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SOME FIXED POINT RESULTS ON UNIFORMLY NON-SQUARE METRIC SPACES

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Abstract. This paper introduces a metric version of Goebel's definition for uniformly non-square Banach spaces. We show that such spaces with a characteristic of convexity strictly less than one possess the uniform normal structure property. This property implies property (R), which in turn is equivalent to reflexivity in Banach spaces. As an application, we explore conditions for finding fixed points of a continuous mapping of uniformly k-Lipschitzian type on a complete bounded uniformly non-square metric space.

Keywords. $CAT_p(0)$ spaces; Convex metric spaces; Fixed points; Uniformly Lipschitzian mappings; Uniformly non-square spaces.

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1. Introduction

Convexity is a known concept related to sets and points in Banach spaces. Indeed, convexity is essential to describe geometric properties of Banach spaces. To be more clear, there are two main types of convexity: strict convexity and uniform convexity; see [8]. Uniformly non-square Banach spaces are a specific type of Banach space with interesting geometric properties, which are related to but distinct from convexity. The concept of uniform non-squareness, which is originated in [17], find numerous applications in various fields in pure mathematics, including operator theory, harmonic analysis, and nonlinear functional analysis. Note that these spaces can be fully characterised by the modulus and the characteristic of convexity. Let E be a Banach space. Recall that the modulus of convexity of E is a function, $\delta : [0,2] \to [0,1]$, defined by [26]

$$\delta_E(\varepsilon) = \inf\{1 - ||\frac{x+y}{2}|| : ||x|| \leq 1, ||y|| \leq 1, ||x-y|| \geq \varepsilon\},$$

where the infimum is taken among all $x, y \in E$.

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From [10], one recalls that the convex characteristic of a Banach space E is $\varepsilon_0(E) = \sup\{\varepsilon : \delta_E(\varepsilon) = 0\}$. Furthermore, one also recalls from [10] that a Banach space E is said to be uniformly convex if and only $\varepsilon_0(E) = 0$; uniformly non-square if and only if $\varepsilon_0(E) < 2$, and strictly convex if and only if $\delta_E(2) = 1$. It is known that every uniformly convex space is a uniformly non-square space. Moreover, every uniformly non-square space is strictly convex space. In the early stages of fixed point theory of Lipschitzian mappings, uniform convexity played a major role. Lipschitzian mappings form classes of mappings which appear not only in many branches of mathematics, such as differential equations, measure theory, nonlinear functional analysis, metric geometry, fractal theory, but also in computer science, such image processing.

Throughout this paper, K is borrowed to stand for a nonempty, convex, closed, and bounded subset of a Banach space E. Recall that a mapping $T : K \to K$ is asymptotically nonexpansive on K if and only if

$$||T^n x - T^n y|| \le k_n ||x - y||, \ x, y \in K, \ n > N.$$

where $\{k_n\}$ with $k_n \to 1$ as $n \to \infty$ is a real sequence.

In 1972, Goebel and Kirk [11] proved that if the framework is uniformly convex, then every asymptotically nonexpansive mapping T on K has a fixed point. This generalizes the fixed point theorems of nonexpansive mappings presented in Browder [4] and Göhde [16] on a uniformly convex Banach space. In a subsequent study, Goebel and Kirk in [12] extended their result for asymptotically nonexpansive mappings [11] to uniformly k-Lipschitzian mappings. They demonstrated that if $T : K \to K$ is a uniformly k-Lipschitzian mapping on set K, that is, there exists a non-negative number k such that $||T^n(x) - T^n(y)|| \le k \, ||x - y||$ for all x and y in set K, then there exists a constant $\gamma > 1$ that is a solution to equation $\gamma[1 - \delta_E(1/\gamma)] = 1$ such that T has a fixed point in K whenever $k < \gamma$. On the other hand, Kirk in [22] extended the fixed point result for asymptotically nonexpansive mappings [11] to non-Lipschitzian mappings of asymptotically nonexpansive type. In particular, Kirk substantially weakened the assumption of asymptotic nonexpansiveness of T to

$$\limsup_{n\to\infty}\{\sup_{y\in K}[||T^n x - T^n y|| - ||x - y||]\} \le 0, \ \forall x \in K,$$

which may hold even if none of the iterates of T is Lipschitzian.

Recall that a mapping $T : K \to K$ is said to be uniformly k-Lipschitzian type if and only if, for each $x \in K$, the sequence $\{c_n(x)\}$ defined by

$$c_n(x) = \max\{\sup_{y\in K}\{||T^n x - T^n y|| - k||x - y||\}, 0\} \tag{1.1}$$

tends to 0 as $n \longrightarrow \infty$. If $k = 1$, this reduces to a mapping of asymptotically nonexpansive type. It is clear that each uniformly $k-$Lipschitzian mapping is of uniformly $k-$Lipschitzian type; see [30]. Based on the method employed by Kirk in [22], Tan and Vuong [30] proved a fixed point result that deals with the mappings of uniformly k-Lipschitzian type defined on some classes of uniformly non-square Banach spaces. These set an extension of the fixed point result in Kirk [22] for mappings of asymptotically nonexpansive type, in particular, the fixed point result of Goebel and Kirk [12] for uniformly Lipschitzian mappings.

Theorem 1.1. [30] *Let T be a continuous mapping of uniformly $k-$Lipschitzian type defined on a subset K of a Banach space E. Let $k < \gamma$, where γ is the solution to $\gamma[1 - \delta_E(1/\gamma)] = 1$. If $\varepsilon_0(E) < 1$, then T has a fixed point.*

The study of fixed point theorems of nonlinear mappings in Banach spaces is now under the spotlight. It is always a hot issue to extend the associated fixed point theorems in a broader context of metric spaces. In this study, we introduce a metric space, which is an analogue to Goebel's definition for uniformly non-square Banach spaces. As an application, we investigate some conditions under which a continuous mapping of uniformly k-Lipschitzian type on a complete, bounded, uniformly non-square metric space possesses a fixed point.

2. Uniform Non-squareness in Metric Spaces

In the classical study of fixed point properties of nonlinear mapping, the attention has always been focused on convex subsets. This convexity play a significant role. Several attempts have been made to generalize the concept of convexity to metric spaces recently. One such convex structure is available in hyperbolic metric spaces [14] (see Definition 2.4).

Let (M,d) be a metric space, and let $\mathbb{R}$ denote the real line. We say that a mapping $c:\mathbb{R}\to M$ is a metric embedding of $\mathbb{R}$ into M if $d(c(s),c(t)) = |s-t|$ for all real s and t in $\mathbb{R}$.

The image of $\mathbb{R}$ under a metric embedding is called a metric segment. The image $c([a,b])\subset M$ of a real interval under a metric embedding is called a metric segment. Let $x,y\in M$. A metric segment $c([a,b])$ is said to join x and y if $c(a)=x$ and $c(b)=y$. We say that (M,d) is of hyperbolic type [13] if M contains a family of metric segments such that, for each pair of distinct points x and y in M, there exists a unique metric segment which joins x and y. the unique metric segment joining the two points x and y from M is denoted $[x,y]$ or $[y,x]$. Clearly, we have $[x,x]=\{x\}$ for any x in M. Next, we present some basic facts about metric spaces of hyperbolic type.

Lemma 2.1. [23] *Let $x,y\in M$, $x\neq y$, and $z,w\in[x,y]$. Then (i) $d(x,z)\leq d(x,y)$; (ii) if $d(x,z)=d(x,w)$, then $z=w$.*

The next result brings us closer to the definition of a convex combination in metric spaces of hyperbolic type.

Lemma 2.2. [23] *Let $c:\mathbb{R}\to M$ be a metric embedding, $a\leq b\in\mathbb{R}$, and $t\in[0,1]$. Then*

(i) $d(c(a),c((1-t)a+tb)) = t\, d(c(a),c(b))$;

(ii) $d(c(b),c((1-t)a+tb)) = (1-t)\, d(c(a),c(b))$.

Proposition 2.3. [23] *Let M be a metric space of hyperbolic type, and let $x,y\in M$. For each $\alpha\in[0,1]$, there exists a unique point $z\in[x,y]$ such that*

$$d(x,z)=\alpha\, d(x,y) \ \text{ and } \ d(y,z)=(1-\alpha)\, d(x,y).$$

Such point is denoted by $(1-\alpha)x\oplus\alpha y$.

Obviously, we have $(1-\alpha)x\oplus\alpha x = x$. If $x\neq y$, then, for any $z\in[x,y]$, $z=(1-\alpha)x\oplus\alpha y$, with $\alpha=\frac{d(x,z)}{d(x,y)}$. This implies that, for any $z\in[x,y]$, $d(x,z)+d(z,y)=d(x,y)$. If, for some $\alpha,\beta\in[0,1]$, $(1-\alpha)x\oplus\alpha y=(1-\beta)x\oplus\beta y$, then $\alpha=\beta$ provided $x\neq y$. Moreover, we have $(1-\alpha)x\oplus\alpha y=\alpha y\oplus(1-\alpha)x$. The following is a direct consequence

$$[x,y]=\big\{(1-\alpha)x\oplus\alpha y;\ \alpha\in[0,1]\big\}.$$

Definition 2.4. [14, 23, 24, 27] Let M be a metric space. M is said to be hyperbolic metric if

$$d\left((1-\alpha)x\oplus\alpha y,(1-\alpha)z\oplus\alpha w\right)\leq(1-\alpha)d(x,z)+\alpha d(y,w),$$

for any $\alpha \in [0,1]$ and all $x,y,z,w \in M$.

Recall that a subset C of a hyperbolic metric space M is said to be convex whenever $[x,y] \subset C$ for any $x,y \in C$. Obviously, normed linear spaces are hyperbolic metric spaces. As nonlinear examples, one can consider the Hadamard manifolds [6], the Hilbert open unit ball equipped with the hyperbolic metric [14], and $CAT_p(0)$ metric spaces [21] (see Example 3.6).

We now turn our attention to the modulus and the characteristic of convexity of hyperbolic metric spaces. Throughout this paper, M is always used to denote a hyperbolic metric space.

Definition 2.5. [20] For any real numbers $r > 0$, $0 \leq \varepsilon \leq 2$, the modulus of convexity of M is

$$\delta_M(r,\varepsilon) = \inf\Big\{1 - \frac{1}{r}\, d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big); d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq r\varepsilon\Big\},$$

where the infimum is taken among all $x,y,a \in M$.

Remark 2.6. Since M is a hyperbolic metric space, it is obvious that $0 \leq \delta_M(r,\varepsilon) \leq 1$ for any real numbers $r > 0$, $0 \leq \varepsilon \leq 2$.

Recall that the characteristic of convexity of M is $\varepsilon_0(M) = \sup\{\varepsilon : \delta_M(r,\varepsilon) = 0, \text{for every } r > 0\}$. Note that the first attempt to generalize the concept of uniform convexity to metric spaces was done in [15]. It is known that M is uniformly convex if $\varepsilon_0(M) = 0$ [20]. Furthermore, if M is uniformly convex, then M is strictly convex, i.e., whenever

$$d\Big(\frac{1}{2}x \oplus \frac{1}{2}y, a\Big) = d(x,a) = d(y,a)$$

for any $x,y,a \in M$, then $x = y$.

A metric formulation of Goebel's geometric definition of uniformly non-square Banach spaces is presented as follows.

Definition 2.7. *M is said to be uniformly non-square metric space if $\varepsilon_0(M) < 2$.*

M is said to have the property (R) if any nonincreasing sequence of nonempty, convex, bounded and closed subsets of M has a nonempty intersection. Recall that the property (R) is a nonlinear metric analogue of the Smulian characterization of reflexivity [29]. Moreover, Khamsi in [18] proved that the uniform normal structure property implies the property (R). Uniform non-squareness is a weaker property that follows from uniform convexity. Furthermore, all uniformly non-square spaces are strictly convex. Furthermore, we demonstrate that these spaces with a characteristic of convexity strictly less than one, exhibit the uniform normal structure property. This property directly implies property (R), which is known to be equivalent to reflexivity in the context of Banach spaces; see [18, 20].

Recall that M is said to have the normal structure (resp. uniform normal structure) if every bounded and convex subset C of M containing more than one point contains a nondiametral point x, i.e., a point such that $\sup\{d(x,y) : y \in C\} < \operatorname{diam} C$ (resp. $\sup\{d(x,y) : y \in C\} \leq \alpha \operatorname{diam} C$, where $\alpha \in (0,1)$ and is independent of x and C). In [18], Khamsi proved that every complete, bounded metric space with uniform normal structure has the property (R). On the other hand, one knows that if $\varepsilon_0(E) < 1$, then Banach spaces have uniform normal structure [1, 19]. A metric analogue is presented as follows.

Theorem 2.8. *Let M be a complete bounded uniformly non-square metric space with $\varepsilon_0(M) <$ 1. Then M has uniform normal structure.*

It is clear because, for any bounded and convex subset C of M, there exists $x \in C$ such that $\sup\{d(x,y) : y \in C\} \leq (1-\delta_M(\operatorname{diam} C, 1)) \operatorname{diam} C$.

Therefore, we can also deduce the following result.

Theorem 2.9. *Every complete bounded uniformly non-square metric space M with $\varepsilon_0(M) < 1$ has the property (R).*

Finally, we conclude this section by posing an interesting question: Is reflexivity a general property of uniformly non-square metric spaces, analogous to the well-established result for Banach spaces?

3. Main Results

In this section, we give several applications of our results. Our findings contribute to the ongoing effort to generalize classical fixed point theorems from Banach spaces to the more general setting of uniformly non-square metric spaces. We collect some basic definitions here. Let $\{x_n\}$ be a bounded sequence in M and C be a nonempty subset. Define $r(., \{x_n\}) : C \to [0,\infty)$ by $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius ρ_C of $\{x_n\}$ with respect to C is given by $\rho_C = \inf\{r(x,\{x_n\}) : x \in C\}$.

The asymptotic radius of $\{x_n\}$ with respect to M is denoted by ρ. A point $\xi \in C$ is said to be an asymptotic center of $\{x_n\}$ with respect to C if

$$r(\xi,\{x_n\}) = r(C,\{x_n\}) = \min\{r(x,\{x_n\}) : x \in C\}.$$

We denote by $A(C,\{x_n\})$ the set of asymptotic centers of $\{x_n\}$ with respect to C. When $C = M$, we call ξ an asymptotic center of $\{x_n\}$ and simply use the notation $A(\{x_n\})$. In general, the set $A(C,\{x_n\})$ of asymptotic centers of a bounded sequence $\{x_n\}$ may be empty or may even contain infinitely many points.

Lemma 3.1. [9] *Let C be a nonempty, convex, and closed subset of a complete uniformly convex metric space M. Then every bounded sequence $\{x_n\}$ in M has a unique asymptotic center with respect to C.*

In fact, it was proved in [9] that the property (R) implies $A(C,\{x_n\})$ is nonempty. Furthermore, strict convexity of M implies $A(C,\{x_n\})$ is a singleton. From this point of view, with the aid of Theorem 2.9, the property (R) and strict convexity of uniformly non-square metric spaces imply the following.

Theorem 3.2. *Let M be a complete bounded uniformly non-square metric space with $\varepsilon_0(M) < 1$. If C is a nonempty convex, and closed subset of M. Then every bounded sequence $\{x_n\}$ in M has a unique asymptotic center with respect to C.*

Now we are in a position to investigate a fixed point result of mappings of uniformly k-Lipschitzian type in complete uniformly non-square metric spaces.

Theorem 3.3. *Let M be a complete and bounded uniformly non-square metric space with $\varepsilon_0(M) < 1$. Let $T : M \to M$ be a continuous mapping of uniformly k-Lipschitzian type with $k < \gamma = \inf_{r>0}\gamma_r$, where γ_r is the solution to the equation $\gamma_r[1-\delta_M(r,1/\gamma_r)] = 1$ for every $r > 0$. Then T has a fixed point.*

Proof. It is not difficult to show that $\varepsilon_0(M) < 1$ implies $\gamma > 1$. Thus we may assume that $k \geq 1$. Now taking any $x_0 \in M$, we denote $x_n = T^n x_0$ for $n : 1,2,\ldots$ By Theorem 3.2, one sees that $\{x_n\}$ has a unique asymptotic center. Denote $z_1 = A(\{x_n\})$ and $r(z_1,\{x_n\}) = r_1$. If $r_1 = 0$, then $T^n x_0 \to z_1$. Since T is continuous, $Tz_1 = z_1$. Thus we may assume $r_1 > 0$. By Equation 1.1 we have

$$d(T^n x, T^n y) \leq kd(x,y) + c_n(x),\ \forall x,y \in M,\ \forall n \geq 1.$$

So, for any $m \geq 1$, we have

$$\limsup_{n\to\infty} d(T^n x_0, T^m z_1) \leq k \limsup_{n\to\infty} d(T^{n-m} x_0, z_1) + c_m(z_1) = kr_1 + c_m(z_1). \tag{3.1}$$

Since M is convex, it is clear that

$$r_1 \leq \limsup_{n\to\infty} d\Big(\frac{1}{2} z_1 \oplus \frac{1}{2} T^m z_1, T^n x_0\Big),\ \forall m \geq 1. \tag{3.2}$$

From the definition of r_1 and (3.1), we have

$$\limsup_{n\to\infty} d(T^n x_0, z_1) = r_1 \leq kr_1 + c_m(z_1), \quad \limsup_{n\to\infty} d(T^n x_0, T^m z_1) \leq kr_1 + c_m(z_1).$$

In light of the definition of $\delta_M(r,\varepsilon)$ and (3.2), we have

$$\begin{aligned} r_1 \leq & \limsup_{m\to\infty} \limsup_{n\to\infty} d\Big(\frac{1}{2} z_1 \oplus \frac{1}{2} T^m z_1, T^n x_0\Big) \\ \leq & \limsup_{m\to\infty} \Big(kr_1 + c_m(z_1)\Big)\Big(1 - \delta_M\Big(kr_1 + c_m(z_1), \frac{d(z_1, T^m z_1)}{kr_1 + c_m(z_1)}\Big)\Big) \\ = & kr_1 \Big(1 - \delta_M\Big(kr_1, \limsup_{m\to\infty} \frac{d(z_1, T^m z_1)}{kr_1}\Big)\Big). \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\frac{1}{k} \leq 1 - \delta_M\Big(kr_1, \limsup_{m\to\infty} \frac{d(z_1, T^m z_1)}{kr_1}\Big).$$

Hence,

$$\delta_M\Big(kr_1, \limsup_{m\to\infty} \frac{d(z_1, T^m z_1)}{kr_1}\Big) \leq 1 - \frac{1}{k}. \tag{3.4}$$

If $\delta_M\Big(kr_1, \limsup_{m\to\infty} \frac{d(z_1, T^m z_1)}{kr_1}\Big) = 0$, then the definition of $\varepsilon_0(M)$ yields

$$\limsup_{m\to\infty} d(z_1, T^m z_1) \leq \varepsilon_0(M) kr_1. \tag{3.5}$$

We show that $\varepsilon_0(M)k < 1$, for which it suffices to prove that $\varepsilon_0(M)\gamma < 1$. Suppose on the contrary that $\varepsilon_0(M)\gamma \geq 1$. Thus

$$1 > \varepsilon_0(M) \geq \frac{1}{\gamma} = 1 - \delta_M(kr_1, 1/\gamma).$$

Consequently, $\delta_M(kr_1, 1/\gamma) > 0$ and $1/\gamma > \varepsilon_0(M)$. Thus one has $\varepsilon_0(M)\gamma < 1$, a contradiction to the above assumption. Now, we suppose that $\delta_M\Big(kr_1, \limsup_{m\to\infty} \frac{d(z_1, T^m z_1)}{kr_1}\Big) > 0$. Then

$$\limsup_{m\to\infty} \frac{d(z_1, T^m z_1)}{kr_1} > \varepsilon_0(M).$$

Since the function $\delta_M(r,\varepsilon)$ restricted on $\varepsilon \in [\varepsilon_0(M),2]$ is invertible and strictly increasing [20] for every fixed $r>0$, we obtain from (3.4) that

$$\limsup_{m\to\infty}\frac{d(z_1,T^m z_1)}{kr_1} \le \delta_M^{-1}\left(1-\frac{1}{k}\right) < \delta_M^{-1}\left(1-\frac{1}{\gamma}\right) = \frac{1}{\gamma},$$

which implies

$$\limsup_{m\to\infty} d(z_1,T^m z_1) < \frac{kr_1}{\gamma}. \tag{3.6}$$

Letting $\alpha = \max\left\{\varepsilon_0(M)k, \dfrac{k}{\gamma}\right\} < 1$, we find from (3.5) and (3.6) that

$$\limsup_{m\to\infty} d(z_1,T^m z_1) \le \alpha r_1. \tag{3.7}$$

Now, we consider $z_2 = A(\{T^n z_1\})$ and denote $r(z_2,\{T^n z_1\}) = r_2$. From (3.7), we have $r_2 \le r(z_1,\{T^n z_1\}) \le \alpha r_1$. Continuing this process, we obtain a sequence $\{z_m\}$ satisfying $z_m = A(\{T^n z_{m-1}\})$ and $r_m = r(z_m,\{T^n z_{m-1}\})$. In this case, we have

$$r_m \le \alpha r_{m-1}, \quad \limsup_{n\to\infty} d(z_m,T^n z_m) \le \alpha r_m. \tag{3.8}$$

It follows that

$$\begin{aligned} d(z_m,z_{m+1}) \le & \limsup_{n\to\infty}\{d(z_{m+1},T^n z_m)+d(z_m,T^n z_m)\} \\ \le & r_{m+1}+\alpha r_m \le 2\alpha r_m \le 2\alpha^m r_1 \end{aligned}$$

Since $\alpha<1$, $\{z_m\}$ is a Cauchy sequence, it converges to some point $z^* \in M$. From (3.8) and the inequality

$$\begin{aligned} d(z^*,T^n z^*) \le & d(z^*,z_m)+d(z_m,T^n z_m)+d(T^n z_m,T^n z^*) \\ \le & (1+k)d(z^*,z_m)+d(z_m,T^n z_m)+c_n(z^*), \end{aligned}$$

we have $\limsup_{n\to\infty} d(z^*,T^n z^*) = 0$, which yields that z^* is a fixed point of T. The proof is complete □

Remark 3.4. Theorem 3.3 gives a metric analogue of Theorem 1.1 for mappings of uniformly $k-$Lipschitzian type in uniformly non-square Banach spaces. This sets an extension of the fixed point result of Kirk [22] for mappings of asymptotically nonexpansive type in uniformly convex Banach spaces. In particular, it implies the fixed point result of Goebel and Kirk [12] for uniformly $k-$Lipschitzian mappings.

Remark 3.5. Theorem 3.3 gives an extension of the fixed point result of Shukri et al [28] for mappings of asymptotically nonexpansive type in uniformly convex metric spaces.

An interesting question raised by Theorem 3.3 is wether or not the solution of $\gamma_r[1-\delta_M(r,1/\gamma_r)] = 1$, for every $r>0$ is the largest number $\gamma = \inf_{r>0}\gamma_r$ for which the theorem holds. In the next example, we give a partial answer in $CAT_p(0)$ metric spaces.

Example 3.6. Let (M,d) be a metric space. A continuous mapping from the interval $[0,1]$ into M is called a path. A path $\gamma:[0,1]\to M$ is called a geodesic or (metric segment) if $d(\gamma(s),\gamma(t)) = |s-t|d(\gamma(0),\gamma(1))$, for every $s,t\in[0,1]$. We say that (M,d) is a geodesic metric space if every two points $x,y\in M$ are connected by a geodesic, i.e., there exists a geodesic $\gamma:[0,1]\to M$ such that $\gamma(0)=x$ and $\gamma(1)=y$. In this case, we denote such geodesic by $[x,y]$.

Note that in general such geodesic is not uniquely determined by its endpoints. The metric space (M,d) is called uniquely geodesic if every two points of M are connected by a unique geodesic. In this case $[x,y]$ denotse the unique geodesic connecting x and y in M.

Definition 3.7. [3] A metric space M is said to be a $CAT(0)$ space if it is a uniquely geodesic metric space, and if every geodesic triangle in M is at least as "thin" as its comparison triangle in the Euclidean plane.

Recently, Khamsi and Shukri in [21] extended the Gromov geometric definition of $CAT(0)$ metric spaces to the case when the comparison triangles belong to the Banach space ℓ_p. Recall that a geodesic triangle in a geodesic metric space is a set of three points connected by geodesic segments. A comparison triangle for a geodesic triangle is a triangle in ℓ_p such that the lengths of the sides of the comparison triangle are equal to the distances between the corresponding vertices of the geodesic triangle [21]. Such triangles always exist for all $p \geq 1$ and unique up to isometry for all $1 < p < \infty$ (see [3, Proposition 4.16]).

Definition 3.8. [21] A $CAT_p(0)$ space is a uniquely geodesic metric space in which the distance between any two points is no greater than the distance between the corresponding points in a comparison triangle in ℓ_p space, for all $1 < p < \infty$.

It is obvious that ℓ_p, $p > 2$, is a $CAT_p(0)$ metric space which is not a $CAT(0)$ metric space; see [21]. Let x, y_1, y_2 be in a $CAT_p(0)$ space M, and $\frac{y_1 \oplus y_2}{2}$ be the midpoint of the geodesic $[y_1, y_2]$. Then, for $p \geq 2$,

$$d^p\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2} d^p(x,y_1) + \frac{1}{2} d^p(x,y_2) - \frac{1}{2^p} d^p(y_1,y_2). \tag{3.9}$$

This inequality is the (CN_p) inequality of Khamsi and Shukri [21]. A direct consequence of the (CN_p) inequality is that any $CAT_p(0)$ metric space for $p \geq 2$ is uniformly non-square metric space, with modulus of convexity [21]

$$\delta(r,\varepsilon) \geq 1 - \left(1 - \frac{\varepsilon^p}{2^p}\right)^{1/p}.$$

Furthermore, $\gamma_r[1 - \delta_M(r, 1/\gamma_r)] = 1$, for every $r > 0$ implies

$$1 < \left(1 + \frac{1}{2^p}\right)^{1/p} \leqslant \gamma = \inf_{r>0} \gamma_r.$$

which suggests an upper bound for k and Theorem 3.3 holds in $CAT_p(0)$ metric spaces. Recall that a type function $\tau : M \to \mathbb{R}_+$ is a function such that there exists a bounded sequence $\{x_n\}$ in M with $\tau(x) = \limsup_{n\to\infty} d(x,x_n)$. In addition, We need the following lemma.

Lemma 3.9. [21] *Let C be a nonempty, convex, bounded, and closed subset of a complete $CAT_p(0)$ metric space M. Let τ be a type defined on C. Then any minimizing sequence $\{x_n\} \subset C$ of τ is convergent and its limit x is the unique minimum of τ, which is called the asymptotic center of $\{x_n\}$, and satisfies*

$$\tau^p(x) + \frac{1}{2^{p-1}}\, d^p(x,z) \leq \tau^p(z), \tag{3.10}$$

for any $z \in C$.

Note that inequality (3.10) is similar to Opial's condition [25]. Now we are in a position to investigate a fixed point of mappings of uniformly $k-$Lipschitzian type in complete $CAT_p(0)$ metric spaces, $p \geq 2$.

Theorem 3.10. *Let M be a complete bounded $CAT_p(0)$ metric space, $p \geq 2$. Let $T : M \to M$ be a continuous mapping of uniformly $k-$Lipschitzian type with $k < \left(1+\frac{1}{2^p}\right)^{1/p}$. Then T has a fixed point.*

Proof. Fixing $x_0 \in M$, we consider the type function τ_1 generated by the sequence $\{T^n x_0\}$. Let z_1 be the minimum point of τ_1 which exists by using Lemma 3.9. Set

$$r_1 = \tau_1(z_1) = \limsup_{n\to\infty} d(z_1, T^n x_0) = \inf\{\tau_1(x); x \in M\}.$$

On the other hand, by (1.1) for any $n > m \geqslant 1$, we have

$$\limsup_{n\to\infty} d(T^m z_1, T^n x_0) \leqslant k \limsup_{n\to\infty} d(z_1, T^{n-m} x_0) + c_m(z_1). \ i.e.,$$

and $\tau_1(T^m z_1) \leqslant kr_1 + c_m(z_1)$. Consequently, one has $d^p(z_1, T^m z_1) \leq 2^{p-1}[(kr_1 + c_m(z_1))^p - r_1^p]$. If $m \longrightarrow \infty$, then

$$\limsup_{m\to\infty} d^p(z_1, T^m z_1) \leq 2^{p-1} r_1^p (k^p - 1).$$

If $r_1 = 0$, then $T^n x_0 \longrightarrow z_1$. Since T is continuous, $Tz_1 = z_1$. Thus, we may assume $r_1 > 0$. Furthermore, if $k < (1+\frac{1}{2^p})^{1/p}$, then the above inequality reduces to

$$\limsup_{m\to\infty} d(z_1, T^m z_1) < (\frac{1}{2})^{1/p} r_1.$$

Now, we consider the type function τ_2 generated by the sequence $\{T^n z_1\}$. Let z_2 be the minimum point of τ_2. Setting

$$r_2 = \tau_2(z_2) = \limsup_{n\to\infty} d(z_2, T^n z_1) = \inf\{\tau_2(x); x \in M\},$$

we have

$$r_2 \leqslant \tau_2(z_1) = \limsup_{m\to\infty} d(z_1, T^m z_1) < \left(\frac{1}{2}\right)^{1/p} r_1.$$

Thus, we can obtain a sequence $\{z_m\} \subset M$ such that z_m is the minimum point of the type function τ_m generated by the sequence $\{T^n z_{m-1}\}$, say $r_m = \tau_m(z_m)$. In this case, $r_m < (\frac{1}{2})^{1/p} r_{m-1}$, and $\limsup_{n\to\infty} d(z_m, T^n z_m) < (\frac{1}{2})^{1/p} r_m$. It follows that

$$\begin{aligned} d(z_{m+1}, z_m) \leqslant& \limsup_{n\to\infty} \{d(z_{m+1}, T^n z_m) + d(z_m, T^n z_m)\} \\ <& r_{m+1} + \left(\frac{1}{2}\right)^{1/p} r_m < 2\left(\frac{1}{2}\right)^{1/p} r_m < 2\left(\frac{1}{2}\right)^{m/p} r_1. \end{aligned}$$

Clearly $\{z_m\}$ is a Cauchy sequence and hence it converges to some point $z* \in M$. Moreover, from

$$\begin{aligned} d(z*, T^n z*) \leqslant& d(z*, z_m) + d(z_m, T^n z_m) + d(T^n z_m, T^n z*) \\ \leqslant& (1+k) d(z*, z_m) + d(z_m, T^n z_m) + c_n(z*), \end{aligned}$$

we have $\limsup_{n\to\infty} d(z*, T^n z*) = 0$, and so $z*$ is a fixed point of T. $\square$

Remark 3.11. Theorem 3.10 is an explicit generalization of the fixed point theorem of Khamsi and Shukri [21] for uniformly $k-$Lipschitzian mappings in complete metric spaces $CAT_p(0)$, $p \geq 2$. Furthermore, it is worth noting that the upper bound for $k < (1+\frac{1}{2^p})^{1/p}$ appears more favorable than the one employed in [21] which relies on the normal structure coefficient [7].

Remark 3.12. Since the class of non-Lipschitzian mappings of asymptotically nonexpansive type contains the class of asymptotically nonexpansive mappings [11]. Then, for $k = 1$, Theorem 3.10 gives an extension of the fixed point result of Bachar and Khamsi for asymptotically nonexpansive mappings [2] in complete metric spaces $CAT_p(0)$, $p \geq 2$.

Remark 3.13. It is obvious that any $CAT_2(0)$ space is a $CAT(0)$ space. In this case, the (CN_p) inequality (3.9) for $p = 2$ reduces to the classical (CN) inequality of Bruhat and Tits [5]. For the Hilbert space [12], the (CN) inequality implies that $k < (1+\frac{1}{2^2})^{1/2} = \frac{\sqrt{5}}{2}$.

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