 = \{x \in H : 0 \in Bx\}$ for all $\lambda > 0$; see [25, 26]. Let $A : C \to H$ be a single-valued operator. If $B : H \to 2^H$ is a maximal monotone operator, then $F\big(J_\lambda^B (I - \lambda A)\big) = (A + B)^{-1} 0$.

The following fundamental results are needed in our proof.

For each $x, y, z \in H$, the following facts are valid for inner product spaces

$$\|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2$$

and

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2,$$

for any $\alpha, \beta, \gamma \in [0,1]$ such that $\alpha + \beta + \gamma = 1$; see [15, 26].

Let C be a convex and closed set in H. For every point $x \in H$, there exists a unique nearest point in C, which is denoted by $P_C x$ here, such that, for all $y \in C$, $\|x - P_C x\| \leq \|x - y\|$. P_C is called a metric projection of H onto C; see [27]. The following property of P_C is known $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$ and $y \in C$.

We also use the following lemmas to prove the main theorems.

Lemma 2.2. [28] *Let C be a convex and closed set in a H, and let $S : C \to C$ be a nonexpansive mapping. Then, $U := I - S$ is demiclosed, that is, $x_n \rightharpoonup x_0$ and $Ux_n \to y_0$ imply $Ux_0 = y_0$.*

Lemma 2.3. [9, 31] *Let $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n + c_n$ for all $n \in \mathbb{N}$, where $\{a_n\}$ is a non-negative real sequence, $\{\alpha_n\}$, $\{b_n\}$, and $\{c_n\}$ are real sequences with (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{\alpha_n\} \subset [0,1]$; (ii) $c_n \geq 0$ and $\sum_{n=1}^{\infty} c_n < \infty$; (iii) $\limsup_{n\to\infty} b_n \leq 0$. Then, $a_n \to 0$ as $n \to \infty$.*

3. Main Results

In this section, we present the following assumptions and our algorithm to solve the monotone inclusion problem and fixed point of nonexpansive mapping, problem (1.2).

(A1) $A : H \to H$ is a κ-Lipschitz continuous and monotone operator;

(A2) $B : H \to 2^H$ is a maximal monotone operator;

(A3) $S : H \to H$ is a nonexpansive mapping;

(A4) $f : H \to H$ is a contraction mapping with coefficient $\eta \in (0,1)$.

Algorithm 3.1. *Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ be sequences in $(0,1)$ with $\alpha_n + \delta_n + \theta_n = 1$ and let the initial $x_1 \in H$ be arbitrary. Define*

$$\begin{aligned} z_n &= J^B_{\lambda_n}(I - \lambda_n A)x_n, \\ w_n &= z_n - \lambda_n(Az_n - Ax_n), \\ y_n &= \beta_n x_n + (1 - \beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n w_n + \theta_n S y_n, \qquad \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\lambda_n\}$ is defined by

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\sigma\|x_n - z_n\|}{\|Ax_n - Az_n\|}, \lambda_n \right\}, & \text{if } Ax_n - Az_n \neq 0; \\ \lambda_n, & \text{otherwise}, \end{cases} \tag{3.1}$$

where $\lambda_1 > 0$ and $\sigma \in (0,1)$.

By using the above assumptions, we have the following Lemmas and the strong convergence theorem (Theorem 3.4).

Lemma 3.2. *The sequence $\{\lambda_n\}$ generated by (3.1) is a non-increasing sequence and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{ \lambda_1, \frac{\sigma}{\kappa} \right\}.$$

Proof. From the definition of $\{\lambda_n\}$ in (3.1), we see that the sequence $\{\lambda_n\}$ is non-increasing. Furthermore, if $Ax_n - Az_n \neq 0$, then

$$\frac{\sigma\|x_n - z_n\|}{\|Ax_n - Az_n\|} \geq \frac{\sigma}{\kappa}.$$

Therefore, it is obvious that $\{\lambda_n\}$ has a lower bound $\min\{\lambda_1, \frac{\sigma}{\kappa}\}$. This completes the proof. $\square$

Lemma 3.3. *Let $\{w_n\}$ be any sequence generated by Algorithm 3.1. Suppose that Assumptions (A1)-(A4) hold and Ω is nonempty. Then,*

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 - \left(1 - \sigma^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n - z_n\|^2, \tag{3.2}$$

for all $p \in \Omega$ and

$$\|w_n - z_n\| \leq \sigma \frac{\lambda_n}{\lambda_{n+1}}\|x_n - z_n\|. \tag{3.3}$$

Proof. If $Ax_n \neq Az_n$, then we find from (3.1) that

$$\lambda_{n+1} = \min\left\{\frac{\sigma\|x_n - z_n\|}{\|Ax_n - Az_n\|}, \lambda_n\right\} \leq \frac{\sigma\|x_n - z_n\|}{\|Ax_n - Az_n\|},$$

for each $n \in \mathbb{N}$. It follows that

$$\|Ax_n - Az_n\| \leq \frac{\sigma\|x_n - z_n\|}{\lambda_{n+1}}. \tag{3.4}$$

Otherwise, we also have (3.4) holds. Observe that

$$\begin{aligned}\|w_n - p\|^2 &= \|z_n - p\|^2 + \lambda_n^2\|Az_n - Ax_n\|^2 - 2\lambda_n\langle z_n - p, Az_n - Ax_n\rangle \\ &\leq \|x_n - p\|^2 - \|x_n - z_n\|^2 - 2\big\langle x_n - z_n - \lambda_n(Ax_n - Az_n), z_n - p\big\rangle + \lambda_n^2\|Az_n - Ax_n\|^2,\end{aligned}$$

for each $n \in \mathbb{N}$. By using (3.4), we obtain

$$\begin{aligned}\|w_n - p\|^2 = \|x_n - p\|^2 - \left(1 - \sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n - z_n\|^2 \\ - 2\big\langle x_n - z_n - \lambda_n(Ax_n - Az_n), z_n - p\big\rangle.\end{aligned} \tag{3.5}$$

Now, we establish that $\big\langle x_n - z_n - \lambda_n(Ax_n - Az_n), z_n - p\big\rangle \geq 0$. From Algorithm 3.1, we have $z_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n$, which yields that

$$(I - \lambda_n A)x_n \in (I + \lambda_n B)z_n.$$

Since B is a maximal monotone operator, there exists $v_n \in Bz_n$ such that $(I - \lambda_n A)x_n = z_n + \lambda_n v_n$. This implies that

$$v_n = \frac{1}{\lambda_n}(x_n - z_n - \lambda_n Ax_n). \tag{3.6}$$

In addition, we have $0 \in (A+B)p$ and $Az_n + v_n \in (A+B)z_n$. By $A+B$ is maximal, we see that

$$\langle Az_n + v_n, z_n - p\rangle \geq 0. \tag{3.7}$$

Substituting (3.6) into (3.7), we obtain $\frac{1}{\lambda_n}\langle x_n - z_n - \lambda_n Ax_n + \lambda_n Az_n, z_n - p\rangle \geq 0$. It follows that

$$\big\langle x_n - z_n - \lambda_n(Ax_n - Az_n), z_n - p\big\rangle \geq 0.$$

From (3.5), we obtain

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 - \left(1 - \sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n - z_n\|^2.$$

Moreover, by using the definition of w_n and (3.4), we have

$$\|w_n - z_n\| \leq \lambda_n \|Az_n - Ax_n\| \leq \sigma \frac{\lambda_n}{\lambda_{n+1}} \|x_n - z_n\|,$$

for each $n \in \mathbb{N}$. This completes the proof. ☐

Theorem 3.4. *Let $\{x_n\}$ be generated by Algorithm 3.1. Suppose that assumptions (A1)-(A4) hold, $\Omega \neq \emptyset$, and the following control conditions are satisfied:*
(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) There exist a positive real number a with $0 < a \leq \delta_n$ and $0 < a \leq \theta_n$, for each $n \in \mathbb{N}$.
Then, $\{x_n\}$ converges strongly to $p \in \Omega$, where $p = P_\Omega f(p)$.

Proof. Firstly, we prove that $\{x_n\}$ is a bounded sequence. Let $z \in \Omega$. Then, $z \in F(S)$ and $z \in (A+B)^{-1}0$, which implies that $J^B_{\lambda_n}(I - \lambda_n A)z = z$. Observe that there exists $n_0 \in \mathbb{N}$ such that

$$1 - \sigma^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \tag{3.8}$$

for each $n \geq n_0$. Thus, from (3.2), we obtain

$$\|w_n - z\| \leq \|x_n - z\|, \tag{3.9}$$

for each $n \geq n_0$. By the definition of y_n and (3.9), we have

$$\|y_n - z\| \leq \beta_n \|x_n - z\| + (1 - \beta_n)\|w_n - z\| \leq \|x_n - z\|,$$

for each $n \geq n_0$. Now, we use the definition of x_{n+1} and (3.9) to derive

$$\begin{aligned}
\|x_{n+1} - z\| &\leq \alpha_n \big\|f(x_n) - z\big\| + \delta_n \|w_n - z\| + \theta_n \|Sy_n - z\| \\
&\leq \alpha_n \big\|f(x_n) - f(z)\big\| + \alpha_n \big\|f(z) - z\big\| + \delta_n \|x_n - z\| + \theta_n \|y_n - z\| \\
&\leq \alpha_n \eta \|x_n - z\| + \alpha_n \big\|f(z) - z\big\| + \delta_n \|x_n - z\| + \theta_n \|x_n - z\| \\
&\leq \big(\alpha_n \eta + \delta_n + \theta_n\big) \|x_n - z\| + \alpha_n \big\|f(z) - z\big\| \\
&= \big(1 - \alpha_n(1-\eta)\big) \|x_n - z\| + \alpha_n (1-\eta) \frac{\big\|f(z) - z\big\|}{1-\eta} \\
&\leq \max\{\|x_n - z\|, \frac{\big\|f(z) - z\big\|}{1-\eta}\} \\
&\dots \\
&\leq \max\{\|x_{n_0} - z\|, \frac{\big\|f(z) - z\big\|}{1-\eta}\},
\end{aligned}$$

for each $n \geq n_0$. Thus, we obtain $\big\{\|x_n - z\|\big\}$ is a bounded sequence, which implies that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$, $\{y_n\}$, and $\{f(x_n)\}$ are also bounded

Next, we note that $P_\Omega f(\cdot)$ is a contraction mapping. Let p be a unique fixed point of $P_\Omega f(\cdot)$, that is, $p = P_\Omega f(p)$. We consider

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \frac{\alpha_n}{2}\left(\left\|f(x_n)-f(p)\right\|^2+\left\|x_{n+1}-p\right\|^2\right)\\
&\quad+\frac{\delta_n}{2}\left(\left\|w_n-p\right\|^2+\left\|x_{n+1}-p\right\|^2\right)+\frac{\theta_n}{2}\left(\left\|Sy_n-p\right\|^2+\left\|x_{n+1}-p\right\|^2\right)\\
&\quad+\alpha_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq\left(\frac{\alpha_n\eta^2+\delta_n+\theta_n}{2}\right)\|x_n-p\|^2+\frac{1}{2}\|x_{n+1}-p\|^2+\alpha_n\langle f(p)-p,x_{n+1}-p\rangle,
\end{aligned}$$

for each $n \in \mathbb{N}$. This gives

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \left(1-\alpha_n(1-\eta^2)\right)\|x_n-p\|^2\\
&\quad+\alpha_n(1-\eta^2)\left(\frac{2}{1-\eta^2}\langle f(p)-p,x_{n+1}-p\rangle\right).
\end{aligned}$$

Now, we consider the following two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n-p\|\}$ is monotonically non-increasing for all $n \geq n_0$. Since $\{\|x_n-p\|\}$ is bounded, it is a convergent sequence. By using (3.2), (3.9) and the definition of y_n, we have

$$\begin{aligned}
\|y_n-p\|^2 &\leq \beta_n^2\|x_n-p\|^2+2\beta_n(1-\beta_n)\langle x_n-p,w_n-p\rangle+(1-\beta_n)^2\|w_n-p\|^2\\
&\leq \beta_n^2\|x_n-p\|^2+2\beta_n(1-\beta_n)\|x_n-p\|\|w_n-p\|+(1-\beta_n)^2\|w_n-p\|^2\\
&\leq (2\beta_n-\beta_n^2)\|x_n-p\|^2+(1-\beta_n)^2\left(\|x_n-p\|^2-\left(1-\sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n-z_n\|^2\right)\\
&\leq \|x_n-p\|^2-\left(1-\sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n-z_n\|^2,
\end{aligned}$$

for each $n \geq n_0$. Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|w_n-p\|^2+\theta_n\|Sy_n-p\|^2\\
&\quad-\alpha_n\delta_n\left\|f(x_n)-w_n\right\|^2-\alpha_n\theta_n\left\|f(x_n)-Sy_n\right\|^2-\delta_n\theta_n\|w_n-Sy_n\|^2\\
&\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2+\theta_n\|y_n-p\|^2\\
&\leq \alpha_n\left\|f(x_n)-p\right\|^2+\delta_n\|x_n-p\|^2\\
&\quad+\theta_n\left(\|x_n-p\|^2-\left(1-\sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n-z_n\|^2\right)\\
&\leq \alpha_n\left\|f(x_n)-p\right\|^2+\|x_n-p\|^2-\theta_n\left(1-\sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n-z_n\|^2,
\end{aligned}$$

for each $n \geq n_0$. Then,

$$\theta_n\left(1-\sigma^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|x_n-z_n\|^2 \leq \alpha_n\left\|f(x_n)-p\right\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2. \tag{3.10}$$

By using (3.8) and conditions (C1) and (C2), we obtain $\lim_{n\to\infty}\|x_n - z_n\| = 0$. From (3.3), it follows that $\lim_{n\to\infty}\|w_n - z_n\| = 0$. Moreover, from $\|w_n - x_n\| \le \|w_n - z_n\| + \|z_n - x_n\|$, we obtain $\lim_{n\to\infty}\|w_n - x_n\| = 0$. Observe that$\|y_n - x_n\| \le (1-\beta_n)\|w_n - x_n\|$. It follows that $\lim_{n\to\infty}\|y_n - x_n\| = 0$. From

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \alpha_n\big\|f(x_n) - p\big\|^2 + \delta_n\|w_n - p\|^2 + \theta_n\|Sy_n - p\|^2 \\ &\quad -\alpha_n\delta_n\|f(x_n) - w_n\|^2 - \alpha_n\theta_n\|f(x_n) - Sy_n\|^2 - \delta_n\theta_n\|w_n - Sy_n\|^2,\end{aligned}$$

we have

$$\begin{aligned}\delta_n\theta_n\|x_n - Sy_n\|^2 &\le \alpha_n\big\|f(x_n) - p\big\|^2 + \delta_n\|x_n - p\|^2 + \theta_n\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\big\|f(x_n) - p\big\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2.\end{aligned}$$

Using conditions (C1) and (C2), we obtain $\lim_{n\to\infty}\|x_n - Sy_n\| = 0$.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Consider

$$\|x_{n+1} - x_n\| \le \alpha_n\big\|f(x_n) - x_n\big\| + \delta_n\|w_n - x_n\| + \theta_n\|Sy_n - x_n\|,$$

for each $n \ge n_0$. By using condition (C1), we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $\{x_n\}$ is bounded on H, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to $x^* \in H$. Next, we show that $x^* \in \Omega$.

To show $x^* \in (A+B)^{-1}0$, we consider

$$\begin{aligned}\big\|x^* - J^B_{\lambda_n}(I-\lambda_n A)x^*\big\|^2 &\le \big\langle x^* - J^B_{\lambda_n}(I-\lambda_n A)x^*, x^* - x_{n_j}\big\rangle \\ &\quad + \big\langle x^* - J^B_{\lambda_n}(I-\lambda_n A)x^*, x_{n_j} - J^B_{\lambda_n}(I-\lambda_n A)x_{n_j}\big\rangle \\ &\quad + \big\langle x^* - J^B_{\lambda_n}(I-\lambda_n A)x^*, J^B_{\lambda_n}(I-\lambda_n A)x_{n_j} - J^B_{\lambda_n}(I-\lambda_n A)x^*\big\rangle,\end{aligned}$$

for each $j \in \mathbb{N}$. Observe that $\lim_{n\to\infty}\|x_n - J^B_{\lambda_n}(I-\lambda_n A)x_n\| = 0$. It follows that

$$\lim_{j\to\infty}\big\|x^* - J^B_{\lambda_n}(I-\lambda_n A)x^*\big\| = 0.$$

Thus, we have $x^* = J^B_{\lambda_n}(I-\lambda_n A)x^*$ and hence $x^* \in (A+B)^{-1}0$.

To show that $x^* \in F(S)$, we consider

$$\begin{aligned}\|x_{n+1} - Sy_n\| &\le \alpha_n\|f(x_n) - Sy_n\| + \delta_n\|w_n - Sy_n\| + \theta_n\|Sy_n - Sy_n\| \\ &\le \alpha_n\|f(x_n) - Sy_n\| + \delta_n\Big(\|w_n - x_n\| + \|x_n - Sy_n\|\Big),\end{aligned}$$

for each $n \ge n_0$. By using condition (C1), we have $\lim_{n\to\infty}\|x_{n+1} - Sy_n\| = 0$. By the following relation

$$\big\|y_n - Sy_n\big\| \le \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - Sy_n\|,$$

for each $n \ge n_0$. It follows that $\lim_{n\to\infty}\big\|y_n - Sy_n\big\| = 0$. Since $y_{n_j} \rightharpoonup x^*$, for each $j \in \mathbb{N}$, we obtain from Lemma 2.2 that $x^* \in F(S)$. From the above results, we have that $x^* \in \Omega$.

Finally, we prove that $\{x_n\}$ converges strongly to $p = P_\Omega f(p)$. Now, we know that $\{x_n\}$ is bounded, and we have that $\|x_{n+1} - x_n\| \to 0$, as $n \to \infty$. With loss of generality, we may assume that a subsequence $\{x_{n_j+1}\}$ of $\{x_{n+1}\}$ converges weakly to $x^* \in H$. Thus, we have

$$\limsup_{n\to\infty}\frac{2}{1-\eta^2}\big\langle f(p) - p, x_{n+1} - p\big\rangle = \frac{2}{1-\eta^2}\big\langle f(p) - p, x^* - p\big\rangle \le 0.$$

By using Lemma 2.3, we can conclude that $\|x_n - p\| \to 0$, as $n \to \infty$. Therefore $x_n \to p$, as $n \to \infty$.

Case 2. Suppose that $\{\|x_n - p\|\}$ is not monotonically decreasing sequence for all $n \geq n_0$. Set $\Gamma_n = \|x_n - p\|$ for all $n \in \mathbb{N}$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \quad \Gamma_k \leq \Gamma_{k+1}\},$$

for all $n \geq n_0$. Then, we have that $\{\tau(n)\}$ is a nondecreasing sequence, with $\tau(n) \to \infty$ as $n \to \infty$ and $0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ for all $n \geq n_0$, see [12]. Now, it follows that $\|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \leq 0$, for each $n \geq n_0$. Thus, from (3.10), we obtain

$$\begin{aligned}\theta_{\tau(n)}\left(1 - \sigma^2 \frac{\lambda_{\tau(n)}^2}{\lambda_{\tau(n)+1}^2}\right)\|x_{\tau(n)} - z_{\tau(n)}\|^2 &\leq \alpha_{\tau(n)}\|f(x_{\tau(n)}) - p\|^2 + \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \\ &\leq \alpha_{\tau(n)}\|f(x_{\tau(n)}) - p\|^2,\end{aligned}$$

for each $n \geq n_0$. Similar as Case 1, we also have

$$\lim_{n\to\infty}\|x_{\tau(n)} - z_{\tau(n)}\| = \lim_{n\to\infty}\|w_{\tau(n)} - z_{\tau(n)}\| = \lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$$

and

$$\limsup_{n\to\infty}\frac{2}{1-\eta^2}\langle f(p) - p, x_{\tau(n)+1} - p\rangle \leq 0.$$

Since $\{x_{\tau(n)}\}$ is bounded, we can find a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to $x^* \in F(S) \cap (A+B)^{-1}0$. Note that

$$\begin{aligned}\|x_{\tau(n)+1} - p\|^2 &\leq (1 - \alpha_{\tau(n)}(1-\eta^2))\|x_{\tau(n)} - p\|^2 \\ &\quad + \alpha_{\tau(n)}(1-\eta^2)\left(\frac{2}{1-\eta^2}\langle f(p) - p, x_{\tau(n)+1} - p\rangle\right),\end{aligned}$$

for each $n \geq n_0$. Consequently, we have

$$\begin{aligned}\alpha_{\tau(n)}(1-\eta^2)\|x_{\tau(n)} - p\|^2 &\leq \|x_{\tau(n)} - p\|^2 - \|x_{\tau(n)+1} - p\|^2 \\ &\quad + \alpha_{\tau(n)}(1-\eta^2)\left(\frac{2}{1-\eta^2}\langle f(p) - p, x_{\tau(n)+1} - p\rangle\right), \\ &\leq \alpha_{\tau(n)}(1-\eta^2)\left(\frac{2}{1-\eta^2}\langle f(p) - p, x_{\tau(n)+1} - p\rangle\right). \quad (3.11)\end{aligned}$$

Since $\alpha_{\tau(n)}(1-\eta^2) > 0$, it follows from (3.11) that $\limsup_{n\to\infty}\|x_{\tau(n)} - p\|^2 \leq 0$. which implies that $\lim_{n\to\infty}\|x_{\tau(n)} - p\|^2 = 0$, and also $\lim_{n\to\infty}\|x_{\tau(n)} - p\| = 0$. Moreover, from $\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$, we have

$$\|x_{\tau(n)+1} - p\| \leq \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - p\| \to 0, \quad (3.12)$$

as $n \to \infty$. In addition, if $\tau(n) < n$, we also have $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, because $\Gamma_{\tau(n)+1} \geq \Gamma_j$ for $\tau(n) + 1 \leq j \leq n$. Consequently, we obtain

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1},$$

for each $n \geq n_0$. By using (3.12), we have $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, we can conclude that $\{x_n\}$ converges strongly to p. This completes the proof. $\square$

4. Applications

1. Variational inequality problem. Recall that the normal cone to C at $u \in C$ is defined as $N_C(u) = \{z \in H : \langle z, y-u\rangle \leq 0, \forall y \in C\}$. We note that N_C is a maximal monotone operator. In the case $B := N_C : H \to 2^H$, we can verify that problem (1.1) is reduced to the problem of finding $x^* \in C$ such that $\langle Ax^*, x-x^*\rangle \geq 0$ for all $x \in C$. We denote its solution set by $VIP(C,A)$. Note that $J_\lambda^B =: P_C$. By the above setting, problem (1.2) is reduced to a problem of finding $x^* \in F(S) \cap VIP(C,A) =: \Omega_{A,S}$. By applying Theorem 3.4, we obtain the following result.

Algorithm 4.1. *Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ be sequences in $(0,1)$ with $\alpha_n+\delta_n+\theta_n = 1$ and the initial $x_1 \in H$ be arbitrary. Define*

$$\begin{aligned} z_n &= P_C(I-\lambda_n A)x_n, \\ w_n &= z_n - \lambda_n(Az_n - Ax_n), \\ y_n &= \beta_n x_n + (1-\beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n w_n + \theta_n S y_n, \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\lambda_n\}$ is defined by

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\sigma\|x_n - z_n\|}{\|Ax_n - Az_n\|}, \lambda_n\right\}, & \text{if } Ax_n - Az_n \neq 0; \\ \lambda_n, & \text{otherwise}, \end{cases}$$

when $\lambda_1 > 0$ and $\sigma \in (0,1)$.

Theorem 4.2. *Let $\{x_n\}$ be generated by Algorithm 4.1. Suppose that assumptions (A1), (A3), and (A4) hold, $\Omega_{A,S} \neq \emptyset$ and the following control conditions are satisfied:*
(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;
(C2) There exist a positive real number a with $0 < a \leq \delta_n$ and $0 < a \leq \theta_n$, for each $n \in \mathbb{N}$.
Then, $\{x_n\}$ converges strongly to $p \in \Omega_{A,S}$, where $p = P_{\Omega_{A,S}} f(p)$.

2. Convex minimization problem
We consider a convex function $g : H \to R$, which is also Fréchet differentiable. Let C be a given convex and closed subset of H. By setting $A := \nabla g$ (the gradient of g) and $B := N_C$, problem (1.1) is equivalent to finding a point $x^* \in C$ such that $\langle \nabla g(x^*), x-x^*\rangle \geq 0$ for all x in C. Note that this problem is equivalent to the following minimization problem: find $x^* \in C$ such that $x^* \in \arg\min_{x\in C} g(x)$. Thus, in this case, the problem (1.2) is reduced to a problem of finding $x^* \in F(S) \cap \arg\min_{x\in C} g(x) =: \Omega_{g,S}$. Then, we obtain the following result.

Algorithm 4.3. *Let $\{\alpha_n\}$, $\{\delta_n\}$, $\{\theta_n\}$, and $\{\beta_n\}$ be sequences in $(0,1)$ with $\alpha_n+\delta_n+\theta_n = 1$ and the initial $x_1 \in H$ be arbitrary. Define*

$$\begin{aligned} z_n &= P_C(I-\lambda_n \nabla g)x_n, \\ w_n &= z_n - \lambda_n(\nabla g z_n - \nabla g x_n), \\ y_n &= \beta_n x_n + (1-\beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n w_n + \theta_n S y_n, \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\lambda_n\}$ *is defined by*

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\sigma\|x_n - z_n\|}{\|\nabla g x_n - \nabla g z_n\|}, \lambda_n\right\}, & \text{if } \nabla g x_n - \nabla g z_n \neq 0; \\ \lambda_n, & \text{otherwise}, \end{cases}$$

when $\lambda_1 > 0$ *and* $\sigma \in (0,1)$.

Theorem 4.4. *Let* $g : H \to \mathbb{R}$ *be convex and Fréchet differentiable,* ∇g *be* κ*-Lipschitz. Let* $\{x_n\}$ *be generated by Algorithm 4.3. Suppose that assumptions (A3) and (A4) hold,* $\Omega_{g,S} \neq \emptyset$ *and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) There exist a positive real number a *with* $0 < a \leq \delta_n$ *and* $0 < a \leq \theta_n$*, for each* $n \in \mathbb{N}$*.*

Then, $\{x_n\}$ *converges strongly to* $p \in \Omega_{g,S}$*, where* $p = P_{\Omega_{g,S}} f(p)$*.*

3. Split feasibility problem.

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be convex and closed set in H_1 and H_2, and let $L : H_1 \to H_2$ be a bounded and linear operator. By setting $B := N_C : H_1 \to 2^{H_1}$, we have $J^B_{\lambda_n} =: P_C$. Thus, we obtain $F(J^B_{\lambda_n}) = F(P_C) = C$. Note that $L^*(I - P_Q)L$ is $\frac{1}{2\|L\|^2} - ism$; see [25]. Now, we set $A =: L^*(I - P_Q)L$, which implies it is $2\|L\|^2$-Lipschitz. We can verify that problem (1.1) is reduced to the split feasibility problem [5], finding $x^* \in C \cap L^{-1}Q$. Thus problem (1.2) is reduced to a problem of finding $x^* \in F(S) \cap C \cap L^{-1}Q =: \Omega_{S,C,Q}$.

Subsequently, by applying Theorem 3.4, we obtain the following result.

Algorithm 4.5. *Let* $\{\alpha_n\}$*,* $\{\delta_n\}$*,* $\{\theta_n\}$ *and* $\{\beta_n\}$ *be sequences in* $(0,1)$ *with* $\alpha_n + \delta_n + \theta_n = 1$ *and the initial* $x_1 \in H$ *be arbitrary. Define*

$$\begin{aligned} z_n &= P_C\big(I - \lambda_n L^*(I - P_Q)L\big)x_n, \\ w_n &= z_n - \lambda_n\big(L^*(I - P_Q)Lz_n - L^*(I - P_Q)Lx_n\big), \\ y_n &= \beta_n x_n + (1 - \beta_n)w_n, \\ x_{n+1} &= \alpha_n f(x_n) + \delta_n w_n + \theta_n S y_n, \forall n \in \mathbb{N}, \end{aligned}$$

where $\{\lambda_n\}$ *is defined by*

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\sigma\|x_n - z_n\|}{\|L^*(I-P_Q)Lx_n - L^*(I-P_Q)Lz_n\|}, \lambda_n\right\}, & \text{if } L^*(I - P_Q)Lx_n - L^*(I - P_Q)Lz_n \neq 0; \\ \lambda_n, & \text{otherwise}, \end{cases}$$

when $\lambda_1 > 0$ *and* $\sigma \in (0,1)$.

Theorem 4.6. *Let* H_1 *and* H_2 *be two real Hilbert spaces. Let* C *and* Q *be convex, closed, and nonempty subsets of* H_1 *and* H_2*, respectively, and let* $L : H_1 \to H_2$ *be a bounded and linear operator. Let* $\{x_n\}$ *be generated by Algorithm 4.5. Suppose that assumptions (A3) and (A4) hold,* $\Omega_{S,C,Q} \neq \emptyset$ *and the following control conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(C2) There exist a positive real number a *with* $0 < a \leq \delta_n$ *and* $0 < a \leq \theta_n$*, for each* $n \in \mathbb{N}$*.*

Then, $\{x_n\}$ *converges strongly to* $p \in \Omega_{S,C,Q}$*, where* $p = P_{\Omega_{S,C,Q}} f(p)$*.*

5. Numerical Experiments

In this section, we consider some numerical experiments to demonstrate the applications of the theorems and the results obtained by applying Theorem 3.4.

Example 5.1. Let $H = \mathbb{R}^2$ be equipped with the Euclidean norm. For each $x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H$, we consider the following two norms:

$$\|x\|_1 = |x_1| + |x_2| \quad \text{and} \quad \|x\|_\infty = \max\big\{|x_1|, |x_2|\big\}.$$

Let a function $g : H \to \mathbb{R}$, which is defined by $g(x) = \|x\|_1$ for all $x \in H$. A subdifferential operator of g is

$$\partial g(x) = \big\{z \in H : \langle x, z\rangle = \|x\|_1, \|z\|_\infty \le 1\big\}, \qquad \forall x \in H.$$

Since g is a convex function, $\partial g(\cdot)$ is a maximal monotone operator. Furthermore, for each $\lambda > 0$, we have

$$J_\lambda^{\partial g}(x) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H : u_i = x_i - \big(\min\{|x_i|, \lambda\}\big) sgn(x_i), \ \text{ for } \ i = 1, 2 \right\},$$

where $sgn(\cdot)$ denotes the signum function.

Next, let $\tilde{x} := \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\bar{x} := \begin{pmatrix} 6 \\ -5 \end{pmatrix}$ be two fixed vectors in H. We consider the 1-Lipschitz operator P_Q, where $Q := \big\{u \in H : \langle \tilde{x}, u\rangle \le -5\big\}$, and the operator P_C, where C is the following nonempty convex subsets of H:

$$C := \big\{u \in H : \langle \bar{x}, u\rangle \le 15\big\}.$$

Now, we have $F(P_C) = C$. In addition, we consider a contraction mapping $f := \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$. Under the above settings, we consider the problem to find a point

$$x^* \in C \cap (P_Q + \partial g)^{-1} 0. \tag{5.1}$$

Note that the solution set of problem (5.1) is

$$\Omega := \left\{ \begin{pmatrix} x \\ \frac{2x-1}{3} \end{pmatrix} \in H : 0.5 \le x \le 5 \right\}.$$

We determine the results by using the stopping criterion by

$$\frac{\|x_{n+1} - x_n\|}{\max\{1, \|x_n\|\}} \le 1.0e^{-06}.$$

We first consider Algorithm 3.1 with three cases of the step size parameters α_n, δ_n and θ_n as follows:

Case 1. $\alpha_n = \dfrac{1}{100n}$, $\delta_n = 0.1 - \dfrac{1}{100n}$, $\theta_n = 1 - (\alpha_n + \delta_n)$;

Case 2. $\alpha_n = \dfrac{1}{100n}$, $\delta_n = 0.5 - \dfrac{1}{100n}$, $\theta_n = 1 - (\alpha_n + \delta_n)$;

Case 3. $\alpha_n = \dfrac{1}{100n}$, $\delta_n = 0.9 - \dfrac{1}{100n}$, $\theta_n = 1 - (\alpha_n + \delta_n)$.

TABLE 1. Numerical experiments for the different stepsize parameters of α_n, δ_n and θ_n to Algorithm 3.1 with some initial points.

Case $\rightarrow$	**Case 1**		**Case 2**		**Case 3**	
# $x_1 \downarrow$	**Iters**	**Sol**	**Iters**	**Sol**	**Iters**	**Sol**
$(0,0)^\top$	207	$\begin{pmatrix}0.499815\\0.000011\end{pmatrix}$	174	$\begin{pmatrix}0.499842\\0.000010\end{pmatrix}$	153	$\begin{pmatrix}0.499858\\0.000009\end{pmatrix}$
$(-1,1)^\top$	208	$\begin{pmatrix}0.499816\\0.000011\end{pmatrix}$	174	$\begin{pmatrix}0.499840\\0.000010\end{pmatrix}$	153	$\begin{pmatrix}0.499858\\0.000009\end{pmatrix}$
$(-1,-1)^\top$	207	$\begin{pmatrix}0.499814\\0.000011\end{pmatrix}$	174	$\begin{pmatrix}0.499840\\0.000010\end{pmatrix}$	153	$\begin{pmatrix}0.499858\\0.000009\end{pmatrix}$
$(1,1)^\top$	5,238	$\begin{pmatrix}1.251762\\0.501176\end{pmatrix}$	5,238	$\begin{pmatrix}1.251844\\0.501230\end{pmatrix}$	5,238	$\begin{pmatrix}1.251904\\0.501270\end{pmatrix}$
$(1,-10)^\top$	207	$\begin{pmatrix}0.499815\\0.000011\end{pmatrix}$	174	$\begin{pmatrix}0.499840\\0.000010\end{pmatrix}$	128	$\begin{pmatrix}0.499844\\0.000009\end{pmatrix}$
$(-10,-10)^\top$	209	$\begin{pmatrix}0.499816\\0.000011\end{pmatrix}$	174	$\begin{pmatrix}0.499840\\0.000010\end{pmatrix}$	153	$\begin{pmatrix}0.499858\\0.000009\end{pmatrix}$
$(10,10)^\top$	5,471	$\begin{pmatrix}4.944810\\2.963201\end{pmatrix}$	5,471	$\begin{pmatrix}4.941488\\2.960989\end{pmatrix}$	5,471	$\begin{pmatrix}4.960181\\2.973452\end{pmatrix}$

We consider seven different initial points: $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-10\end{pmatrix}, \begin{pmatrix}-10\\-10\end{pmatrix}$, and $\begin{pmatrix}10\\10\end{pmatrix}$. The results are presented in Table 1, with $\beta_n = 0.5, \sigma = 0.5$, and $\lambda_1 = 0.5$ fixed values.

From each initial point in Table 1, we see that case 3 shows a better convergence rate than the other cases, except for points $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}10\\10\end{pmatrix}$, where all three cases have the same value.

In Table 2, we focus to the parameter β_n and consider the initial point $\begin{pmatrix}1\\-10\end{pmatrix}$ with the three cases of parameters α_n, δ_n and θ_n as above. We will consider different five values of β_n that are $\beta_n = 0.1$, $\beta_n = 0.3$, $\beta_n = 0.5$, $\beta_n = 0.7$ and $\beta_n = 0.9$. From the results in Table 2, we found that Case 3 with $\beta_n = 0.9$ provides faster convergence.

From Table 3, we compare the two algorithms between Algorithm 3.1 and Algorithm (1.3) with the initial point $\begin{pmatrix}1\\-10\end{pmatrix}$ and also consider the three cases of parameters α_n, δ_n, and θ_n as above. In Algorithm 3.1, we set the parameter $\beta_n = 0$. Under the same fixed settings in Table 1, we can see that Algorithm 3.1 shows faster convergence than Algorithm (1.3).

6. CONCLUSIONS

In this paper, we presented a modified algorithm for solving a monotone inclusion problem and a fixed point problem of nonexpansive mappings in Hilbert spaces. The proposed algorithm, Algorithm 3.1, is designed by including the Mann viscosity method and Tseng type method. By providing suitable control conditions to the process, we obtained a strong convergence theorem, Theorem 3.4. In applications, we apply our result to the variational inequality,

TABLE 2. Influence of the parameter β_n of Algorithm 3.1 for different cases of parameters α_n, δ_n and θ_n with the initial point $(1,-10)^\top$.

Case $\rightarrow$	**Case 1**		**Case 2**		**Case 3**	
# $\beta_n \downarrow$	**Iters**	**Sol**	**Iters**	**Sol**	**Iters**	**Sol**
$\beta_n = 0.1$	157	$\begin{pmatrix}0.499855\\0.000009\end{pmatrix}$	153	$\begin{pmatrix}0.499858\\0.000009\end{pmatrix}$	96	$\begin{pmatrix}0.499798\\0.000012\end{pmatrix}$
$\beta_n = 0.3$	177	$\begin{pmatrix}0.499839\\0.000010\end{pmatrix}$	163	$\begin{pmatrix}0.499850\\0.000009\end{pmatrix}$	119	$\begin{pmatrix}0.499836\\0.000010\end{pmatrix}$
$\beta_n = 0.5$	207	$\begin{pmatrix}0.499815\\0.000011\end{pmatrix}$	174	$\begin{pmatrix}0.499840\\0.000010\end{pmatrix}$	128	$\begin{pmatrix}0.499844\\0.000009\end{pmatrix}$
$\beta_n = 0.7$	264	$\begin{pmatrix}0.499779\\0.000013\end{pmatrix}$	188	$\begin{pmatrix}0.499828\\0.000010\end{pmatrix}$	80	$\begin{pmatrix}0.499742\\0.000015\end{pmatrix}$
$\beta_n = 0.9$	436	$\begin{pmatrix}0.499724\\0.000017\end{pmatrix}$	206	$\begin{pmatrix}0.499814\\0.000011\end{pmatrix}$	47	$\begin{pmatrix}0.499541\\0.000027\end{pmatrix}$

TABLE 3. Comparison between Algorithm 3.1 and Algorithm (1.3) for different cases of parameters α_n, δ_n and θ_n with the initial point $(1,-10)^\top$.

Algorithm $\rightarrow$	**Alg. 3.1**		**Alg. (1.3)**	
# Case $\downarrow$	**Iters**	**Sol**	**Iters**	**Sol**
Case 1	149	$\begin{pmatrix}0.499861\\0.000008\end{pmatrix}$	158	$\begin{pmatrix}0.499854\\0.000009\end{pmatrix}$
Case 2	149	$\begin{pmatrix}0.499861\\0.000008\end{pmatrix}$	217	$\begin{pmatrix}0.499805\\0.000012\end{pmatrix}$
Case 3	82	$\begin{pmatrix}0.499765\\0.000014\end{pmatrix}$	662	$\begin{pmatrix}0.499628\\0.000023\end{pmatrix}$

convex minimization, and split feasibility problems. Furthermore, in numerical examples, our results are better on the convergence rate in different cases, including the comparison with other related work.

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