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NORMALIZED SOLUTIONS FOR P-KIRCHHOFF EQUATIONS WITH PRESCRIBED MASS AND GENERAL NONLINEARITIES

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Abstract. In this paper, we study the existence of normalized solutions to the following p-Kirchhoff equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^p dx\right)\Delta_p u+\lambda u=g(u), \quad in \quad \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2 dx=m, \end{cases}$$

where $a>0$, $b\geq 0$, and $m>0$ are constants, $p^*=\frac{Np}{N-p}$ is the critical Sobolev exponent, λ is a Lagrange multiplier, $\Delta_p u=div(|\nabla u|^{p-2}\nabla u)$, $2<p<N=3$, and the nonlinear term g satisfies certain conditions. Under the right conditions, we study that the $p-$Kirchhoff equation has a positive normalized solution $(\lambda,u_\lambda)\in(0,+\infty)\times W^{1,p}_{rad}(\mathbb{R}^3)$ for any $m>0$.

Keywords. Mass subcritical, Normalized solution; p-Kirchhoff equation.
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1. Introduction And Main Results

In this paper, we mainly consider the following equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^p dx\right)\Delta_p u+\lambda u=g(u), \quad in \quad \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2 dx=m, \end{cases} \tag{1.1}$$

which possesses a positive normalized solution under the mass subcritical case, where $a>0$, $b\geq 0$, and $m>0$ are constants, $p^*=\frac{Np}{N-p}$ is the critical Sobolev exponent, λ is a Lagrange multiplier, $\Delta_p u=div(|\nabla u|^{p-2}\nabla u)$, and $2<p<N=3$.

The Kirchhoff equation was first proposed in 1883 by Kirchhoff in order to describe the transversal free vibrations of a clamped string [7]. It is related to the free vibration problem of elastic strings in the famous D'Alembert wave equation. It is worth mentioning that a functional

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analysis approach was introduced by Lions in the pioneer work [8]. In [12, 13], Ye studied the existence of the canonical ground state solution of the Kirchhoff equation and pointed out that L^2 critical index $\bar{p} := 2+\frac{8}{N}$ was the threshold index of the Kirchhoff equation at $f(u) = |u|^{p-2}$. In 2014, Liang used the topological degree theory and variational methods to study the existence of positive solutions to a class of Kirchhoff type differential equations with asymptotic nonlinear terms [9]. In [4], the normalized solution of the Kirchhoff equation was studied when the nonlinear term was non-homogeneous and the mass was supercritical. In 2021, Chen et al. [2] studied the normalized solution of the Non-autonomous Kirchhoff equation under subcritical and supercritical conditions. Zhang et al. [14] used the Pohozaev identity to study the existence of gauge solutions for a class of critical Kirchhoff type equations in $\mathbb{R}^3$. In addition, many scholars focused their attention on the situation of the $p-$Kirchhoff equation. For example, Huang et al. [5] applied the fountain theorem to obtain the existence result of infinitely many solutions for a class of $p-$Kirchhoff type equations. Rasouli proved the existence of the sign change solution of the $P-$Kirchhoff equation by using the deformation lemma [11]. Zhou and Song [15] used the classical Kranoselski genus method to study a class of critically growing $p-$Kirchhoff type problems, and obtained an infinite number of non-trivial solutions to the problems.

In particular, if $a > 0$, $b > 0$, $m > 0$, and $p = 2$, then equation (1.1) is reduced to the Kirchhoff equation below

$$\begin{cases} -\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+\lambda u = g(u), \quad in \quad \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2dx = m. \end{cases} \tag{1.2}$$

Zhang et al. [16] studied that Kirchhoff equation (1.2) possesses a positive normalized solution $(\lambda, u) \in \times H^1(\mathbb{R}^N)$ based on a global branch approach and the Azzollini's correspondence.

And if $a = 1$, $b = 0$, $m > 0$, then equation (1.1) is reduced to the $p-$Laplace equation below

$$\begin{cases} -\Delta_p u+\lambda u = g(u), \quad in \quad \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|u|^2dx = m. \end{cases} \tag{1.3}$$

In [10], Li and Niu mainly studied the existence of non-trivial solutions to the following nonlinear $p-$Kirchhoff type equations with critical growth

$$\begin{cases} -\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^pdx\Big)\Delta_p u = |u|^{p^*-2}u+\mu f(x)|u|^{q-2}u, \quad x\in\mathbb{R}^N, \\ u\in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $a \geq 0$, $b > 0$, $1 < p < N$, $1 < q < p$, $p^* = \frac{Np}{N-p}$, $\mu \geq 0$, and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$. They proved it using Ekeland's variational principle and the Mountain pass theorem equation (1.4) has at least two nontrivial solutions under appropriate conditions.

We also study problem (1.1) by searching for its solutions satisfying the mass constraint. From thie viewpoint, the mass $m > 0$ is prescribed, while the frequency λ is unknown and will come out as a Lagrange multiplier. Hence, we call it fixed mass problem and the solution u is called a normalized solution. Normalized solutions to (1.1) are critical points of the energy functional

$$J(u) := \frac{a}{p}\int_{\mathbb{R}^3}|\nabla u|^pdx+\frac{b}{2p}\Big(\int_{\mathbb{R}^3}|\nabla u|^pdx\Big)^2-\int_{\mathbb{R}^3}G(u)dx,$$

on the constraint

$$S_m := \{u \in W^{1,p}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = m > 0\}.$$

In studying the positive normalized solutions via a global branch approach, the main ingredients are the study of the asymptotic behaviors of the positive solutions as $\lambda \to 0^+$ or $\lambda \to +\infty$ and the existence of an unbounded continuum of solutions in $(0,+\infty)\times W^{1,p}(\mathbb{R}^3)$. We remark that under assumptions (G1)-(G3), the authors in [6] obtained a connected positive solutions set

$$\tilde{S}_1 := \{(\lambda(s), v(s,x)) \in \mathbb{R}^+ \times H^1_{rad}(\mathbb{R}^N) : s \in \mathbb{R}, \lim_{s\to-\infty} \lambda(s) = 0, \lim_{s\to+\infty} \lambda(s) = +\infty\}.$$

We also can obtain a connected positive solutions set

$$\tilde{S}_2 := \{(\lambda(s), v(s,x)) \in \mathbb{R}^+ \times W^{1,p}_{rad}(\mathbb{R}^3) : s \in \mathbb{R}, \lim_{s\to-\infty} \lambda(s) = 0, \lim_{s\to+\infty} \lambda(s) = +\infty\} \tag{1.5}$$

to equation (1.3). And we also make clear the behavior of positive solutions of equation (1.3) as $\lambda \to 0^+$ as well as $\lambda \to +\infty$ in Section 2.

For the convenience of discussion, we put out the conditions involved in this work. For $g(s)$, the nonlinear term, we always assume the conditions as follows

(G1) $g \in C^1([0,+\infty))$, $g(s) > 0$ for $s > 0$;

(G2) There exists some $(\alpha,\beta) \in \mathbb{R}^2_+$ satisfying

$$p < \alpha, \beta < p^* := \frac{3p}{3-p} \quad if \quad 3 > p,$$

such that

$$\lim_{s\to0^+} \frac{g'(s)}{s^{\alpha-2}} = \mu_1(\alpha-1) > 0 \quad and \quad \lim_{s\to+\infty} \frac{g'(s)}{s^{\beta-2}} = \mu_2(\beta-1) > 0.$$

(G3) $-\Delta_p u = g(u)$ has no positive radial decreasing classical solution in $\mathbb{R}^3$.

Remark 1.1. It is easy to check that the following function $g(t)$ satisfies the conditions of G(1)-G(3),

$$g(t) = \begin{cases} |t|^{\alpha-2}t, & 0 < |t| < \theta, \\ |t|^{\beta-2}t, & |t| > \theta, \end{cases}$$

where θ is a constant.

We also denote

$$I_1 = \left(p, p + \frac{p^2}{3}\right) \quad and \quad J_1 = (p, p + \frac{2p^2}{3}).$$

For simplicity, we write $K_{11} := I_1 \times J_1$.

Now we state our main result as follows.

Theorem 1.2. *Let $a > 0, b > 0, 2 < p < N = 3$ and (G1)-(G3) hold. If $(\alpha,\beta) \in K_{11}$, then ,for any give $m > 0$, equation (1.1) possesses a positive normalized solution $(\lambda, u_\lambda) \in (0,+\infty) \times W^{1,p}_{rad}(\mathbb{R}^3)$.*

In this work, our arguments are based on a global branch approach and the Azzollini's correspondence. We establish some preliminaries for $N = 3$ and give the proof to Theorem 1.2 in Section 3.

2. PRELIMINARIES

Referring to the method in [1], we have the solution of the $p-$Kirchhoff equation (1.1) as follows.

Theorem 2.1. *Fix $a>0$, $b>0$, $\lambda>0$, and $2<p<N=3$. $u\in C^2(\mathbb{R}^3)\cap W^{1,p}(\mathbb{R}^3)$ is a solution to equation (1.1) if and only if there exist $v\in C^2(\mathbb{R}^3)\cap W^{1,p}(\mathbb{R}^3)$ solving $-\Delta_p v+\lambda v=g(v)$ in $\mathbb{R}^3$ and $t>0$ such that*

$$t^p a+t^{2p-3}b\int_{\mathbb{R}^3}|\nabla v|^p dx=1 \quad \text{and} \quad u(x)=v(tx).$$

Proof. We first prove that u is the solution to equation (1.1). Suppose $v\in C^2(\mathbb{R}^3)\cap W^{1,p}(\mathbb{R}^3)$ and $t>0$ are as in the statement of the theorem and set $u(x)=v(tx)=v_t(x)\in C^2(\mathbb{R}^3)\cap W^{1,p}(\mathbb{R}^3)$. We compute

$$\begin{aligned}
-\Delta_p u(x) &= -t^p\Delta_p v(tx)\\
&= t^p(g(v(tx))-\lambda v(tx))\\
&= \frac{g(v(tx))-\lambda v(tx)}{a+bt^{p-3}\int_{\mathbb{R}^3}|\nabla v|^p dx}\\
&= \frac{g(u(x))-\lambda u(x)}{a+b\int_{\mathbb{R}^3}|\nabla u|^p dx}.
\end{aligned}$$

Hence, u is the solution to equation (1.1).

Now we prove the $t^p a+t^{2p-3}b\int_{\mathbb{R}^3}|\nabla v|^p dx=1$. Suppose that $u\in C^2(\mathbb{R}^3)\cap W^{1,p}(\mathbb{R}^3)$ is a solution of equation (1.1) and set $t^p=\frac{1}{a+b\int_{\mathbb{R}^3}|\nabla u|^p dx}$, and $v(x)=u\left(\frac{1}{t}x\right)=u_{\frac{1}{t}}(x)$, where $t>0$. Moreover

$$\begin{aligned}
-\Delta_p v(x) &= -\frac{1}{t^p}\Delta_p u\left(\frac{1}{t}x\right)\\
&= \frac{1}{t^p}\frac{g(u(\frac{1}{t}x))-\lambda u(\frac{1}{t}x)}{a+b\int_{\mathbb{R}^3}|\nabla u(\frac{1}{t}x)|^p d(\frac{1}{t}x)}\\
&= g(v(x))-\lambda v(x)
\end{aligned}$$

and

$$t^p=\frac{1}{a+b\int_{\mathbb{R}^3}|\nabla u|^p dx}=\frac{1}{a+bt^{p-3}\int_{\mathbb{R}^3}|\nabla v|^p dx}.$$

Hence, $t^p a+t^{2p-3}b\int_{\mathbb{R}^3}|\nabla v|^p dx=1$. □

Theorem 2.2. *Let $N=3$ and assume that (G1)-(G3) hold. Let $U\in C_{r,0}(\mathbb{R}^3)$ be the unique positive solution to*

$$-\Delta_p U+U=\mu_1 U^{\alpha-1} \quad in \quad \mathbb{R}^3 \tag{2.1}$$

and $V\in C_{r,0}(\mathbb{R}^3)$ be the unique positive solution to

$$-\Delta_p V+V=\mu_2 V^{\beta-1} \quad in \quad \mathbb{R}^3, \tag{2.2}$$

where $C_{r,0}(\mathbb{R}^3)$ denotes the space of continuous radial functions vanishing at ∞. Then it holds,

(i) If $\{u_n\}_{n=1}^{\infty}\subset W^{1,p}(\mathbb{R}^3)$ are positive solutions to (1.3) with $\lambda=\lambda_n\to 0^+$, then

$$\lim_{n\to\infty}\|\nabla u_n\|_p=0$$

and

$$\lim_{s\to+\infty}\|u_n\|_p^p=\begin{cases}0, & \alpha<p+\frac{p^2}{3},\\ \|U\|_p^p, & \alpha=p+\frac{p^2}{3},\\ +\infty, & \alpha>p+\frac{p^2}{3}.\end{cases}$$

(ii) If $\{u_n\}_{n=1}^{\infty}\subset W^{1,p}(\mathbb{R}^3)$ are positive solutions to (1.3) with $\lambda=\lambda_n\to+\infty$, then

$$\lim_{n\to\infty}\|\nabla u_n\|_p=+\infty.$$

Proposition 2.3. *Let $N=3$ and assume that (G1)-(G2) hold. Let u be a non-negative classical solution to (1.3) with $\lambda>0$, which vanishes at infinity. Then, up to a translation, u is radially symmetric and decreases with respect to $|x|$.*

Proof. We assume $u(x_0)=\max_{x\in R^3}u(x)$. Since u is a non-negative classical solution to (1.3), we can obtain that

$$-\Delta_p u(x_0)+\lambda u(x_0)=g(u(x_0)).$$

Since $u(x_0)$ is the maximum, we have $\nabla u(x_0)=0$. Then $\Delta_p u(x_0)\le 0$ and $\lambda u(x_0)\le g(u(x_0))$. We have $g(u)>0$ for $u>0$. Hence, $u(x_0)>0$. Since u reaches its maximum at x_0 and vanishes at infinity, applied the maximum principle, we obtain that u is constant on the sphere centered on x_0. Then, up to a translation, such that x_0 is the origin. Therefore, u is constant on the sphere centered at the origin. We have that u is radially symmetric. We can express u as $u(x)=u(r)$, where $r=|x|$. Since u is a radial solution, then $u(r)$ is positive in $[0,+\infty)$ by the maximum principle and it solves

$$-((|u'(r)|^{p-2}u'(r))'-\frac{2}{r}(|u'(r)|^{p-2}u'(r))+\lambda u(r)=g(u(r))>0,$$

which implies that $(r^2|u'(r)|^{p-2}u'(r))'<0$. Hence, $r^2|u'(r)|^{p-2}u'(r)$ is decreasing in $[0,+\infty)$. By $r^2|u'(r)|^{p-2}u'(r)|_{r=0}=0$, we see that $r^2|u'(r)|^{p-2}u'(r)<0$ for $r>0$. Hence, $u'(r)<0$ for $r>0$. □

Proposition 2.4. *Let $1<q\le p_{sg}$ and $p_{ps}=\frac{N(p-1)}{N-p}=\frac{3(p-1)}{3-p}$. Then the inequality*

$$-\Delta_p u\ge u^q,\quad x\in\mathbb{R}^3 \tag{2.3}$$

does not possess any positive classical solution.

Proof. Take $\xi\in W(B_1), 0\le\xi\le 1$, with $\xi=1$ for $|x|\le\frac{1}{2}$, and let $m=\frac{pq}{pq-q-1}$. Fix $R>0$ and define $\varphi_R(x)=\xi^m(x/R)$. We observe that

$$\Delta_p\varphi_R=m^{p-1}R^{-p}[(m-1)(p-1)\xi^{(m-1)(p-1)-1}|\nabla\xi|^p+\xi^{(m-1)(p-1)}\Delta_p\xi](x/R).$$

Hence

$$|\Delta_p\varphi_R|\le CR^{-p}\xi^{(m-1)(p-1)-1}(x/R)_{\chi\{|x|>R/2\}}=CR^{-p}\varphi_R^{1/q}{}_{\chi\{|x|>R/2\}}.$$

Multiplying (2.3) by φ_R, integrating by parts, and using Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} u^q \varphi_R dx &\leq -\int_{\mathbb{R}^3} u \Delta_p \varphi_R dx \\ &\leq CR^{-p} \int_{R/2<|x|<R} u \varphi_R^{1/q} dx \\ &\leq CR^{3(q-1)/q-p} \Big(\int_{R/2<|x|<R} u^q \varphi_R dx \Big)^{1/q}.\end{aligned} \tag{2.4}$$

In particular, it follows that

$$\int_{\mathbb{R}^3} u^q \varphi_R dx \leq CR^{3-pq/(q-p+1)}. \tag{2.5}$$

If $q < p_{sg}$, i.e., $3 - pq/(q-p+1) < 0$, then this implies $u \equiv 0$ upon letting $R \to \infty$. If $q = p_{sg}$, then (2.5) implies $\int_{\mathbb{R}^3} u^q dx < \infty$. Therefore, the right-hand side of (2.4) goes to 0 as $R \to \infty$ and we conclude that $u \equiv 0$. □

The case of $\lambda \to 0^+$. Our first lemma gathers some properties of g that hold under (G1)-(G2). Its proof is by direct computations.

Lemma 2.5. *Let $N = 3$ and assume that (G1)-(G2) hold,*

(i) $\lim_{s\to 0^+} \frac{g(s)}{s^{\alpha-1}} = \mu_1$ *and* $\lim\limits_{s\to+\infty} \frac{g(s)}{s^{\beta-1}} = \mu_2$.

(ii) There exists some $C > 0$ such that $g(s) \leq C(s^{\alpha-1} + s^{\beta-1})$, $s \in \mathbb{R}^+$ and for any $M > 0$, there exists some $C_{M,1} \geq C_{M,2} > 0$ such that $C_{M,2}s^{\alpha-1} \leq g(s) \leq C_{M,1}s^{\beta-1}$ for all $s \in [0, M]$.

(iii) There exists some $\tilde{C} > 0$ such that $G(s) \leq \tilde{C}(s^\alpha + s^\beta)$, $s \in \mathbb{R}^+$, where $G(s) := \int_0^s g(t)dt$.

(iv) If $\alpha > \beta$, we have, for some $D > 0$, $g(s) \leq Ds^{\beta-1}$.

Lemma 2.6. *Let $N = 3$ and $g \in (\mathbb{R}^+, \mathbb{R}^+)$ be such that,*

$$\liminf_{s\to 0+} \frac{g(s)}{s^{\gamma-1}} > 0, \quad \textit{for some} \quad \gamma > p. \tag{2.6}$$

(i) If $N = 3$ assuming that $\gamma - 1 \leq \frac{N(p-1)}{N-p}$, (1.3) has no nontrivial non-negative classical solution when $\lambda = 0$.

(ii) If $N = 3$ assuming that $\gamma - 1 \leq \frac{Np-N+p}{N-p}$ and $sg(s) \leq p^* G(s)$ in $\mathbb{R}^+$, (1.3) has no nontrivial non-negative radial classical solution when $\lambda = 0$.

Proof. (i) Suppose that $u \geq 0$ is a classical solution to (1.3) with $\lambda = 0$. From (2.6) and the fact that u belongs to $L^\infty(\mathbb{R}^3)$ we can find a constant $C > 0$ such that $-\Delta_p u \geq Cu^{\gamma-1}$ in $\mathbb{R}^3$. Under our assumption on $\gamma > p$, the conclusion follows by Proposition 2.4.

(ii) We first remark that if $u \neq 0$ is a radial solution, then $u(r)$ is positive in $[0, +\infty)$ by the maximum principle and it solves

$$\begin{cases} -(|u'|^{p-2}u')' - \frac{2}{r}(|u'|^{p-2}u') = g(u), & r > 0, \\ u'(0) = 0. \end{cases} \tag{2.7}$$

Also, u decreases with respect to $r \in [0, +\infty)$. Indeed,

$$-((|u'(r)|^{p-2}u'(r))' + \frac{2}{r}(|u'(r)|^{p-2}u'(r))) = g(u(r)) > 0,$$

which implies that $(r^2|u'(r)|^{p-2}u'(r))' < 0$. Hence, $r^2|u'(r)|^{p-2}u'(r)$ is decreasing in $[0,+\infty)$. By $r^2|u'(r)|^{p-2}u'(r)|_{r=0} = 0$, we see that $r^2|u'(r)|^{p-2}u'(r) < 0$ for $r > 0$. Hence, $|u'(r)|^{p-2}u'(r) < 0$ for $r > 0$. This implies that $\tau := \lim\limits_{r\to+\infty} u(r) \geq 0$ exists. Since

$$g(\tau) = \lim_{r\to+\infty} -((|u'(r)|^{p-2}u'(r))' + \frac{2}{r}(|u'(r)|^{p-2}u'(r))) = 0,$$

we have that $\tau = 0$, i.e., $\lim_{r\to+\infty} u(r) = 0$. The corresponding Pohozaev function is defined by

$$P(r) := r^3\Big(\frac{u'(r)^p}{p} + G(u(r)) + \frac{3-p}{p}\frac{u(r)(|u'(r)|^{p-2}u'(r))}{r}\Big). \tag{2.8}$$

It is of class $C^1[0,+\infty)$ and satisfies $P(0) = 0$. A direct computation shows that

$$P'(r) := r^2\Big(3G(u(r)) - \frac{3-p}{p}u(r)g(u(r))\Big), \quad \forall r > 0.$$

Hence, under our assumption $sg(s) \leq p^*G(s)$ in $\mathbb{R}^+$, we see that $P(r)$ is non-decreasing in $[0,+\infty)$.

Let us now prove that $r(|u'(r)|^{p-2}u'(r)) + u(r) \geq 0$. It holds for $r = 0$. Suppose there exists some $r_0 > 0$ such that

$$T := r_0(|u'(r_0)|^{p-2}u'(r_0)) + u(r_0) < 0.$$

By $-\Delta_p u \geq 0$ and $u'(r) < 0$, we obtain $[r(|u'(r)|^{p-2}u'(r)) + u(r)]' \leq 0$. That is, $r(|u'(r)|^{p-2}u'(r)) + u(r)$ is non-increasing in $\mathbb{R}^+$. Thus

$$r(|u'(r)|^{p-2}u'(r)) + u(r) \leq T < 0, \quad \forall r > r_0.$$

For $s > t > r_0$, since u is positive, we have

$$-u(t) \leq \int_t^s |u'(\tau)|^{p-2}u'(\tau)d\tau \leq \int_t^s \frac{T}{\tau^{\frac{p-2}{p-1}}}d\tau = T(p-1)(s^{\frac{1}{p-1}} - t^{\frac{1}{p-1}}).$$

Let $s \to +\infty$. Since $T < 0$, we obtain a contradiction. Hence, $r(|u'(r)|^{p-2}u'(r)) + u(r) \geq 0$ for all $r \in [0,+\infty)$ and thus

$$\big||u'(r)|^{p-2}u'(r)\big| \leq \frac{u(r)}{r}, \quad r > 0. \tag{2.9}$$

Using the fact $u(r)$ is decreasing in $[0,+\infty)$, it follows from (2.7) and (2.9) that

$$\begin{aligned} G(u(r)) &= 2\int_r^\infty \frac{u'(t)^p}{t}dt - u'(r)^{p-1}u(r) \\ &\leq 2\int_r^\infty \frac{u'(t)^p}{t}dt \\ &\leq C\int_r^\infty \frac{u(t)^{\frac{p}{p-1}}}{t^{\frac{2p-1}{p-1}}}dt \leq C\frac{u(r)^{\frac{p}{p-1}}}{r^{\frac{p}{p-1}}}. \end{aligned} \tag{2.10}$$

Since $u \in L^\infty(\mathbb{R}^3)$, under our assumption on $\gamma > p$, there exists a constant $C > 0$ such that $G(u(r)) \geq Cu(r)^\gamma$, for all $r > 0$. Thus, using (2.10), we obtain that

$$u(r) \leq C_0 r^{-\frac{p}{\gamma(p-1)-p}} \tag{2.11}$$

for r large enough. Then by (2.9), we have that

$$\left||u'(r)|^{p-2}u'(r)\right| \leq C_1 r^{-\frac{\gamma(p-1)}{\gamma(p-1)-p}} \tag{2.12}$$

for $r>0$ large enough. From (2.8), (2.10), (2.11), (2.12) and $\gamma<\frac{3p}{3-p}$, we conclude $\lim\limits_{r\to+\infty}P(r)=0$. Hence, the monotonicity of $P(r)$ implies that $P(r)\equiv 0$. Thus $P'(r)\equiv 0$ in $r\in[0,+\infty)$. Namely,

$$u(r)g(u(r)) \equiv p^*G(u(r)), \quad r\in[0,+\infty). \tag{2.13}$$

By the Hospital's rule, (2.11) and (2.13) lead to

$$\begin{aligned} 0<b:=\lim_{s\to0+}\frac{sg(s)}{s^\gamma} &= \lim_{s\to0+}\frac{\gamma G(s)}{s^\gamma} = \lim_{r\to+\infty}\frac{\gamma G(u(r))}{u(r)^\gamma} \\ &= \lim_{r\to+\infty}\frac{\frac{\gamma}{p^*}g(u(r))u(r)}{u(r)^\gamma} = \lim_{s\to0+}\frac{\frac{\gamma}{p^*}g(s)s}{s^\gamma} = \frac{\gamma}{p^*}b, \end{aligned}$$

which implies that $\gamma=p^*$, a contradiction. □

Lemma 2.7. *Let $N=3$ and assume that (G1)-(G2) hold. Let $\{u_n\}_{n=1}^\infty\subset W^{1,p}(\mathbb{R}^3)$ be positive solutions to (1.3) with $\lambda=\lambda_n\to0^+$. Then* $\limsup_{n\to+\infty}\|u_n\|_\infty<+\infty$.

Proof. For any $M>0$, there exists $C_M>0$ such that for any non-negative solution $u\in W^{1,p}(\mathbb{R}^3)$ to (1.3) $\lambda\in(0,M)$. We proceed by assuming that there exists a sequence $(\lambda_n,u_n)\in(0,M)\times W^{1,p}(\mathbb{R}^3)$, where $u_n\in W^{1,p}(\mathbb{R}^3)$ is solution to (1.3) with $\lambda=\lambda_n$ and

$$\max_{x\in\mathbb{R}^3}u_n(x)\to+\infty \quad as \quad n\to+\infty.$$

We follow a blow up procedure introduced by Gidas and Spruck [3]. Without loss of generality, we may assume that $M_n:=u_n(0)=\max\limits_{x\in\mathbb{R}^3}u_n(x)$. Now we perform a rescaling, setting $x=\frac{y}{M_n^{\frac{\beta-p}{p}}}$ and defining $\tilde{u}_n:\mathbb{R}^3\to\mathbb{R}$ by

$$\tilde{u}_n(y):=\frac{1}{M_n}u_n\left(\frac{y}{M_n^{\frac{\beta-p}{p}}}\right).$$

Then $\tilde{u}_n(0)=\max\limits_{x\in\mathbb{R}^3}\tilde{u}_n(y)=1$ and

$$-\Delta_p\tilde{u}_n=\frac{g(M_n\tilde{u}_n)}{M_n^{\beta-1}}-\frac{\lambda_n}{M_n^{\beta-2}}\tilde{u}_n.$$

If $\alpha\leq\beta$, since $\|\tilde{u}_n\|_\infty=1$ and $M_n\to+\infty$, we obtain from Lemma 2.5 that

$$\frac{g(M_n\tilde{u}_n)}{M_n^{\beta-1}}\leq C(M_n^{\alpha-\beta}\tilde{u}_n^{\alpha-1}+\tilde{u}_n^{\beta-1})\leq C(\tilde{u}_n^{\alpha-1}+\tilde{u}_n^{\beta-1})\in L^\infty(\mathbb{R}^3).$$

If $\alpha>\beta$, we also have that

$$\frac{g(M_n\tilde{u}_n)}{M_n^{\beta-1}}\leq\frac{C(M_n\tilde{u}_n)^{\beta-1}}{M_n^{\beta-p+1}}=C\tilde{u}_n^{\beta-1}.$$

We also remark that

$$\frac{\lambda_n}{M_n^{\beta-2}}\tilde{u}_n \in L^\infty(\mathbb{R}^3).$$

Hence, applying standard elliptic estimates, and passing to a subsequence if necessary, we may assume that $\tilde{u}_n \to \tilde{u}$ in $W^{1,p}_{loc}(\mathbb{R}^3)$, where $\tilde{u}$ is a nontrivial and non-negative bounded radial solution to $-\Delta_p \tilde{u} = \mu_2 \tilde{u}^{\beta-1}$ in $\mathbb{R}^3$. At this point, Lemma 2.6 provides a contradiction. We have that $\max\limits_{x\in\mathbb{R}^3} u(x) \le C_M$. □

Lemma 2.8. *Let $N = 3$ and assume that (G1)-(G2) hold. Let $\{u_n\}_{n=1}^\infty \subset W^{1,p}(\mathbb{R}^3)$ be positive solutions to (1.3) with $\lambda = \lambda_n \to 0^+$. Then $\liminf_{n\to+\infty} \frac{\|u_n\|_\infty^{\alpha-p}}{\lambda_n} > 0$. Furthermore, if (G3) holds, then*

$$\|u_n\|_\infty \to 0 \quad as \quad n \to +\infty. \tag{2.14}$$

Proof. For any sequence (λ_n, u_n) with $\lambda_n \to 0^+$ and

$$-\Delta_p u_n + \lambda_n u_n = g(u_n) \quad in \quad \mathbb{R}^3, \quad 0 < u_n \in W^{1,p}(\mathbb{R}^3), \tag{2.15}$$

thanks to Lemma 2.7 and the invariance of (1.3) under translation, we may suppose that $u_n(0) = \|u_n\|_\infty \le M, \forall n \in \mathbb{N}$. Setting

$$\bar{u}_n(x) := \frac{1}{u_n(0)} u_n\left(\frac{x}{\sqrt[p]{\lambda_n}}\right),$$

we have $\bar{u}_n(0) = \|\bar{u}_n\|_\infty = 1$ and

$$-\Delta_p \bar{u}_n + u_n(0)^{2-p}\bar{u}_n = \frac{1}{\lambda_n u_n(0)^{p-1}} g(u_n(0)\bar{u}_n) \ in \ \mathbb{R}^3. \tag{2.16}$$

By standard regularity arguments, $\bar{u}_n$ is a classical solution to (1.3). Taking $x = 0$, we obtain

$$\begin{aligned} u_n(0)^{2-p} &= u_n(0)^{2-p}\bar{u}_n(0) \le -\Delta_p \bar{u}_n(0) + u_n(0)^{2-p}\bar{u}_n(0) \\ &= \frac{1}{\lambda_n u_n(0)^{p-1}} g(u_n(0)) \\ &\le C_{M,1}\frac{u_n(0)^{\alpha-p}}{\lambda_n}. \end{aligned}$$

Hence,

$$\liminf_{n\to+\infty} \frac{\|u_n\|_\infty^{\alpha-p}}{\lambda_n} > \frac{u_n(0)^{2-p}}{C_{M,1}} > 0.$$

Suppose now that $\liminf_{n\to+\infty} u_n(0) > 0$. Since $g(u_n) - \lambda_n u_n \in L^\infty(\mathbb{R}^3)$ in (2.15), passing to a subsequence, we may assume that $u_n \to u$ in $W^{1,p}_{loc}(\mathbb{R}^3)$ with $u(0) = \max_{x\in\mathbb{R}^3} u(x) > 0$ and u a non-negative bounded radial decreasing function which solves $-\Delta_p u = g(u)$ in $\mathbb{R}^3$. Hence, by (G3), we obtain that $u \equiv 0$, a contradiction to $u(0) > 0$. □

Lemma 2.9. *Let $N = 3$ and assume that (G1)-(G2) hold. Let $\{u_n\}_{n=1}^\infty \subset W^{1,p}(\mathbb{R}^3)$ be positive solutions to (1.3) with $\lambda = \lambda_n \to 0^+$. Then $\limsup_{n\to+\infty} \frac{\|u_n\|_\infty^{\alpha-p}}{\lambda_n} < +\infty$.*

Proof. We argue by contradiction and assume there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}(\mathbb{R}^3)$ of positive solutions to (1.3) with $\lambda = \lambda_n \to 0^+$ such that

$$\frac{\|u_n\|_\infty^{\alpha-p}}{\lambda_n} \to +\infty.$$

Without loss of generality, we can suppose that $u_n(0) = \|u_n\|_\infty, \forall n \in \mathbb{N}$. Setting

$$\tilde{u}_n(x) := \frac{1}{\|u_n\|_\infty} u_n\left(\frac{x}{\|u_n\|_\infty^{\frac{\alpha-p}{p}}}\right),$$

we have $\tilde{u}_n(0) = \|\tilde{u}_n\|_\infty = 1$ and

$$-\Delta_p \tilde{u}_n(x) = \frac{g(\|u_n\|_\infty \tilde{u}_n)}{\|u_n\|_\infty^{\alpha-1}} - \frac{\lambda_n}{\|u_n\|_\infty^{\alpha-2}} \tilde{u}_n. \tag{2.17}$$

In view of Lemmas 2.5 and 2.7, the right hand side of (2.17) is in $L^\infty(\mathbb{R}^3)$. Passing to a subsequence if necessary, we may assume that $\tilde{u}_n \to \tilde{u}$ in $W^{1,p}_{loc}(\mathbb{R}^3)$. Using (2.14), we obtain that $\tilde{u}_n$ is a non-negative bounded solution to $-\Delta_p \tilde{u}_n = \mu_1 \tilde{u}^{\alpha-1}$ in $\mathbb{R}^3$. Thus Lemma 2.6 provides a contradiction to $\tilde{u}(0) = \|\tilde{u}\|_\infty = 1$. □

Lemma 2.10. *Let $N = 3$ and assume that (G1)-(G3) hold. Let $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}(\mathbb{R}^3)$ be positive radial solutions to (1.3) with $\lambda = \lambda_n \to 0^+$. Defining*

$$\bar{u}_n(x) := \frac{1}{u_n(0)} u_n\left(\frac{x}{\sqrt[p]{\lambda_n}}\right), \tag{2.18}$$

then $\bar{u}_n(x) \to 0$ as $|x| \to +\infty$ uniformly in $n \in \mathbb{N}$.

Proof. By Lemmas 2.8 and 2.9, up to a subsequence, we may assume that

$$\frac{\|u_n\|_\infty^{\alpha-p}}{\lambda_n} \to C_0 > 0.$$

Take $\varepsilon > 0$ small such that $C_0 \mu_1 \varepsilon^{\alpha-1} < 1$. We argue by contradiction. We suppose there exists a sequence $r_n \to +\infty$ such that $\bar{u}_n(r_n) = \varepsilon$. By changing the origin to r_n and passing to the limit, we obtain a nontrivial solution $\bar{u}$ of the following equation,

$$-\Delta_p \bar{u}_n + u_n(0)^{2-p} \bar{u}_n = C_0 \mu_1 \bar{u}^{\alpha-1}, \quad r \in \mathbb{R}, \tag{2.19}$$

with $\bar{u}(0) = \varepsilon$, $\bar{u} \geq 0$ and bounded. By Proposition 2.3, we obtain that $\bar{u}$ is decreasing on $\mathbb{R}$. Hence, $\bar{u}$ has a limit $\bar{u}_+$ at $r = +\infty$ and a limit $\bar{u}_-$ at $r = -\infty$. In particular, $\bar{u}_\pm$ solve

$$-u_n(0)^{2-p} \bar{u}_\pm + C_0 \mu_1 \bar{u}_\pm^{\alpha-1} = 0, \quad r \in \mathbb{R}.$$

So by $\bar{u}_+ < \varepsilon \leq \bar{u}_-$, we obtain that $\bar{u}_+ = 0$ and

$$\bar{u}_- = \left(\frac{C_0 \mu_1}{u_n(0)^{2-p}}\right)^{\frac{1}{2-\alpha}}.$$

Then, from (2.19), we have that $-\Delta_p \bar{u}_n \leq 0$ on $\mathbb{R}$ necessarily $\bar{u}'(0) < 0$. Using again that $-\Delta_p \bar{u}_n \leq 0$ on $(-\infty, 0]$ we get a contradiction with the fact that $\bar{u}$ is bounded. □

Lemma 2.11. *Let $N=3$ and assume that (G1)-(G3) hold. Let $\{u_n\}_{n=1}^{\infty}$ be positive radial solutions to (1.3) with $\lambda=\lambda_n\to 0^+$. Defining $v_n(x):=\lambda_n^{\frac{1}{p-\alpha}}u_n\left(\frac{x}{\sqrt[p]{\lambda_n}}\right)$ and passing to a subsequence if necessary, it holds $v_n\to U$ in $C_{r,0}(\mathbb{R}^3)$ as $n\to+\infty$, where U is given by Theorem 2.2.*

Proof. By Lemmas 2.8 and 2.9, we have

$$0<\liminf_{n\to\infty}\frac{u_n(0)^{\alpha-p}}{\lambda_n}\le\limsup_{n\to\infty}\frac{u_n(0)^{\alpha-p}}{\lambda_n}<+\infty. \tag{2.20}$$

Note that v_n solves the equation

$$-\Delta_p v_n+\lambda_n^{\frac{2-p}{\alpha-p}}v_n=\frac{g(\lambda_n^{\frac{1}{\alpha-p}}v_n)}{\lambda_n^{\frac{\alpha-1}{\alpha-p}}}\quad in\quad \mathbb{R}^3. \tag{2.21}$$

By standard regularity argument, it is easy to see that $\{v_n\}$ is equi-continuous on bounded sets. On the other hand, we remark that $v_n(x)=u_n(0)\lambda_n^{\frac{-1}{\alpha-p}}\bar{u}_n(x)$, where $\bar{u}_n(x)$ is given by (2.18). Hence, by Lemma 2.10 and (2.20), we see that $\{v_n\}$ decay to 0 uniformly at ∞. Hence, $\{v_n\}$ is pre-compact in $C_{r,0}(\mathbb{R}^3)$. Passing to a subsequence if necessary, we may assume that $v_n\to U\in C_{r,0}(\mathbb{R}^3)$, where $U\in C_{r,0}(\mathbb{R}^3)$ solves (2.1) with $U(0)=\max_{x\in\mathbb{R}^3}U(x)>0$. □

Lemma 2.12. *Let $N=3$ and assume that (G1)-(G3) hold. Let $\{u_n\},\{\lambda_n\},\{v_n\}$, and U be given by Lemma 2.11. Then $v_n\to U$ in $W^{1,p}(\mathbb{R}^3)$.*

Proof. We can rewrite (2.21) as

$$-\Delta_p v_n+\lambda_n^{\frac{2-p}{\alpha-p}}\left(1-\frac{g(\lambda_n^{\frac{1}{\alpha-p}}v_n)}{\lambda_n^{\frac{\alpha-p+1}{\alpha-p}}v_n}\right)v_n=0\quad in\quad\mathbb{R}^3. \tag{2.22}$$

By the proof of Lemma 2.11, we know that v_n is pre-compact in $C_{r,0}(\mathbb{R}^3)$. In view of Lemma 2.5, we can find some $R>0$ large enough such that

$$1-\frac{g(\lambda_n^{\frac{1}{\alpha-p}}v_n)}{\lambda_n^{\frac{\alpha-p+1}{\alpha-p}}v_n}>\frac{1}{2},\quad\forall|x|\ge R,\quad\forall n\in\mathbb{N}.$$

Then we have

$$-\Delta_p v_n(x)+\lambda_n^{\frac{2-p}{\alpha-p}}\frac{1}{2}v_n(x)\le 0,\quad\forall|x|\ge R,\quad\forall n\in\mathbb{N}.$$

It follows by the comparison principle, Lemma 2.11, and $v_n\to\tilde{U}$ in $C_{r,0}(\mathbb{R}^3)$ that one can find some $C_1,C_2>0$ such that $v_n(x)\le C_1e^{-C_2|x|}$ in $\mathbb{R}^3$ for all $n\in\mathbb{N}$. Then, for n large, we can write (2.22) as

$$-\Delta_p v_n+\lambda_n^{\frac{2-p}{\alpha-p}}v_n(1-(\mu_1+o(1))u_n^{\alpha-2}\lambda_n^{p-2})=0$$

By (2.16), we compute

$$\|\nabla v_n\|_{L^p(\mathbb{R}^3)}^p=\int_{\mathbb{R}^3}\left|\nabla\lambda_n^{\frac{1}{p-\alpha}}u_n\left(\frac{x}{\sqrt[p]{\lambda_n}}\right)\right|^p dx=\lambda_n^{\frac{p}{p-\alpha}}\frac{1}{\lambda_n}\int_{\mathbb{R}^3}|\nabla u_n(x)|^p dx=\lambda_n^{\frac{\alpha}{p-\alpha}+\frac{3}{p}}\|\nabla u_n\|_p^p$$

and

$$\|v_n\|_p^p = \int_{\mathbb{R}^3} |v_n|^p dx = \int_{\mathbb{R}^3} \left| \lambda_n^{\frac{1}{p-\alpha}} u_n\left(\frac{x}{\sqrt[p]{\lambda_n}}\right)\right|^p dx = \lambda_n^{\frac{p}{p-\alpha}+\frac{3}{p}} \|u_n\|_p^p.$$

Since we can obtain that

$$\begin{cases} \|\nabla u_n\|_p^p = \lambda_n^{1+\frac{p}{\alpha-p}-\frac{3}{p}} \|\nabla v_n\|_p^p, \\ \|u_n\|_p^p = \lambda_n^{\frac{p}{\alpha-p}-\frac{3}{p}} \|v_n\|_p^p. \end{cases} \tag{2.23}$$

We known that $\|v_n\|_p^p$ and $\|\nabla v_n\|_p^p$ are comparable. There exist some $C_3, C_4 > 0$ and $C_5, C_6 > 0$ such that

$$C_3 \leq \|v_n\|_p^p \leq C_4, \quad \forall N \in \mathbb{N} \quad and \quad C_5 \leq \|\nabla v_n\|_p^p \leq C_6.$$

which implies that v_n is bounded in $W^{1,p}(\mathbb{R}^3)$. Without loss of generality, we can assume that $v_n \rightharpoonup U$ in $W^{1,p}(\mathbb{R}^3)$. By the Lebesgue dominated convergence theorem, we see that $v_n \to \tilde{U}$ in $L^p(\mathbb{R}^3)$. So $\tilde{U}$ solves

$$-\Delta_p \tilde{U} + \tilde{U} = \mu_1 \tilde{U}^{\alpha-1} \quad in \quad \mathbb{R}^3 \quad with \quad \lim_{|x|\to+\infty} \tilde{U}(x) = 0.$$

Hence $\tilde{U} = U$ and it follows by (2.23) that $\|\nabla v_n\|_p^p \to \|\nabla U\|_p^p$. which implies that $v_n \to U$ in $W^{1,p}(\mathbb{R}^3)$.

Proof of Theorem 2.2 (i)

Step 1. Let us prove that

$$\lim_{s\to+\infty} \|u_n\|_p^p = \begin{cases} 0, & \alpha < p + \frac{p^2}{3}, \\ \|U\|_p^p, & \alpha = p + \frac{p^2}{3}, \\ +\infty, & \alpha > p + \frac{p^2}{3}. \end{cases}$$

By (2.23), we have that $\|u_n\|_p^p = \lambda_n^{\frac{p}{\alpha-p}-\frac{3}{p}} \|v_n\|_p^p$. Let us further analyze according to the value range of α

$$\lim_{s\to+\infty} \|u_n\|_p^p = \begin{cases} 0, & \alpha < p + \frac{p^2}{3} \Rightarrow \frac{p}{\alpha-p} - \frac{3}{p} > 0, \\ \|U\|_p^p, & \alpha = p + \frac{p^2}{3} \Rightarrow \frac{p}{\alpha-p} - \frac{3}{p} = 0, \\ +\infty, & \alpha > p + \frac{p^2}{3} \Rightarrow \frac{p}{\alpha-p} - \frac{3}{p} < 0. \end{cases}$$

Step 2. $\lim_{n\to+\infty} \|\nabla u_n\|_p = 0$. By Lemma 2.11, we obtain that

$$\|\nabla v_n\|_p \to \|\nabla U\|_p \quad in \quad C_{r,0}(\mathbb{R}^3) \quad as \quad n \to +\infty$$

and $U(x)$ is a positive solution. Hence, we have that $\|\nabla U\|_p$ is finite. By (2.23), we have that $\|\nabla u_n\|_p^p = \lambda_n^{1+\frac{p}{\alpha-p}-\frac{3}{p}} \|\nabla v_n\|_p^p$ and $\lambda_n^{1+\frac{p}{\alpha-p}-\frac{3}{p}} \to 0$ as $\lambda_n \to 0^+$. Hence, $\|\nabla u_n\|_p^p \to 0$. We obtain that $\|\nabla u_n\|_p \to 0$. □

The case of $\lambda \to +\infty$

Lemma 2.13. *Let $N = 3$ and assume that (G1)-(G2) hold. Let $\{u_n\}_{n=1}^\infty \subset W^{1,p}(\mathbb{R}^3)$ be positive solutions to (1.3) with $\lambda = \lambda_n \to +\infty$. Then*

$$\liminf_{n\to+\infty} \|u_n\|_\infty = +\infty \tag{2.24}$$

and

$$\liminf_{n\to+\infty}\frac{\|u_n\|_\infty^{\beta-p}}{\lambda_n}>0. \tag{2.25}$$

Proof. By regularity, for any fixed n, $u_n\in W^{1,p}(\mathbb{R}^3)$, we may suppose that $u_n(0)=\|u_n\|_\infty$. We let $\bar{u}_n$ be given as in the proof of Lemma 2.8. Thus (2.16) holds. Taking $x=0$, by Lemma 2.5, we obtain that

$$\begin{aligned}u_n(0)^{2-p}&=u_n(0)^{2-p}\bar{u}_n(0)\le-\Delta_p\bar{u}_n(0)+u_n(0)^{2-p}\bar{u}_n(0)\\&\le\frac{1}{\lambda_n u_n(0)^{p-1}}g(u_n(0))\\&\le C\frac{u_n(0)^{\alpha-p}+u_n(0)^{\beta-p}}{\lambda_n}.\end{aligned}$$

Since $\lambda_n\to+\infty$ and $\min\{\alpha,\beta\}>p$, we obtain (2.24). Furthermore, if $\alpha\le\beta$, (2.25) trivially holds. If $\alpha>\beta$, by Lemma 2.5, $g(s)\le Cs^{\beta-1},s\in\mathbb{R}^+$, and we obtain (2.25). □

Lemma 2.14. *Let $N=3$ and assume that (G1)-(G2) hold. Let $\{u_n\}_{n=1}^\infty\subset W^{1,p}(\mathbb{R}^3)$ be positive solutions to (1.3) with $\lambda=\lambda_n\to+\infty$. Then* $\limsup_{n\to+\infty}\frac{\|u_n\|_\infty^{\beta-p}}{\lambda_n}<+\infty$.

Proof. As in the proof of Lemma 2.9, we argue by assuming that there exists a sequence $\{u_n\}_{n=1}^\infty\subset W^{1,p}(\mathbb{R}^3)$ of positive solutions to (1.3) with $\lambda=\lambda_n\to+\infty$ such that

$$\frac{\|u_n\|_\infty^{\beta-p}}{\lambda_n}\to+\infty. \tag{2.26}$$

Without loss of generality, we suppose that $u_n(0)=\|u_n\|_\infty,\forall n\in\mathbb{N}$. Setting

$$\tilde{u}_n(x):=\frac{1}{\|u_n\|_\infty}u_n\left(\frac{x}{\|u_n\|_\infty^{\frac{\beta-p}{p}}}\right),$$

we have $\tilde{u}_n(0)=\|\tilde{u}_n\|_\infty=1$ and

$$-\Delta_p\tilde{u}_n(x)=\frac{g(\|u_n\|_\infty\tilde{u}_n)}{\|u_n\|_\infty^{\beta-1}}-\frac{\lambda_n}{\|u_n\|_\infty^{\beta-2}}\tilde{u}_n. \tag{2.27}$$

By Lemma 2.5, the right hand side of (2.27) is in $L^\infty(\mathbb{R}^3)$. Passing to the limit, we may assume that $\tilde{u}_n\to\tilde{u}$ in $W^{1,p}_{loc}(\mathbb{R}^3)$, where $\tilde{u}$ is a non negative bounded solution to $-\Delta_p\tilde{u}_n=\mu_2\tilde{u}^{\beta-1}$ in $\mathbb{R}^3$. Here, Lemma 2.6 provides a contradiction to $\tilde{u}(0)=1$. □

Lemma 2.15. *Let $N=3$ and assume that (G1)-(G2) hold. Let $\{u_n\}_{n=1}^\infty\subset W^{1,p}(\mathbb{R}^3)$ be positive radial solutions to (1.3) with $\lambda=\lambda_n\to+\infty$. Defining*

$$v_n(x):=\lambda_n^{\frac{1}{p-\beta}}u_n\left(\frac{x}{\sqrt[p]{\lambda_n}}\right) \tag{2.28}$$

and passing to a subsequence if necessary, one has $v_n\to V$ in $C_{r,0}(\mathbb{R}^3)$ as $n\to+\infty$, where V is given by Theorem 2.2.

Proof. By Lemmas 2.13 and 2.14, one has

$$0 < \liminf_{n\to\infty} \frac{u_n(0)^{\beta-p}}{\lambda_n} \le \limsup_{n\to\infty} \frac{u_n(0)^{\beta-p}}{\lambda_n} < +\infty,$$

which implies $\{v_n\}$ is uniformly bounded in $L^\infty(\mathbb{R}^3)$. We note that v_n solves

$$-\Delta_p v_n + \lambda_n^{\frac{2-p}{\beta-p}} v_n = \frac{g(\lambda_n^{\frac{1}{\beta-p}} v_n)}{\lambda_n^{\frac{\beta-1}{\beta-p}}} \quad in \quad \mathbb{R}^3. \tag{2.29}$$

If $\alpha \le \beta$, by Lemma 2.5 and $\lambda_n \to +\infty$, we have that

$$\frac{g(\lambda_n^{\frac{1}{\beta-p}} v_n)}{\lambda_n^{\frac{\beta-1}{\beta-p}}} \le C(\lambda_n^{\frac{\alpha-\beta}{\beta-p}} v_n^{\alpha-1} + v_n^{\beta-1}),$$

which implies the right hand side of (2.29) is in $L^\infty(\mathbb{R}^3)$. If $\alpha > \beta$, still by Lemma 2.5, we have that

$$\frac{g(\lambda_n^{\frac{1}{\beta-p}} v_n)}{\lambda_n^{\frac{\beta-1}{\beta-p}}} \le C v_n^{\beta-1},$$

the right hand side of (2.29) is also of $L^\infty(\mathbb{R}^3)$. Thus, in view of standard elliptic estimates, we may assume that $v_n \to V$ in $W^{1,p}_{loc}(\mathbb{R}^3)$. Hence, $\{v_n\}$ is equi-continuous on bounded sets. Also, proceeding exactly as in the proof of Lemma 2.10 we may check that $\{v_n\}$ decay to 0 uniformly at ∞. Hence, $\{v_n\}$ is pre-compact in $C_{r,0}(\mathbb{R}^3)$ and $v_n \to V$ in $C_{r,0}(\mathbb{R}^3)$ where $0 < V \in C_{r,0}(\mathbb{R}^3)$ solves (2.2). □

Now, adapting in a straightforward way the proof of Lemma 2.12, we also have the following.

Lemma 2.16. *Let $N = 3$ and assume that (G1)-(G2) hold. Let $\{u_n\}, \{\lambda_n\}, \{v_n\}$ and V be given by Lemma 2.15. Then $v_n \to V \quad in \quad W^{1,p}(\mathbb{R}^3)$.*

Proof of the Theorem 2.2 (ii)

By Lemma 2.15, we obtain that $\|\nabla v_n\|_p \to \|\nabla V\|_p$ in $C_{r,0}(\mathbb{R}^3)$ as $n \to +\infty$ and $V(x)$ is a positive solution. Hence, we have that $\|\nabla V\|_p$ is finite. By (2.28), we have that $\|\nabla u_n\|_p^p = \lambda_n^{1+\frac{p}{\beta-p}-\frac{3}{p}} \|\nabla v_n\|_p^p$ and $\lambda_n^{1+\frac{p}{\beta-p}-\frac{3}{p}} \to +\infty$ as $\lambda_n \to +\infty$. It follows that $\|\nabla u_n\|_p^p \to +\infty$. We obtain that $\|\nabla u_n\|_p \to +\infty$.

3. Proof of The Main Results

Lemma 3.1. *Let $a > 0, b > 0$, and $2 < p < N = 3$. For any $v \in W^{1,p}(\mathbb{R}^3)$, there exists a unique $t_v > 0$ such that t_v solves $t^p a + t^{2p-3} b \int_{\mathbb{R}^3} |\nabla v|^p dx - 1 = 0$. Furthermore, t_v depends continuously on $v \in W^{1,p}(\mathbb{R}^3)$ and $t_v^p = \frac{1}{a}(1+o(1))$ as $\|\nabla v\|_p \to 0$, while $t_v^{2p-3} b \int_{\mathbb{R}^3} |\nabla v|^p dx = 1 + o(1)$ as $\|\nabla v\|_p \to +\infty$.*

Proof. For any $v \in W^{1,p}(\mathbb{R}^3)$, we set

$$f(t) = t^p a + t^{2p-3} b \int_{\mathbb{R}^3} |\nabla v|^p dx - 1 \in C^1([0, +\infty)).$$

Since $a>0$, $b>0$, $2<p<N=3$, we have

$$f'(t)=pat^{p-1}+(2p-3)t^{2p-4}b\int_{\mathbb{R}^3}|\nabla v|^p dx>0,\quad t>0,$$

which implies that $f(t)$ increases strictly in $[0,+\infty)$. On the other hand, by $N=3$, we see that $f(0)=-1<0$ and $\lim_{t\to+\infty}f(t)=+\infty$. Hence, there exists a unique $t_v>0$ such that $f(t_v)=0$, $f(t)<0$ for $t\in[0,t_v)$, and $f(t)>0$ for $t>t_v$.

Let $v_n\to v_0$ in $W^{1,p}(\mathbb{R}^3)$, we have that $\|\nabla v_n\|_p^p\to\|\nabla v_0\|_p^p$ as $n\to\infty$. By the definition of t_{v_n}, it is easy to see that $\{t_{v_n}\}$ is bounded in $\mathbb{R}$. Up to a subsequence, still denoted by t_{v_n}, we may assume that $t_{v_n}\to\bar{t}\geq 0$. Hence,

$$a\bar{t}^p+\bar{t}^{2p-3}b\int_{\mathbb{R}^3}|\nabla v_0|^p dx-1=\lim_{n\to+\infty}\left[at_{v_n}^p+t_{v_n}^{2p-3}b\int_{\mathbb{R}^3}|\nabla v_n|^p dx-1\right]=0,$$

which implies that $t_{v_0}=\bar{t}$. That is, $t_{v_n}\to t_{v_0}$.

In particular, if $v_n\to 0$ in $W^{1,p}(\mathbb{R}^3)$, we have that

$$bt_v^{2p-3}\int_{\mathbb{R}^3}|\nabla v_n|^p dx=o(1)$$

since $\{t_{v_n}\}$ is bounded. Thus $t_{v_n}^p=\frac{1}{a}(1+o(1))$. If $\|\nabla v\|_p\to+\infty$, then by $b>0$ we have that $t_{v_n}\to 0$. Thus, $bt_v^{2p-3}\int_{\mathbb{R}^3}|\nabla v_n|^p dx\to 1$ as $n\to\infty$. Assume that $a>0$, $b>0$, and $2<p<N=3$. Let $\tilde{S}_2$ be the connected positive solutions branch given by equation (1.5). For any $s>0$, let $t_{v(s,x)}$ be the unique number determined by Lemma 3.1 and denoted by t_s for simplicity. Put $u(s,x):=v(s,t_sx)$. Let $v(s,x)\in W^{1,p}(\mathbb{R}^3)$ be given in equation (1.5). There exists some $s^*\in\mathbb{R}$. Let us introduce the function

$$\rho:\mathscr{C}\to\mathbb{R}^+,\quad(\lambda(s),u(s,\cdot))\mapsto\|u(s,\cdot)\|_p^p.$$

□

We divide the proof of Theorem 1.2 into two steps.

Step 1. $\mathscr{C}:=\{(\lambda(s),u(s,x)):s\in\mathbb{R}\}$ is a connected positive solution branch of equation (1.1).

For any $s\in\mathbb{R}$, since $(\lambda(s),v(s,x))$ is a positive solution to equation (1.3), by the definition of t_s and $u(s,x)$, Theorem 2.1 implies that $(\lambda(s),u(s,x))$ is a positive solution to equation (1.1).

Next, we prove that $\mathscr{C}$ is connected in $\mathbb{R}\times W^{1,p}(\mathbb{R}^3)$. Indeed, for any $s^*\in\mathbb{R}$ and $s_n\to s^*$, it follows that $\lambda(s_n)\to\lambda(s^*)$ since $\tilde{S}_2$ is connected. Furthermore, $v(s_n,x)\to v(s^*,x)$ in $W^{1,p}(\mathbb{R}^3)$. So $\|\nabla v(s_n,\cdot)\|_p^p\to\|\nabla v(s^*,\cdot)\|_p^p$ as $n\to+\infty$. It follows by Lemma 3.1 that $t_{s_n}\to t_{s*}$ as $n\to+\infty$. Combining with $v(s_n,x)\to v(s^*,x)$ in ,$W^{1,p}(\mathbb{R}^3)$, we obtain that $u(s_n,x)\to u(s^*,x)$ in $W^{1,p}(\mathbb{R}^3)$. Hence, $\mathscr{C}$ is connected in $\mathbb{R}\times W^{1,p}(\mathbb{R}^3)$.

Step 2. ρ is onto. In other words, $\forall m>0$, $\exists(\lambda,u)\in\tilde{S}_2$ such that $\rho(\lambda,u)=m$.

For $(\alpha,\beta)\in K_{11}$, by Theorem 2.2, we have that $\|\nabla v(s,\cdot)\|_p\to 0$ as $s\to-\infty$. By Lemma 3.1, we have that $t_s^p\to\frac{1}{a}$ as $s\to-\infty$. A direct computation shows that

$$\|\nabla u(s,\cdot)\|_p^p=\int_{\mathbb{R}^3}|\nabla u(s,\cdot)|^p dx=\int_{\mathbb{R}^3}|\nabla v(s,t\cdot)|^p dx=t_s^{p-3}\|\nabla v(s,\cdot)\|_p^p$$

and

$$\|u(s,\cdot)\|_p^p=\int_{\mathbb{R}^3}|u(s,\cdot)|^p dx=\int_{\mathbb{R}^3}|v(s,t\cdot)|^p dx=t_s^{-3}\|v(s,\cdot)\|_p^p. \tag{3.1}$$

Combining with $\|\nabla v(s,\cdot)\|_p \to 0$ as $s \to -\infty$, we have $\|\nabla u(s,\cdot)\|_p^p \to 0$ as $s \to -\infty$. By Theorem 2.2, we have that

$$\lim_{s\to+\infty} \|v(s,\cdot)\|_p^p = \begin{cases} 0, & \alpha < p + \frac{p^2}{3}, \\ \|U\|_p^p, & \alpha = p + \frac{p^2}{3}, \\ +\infty, & \alpha > p + \frac{p^2}{3}. \end{cases}$$

By equation (3.1), we obtain that

$$\lim_{s\to-\infty} \|u(s,\cdot)\|_p^p = \begin{cases} 0, & \alpha < p + \frac{p^2}{3}, \\ a^{\frac{3}{p}}\|U\|_p^p, & \alpha = p + \frac{p^2}{3}, \\ +\infty, & \alpha > p + \frac{p^2}{3}. \end{cases}$$

Hence, there exist $\{s_n\} \subset \mathbb{R}^3$ with $s_n \to -\infty$ and $\|u(s_n,\cdot)\|_p^p \to 0$.

Next, by Theorem 2.2, we have that $\|\nabla v(s,\cdot)\|_p^p \to +\infty$ as $s \to +\infty$. Thus Lemma 3.1 implies that

$$\frac{t_s}{\left(\frac{1}{b\int_{\mathbb{R}^3}|\nabla v(s,\cdot)|^p dx}\right)^{\frac{1}{2p-3}}} \to 1 \quad as \quad s \to +\infty.$$

Recall from equation (2.28) and Lemma 2.16 that

$$\phi(s,x) := \lambda(s)^{\frac{1}{p-\beta}} v\left(s, \left(\frac{x}{\sqrt[p]{\lambda(s)}}\right)\right) \to V \quad in \quad W^{1,p}(\mathbb{R}^3) \quad as \quad s \to +\infty$$

and a direct calculation shows that

$$\begin{aligned} \|\nabla\phi(s,\cdot)\|_p^p &= \int_{\mathbb{R}^3} |\nabla\phi(s,\cdot)|^p dx \\ &= \int_{\mathbb{R}^3} \left|\nabla\lambda(s)^{\frac{1}{p-\beta}} v\left(s, \left(\frac{x}{\sqrt[p]{\lambda(s)}}\right)\right)\right|^p dx \\ &= \lambda(s)^{\frac{p}{p-\beta}} \frac{1}{\lambda(s)} \int_{\mathbb{R}^3} |\nabla v(s,\cdot)|^p dx \\ &= \lambda(s)^{\frac{\beta}{p-\beta}+\frac{3}{p}} \|\nabla v(s,\cdot)\|_p^p \end{aligned}$$

and

$$\begin{aligned} \|\phi(s,\cdot)\|_p^p &= \int_{\mathbb{R}^3} |\phi(s,\cdot)|^p dx \\ &= \int_{\mathbb{R}^3} \left|\lambda(s)^{\frac{1}{p-\beta}} v\left(s, \frac{x}{\sqrt[p]{\lambda_n}}\right)\right|^p dx \\ &= \lambda(s)^{\frac{p}{p-\beta}+\frac{3}{p}} \|v(s,\cdot)\|_p^p. \end{aligned}$$

Observe that

$$\begin{cases} \|\nabla v(s,\cdot)\|_p^p = \lambda(s)^{1+\frac{p}{\beta-p}-\frac{3}{p}} \|\nabla\phi(s,\cdot)\|_p^p, \\ \|v(s,\cdot)\|_p^p = \lambda(s)^{\frac{p}{\beta-p}-\frac{3}{p}} \|\phi(s,\cdot)\|_p^p. \end{cases}$$

Then,

$$\begin{aligned}\lim_{s\to+\infty}\|\nabla u(s,\cdot)\|_p^p &= \lim_{s\to+\infty} t_s^{p-3}\|\nabla v(s,\cdot)\|_p^p\\ &= \lim_{s\to+\infty}\left(b\int_{\mathbb{R}^3}|\nabla v(s,\cdot)|^p dx\right)^{\frac{3-p}{2p-3}}\|\nabla v(s,\cdot)\|_p^p\\ &= \lim_{s\to+\infty} b^{\frac{3-p}{2p-3}}\left(\int_{\mathbb{R}^3}|\nabla v(s,\cdot)|^p dx\right)^{\frac{p}{2p-3}}\\ &= b^{\frac{3-p}{2p-3}}(\|\nabla V\|_p^p)^{\frac{p}{2p-3}}\lim_{s\to+\infty}\lambda(s)^{(1+\frac{p}{\beta-p}-\frac{3}{p})\cdot\frac{p}{2p-3}}\\ &= +\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{s\to+\infty}\|u(s,\cdot)\|_p^p &= \lim_{s\to+\infty} t_s^{-3}\|v(s,\cdot)\|_p^p\\ &= \lim_{s\to+\infty}\left(\frac{1}{b\int_{\mathbb{R}^3}|\nabla v(s,\cdot)|^p dx}\right)^{\frac{-3}{2p-3}}\|v(s,\cdot)\|_p^p\\ &= b^{\frac{3}{2p-3}}\lim_{s\to+\infty}\left(\int_{\mathbb{R}^3}|\nabla v(s,\cdot)|^p dx\right)^{\frac{3}{2p-3}}\|v(s,\cdot)\|_p^p\\ &= b^{\frac{3}{2p-3}}\lim_{s\to+\infty}\left(\int_{\mathbb{R}^3}|\nabla V|^p dx\right)^{\frac{3}{2p-3}}[\lambda(s)^{1+\frac{p}{\beta-p}-\frac{3}{p}}]^{\frac{3}{2p-3}}\int_{\mathbb{R}^3}|V|^p dx\lambda(s)^{\frac{p}{\beta-p}-\frac{3}{p}}\\ &= b^{\frac{3}{2p-3}}\left(\int_{\mathbb{R}^3}|\nabla V|^p dx\right)^{\frac{3}{2p-3}}\int_{\mathbb{R}^3}|V|^p dx\lim_{s\to+\infty}\lambda(s)^{(1+\frac{p}{\beta-p}-\frac{3}{p})\cdot\frac{3}{2p-3}+\frac{p}{\beta-p}-\frac{3}{p}}\\ &= b^{\frac{3}{2p-3}}\left(\int_{\mathbb{R}^3}|\nabla V|^p dx\right)^{\frac{3}{2p-3}}\int_{\mathbb{R}^3}|V|^p dx\lim_{s\to+\infty}\lambda(s)^{\sigma},\end{aligned}$$

where

$$\sigma := \left(1+\frac{p}{\beta-p}-\frac{3}{p}\right)\cdot\frac{3}{2p-3}+\frac{p}{\beta-p}-\frac{3}{p} = \frac{2p^2-3(\beta-p)}{(2p-3)(\beta-p)}.$$

Note that

$$\sigma\begin{cases} >0, & \beta<p+\frac{2p^2}{3},\\ =0, & \beta=p+\frac{2p^2}{3},\\ <0, & \beta>p+\frac{2p^2}{3}.\end{cases}$$

Hence,

$$\lim_{s\to+\infty}\|u(s,\cdot)\|_p^p = \begin{cases} +\infty, & \beta<p+\frac{2p^2}{3},\\ b^{\frac{3}{2p-3}}(\|\nabla V\|_p^p)^{\frac{3}{2p-3}}\|V\|_p^p, & \beta=p+\frac{2p^2}{3},\\ 0, & \beta>p+\frac{2p^2}{3}.\end{cases}$$

Hence, there exists $\{s_n'\}\subset\mathbb{R}$ with $s_n'\to+\infty$ and $\|u(s_n',\cdot)\|_p^p\to+\infty$. Since $\mathscr{C}$ is connected, it follows that ρ is onto. In other words, $\forall m>0$, $\exists(\lambda,u)\in\tilde{S}_2$ such that $\rho(\lambda,u)=m$.

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