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# COMPLETELY LI-YORKE CHAOTIC HOMEOMORPHISMS WITH POSITIVE ENTROPY

ZIJIE LIN[1,*], XIANGDONG YE[1]

School of Mathematical sciences, University of Science and Technology of China, Hefei 230026, China

**Abstract.** It is an open problem whether a homeomorphism on a compact metric space satisfying that each proper pair is either positively or negatively Li-Yorke, called completely Li–Yorke chaotic, can have positive entropy. In this paper, an affirmative answer to this question is given. Indeed, for each homeomorphism $T$ with positive entropy such that each proper pair is not two-sided asymptotic, a completely Li-Yorke chaotic homeomorphism with positive entropy associated with the given homeomorphism can be constructed.

**Keywords.** Li-Yorke chaos; Proximal; Topological entropy.

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## 1. Introduction

A pair $(X,T)$ is called a *topological dynamical system* (t.d.s. for short) if $X$ is a compact metric space with a metric $d_X$ and $T: X \to X$ is a homeomorphism. A *proper* pair of $X$ is a pair $(x,y) \in X \times X$ with $x \neq y$. Below we state our main results and the ideas of the construction.

1.1. **Main results.** Chaotic phenomenon is one of the most important subjects in the study of dynamical systems. Li and Yorke [7] first introduced the notion of Li–Yorke chaos. Let $(X,T)$ be a t.d.s. A pair $(x,y)$ of $X$ is called a *positively proximal pair for* $T$ if $\liminf_{n\to\infty} d_X(T^n x, T^n y) = 0$, and called a *positively asymptotic pair for* $T$ if $\lim_{n\to\infty} d_X(T^n x, T^n y) = 0$. A proper pair is called a *positively scrambled pair (or Li–Yorke pair) for* $T$ if it is a positively proximal pair for $T$ but not a positively asymptotic pair for $T$. A t.d.s. is called *Li–Yorke chaotic* if there exists an uncountable *scrambled set* $S \subset X$, that is, for any two distinct points $x, y \in S$, $(x,y)$ is a positively scrambled pair.

Now we give a short list of works related to the study of scrambled sets. Li and Yorke [7] showed that a map on a closed interval with a periodic point of period three is Li–Yorke

*Corresponding author.

E-mail address: zjlin137@126.com (Zijie Lin), yexd@ustc.edu.cn (Xiangdong Ye).

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chaotic. Misiurewicz [10] proved that there is a continuous map on a closed interval such that it has a scrambled set with full Lebesgue measure. Mai [9] constructed a continuous map on the non-compact space $(0,1)^n$ such that the whole space becomes a scrambled set. Huang and Ye [5] showed that there is a t.d.s. such that the whole space becomes a scrambled set, called *positively completely scrambled*. Such an example can be given by using the graph covers; see Shimomura [11] or by considering the so-called almost equicontinuous systems with a unique minimal point; see Glasner and Weiss [3]. Answering some question asked in [5], Boroǹski, Kupka, and Oprocha [2] showed that, for any integer $n \geq 1$, there exists a topological mixing homeomorphism on an $n$-dimensional continuum, which is positively completely scrambled.

Another question posed by the authors in [5] is whether a positively completely scrambled homeomorphism can have positive entropy. As a by-product the question was answered negatively by Blanchard, Host and Ruette [1] who proved that positive entropy implies the existence of a proper positively asymptotic pair. This result shows that if the whole space is a scrambled set, that is, all the proper pairs are positively scrambled pairs, then the map should have zero entropy.

This leads us to consider the natural question for positive Li-Yorke pairs or negative Li-Yorke pairs. There are examples that have positive entropy and also have the following property: there is no proper pair which is both positively asymptotic and negatively asymptotic, see for example the work by Lind and Schmidt [8].

A t.d.s. $(X,T)$ is called *completely scrambled* or *completely Li–Yorke chaotic* if each proper pair is either a positively or a negatively scrambled pair. This means that some of proper pairs are positively scrambled pairs and the complement of them are negatively scrambled pairs.

Unlike the positively completely scrambled situation, we find out that a completely Li–Yorke chaotic homeomorphism with positive entropy exists, which gives a positive answer to a problem in [6, Problem 1]. That is,

**Theorem 1.1.** *There is a t.d.s. $(X,T)$ such that it is completely Li–Yorke chaotic and has positive topological entropy.*

In fact, such an example can be constructed from any given t.d.s. with a mild condition, i.e., the system has positive entropy and has no proper pairs which are both positively and negatively asymptotic.

For the system $(X,T)$ in Theorem 1.1, if we take an ergodic $T$-invariant probability Borel measure $\mu$ on $(X,T)$ with positive measure-theoretic entropy, then the support of $\mu$ is a transitive subsystem of $(X,T)$ with positive topological entropy, which is also completely Li–Yorke chaotic. Thus we have the following corollary.

**Corollary 1.2.** *There is a transitive t.d.s. such that it is completely Li–Yorke chaotic and has positive topological entropy.*

We remark that if we require that each pair in a t.d.s. $(X,T)$ is either positively or negatively recurrent, then the entropy should be zero, see the recent work by Hochman [4].

1.2. **Main ideas.** Our main idea is to construct a proximal system from a given one such that this proximal system can carry the information of each orbit in the given system.

Namely, given a t.d.s. $(X,T)$, we construct a subsystem of $(X_*^{\mathbb{Z}},\sigma)$, where $X_*$ is the space by adding an isolating fixed point $*$ to $X$ and $\sigma$ is the left shift map. Choose a suitable subsequence

$\{n_k\}_{k=0}^{\infty}$ of $\mathbb{N}$. For each $n_k$, we divide the integer interval $\{0,1,2,\ldots,n_k-1\}$ into two parts: the "proximal part" and the "orbit part". We put the fixed point $*$ into the proximal part, and put the orbit in $X$ into the orbit part. And then we obtain a closed subset of $X_*^{\mathbb{Z}}$, which relies only on $n_k$ coordinates. Finally, collecting all those points and their orbits under the left shift map, we construct a proximal system $Y \subset X_*^{\mathbb{Z}}$ depending on $(X,T)$.

The proximal part is required to intersect with its translation, since every point must be proximal to its shift when $(Y,\sigma)$ is a proximal system. On the other hand, the orbit part is needed to be large enough so that $(Y,\sigma)$ preserves positive topological entropy. This means that the proximal part should be small enough. We find out that such a partition exists for large enough integer $n$ (see Lemma 3.1). So we can show that the constructed system $Y \subset X_*^{\mathbb{Z}}$ has positive topological entropy when so does $(X,T)$, and all the pairs in $Y$ are positively proximal pairs for both $\sigma$ and $\sigma^{-1}$.

Finally, choose a suitable system as $(X,T)$, that is, a homeomorphism with positive topological entropy satisfying that all the proper pairs are positively non-asymptotic pairs for $T$ or $T^{-1}$. Since the orbit part contains the information of orbits in $X$ and all the proper pairs in $X$ are positively non-asymptotic pairs for $T$ or $T^{-1}$, we can show that it is same for $(Y,\sigma)$. Thus the constructed system $(Y,\sigma)$ is completely Li–Yorke chaotic and has positive topological entropy.

## 2. PRELIMINARIES

For subsets $A,B \subset \mathbb{Z}$ and $n \in \mathbb{Z}$, we write

$$A+B := \{a+b \in \mathbb{Z} : a \in A, b \in B\},$$

$$n+A := \{n+a \in \mathbb{Z} : a \in A\} \text{ and } nA := \{na \in \mathbb{Z} : a \in A\}.$$

For a t.d.s. $(X,T)$ and $x \in X$, we denote by $\overline{O}_+(x,T)$ the closure of the set $\{T^n x : n \geq 0\}$, called the *positive orbit closure for $T$* of $x$. We denote by $\overline{O}_{\mathbb{Z}}(x,T)$ the closure of the set $\{T^n x : n \in \mathbb{Z}\}$, called the *$\mathbb{Z}$-orbit closure for $T$* of $x$. A point $x \in X$ is called a *fixed point* if $Tx = x$. A point $x \in X$ is called a *minimal point* if its orbit closure $\overline{O}_+(x,T)$ does not contain any proper closed $T$-invariant subset. A fixed point is a minimal point.

For a t.d.s. $(X,T)$, we define the product space

$$X^{\mathbb{Z}} := \{(x_i)_{i\in\mathbb{Z}} : x_i \in X, i \in \mathbb{Z}\}$$

with the product topology, and the shift map $\sigma : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ by $\sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}$ for $x = (x_i)_{i\in\mathbb{Z}}$. So $(X^{\mathbb{Z}},\sigma)$ is a t.d.s. For $x = (x_i)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$ and $p < q \in \mathbb{Z}$, we write

$$x|_{[p,q)} = (x_p, x_{p+1}, \ldots, x_{q-1}).$$

For $z \in X$, we write $z^{\mathbb{Z}} := (z)_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$ and $z^N = (z,z,\ldots,z) \in X^N$ for any integer $N \geq 1$.

### 2.1. **Pairs in topological dynamical systems.**
In this subsection, we give some definitions related to a t.d.s.

**Definition 2.1.** Let $(X,T)$ be a t.d.s. A pair $(x,y) \in X \times X$ is called

- *positively proximal for $T$* if $\liminf_{n\to+\infty} d_X(T^n x, T^n y) = 0$;
- *negatively proximal for $T$* if it is positively proximal for $T^{-1}$;
- *positively asymptotic for $T$* if $\lim_{n\to+\infty} d_X(T^n x, T^n y) = 0$;
- *negatively asymptotic for $T$* if it is positively asymptotic for $T^{-1}$;
- *two-sided asymptotic for $T$* if it is both positively and negatively asymptotic for $T$;

- a *positively scrambled (or Li–Yorke) pair for $T$* if it is positively proximal and
$$\limsup_{n\to+\infty} d_X(T^n x, T^n y) > 0;$$
- a *negatively scrambled (or Li–Yorke) pair for $T$* if it is a positively scrambled pair for $T^{-1}$.
- a *scrambled (or Li–Yorke) pair for $T$* if it is either a positively or a negatively scrambled pair for $T$.

Notice that a pair $(x,y)$ is not two-sided asymptotic if and only if there is some $\delta > 0$ such that $\#\{n \in \mathbb{Z} : d_X(T^n x, T^n y) > \delta\} = \infty$ where $\#(\cdot)$ stands for the cardinality of a set.

2.2. **Topological entropy.** In this subsection, we give some notations concerning topological entropy.

Let $(X,T)$ be a t.d.s. For $\varepsilon > 0$ and integer $n > 0$, a subset $E \subset X$ is called $(n,\varepsilon)$-*separated* if for any distinct $x,y \in E$, there exists $0 \le n' < n$ such that $d_X(T^{n'}x, T^{n'}y) > \varepsilon$. Denote by $s_n(X,T,\varepsilon)$ the greatest number of all $(n,\varepsilon)$-separated sets. The *topological entropy* of $(X,T)$ is defined by
$$h_{\text{top}}(X,T) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log s_n(X,T,\varepsilon)}{n} = \lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{\log s_n(X,T,\varepsilon)}{n}.$$

2.3. **Completely Li-Yorke chaotic homeomorphisms.** In this subsection, we introduce some basic properties of completely Li–Yorke chaotic homeomorphisms. The following is well known.

**Lemma 2.2.** *Let $(X,T)$ be a t.d.s. If any pair $(x,y)$ is either positively or negatively proximal for $T$, then there exists a unique fixed point $p$ for $T$. Moreover, $p$ is the unique minimal point of $(X,T)$.*

*Proof.* The proof is similar to Proposition 2.2 in [5]. Choosing any $x \in X$ , we have that $(x,Tx)$ is a positively or negatively proximal pair for $T$. If $(x,Tx)$ is a positively proximal pair for $S \in \{T,T^{-1}\}$, then we choose a sequence $\{n_k\}$ of positive integers such that $S^{n_k}x \to p$ for some $p \in X$ and $\lim_{k\to\infty} d_X(S^{n_k}x, S^{n_k+1}x) = 0$. Thus $Tp = p$, that is, $p$ is a fixed point. If there exists another minimal point $x \neq p$ for $(X,T)$, then $p \notin \overline{O}_{\mathbb{Z}}(x,T)$. Thus
$$d_X(p, \overline{O}_{\mathbb{Z}}(x,T)) := \inf_{z\in\overline{O}_{\mathbb{Z}}(x,T)} d_X(p,z) > 0.$$
Hence, $(x,p)$ is not a positively nor negatively proximal pair for $T$, a contradiction. □

**Proposition 2.3.** *Let $(X,T)$ be a t.d.s. If any $(x,y)$ is either a positively or negatively proximal pair for $T$, then all the pairs of $X$ are both positively and negatively proximal pairs for $T$.*

*Proof.* Fix any $x \neq y \in X$, there exists a minimal point $(z,z') \in \overline{O}_+((x,y), T\times T)$ for $T\times T$. By Lemma 2.2, the fixed point $p \in X$ is the unique minimal point. Then the fixed point $(p,p) \in X\times X$ is also the unique minimal point of $(X\times X, T\times T)$. Thus $(z,z') = (p,p) \in \overline{O}_+((x,y), T\times T)$, and then $(x,y)$ is a positively proximal pair for $T$. By a similar argument, we see that $(x,y)$ is also a negatively proximal pair for $T$. □

## 3. Induce a Proximal System

In this section, we construct an induced proximal system from a given t.d.s. and show the relationship between their topological entropy.

Let $(X,T)$ be a t.d.s. Let $X_* = X \cup \{*\}$, where $*$ is a point which does not belong to $X$. We can make $X_*$ to be a compact metric space by extending the metric $d_X$ on $X$ to a metric $d_{X_*}$ defined as:

$$d_{X_*}(*,*) = 0,\, d_{X_*}(x,y) = d_X(x,y),\, d_{X_*}(x,*) = \operatorname{diam}(X)+1 \text{ for all } x,y \in X$$

where $\operatorname{diam}(X)$ is the diameter of $X$ under the metric $d_X$. Extend $T$ to a homomorphism $T_* : X_* \to X_*$ by

$$T_*(*) = * \text{ and } T_*(x) = T(x) \text{ for } x \in X.$$

Then $(X_*, T_*)$ is also a t.d.s.

Next, we construct a proximal subsystem $(Y,\sigma)$ of $(X_*^{\mathbb{Z}}, \sigma)$ and show the relationship between the topological entropy of $(X,T)$ and $(Y,\sigma)$. To do so, the following lemma is needed.

**Lemma 3.1.** *For any $\varepsilon > 0$, there exists a positive integer $N$ such that, for any $n > N$, there exists a disjoint union $\{0,1,2,\dots,n-1\} = A \cup B$ with $0, n-1 \in B$ satisfying the following statements:*
  (i) *$2 \le \#A < n\varepsilon$;*
  (ii) *for any integer $0 \le i < n$, there is $a \in A$ such that $a+i \in A \cup (n+A)$;*
  (iii) *for any integer $0 < i \le \frac{n}{2}+1$, $\#((A+i)\cap B) \ge \sqrt{\frac{n}{2}}-1$.*

*Proof.* Fix $\varepsilon > 0$. Choose $N$ such that, for any $n > N$,

$$4 \le \sqrt{\frac{n}{2}} \text{ and } \sqrt{\frac{2}{n}} + \frac{3}{n} < \varepsilon. \tag{3.1}$$

Fix $n > N$ and set

$$p = \left\lceil \sqrt{\frac{n}{2}} \right\rceil \text{ and } q = \left\lceil \frac{n}{2p} \right\rceil,$$

where $\lceil x \rceil := \min\{n \in \mathbb{Z} : n \ge x\}$. Thus $n \le 2pq$ and

$$\sqrt{n/2} - 1 < \frac{n}{2(\sqrt{n/2}+1)} < q < \frac{n}{2p} + 1 \le \sqrt{n/2} + 1. \tag{3.2}$$

Let $n = k_0 p + r_0$ for some integers $k_0 > 0$ and $0 \le r_0 < p$. It follows that

$$2q-3 = 2(q-1)-1 < n/p - 1 < k_0 \le n/p \le 2q. \tag{3.3}$$

Let

$$A = \{1,2,\dots,p-1,p\} \cup \{p+r_0, 2p+r_0, \dots, qp+r_0, (q+1)p+r_0\}.$$

By the first inequality of (3.1) and (3.2), we know that $4 \le q$. By (3.3), we have $k_0 > 2q-3 \ge q+1$. So $(q+1)p + r_0 < k_0 p + r_0 = n$. Therefore, we have $A \subset \{0,1,2,\dots,n-1\}$ and $n-1 \notin A$ by $p > 1$. Statement (i) holds since

$$\begin{aligned} 2 \le \#A \le p+q+1 &< 2\sqrt{\frac{n}{2}} + 3 \quad (\text{by } (3.2)) \\ &= n\left(\sqrt{\frac{2}{n}} + \frac{3}{n}\right) \\ &< n\varepsilon. \quad (\text{by the second inequality of } (3.1)) \end{aligned}$$

Let $B = \{0,1,2,\dots,n-1\}\setminus A$. So $0, n-1 \in B$.

To prove (ii), we fix $0 \le i < n$. Let $i = kp + r$ for some integers $k$ and $0 \le r < p$. So $0 \le k \le k_0 \le 2q$. There are two cases:

- $0 \le k \le q$: if $0 \le r < r_0$ then we have $r_0 - r \in A$ and $i + r_0 - r = kp + r_0 \in A$. If $r_0 \le r < p$, then $p + r_0 - r \in A$ and $i + p + r_0 - r = (k+1)p + r_0 \in A$.
- $q < k \le k_0$: if $r = 0$, then $0 < k_0 - k + 1 \le q$ by $k_0 \le 2q$. Thus $(k_0 - k + 1)p + r_0 \in A$ and $(k_0 - k + 1)p + r_0 + i = n + p \in A + n$. If $r > 0$, then $0 \le k_0 - k < q$ by $k_0 \le 2q$. So $(k_0 - k)p + r_0 \in A$ and $(k_0 - k)p + r_0 + i = n + r \in A + n$.

Combining with the above two cases, we prove that (ii) holds.

To prove (iii), we fix $0 < i \le \frac{n}{2} + 1$ and divide into two cases.

**Case 1:** if $0 < i < p$, then $i + jp + r_0 \in B$ for all $1 \le j \le q$. Since $jp + r_0 \in A$ for all $1 \le j \le q$, we have

$$\#((A+i)\cap B) \ge q > \sqrt{\frac{n}{2}} - 1$$

where the last inequality holds by (3.2).

**Case 2:** if $p \le i \le \frac{n}{2} + 1$, by the first inequality of (3.1), $p < \frac{n}{2} - 1$ and $i + p < n$. Since $i + 1 > p$, then there exists at most one element of $i + \{1,2,\dots,p\}$ belongs to $A$. Since $\{1,2,\dots,p\} \subset A$, we have

$$\#((A+i)\cap B) \ge p - 1 \ge \sqrt{\frac{n}{2}} - 1.$$

Combining with the above cases, (iii) is proved. $\square$

3.1. **The construction.** Fix a decreasing sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ with $0 < \varepsilon_k < 1/2$ for all $k \ge 0$. By the above lemma, we inductively define sequences $\{n_k\}_{k=0}^{\infty}$, $\{p_k\}_{k=0}^{\infty}$, and for $k \ge 0$, a disjoint union $\{0,1,2,\dots,n_k - 1\} = A_k \cup B_k$ with $0, n_k - 1 \in B_k$.

For $k = 0$, by Lemma 3.1 for $\varepsilon_0$, there exist an integer $n_0$ and a disjoint union $\{0,1,2,\dots,n_0 - 1\} = A_0 \cup B_0$ with $0, n_0 - 1 \in B_0$ satisfying the (i), (ii), and (iii) in Lemma 3.1. For each integer $k \ge 1$, by Lemma 3.1 for $\varepsilon_k$, there exist an integer $n_k > n_{k-1}$ and a disjoint union $\{0,1,2,\dots,n_k - 1\} = A_k \cup B_k$ with $0, n_k - 1 \in B_k$ satisfying the (i), (ii) and (iii) in Lemma 3.1. Set $p_{-1} = 1$ and $p_k = n_k p_{k-1}$ for all $k \ge 0$.

Inductively for $k \ge 0$, assume that we have defined $n_k$, $p_k = n_k p_{k-1}$ (where $p_{-1} = 1$) and a disjoint union $\{0,1,2,\dots,n_k - 1\} = A_k \cup B_k$ with $0, n_k - 1 \in B_k$. Let $b_{-1} = 1$ and $b_k = \#B_k$ for $k \ge 0$. Let $\tau_k : B_k \to \{0,1,\dots,b_k - 1\}$ be a strictly increasing bijection. So, for all $k \ge 0$, we have $\tau_k(0) = 0$, $\tau_k(n_k - 1) = b_k - 1$ and $|\tau_k^{-1}(i) - \tau_k^{-1}(j)| \ge |i - j|$ for any $i, j \in \{0,1,\dots,b_k - 1\}$. For $k \ge 0$, we define $f_k : B_k \to (-(b_k - 2), 0] \cap \mathbb{Z}$ by

$$f_k(i) = \begin{cases} 0, & i = 0, \\ -(b_k - 3), & i = n_k - 1, \\ \tau_k(i) - (b_k - 2), & otherwise. \end{cases}$$

More explicitly, see Figure 1. So

$$f_k|_{B_k \setminus \{0, n_k - 1\}} : B_k \setminus \{0, n_k - 1\} \to (-(b_k - 2), 0] \cap \mathbb{Z}$$

is a bijection. Set $b'_{-1} = 1$, $b'_0 = \#B_0 - 2$, $I_0 = (-b'_0, 0] \cap \mathbb{Z}$ and $I_k = f_k(B_k)\#I_{k-1} + I_{k-1}$ for $k \ge 1$. So each $I_k = (-b'_k, 0] \cap \mathbb{Z}$ is an interval of $\mathbb{Z}$ where $b'_k = b'_{k-1}\#f_k(B_k)$ for $k \ge 0$. For each

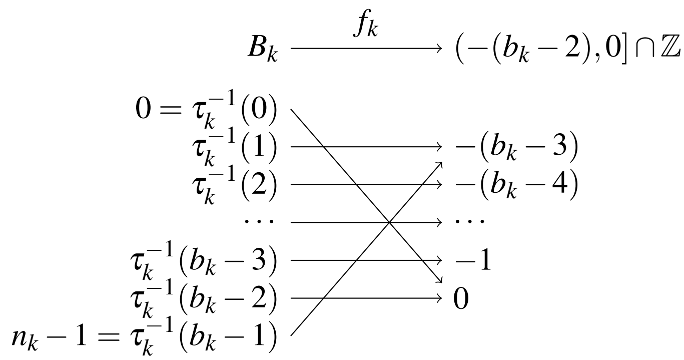



FIGURE 1. The map $f_k$

$k\geq 0$, we define $g_k : B_k\times X_*\to X_*$ by $g_k(i,x)=T_*^{b'_{k-1}f_k(i)}x$. If we list the $g_k(i,x)$, $i\in B_k$, then the form is

$$x,T_*^{-b'_{k-1}(b_k-3)}x,T_*^{-b'_{k-1}(b_k-4)}x,\dots,T_*^{-2b'_{k-1}}x,T_*^{-b'_{k-1}}x,x,T_*^{-b'_{k-1}(b_k-3)}x,$$

which we add the first point of the orbit $T_*^{-b'_{k-1}(b_k-3)}x,\dots,T_*^{-2b'_{k-1}}x,T_*^{-b'_{k-1}}x,x$ to the end of it, and add the last point of the same orbit to the start of it. For $k\geq 0$, we set

$$Q_k=\{x=(x_i)_{i\in\mathbb{Z}}\in X_*^{\mathbb{Z}} : x_{ip_{k-1}}=g_k(i,x_0), i\in B_k\}$$

and

$$P_k=\{x=(x_i)_{i\in\mathbb{Z}}\in X_*^{\mathbb{Z}} : x_{ip_{k-1}}=*,\ i\in A_k\}.$$

Then, for $k\geq 0$, $P_k$ and $Q_k$ are closed subsets of $X_*^{\mathbb{Z}}$ since $g_k$ is continuous. Noticing that for $x=(x_i)_{i\in\mathbb{Z}}\in Q_k$, points $x_{ip_{k-1}}$, $i\in B_k$ belong to the orbit of $x_0$, it is shown that one of points $x_{ip_{k-1}}$, $i\in B_k$ is $*$ if and only if so are all of them. Let $Y_{-1}=X_*^{\mathbb{Z}}$. For $k\geq 0$,

$$Y_k=Y_{k-1}\cap\bigcap_{n\in\mathbb{Z}}\sigma^{np_k}(P_k\cap Q_k)$$

and

$$Y=\bigcap_{k=0}^{\infty}\bigcup_{r=0}^{p_k-1}\sigma^{-r}Y_k. \tag{3.4}$$

Since $Y_k$ is closed and $\sigma^{p_k}$-invariant, $\bigcup_{r=0}^{p_k-1}\sigma^{-r}Y_k$ is closed and $\sigma$-invariant. Then $Y$ is closed and $\sigma$-invariant. So $(Y,\sigma|_Y)$ is a subsystem of $(X_*^{\mathbb{Z}},\sigma)$. Notice that $*^{\mathbb{Z}}\in Y$. Now, we give another point $x=(x_i)_{i\in\mathbb{Z}}\in Y$, which shows the form of some points in $Y$.

**Lemma 3.2.** *For any $z\in X$, there exists $x=(x_i)_{i\in\mathbb{Z}}\in\bigcap_{k=0}^{\infty}Y_k$ with $x_0=z$.*

*Proof.* Fix $z\in X$ and let $x_{-1}=x_0=z\in X$. For $1\leq i<n_0=p_0$, we set

$$x_i=\begin{cases}*, & i\in A_0,\\ g_0(i,x_0), & i\in B_0.\end{cases}$$

Let $x_{-p_0} \in X$ such that $g_0(n_0-1, x_{-p_0}) = x_{-1}$. For $-n_0 \le i < 0$, we set

$$x_i = \begin{cases} *, & i+n_0 \in A_0, \\ g_0(i+n_0, x_{-p_0}), & i+n_0 \in B_0. \end{cases}$$

Indeed, to ensure $x \in Y_0$, it is shown that, for any $n \in \mathbb{Z}$, all of $x_{np_0+i}$, $0 \le i < n_0$ are fixed if one of $x_{np_0+i}$, $i \in B_0$ is defined. So, for any $n \in \mathbb{Z}$, whenever $x_{np_0}$ have been defined, all of $x_{np_0+i}$, $0 \le i < p_0$ are defined. Thus we have defined $x|_{[-p_0,p_0)}$. Similarly, to ensure $x \in Y_1$, for any $n \in \mathbb{Z}$, all of $x_{np_1+ip_0}$, $0 \le i < n_1$ are fixed if one of $x_{np_1+ip_0}$, $i \in B_1$ is defined. So, for $1 \le i < n_1$, we set

$$x_{ip_0} = \begin{cases} *, & i \in A_1, \\ g_1(i, x_0), & i \in B_1. \end{cases}$$

Let $x_{-p_1}$ such that $g_1(n_1-1, x_{-p_1}) = x_{-p_0}$. For $-n_1 \le i < 0$, we set

$$x_{ip_0} = \begin{cases} *, & i+n_1 \in A_1, \\ g_1(i+n_1, x_{-p_1}), & i+n_1 \in B_1. \end{cases}$$

By defining $x_{ip_0}$ for $ip_0 \in [-p_1, p_1)$, it has defined $x_{ip_0+j}$ for $0 \le j < p_0$ and $ip_0 \in [-p_1, p_1)$, and then we have defined $x|_{[-p_1,p_1)}$. Suppose that $x|_{[-p_k,p_k)}$ has been defined for some $k \ge 0$. To ensure $x \in Y_{k+1}$, all of $x_{np_{k+1}+ip_k}$, $0 \le i < n_{k+1}$ are fixed if one of $x_{np_{k+1}+ip_k}$, $i \in B_{k+1}$ is defined. So, for $1 \le i < n_{k+1}$, we set

$$x_{ip_k} = \begin{cases} *, & i \in A_{k+1}, \\ g_{k+1}(i, x_0), & i \in B_{k+1}. \end{cases}$$

Let $x_{-p_{k+1}}$ such that $g_{k+1}(n_{k+1}-1, x_{-p_{k+1}}) = x_{-p_k}$. For $-n_{k+1} \le i < 0$, we set

$$x_{ip_k} = \begin{cases} *, & i+n_{k+1} \in A_{k+1}, \\ g_{k+1}(i+n_{k+1}, x_{-p_{k+1}}), & i+n_{k+1} \in B_{k+1}. \end{cases}$$

By defining $x_{ip_k}$ for $ip_k \in [-p_{k+1}, p_{k+1})$, it has defined $x_{ip_k+j}$ for $0 \le j < p_k$ and $ip_k \in [-p_{k+1}, p_{k+1})$, and then we have defined $x|_{[-p_{k+1},p_{k+1})}$. By the induction process, we have defined $x = (x_i)_{i\in\mathbb{Z}} \in \bigcap_{k=0}^{\infty} Y_k$. $\square$

The following lemma shows the form of points in $Y$.

**Lemma 3.3.** *Let $x = (x_i)_{i\in\mathbb{Z}} \in Y$. If $x_m \neq *$ for some $m \in \mathbb{Z}$, then there exists a unique sequence $\{r_k\}_{k=0}^{\infty}$ with $0 \le r_k < p_k$ such that $r_{k+1} \equiv r_k \mod p_k$, $k \ge 0$ and*

$$x \in \bigcap_{k=0}^{\infty} \sigma^{-r_k} Y_k.$$

*Proof.* Fix $x = (x_i)_{i\in\mathbb{Z}} \in Y$ with $x_m \neq *$ for some $m \in \mathbb{Z}$. Then there exists a sequence $\{r_k\}_{k=0}^{\infty}$ such that $0 \le r_k < p_k$ and $x \in \sigma^{-r_k} Y_k$ for all $k \ge 0$.

We prove the uniqueness of $0 \le r_k < p_k$ and $r_{k+1} \equiv r_k \mod p_k$ for $k \ge 0$ by induction.

**Step** $k = 0$**:** First, we prove the uniqueness of $r_0$. If there exist two distinct integers $0 \le r_0 \neq r_0' < p_0$ such that $x \in \sigma^{-r_0} Y_0$ and $x \in \sigma^{-r_0'} Y_0$. Without loss of generality, we assume that $r_0' > r_0$. Take $m_0 \in \mathbb{Z}$ with $r_0 + m_0p_0 \le m < r_0 + (m_0+1)p_0$. Since $\sigma^{r_0}x \in Y_0$, we have $m \in r_0 + m_0p_0 + B_0$ and

$$x_i \begin{cases} = *, & i \in r_0 + m_0p_0 + A_0, \\ \neq *, & i \in r_0 + m_0p_0 + B_0. \end{cases} \tag{3.5}$$

In particular, $x_{r_0+m_0p_0} \neq *$ and $x_{r_0+m_0p_0+p_0-1} \neq *$. Since $r_0' + (m_0-1)p_0 \le r_0 + m_0p_0 < r_0' + m_0p_0$ and $r_0' + m_0p_0 \le r_0 + m_0p_0 + p_0 - 1 < r_0' + (m_0+1)p_0$, by $\sigma^{r_0'}x \in Y_0$, we have

$$x_i \begin{cases} = *, & i \in (r_0' + (m_0-1)p_0 + A_0) \cup (r_0' + m_0p_0 + A_0), \\ \neq *, & i \in (r_0' + (m_0-1)p_0 + B_0) \cup (r_0' + m_0p_0 + B_0). \end{cases} \tag{3.6}$$

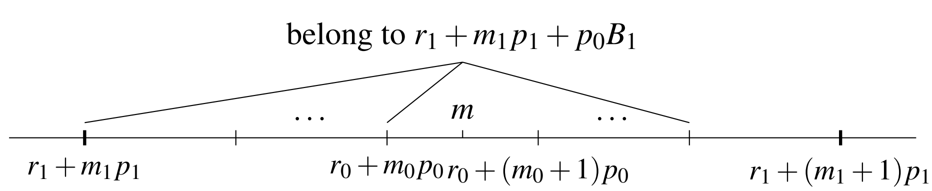
Depending on whether $r_0' - r_0 \le \frac{n_0}{2}$, there are two cases:

- $r_0' - r_0 \le \frac{n_0}{2}$: By (iii) of Lemma 3.1, $(A_0 + r_0' - r_0) \cap B_0 \neq \emptyset$. So
$$(r_0' + m_0p_0 + A_0) \cap (r_0 + m_0p_0 + B_0) = r_0 + m_0p_0 + (A_0 + r_0' - r_0) \cap B_0 \neq \emptyset,$$
a contradiction to (3.5) and (3.6).
- $r_0' - r_0 > \frac{n_0}{2}$: Then $p_0 - (r_0' - r_0) < \frac{n_0}{2}$. By (iii) of Lemma 3.1, $(A_0 + p_0 - (r_0' - r_0)) \cap B_0 \neq \emptyset$. So
$$\begin{aligned} &(r_0 + m_0p_0 + A_0) \cap (r_0' + (m_0-1)p_0 + B_0) \\ =& r_0' + (m_0-1)p_0 + (A_0 + p_0 - (r_0' - r_0)) \cap B_0 \neq \emptyset, \end{aligned}$$
a contradiction to (3.5) and (3.6).

So we can conclude that there exists a unique $0 \le r_0 < p_0$ such that $x \in \sigma^{-r_0}Y_0$.

**Step** $k = 1$**:** We show that $r_1 \equiv r_0 \mod p_0$ and prove the uniqueness of $r_1$. Since $x \in \sigma^{-r_1}Y_1 \subset \sigma^{-r_1}Y_0$, by the uniqueness of $r_0$, we have $r_1 \equiv r_0 \mod p_0$.

If there are two distinct $0 \le r_1 < r_1' < p_1$ such that $x \in \sigma^{-r_1}Y_1$ and $x \in \sigma^{-r_1'}Y_1$, then $r_1 \equiv r_1' \equiv r_0 \mod p_0$. Take $m_1 \in \mathbb{Z}$ with $r_1 + m_1p_1 \le m < r_1 + (m_1+1)p_1$. And by $r_0 + m_0p_0 \le m < r_0 + (m_0+1)p_0$ and $r_1 \equiv r_0 \mod p_0$, we have $[r_0 + m_0p_0, r_0 + (m_0+1)p_0) \subset [r_1 + m_1p_1, r_1 + (m_1+1)p_1)$ and $r_0 + m_0p_0 \in r_1 + m_1p_1 + p_0\{0,1,2,\dots,n_1-1\}$.




By $x_{r_0+m_0p_0} \neq *$, we have $r_0 + m_0p_0 \in r_1 + m_1p_1 + p_0B_1$ and

$$x_i \begin{cases} = *, & i \in r_1 + m_1p_1 + p_0A_1, \\ \neq *, & i \in r_1 + m_1p_1 + p_0B_1. \end{cases} \tag{3.7}$$

In particular, $x_{r_1+m_1p_1} \neq *$ and $x_{r_1+(m_1+1)p_1-p_0} \neq *$. Since $r_1' + (m_1-1)p_1 \le r_1 + m_1p_1 < r_1' + m_1p_1$, $r_1' + m_1p_1 \le r_1 + (m_1+1)p_1 - p_0 < r_1' + (m_1+1)p_1$ and $r_1 \equiv r_1' \equiv r_0 \mod p_0$, we have $r_1+m_1p_1 \in r_1'+(m_1-1)p_1+p_0\{0,1,2,\dots,n_1-1\}$ and $r_1+(m_1+1)p_1-p_0 \in r_1'+m_1p_1+p_0\{0,1,2,\dots,n_1-1\}$. So we have

$$x_i \begin{cases} =*, & i \in (r_1'+(m_1-1)p_1+p_0A_1)\cup(r_1'+m_1p_1+p_0A_1), \\ \neq *, & i \in (r_1'+(m_1-1)p_1+p_0B_1)\cup(r_1'+m_1p_1+p_0B_1). \end{cases} \tag{3.8}$$

Depending on whether $i_1 := \frac{r_1'-r_1}{p_0} \leq \frac{n_1}{2}$, there are two cases:

- $i_1 \leq \frac{n_1}{2}$: By (iii) of Lemma 3.1, $(A_1+i_1)\cap B_1 \neq \emptyset$. So
$$(r_1'+m_1p_1+p_0A_1)\cap(r_1+m_1p_1+p_0B_1) = r_1+m_1p_1+p_0((A_1+i_1)\cap B_1) \neq \emptyset,$$
a contradiction to (3.7) and (3.8).
- $i_1 > \frac{n_1}{2}$: Then $n_1-i_1 < \frac{n_1}{2}$. By (iii) of Lemma 3.1, $(A_1+n_1-i_1)\cap B_1 \neq \emptyset$. So
$$\begin{aligned} &(r_1+m_1p_1+p_0A_1)\cap(r_1'+(m_1-1)p_1+p_0B_1) \\ =&r_1'+(m_1-1)p_1+p_0((A_1+n_1-i_1)\cap B_1) \neq \emptyset, \end{aligned}$$
a contradiction to (3.7) and (3.8).

So we can conclude that there exists a unique $0 \leq r_1 < p_1$ such that $x \in \sigma^{-r_1}Y_1$.

**Step $k+1$:** Suppose that, for some $k \geq 1$, we have proved that, for any $1 \leq K \leq k$, $0 \leq r_K < p_K$ is unique, $r_K \equiv r_{K-1} \mod p_{K-1}$, $r_K+m_Kp_K \leq m < r_K+(m_K+1)p_K$ and $x_{r_K+m_Kp_K} \neq *$. We show that $r_{k+1} \equiv r_k \mod p_k$ and $r_{k+1}$ is unique. Since $x \in \sigma^{-r_{k+1}}Y_{k+1} \subset \sigma^{-r_{k+1}}Y_k$, by the uniqueness of $r_k$, we have $r_{k+1} \equiv r_k \mod p_k$.

If there exist two distinct $0 \leq r_{k+1} < r_{k+1}' < p_{k+1}$ such that $x \in \sigma^{-r_{k+1}}Y_{k+1}$ and $x \in \sigma^{-r_{k+1}'}Y_{k+1}$, then $r_{k+1} \equiv r_{k+1}' \equiv r_k \mod p_k$. Take $m_{k+1} \in \mathbb{Z}$ such that $r_{k+1}+m_{k+1}p_{k+1} \leq m < r_{k+1}+(m_{k+1}+1)p_{k+1}$. By $r_k+m_kp_k \leq m < r_k+(m_k+1)p_k$ and $r_{k+1} \equiv r_k \mod p_k$, we have

$$[r_k+m_kp_k, r_k+(m_k+1)p_k) \subset [r_{k+1}+m_{k+1}p_{k+1}, r_{k+1}+(m_{k+1}+1)p_{k+1})$$

and $r_k+m_kp_k \in r_{k+1}+m_{k+1}p_{k+1}+p_k\{0,1,2,\dots,n_{k+1}-1\}$. By $x_{r_k+m_kp_k} \neq *$, we have $r_k+m_kp_k \in r_{k+1}+m_{k+1}p_{k+1}+p_kB_{k+1}$ and

$$x_i \begin{cases} =*, & i \in r_{k+1}+m_{k+1}p_{k+1}+p_kA_{k+1}, \\ \neq *, & i \in r_{k+1}+m_{k+1}p_{k+1}+p_kB_{k+1}. \end{cases} \tag{3.9}$$

In particular, $x_{r_{k+1}+m_{k+1}p_{k+1}} \neq *$ and $x_{r_{k+1}+(m_{k+1}+1)p_{k+1}-p_k} \neq *$. Since

$$r_{k+1}'+(m_{k+1}-1)p_{k+1} \leq r_{k+1}+m_{k+1}p_{k+1} < r_{k+1}'+m_{k+1}p_{k+1},$$

$$r_{k+1}'+m_{k+1}p_{k+1} \leq r_{k+1}+(m_{k+1}+1)p_{k+1}-p_k < r_{k+1}'+(m_{k+1}+1)p_{k+1}$$

and $r_{k+1} \equiv r_{k+1}' \equiv r_k \mod p_k$, we have

$$r_{k+1}+m_{k+1}p_{k+1} \in r_{k+1}'+(m_{k+1}-1)p_{k+1}+p_k\{0,1,2,\dots,n_{k+1}-1\}$$

and

$$r_{k+1}+(m_{k+1}+1)p_{k+1}-p_k \in r_{k+1}'+m_{k+1}p_{k+1}+p_k\{0,1,2,\dots,n_{k+1}-1\}.$$

It follows that

$$x_i \begin{cases} =*, & i \in (r_{k+1}'+(m_{k+1}-1)p_{k+1}+p_kA_{k+1})\cup(r_{k+1}'+m_{k+1}p_{k+1}+p_kA_{k+1}), \\ \neq *, & i \in (r_{k+1}'+(m_{k+1}-1)p_{k+1}+p_kB_{k+1})\cup(r_{k+1}'+m_{k+1}p_{k+1}+p_kB_{k+1}). \end{cases} \tag{3.10}$$

Depending on whether $i_{k+1} := \frac{r_{k+1}'-r_{k+1}}{p_k} \leq \frac{n_{k+1}}{2}$, there are two cases:

- $i_{k+1} \le \frac{n_{k+1}}{2}$: By (iii) of Lemma 3.1, $(A_{k+1}+i_{k+1})\cap B_{k+1} \neq \emptyset$. So
$$\begin{aligned}&(r'_{k+1}+m_{k+1}p_{k+1}+p_kA_{k+1})\cap(r_{k+1}+m_{k+1}p_{k+1}+p_kB_{k+1})\\ =&r_{k+1}+m_{k+1}p_{k+1}+p_k((A_{k+1}+i_{k+1})\cap B_{k+1})\neq\emptyset,\end{aligned}$$
a contradiction to (3.9) and (3.10).
- $i_{k+1} > \frac{n_{k+1}}{2}$: Then $n_{k+1}-i_{k+1} < \frac{n_{k+1}}{2}$. By (iii) of Lemma 3.1, $(A_{k+1}+n_{k+1}-i_{k+1})\cap B_{k+1}\neq\emptyset$. So
$$\begin{aligned}&(r_{k+1}+m_{k+1}p_{k+1}+p_kA_{k+1})\cap(r'_{k+1}+(m_{k+1}-1)p_{k+1}+p_kB_{k+1})\\ =&r'_{k+1}+(m_{k+1}-1)p_{k+1}+p_k((A_{k+1}+n_{k+1}-i_{k+1})\cap B_{k+1})\neq\emptyset,\end{aligned}$$
a contradiction to (3.9) and (3.10).

Now, we have proved that the sequence $\{r_k\}$ is unique for $0\le r_k<p_k$, $k\ge 0$, and $r_{k+1}\equiv r_k \mod p_k$ for all $k\ge 0$. □

By Lemma 3.3, for any $x\in Y$ with $x_m\neq *$ for some $m\in\mathbb{Z}$, we can always assume that $x\in\bigcap_{k=0}^{\infty}\sigma^{-r_k}Y_k$ for some sequence $\{r_k\}_{k=0}^{\infty}$ with $0\le r_k<p_k$ and $r_{k+1}\equiv r_k \mod p_k$ for all $k\ge 0$.

3.2. **Proximality of $Y$.** In this subsection, we show that all pairs of $Y$ are both positively and negatively proximal. First, we need the following lemma.

**Lemma 3.4.** *Let $x=(x_i)_{i\in\mathbb{Z}}\in\bigcap_{k=0}^{\infty}\sigma^{-r_k}Y_k$ for some sequence $\{r_k\}_{k=0}^{\infty}$ with $0\le r_k<p_k$ and $r_{k+1}\equiv r_k \mod p_k$ for all $k\ge 0$. Then, for any $k\ge 0$,*

(i) *$\{i\in\mathbb{Z}: x_i=*\}\supset r_k+p_k\mathbb{Z}+p_{k-1}A_k$;*

(ii) *if $x_{r_k+np_k}=*$ for some $n\in\mathbb{Z}$, then $x_{r_k+np_k+i}=*$, $i=0,1,2,\dots,p_k-1$.*

*Moreover,*
$$\{i\in\mathbb{Z}: x_i=*\}\supset\bigcup_{k=0}^{\infty}(r_k+p_k\mathbb{Z}+p_{k-1}A_k+[0,p_{k-1})\cap\mathbb{Z}). \tag{3.11}$$

*Proof.* We prove by induction. For $k=0$, we have $\sigma^{r_0}x\in Y_0\subset\bigcap_{n\in\mathbb{Z}}\sigma^{np_0}P_0$. So, for any $n\in\mathbb{Z}$, $x_{r_0+np_0+i}=*$ for all $i\in A_0$. Therefore, $\{i\in\mathbb{Z}:x_i=*\}\supset r_0+p_0\mathbb{Z}+A_0$. Moreover, by $\sigma^{r_0}x\in Y_0\subset\bigcap_{n\in\mathbb{Z}}\sigma^{np_0}(P_0\cap Q_0)$, if $x_{r_0+np_0}=*$ for some $n\in\mathbb{Z}$, then $x_{r_0+np_0+i}=*$, $i=0,1,2,\dots,p_0-1$. So (i) and (ii) hold for $k=0$.

Suppose that (i) and (ii) hold for some $k\ge 0$. Since $\sigma^{r_{k+1}}x\in Y_{k+1}\subset\bigcap_{n\in\mathbb{Z}}\sigma^{np_{k+1}}P_{k+1}$ for any $n\in\mathbb{Z}$, then $x_{r_{k+1}+np_{k+1}+ip_k}=*$ for all $i\in A_{k+1}$. Since (ii) holds for $k$ and $r_{k+1}+np_{k+1}+ip_k\in r_k+p_k\mathbb{Z}$ by $r_{k+1}\equiv r_k \mod p_k$, we have $x_{r_{k+1}+np_{k+1}+ip_k+j}=*$ for all $i\in A_{k+1}$ and $j=0,1,2,\dots,p_k-1$. Therefore, we have
$$\{i\in\mathbb{Z}:x_i=*\}\supset r_{k+1}+p_{k+1}\mathbb{Z}+p_kA_{k+1}+[0,p_k)\cap\mathbb{Z}.$$
Moreover, by $\sigma^{r_{k+1}}x\in Y_{k+1}\subset\bigcap_{n\in\mathbb{Z}}\sigma^{np_{k+1}}(P_{k+1}\cap Q_{k+1})$, if $x_{r_{k+1}+np_{k+1}}=*$ for some $n\in\mathbb{Z}$, then $x_{r_{k+1}+np_{k+1}+ip_k}=*$ for all $i=0,1,2,\dots,n_k-1$. So (i) and (ii) hold for $k+1$.

Combining with (i) and (ii) for all $k\ge 0$, (3.11) is proved. □

In Lemma 3.4, by (3.11), we also have
$$\{i\in\mathbb{Z}:x_i\neq *\}\subset\bigcap_{k=0}^{\infty}(r_k+p_k\mathbb{Z}+p_{k-1}B_k+[0,p_{k-1})\cap\mathbb{Z}). \tag{3.12}$$

**Proposition 3.5.** *For any $x,y\in Y$, $(x,y)$ is both positively and negatively proximal.*

*Proof.* Fix $x=(x_i)_{i\in\mathbb{Z}}, y=(y_i)_{i\in\mathbb{Z}}\in Y$. By Lemma 3.3, there exist sequences $\{r_k\}_{k=0}^{\infty}$ and $\{r'_k\}_{k=0}^{\infty}$ such that
$$x\in\bigcap_{k=0}^{\infty}\sigma^{-r_k}Y_k \text{ and } y\in\bigcap_{k=0}^{\infty}\sigma^{-r'_k}Y_k.$$
If $r_k=r'_k$ for all $k\ge 0$, then by (3.11), $(x,y)$ is both positively and negatively proximal. If there exists some $k_0\ge 0$ such that $r_{k_0}\ne r'_{k_0}$. By $r_{k+1}\equiv r_k \mod p_k$ and $r'_{k+1}\equiv r'_k \mod p_k$ for all $k\ge 0$, we have $r_k\ne r'_k$ for all $k\ge k_0$.

Next, we show that, for any $N\ge 0$, there exist $n^+>N$ and $n^-<-N$ such that
$$x|_{[n^--N,n^-)}=y|_{[n^--N,n^-)}=x|_{[n^+,n^++N)}=y|_{[n^+,n^++N)}=*^N, \tag{3.13}$$
which implies that $(x,y)$ is both positively and negatively proximal. Fix $N\ge 0$ and choose $k>k_0$ with $p_k>2N$. Without loss of generality, we assme that $0\le r_{k+1}<r'_{k+1}<p_{k+1}$. Set $r'_{k+1}-r_{k+1}=ap_k+b$ for some integers $0\le a<n_{k+1}$ and $0\le b<p_k$. We divide $b$ into two cases.

**Case 1:** if $0\le b<p_k/2$, then by (ii) of Lemma 3.1, there exists $a_1\in A_{k+1}$ such that $a_1+a\in A_{k+1}\cup(A_{k+1}+n_{k+1})$. By (3.11),
$$\begin{aligned}\{i\in\mathbb{Z}:y_i=*\}\supset& r'_{k+1}+p_{k+1}\mathbb{Z}+a_1p_k+[0,p_k)\cap\mathbb{Z}\\ =&r_{k+1}+p_{k+1}\mathbb{Z}+(a_1+a)p_k+b+[0,p_k)\cap\mathbb{Z}\\ \supset& r_{k+1}+p_{k+1}\mathbb{Z}+(a_1+a)p_k+[b,b+N)\cap\mathbb{Z}\end{aligned}$$
where the last statement holds by $N<\frac{p_k}{2}$. Since $a_1+a\in A_{k+1}\cup(A_{k+1}+n_{k+1})$ and $b+N<p_k$, we have
$$\begin{aligned}\{i\in\mathbb{Z}:x_i=*\}\supset& r_{k+1}+p_{k+1}\mathbb{Z}+p_kA_{k+1}+[0,p_k)\cap\mathbb{Z}\quad(\text{by } (3.11))\\ \supset& r_{k+1}+p_{k+1}\mathbb{Z}+(a_1+a)p_k+[b,b+N)\cap\mathbb{Z}\end{aligned}$$
where the last statement holds by $b<\frac{p_k}{2}$ and $N<\frac{p_k}{2}$. So
$$\{i\in\mathbb{Z}:x_i=y_i=*\}\supset r_{k+1}+p_{k+1}\mathbb{Z}+(a_1+a)p_k+[b,b+N)\cap\mathbb{Z},$$
which implies (3.13).

**Case 2:** if $p_k/2\le b<p_k$, then $0<p_k-b<p_k/2$. By (ii) of Lemma 3.1, there exists $a_2\in A_{k+1}$ such that $a_2+(n_{k+1}-1-a)\in A_{k+1}\cup(A_{k+1}+n_{k+1})$. So, by (3.11),
$$\begin{aligned}\{i\in\mathbb{Z}:x_i=*\}\supset& r_{k+1}+p_{k+1}\mathbb{Z}+a_2p_k+[0,p_k)\cap\mathbb{Z}\\ =&r'_{k+1}+p_{k+1}\mathbb{Z}+(a_2+n_{k+1}-1-a)p_k+(p_k-b)+[0,p_k)\cap\mathbb{Z}\\ \supset& r'_{k+1}+p_{k+1}\mathbb{Z}+(a_2+n_{k+1}-1-a)p_k+[p_k-b,p_k-b+N)\cap\mathbb{Z}\end{aligned}$$
where the last statement holds by $N<\frac{p_k}{2}$. Since $p_k-b+N<p_k$ and $a_2+n_{k+1}-1-a\in A_{k+1}\cup(A_{k+1}+n_{k+1})$, we have
$$\begin{aligned}\{i\in\mathbb{Z}:y_i=*\}\supset& r'_{k+1}+p_{k+1}\mathbb{Z}+p_kA_{k+1}+[0,p_k)\cap\mathbb{Z}\quad(\text{by } (3.11))\\ \supset& r'_{k+1}+p_{k+1}\mathbb{Z}+(a_2+n_{k+1}-1-a)p_k+[p_k-b,p_k-b+N)\cap\mathbb{Z}\end{aligned}$$
where the last statement holds by $p_k-b<\frac{p_k}{2}$ and $N<\frac{p_k}{2}$. So
$$\{i\in\mathbb{Z}:x_i=y_i=*\}\supset r'_{k+1}+p_{k+1}\mathbb{Z}+(a_2+n_{k+1}-1-a)p_k+[p_k-b,p_k-b+N)\cap\mathbb{Z},$$
which implies (3.13). ∎

3.3. **Entropy of $Y$.** In this subsection, we show the relationship between the entropy of $X$ and $Y$. In the proof of Lemma 3.2, we can see that $x$ contains the orbit of $x_0$. Indeed, this also holds for any $x \in Y_k$. For $k \geq 0$, let $C_k = B_k \setminus \{0, n_k - 1\}$. So $\#C_k = b_k - 2$, and $f_k|_{C_k} : C_k \to (-(b_k - 2), 0] \cap \mathbb{Z}$ is a bijection. For convenience, we write $f_k^{-1} := (f_k|_{C_k})^{-1} : (-(b_k-2), 0] \cap \mathbb{Z} \to C_k$ for all $k \geq 0$. For $K \geq 0$, we set

$$J_K = \mathbb{Z} \cap [0, p_K) \cap \bigcap_{k=0}^{K} (p_k\mathbb{Z} + p_{k-1}C_k + [0, p_{k-1})).$$

So $J_0 = C_0$. For $K \geq 0$,

$$\begin{aligned}
J_{K+1} =& \mathbb{Z} \cap [0, p_{K+1}) \cap \bigcap_{k=0}^{K+1} (p_k\mathbb{Z} + p_{k-1}C_k + [0, p_{k-1})) \\
=& \mathbb{Z} \cap (p_K C_{K+1} + [0, p_K)) \cap \bigcap_{k=0}^{K} (p_k\mathbb{Z} + p_{k-1}C_k + [0, p_{k-1})) \\
=& \mathbb{Z} \cap \left( \bigcup_{i \in C_{K+1}} (ip_K + [0, p_K)) \right) \cap \bigcap_{k=0}^{K} (p_k\mathbb{Z} + p_{k-1}C_k + [0, p_{k-1})) \\
=& \bigcup_{i \in C_{K+1}} \left( ip_K + \mathbb{Z} \cap [0, p_K) \cap \bigcap_{k=0}^{K} (-ip_K + p_k\mathbb{Z} + p_{k-1}C_k + [0, p_{k-1})) \right) \\
=& p_K C_{K+1} + \mathbb{Z} \cap [0, p_K) \cap \bigcap_{k=0}^{K} (p_k\mathbb{Z} + p_{k-1}C_k + [0, p_{k-1})),
\end{aligned}$$

where the last equality holds by $p_K \in \bigcap_{k=0}^{K} p_k\mathbb{Z}$. So $J_{k+1} = p_k C_{k+1} + J_k$ for all $k \geq 0$. Recall that $I_k = (-b'_k, 0] \cap \mathbb{Z}$ and $I_{k+1} = f_{k+1}(B_{k+1})b'_k + I_k$ for $k \geq 0$.

**Lemma 3.6.** *There exist strictly increasing bijections $\phi_k : I_k \to J_k$, $k \geq 0$ such that, for all $k \geq 0$,*

(i) *for any $x \in Y_k$, $x_{\phi_k(n)} = T_*^n x_0$, $n \in I_k$, and*
(ii) *for any $j \in f_{k+1}(B_{k+1})$, $\phi_{k+1}(jb'_k + I_k) = f_{k+1}^{-1}(j)p_k + J_k$.*

*Proof.* Recall that $f_k^{-1} := (f_k|_{C_k})^{-1}$ for all $k \geq 0$. We prove by induction. For $k = 0$, define a map $\phi_0 : I_0 \to J_0 = C_0$ by $\phi_0(n) = f_0^{-1}(n)$ for $n \in I_0$. Then, for any $x \in Y_0$, by $x \in Y_0 \subset Q_0$, we have $x_{\phi_0(n)} = g_0(f_0^{-1}(n), x_0) = T_*^n x_0$ for all $n \in I_0$. Note that $f_0^{-1}$ is a strictly increasing bijection, so is $\phi_0$. Suppose that we have defined $\phi_k$ as required for some $k \geq 0$. Since $\#I_k = b'_k$, then the translations $jb'_k + I_k$, $j \in \mathbb{Z}$ of $I_k$ are disjoint. So we define a map $\phi_{k+1} : I_{k+1} \to J_{k+1}$ by

$$\phi_{k+1}(n) = f_{k+1}^{-1}(j)p_k + \phi_k(n - jb'_k), n \in jb'_k + I_k, \ j \in f_{k+1}(B_{k+1}).$$

Since $\phi_k(I_k) = J_k \subset [0, p_k)$, $\phi_k$ and $f_{k+1}^{-1}$ are strictly increasing bijections, and $J_{k+1} = p_k C_{k+1} + J_k$, we have $\phi_{k+1}$ is also a strictly increasing bijection and (ii) holds. For any $x \in Y_{k+1}$, by $x \in Y_{k+1} \subset Y_k$, we have $\sigma^{f_{k+1}^{-1}(j)p_k} x \in Y_k$ for all $j \in f_{k+1}(B_{k+1})$. So, for all $j \in f_{k+1}(B_{k+1})$, by the induction hypothesis,

$$x_{f_{k+1}^{-1}(j)p_k + \phi_k(n)} = T_*^n x_{f_{k+1}^{-1}(j)p_k} \tag{3.14}$$

for all $n \in I_k$. By $x \in Y_{k+1} \subset Q_{k+1}$,

$$x_{f_{k+1}^{-1}(j)p_k} = g_{k+1}(f_{k+1}^{-1}(j), x_0) = T_*^{jb'_k} x_0 \tag{3.15}$$

for all $j \in f_{k+1}(B_{k+1})$. By (3.14) and (3.15), we have $x_{\phi_{k+1}(n)} = T^n x_0$ for all $n \in I_{k+1}$. Therefore, it ends the proof. □

By the relationship between the orbits in $X$ and points in $Y$, we are able to estimate the entropy of $Y$.

**Proposition 3.7.**

$$h_{\text{top}}(Y,\sigma) \geq h_{\text{top}}(X,T) \cdot \prod_{k=0}^{\infty} (1-2\varepsilon_k).$$

*Proof.* Since $J_k \subset [0, p_k)$ for all $k \geq 0$, by (i) of Lemma 3.6, the key is to estimate $\frac{\#I_k}{p_k} = \frac{\#J_k}{p_k}$. We prove the following inequality by induction:

$$\#J_K \geq p_K \prod_{k=0}^{K} (1-2\varepsilon_k),\ K \geq 0. \tag{3.16}$$

For $K \geq 0$, $\#J_0 = \#C_0 = p_0 - \#A_0 - 2 > p_0(1-2\varepsilon_0)$ by (i) of Lemma 3.1. Suppose that (3.16) holds for some $K \geq 0$. Recalling that $J_{K+1} = p_K C_{K+1} + J_K$, we have

$$\begin{aligned}
\#J_{K+1} =& \#C_{K+1} \cdot \#J_K \\
\geq& (n_{K+1} - \#A_{K+1} - 2) \cdot p_K \prod_{k=0}^{K} (1-2\varepsilon_k) \\
\geq& n_{K+1}(1-2\varepsilon_{K+1}) \cdot p_K \prod_{k=0}^{K} (1-2\varepsilon_k) \quad \text{(by (i) of Lemma 3.1)} \\
=& p_{K+1} \prod_{k=0}^{K+1} (1-2\varepsilon_k) \quad \text{(by } p_{K+1} = n_{K+1} p_K\text{)}.
\end{aligned}$$

So (3.16) is proved. For any $\eta > 0$, there exists $\delta > 0$ such that $d_Y(x,y) > \delta$ whenever $d_{X_*}(x_0, y_0) > \eta$ for any $x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in Y$. By Lemma 3.2 and (i) of Lemma 3.6, we have $s_{p_K}(Y, \sigma, \delta) \geq s_{\#I_K}(X, T, \eta)$ for all $K \geq 0$. Since $\#I_K = \#J_K$, we have

$$\frac{\log s_{p_K}(Y,\sigma,\delta)}{p_K} > \frac{\log s_{p_K}(Y,\sigma,\delta)}{\#I_K} \cdot \prod_{k=0}^{K} (1-2\varepsilon_k) \geq \frac{\log s_{\#I_K}(X,T,\eta)}{\#I_K} \cdot \prod_{k=0}^{K} (1-2\varepsilon_k).$$

Let $K \to \infty$. By the arbitrariness of $\eta$, we have the desired conclusion immediately. □

By the above lemma, if $(X,T)$ has positive entropy, so does $(Y,\sigma)$ when we take $\{\varepsilon_k\}_{k=0}^{\infty}$ satisfying $\prod_{k=0}^{\infty}(1-2\varepsilon_k) > 0$.

## 4. Completely Li–Yorke Chaotic Systems with Positive Entropy

In this section, we give a completely Li–Yorke chaotic system by the constructed system in Section 3. We mainly show that each proper pair of $Y$ is not two-sided asymptotic for $\sigma$ where $Y$ is induced as in Section 3 by some system $(X,T)$. We divide the proper pairs of $Y$ into three cases, where the first two cases do not have any requirement for $X$, but the last one case does.

First, we show two typical cases for $Y$ induced by an arbitrary t.d.s. $(X,T)$.

**Lemma 4.1.** *Let $(Y,\sigma)$ be as (3.4) in Section 3. Then, for any $x \neq *^{\mathbb{Z}} \in Y$, $(x, *^{\mathbb{Z}})$ is not a two-sided asymptotic pair.*

*Proof.* Choose $m \in \mathbb{Z}$ such that $x_m \neq *$. By Lemma 3.3, there exists a unique sequence $\{r_k\}_{k=0}^{\infty}$ such that $0 \leq r_k < p_k$, $x \in \sigma^{-r_k} Y_k$ and $r_{k+1} \equiv r_k \mod p_k$. Fix any $k \geq 0$. By (3.12), $m \in r_k + p_k\mathbb{Z} + p_{k-1}B_k + [0, p_{k-1}) \cap \mathbb{Z}$. Let $m \in r_k + m_k p_k + i_k p_{k-1} + [0, p_{k-1}) \cap \mathbb{Z}$ for some $m_k \in \mathbb{Z}$ and $i_k \in B_k$. Since $x_m \neq *$, by (ii) of Lemma 3.4, we have $x_{r_k + m_k p_k + i_k p_{k-1}} \neq *$. Since $\sigma^{r_k} x \in Y_k$, $x_{r_k + m_k p_k + i p_{k-1}} \neq *$ for all $i \in B_k$. Since $\#B_k \to \infty$ as $k \to \infty$, $(x, *^{\mathbb{Z}})$ is not a two-sided asymptotic pair. □

By Lemma 3.3, for any $x \in Y \setminus \{*^{\mathbb{Z}}\}$, there exists a unique sequence $\{r_k\}_{k=0}^{\infty}$ with $0 \leq r_k < p_k$ such that $x \in \sigma^{-r_k} Y_k$ and $r_{k+1} \equiv r_k \mod p_k$ for all $k \geq 0$. So we can define maps $r_k : Y \setminus \{*^{\mathbb{Z}}\} \to [0, p_k) \cap \mathbb{Z}$, $k \geq 0$, where $r_k(x)$ is as in Lemma 3.3 for all $k \geq 0$.

**Lemma 4.2.** *Let $(Y,\sigma)$ be as (3.4) in Section 3. Then, for any $x \neq y \in Y \setminus \{*^{\mathbb{Z}}\}$, if there exists some $K \geq 0$ such that $r_K(x) \neq r_K(y)$, then $(x,y)$ is not a two-sided asymptotic pair.*

*Proof.* Fix $x = (x_i)_{i\in\mathbb{Z}} \neq y = (y_i)_{i\in\mathbb{Z}}$ with $r_K(x) \neq r_K(y)$ for some $K \geq 0$. By Lemma 3.3, $r_{k+1}(x) \equiv r_k(x) \mod p_k$ for all $k \geq 0$, which is same for $y$. So $r_k(x) \neq r_k(y)$ for all $k \geq K$. Fix $k \geq K$. Without loss of generality, we assume that $0 \leq r_k(x) < r_k(y) < p_k$. We next show that

$$\#\{n \in \mathbb{Z} : d_{X_*}(x_n, y_n) = d_{X_*}(X, *)\} \geq \sqrt{\frac{n_k}{2}} - 1. \tag{4.1}$$

Set $r_k(y) - r_k(x) = a p_{k-1} + b$, where $0 \leq a < n_k$ and $0 < b \leq p_{k-1}$. We divide $a$ into two cases.

- **Case 1:** if $0 \leq a \leq \frac{n_k}{2}$, choose $m \in \mathbb{Z}$ with $x_m \neq *$. By (3.12), we have $m \in r_k(x) + m_k p_k + i_k p_{k-1} + [0, p_{k-1}) \cap \mathbb{Z}$ for some $m_k \in \mathbb{Z}$ and $i_k \in B_k$. By (ii) of Lemma 3.4, we have $x_{r_k(x) + m_k p_k + i_k p_{k-1}} \neq *$. Since $\sigma^{r_k(x) + m_k p_k} x \in Q_k$ and $i_k \in B_k$, we have

$$x_{r_k(x) + m_k p_k + i p_{k-1}} \neq * \text{ for all } i \in B_k. \tag{4.2}$$

  Then for any $i \in (A_k + a + 1) \cap B_k$,

$$\begin{aligned} & r_k(x) + m_k p_k + i p_{k-1} \\ =& r_k(y) - a p_{k-1} - b + m_k p_k + i p_{k-1} \\ =& r_k(y) + m_k p_k + (i - a - 1) p_{k-1} + p_{k-1} - b. \end{aligned}$$

  By $i - a - 1 \in A_k$, $p_{k-1} - b \in [0, p_{k-1})$ and (3.11), $y_{r_k(x) + m_k p_k + i p_{k-1}} = *$. So by (iii) of Lemma 3.1 and (4.2),

$$\#\{n \in \mathbb{Z} : d_{X_*}(x_n, y_n) = d_{X_*}(X, *)\} \geq \#((A_k + a + 1) \cap B_k) \geq \sqrt{\frac{n_k}{2}} - 1.$$

- **Case 2:** if $\frac{n_k}{2} < a < n_k$, choose $m' \in \mathbb{Z}$ with $y_{m'} \neq *$. By (3.12), we have $m' \in r_k(y) + m'_k p_k + i'_k p_{k-1} + [0, p_{k-1}) \cap \mathbb{Z}$ for some $m'_k \in \mathbb{Z}$ and $i'_k \in B_k$. By (ii) of Lemma 3.4, we have $y_{r_k(y) + m'_k p_k + i'_k p_{k-1}} \neq *$. Since $\sigma^{r_k(y) + m'_k p_k} y \in Q_k$ and $i'_k \in B_k$, we have

$$y_{r_k(y) + m'_k p_k + i p_{k-1}} \neq * \text{ for all } i \in B_k. \tag{4.3}$$

Then if $0<b<p_{k-1}$, for $i\in (n_k-a+A_k)\cap B_k$,

$$\begin{aligned}&r_k(y)+m_k'p_k+ip_{k-1}\\=&r_k(x)+ap_{k-1}+b+m_k'p_k+ip_{k-1}\\=&r_k(x)+(m_k'+1)p_k+(i-(n_k-a))p_{k-1}+b.\end{aligned}$$

By $i-(n_k-a)\in A_k$, $b\in[0,p_{k-1})$ and (3.11), $x_{r_k(y)+m_k'p_k+ip_{k-1}}=*$. So by (iii) of Lemma 3.1 and (4.3),

$$\#\{n\in\mathbb{Z}:d_{X_*}(x_n,y_n)=d_{X_*}(X,*)\}\geq\#((A_k+(n_k-a))\cap B_k)\geq\sqrt{\frac{n_k}{2}}-1.$$

If $b=p_{k-1}$, then $a<n_k-1$ by $r_k(y)=r_k(x)+ap_{k-1}+b$. So for $i\in(n_k-a-1+A_k)\cap B_k$,

$$r_k(y)+m_k'p_k+ip_{k-1}=r_k(x)+(m_k'+1)p_k+(i-(n_k-a-1))p_{k-1}.$$

Since $i-(n_k-a-1)\in A_k$ and (3.11), we have $x_{r_k(y)+m_k'p_k+ip_{k-1}}=*$. So by (iii) of Lemma 3.1 and (4.3),

$$\#\{n\in\mathbb{Z}:d_{X_*}(x_n,y_n)=d_{X_*}(X,*)\}\geq\#((A_k+(n_k-a-1))\cap B_k)\geq\sqrt{\frac{n_k}{2}}-1.$$

To sum up, (4.1) is proved. Since $n_k\to\infty$ as $k\to\infty$, it ends the proof. $\square$

Finally, for the last case of $x\neq y$ with the same sequence $\{r_k\}_{k=0}^{\infty}$, we will show more about points in $Y$. Recall that $C_k=B_k\setminus\{0,n_k-1\}$, $f_k|_{C_k}:C_k\to f_k(B_k)$ is a bijection and $f_k^{-1}: f_k(B_k)\to C_k$ is the inverse of the bijection $f_k|_{C_k}$.

**Lemma 4.3.** *Let $(Y,\sigma)$ be as (3.4) in Section 3, and $k\geq 0$. For any $x=(x_i)_{i\in\mathbb{Z}},y=(y_i)_{i\in\mathbb{Z}}\in Y_k$, we have*

(i) *if $x_0=y_0$, then $x|_{[0,p_k)}=y|_{[0,p_k)}$;*
(ii) *for any $i\in B_k$, $(\sigma^{ip_{k-1}}x)|_{[0,p_{k-1})}=(\sigma^{f_k^{-1}(f_k(i))p_{k-1}}x)|_{[0,p_{k-1})}$.*

*Proof.* We prove (i) by induction. For $k=0$, let $x,y\in Y_0$. So we have $x_i=*=y_i$ for $i\in A_0$, and $x_i=T_*^{f_0(i)}x_0$ and $y_i=T_*^{f_0(i)}y_0$ for $i\in B_0$. So (i) holds for $k=0$.

Suppose that (i) holds for some $k\geq 0$. For any $x,y\in Y_{k+1}$ with $x_0=y_0$, we have $x_{ip_k}=*=y_{ip_k}$ for $i\in A_{k+1}$, and $x_{ip_k}=T_*^{b_k'f_{k+1}(i)}x_0$ and $y_{ip_k}=T_*^{b_k'f_{k+1}(i)}y_0$ for $i\in B_{k+1}$. So $x_{ip_k}=y_{ip_k}$ for $0\leq i<n_{k+1}$. Since $\sigma^{ip_k}x,\sigma^{ip_k}y\in Y_k$, by the induction hypothesis, we have $x|_{[ip_k,(i+1)p_k)}=y|_{[ip_k,(i+1)p_k)}$ for $0\leq i<n_{k+1}$. So (i) holds for $k+1$.

For any $x\in Y_k\subset Q_k$, noticing that $f_k\circ f_k^{-1}=Id_{f_k(B_k)}$ is the identity of $f_k(B_k)$, we have $x_{f_k^{-1}(f_k(i))p_{k-1}}=T^{b_{k-1}'f_k(i)}x_0=x_{ip_{k-1}}$ for $i\in B_k$. Then by (i), (ii) is proved. $\square$

To prove the last case, we also need the following t.d.s.

**Theorem 4.4.** [8, Example 3.4] *There is a t.d.s. $(X,T)$ such that $h_{\mathrm{top}}(X,T)>0$ and all the proper pairs are not two-sided asymptotic pairs.*

If we take $(X,T)$ be as in Theorem 4.4 and construct $(Y,\sigma)$ as in Section 3, then we can show that all the proper pairs of $(Y,\sigma)$ are not two-sided asymptotic pairs.

**Proposition 4.5.** *Let $(X,T)$ be a t.d.s. satisfying that all proper pairs of $(X,T)$ are not two-sided asymptotic pairs, and define $(Y,\sigma)$ as (3.4) in Section 3. Then for any $x \neq y \in Y$, $(x,y)$ is not two-sided asymptotic.*

*Proof.* Fix $x=(x_i)_{i\in\mathbb{Z}} \neq y=(y_i)_{i\in\mathbb{Z}} \in Y$. By Lemma 4.1 and Lemma 4.2, we can assume that $x,y \in Y\setminus\{*^{\mathbb{Z}}\}$ and $x,y \in \bigcap_{k=0}\sigma^{-r_k}Y_k$ for the same sequence $\{r_k\}_{k=0}^{\infty}$ with $0\le r_k<p_k$ and $r_{k+1}\equiv r_k \mod p_k$ for all $k\ge 0$. Without loss of generality, we assume that $x_m \neq *$ and $x_m\neq y_m$ for some $m\in\mathbb{Z}$. By (3.12), $m\in r_k+p_k\mathbb{Z}+p_{k-1}B_k+[0,p_{k-1})\cap\mathbb{Z}$ for all $k\ge 0$. For each $k\ge 0$, there exist unique $m_k\in\mathbb{Z}$ and unique $i_k\in B_k$ such that

$$m\in r_k+m_kp_k+i_kp_{k-1}+[0,p_{k-1})\cap\mathbb{Z}\subset r_k+m_kp_k+[0,p_k)\cap\mathbb{Z}.$$

The integer $m_k$ is also unique such that $m\in r_k+m_kp_k+[0,p_k)\cap\mathbb{Z}$.

Next, we show by induction that for all $k\ge 0$,

$$m=r_k+m_kp_k+\sum_{k'=0}^{k} i_{k'}p_{k'-1}. \tag{4.4}$$

For $k=0$, $m=r_0+m_0p_0+i_0$ by $p_{-1}=1$. Suppose that (4.4) holds for some $k\ge 0$. Since

$$\begin{aligned} m \in & r_{k+1}+m_{k+1}p_{k+1}+i_{k+1}p_k+[0,p_k)\cap\mathbb{Z}\\ =& r_k+\left(m_{k+1}n_{k+1}+\frac{r_{k+1}-r_k}{p_k}+i_{k+1}\right)p_k+[0,p_k)\cap\mathbb{Z}\end{aligned}$$

where $\frac{r_{k+1}-r_k}{p_k}\in\mathbb{Z}$ by $r_{k+1}\equiv r_k \mod p_k$, we have $r_{k+1}+m_{k+1}p_{k+1}+i_{k+1}p_k=r_k+m_kp_k$ by the uniqueness of $m_k$. So by the induction hypothesis,

$$m=r_k+m_kp_k+\sum_{k'=0}^{k} i_{k'}p_{k'-1}=r_{k+1}+m_{k+1}p_{k+1}+\sum_{k'=0}^{k+1} i_{k'}p_{k'-1},$$

that is, (4.4) holds for $k+1$.

Let $S=\{k\ge 0:\tau_k(i_k)\in\{0,1,b_k-2,b_k-1\}\}$. The rest of proof is divided into two cases.

**Case 1:** $\#S=\infty$. In this case, we show that $x_m$ and $y_m$ appear infinitely many times at the same coordinate of $x$ and $y$, respectively. Notice that, for $i\in\tau_k^{-1}\{0,1,b_k-2,b_k-1\}$, $f_k(i)$ has two $f_k$-preimages. For $k\in S$, let $i'_k\neq i_k$ be a $f_k$-preimage of $f_k(i_k)$. By (ii) of Lemma 4.3,

$$(\sigma^{r_k+m_kp_k+i_kp_{k-1}}x)|_{[0,p_{k-1})}=(\sigma^{r_k+m_kp_k+i'_kp_{k-1}}x)|_{[0,p_{k-1})}.$$

In particular, $x_m=x_{m^{(k)}}$ where $m^{(k)}=m+(i'_k-i_k)p_{k-1}$. Similarly, $y_m=y_{m^{(k)}}$. Since $\#S=\infty$ and $|m^{(k)}-m|=|i'_k-i_k|p_{k-1}\ge p_{k-1}\to\infty$ as $k\to\infty$, $(x,y)$ is not two-sided asymptotic.

**Case 2:** $\#S<\infty$. In this case, we show that $x$ and $y$ contain the $\mathbb{Z}$-orbit of $x_m$ and $y_m$, respectively. First, we find another integer $m'$ instead of $m$. For $k\in S$, since $f_k|_{C_k}:C_k\to B_k$ is a bijection, there is $i'_k\in C_k$ such that $f_k(i'_k)=f_k(i_k)$. For $k\notin S$, we set $i'_k=i_k\in C_k\setminus\{\tau_k^{-1}(1),\tau_k^{-1}(b_k-2)\}$. Let

$$m':=m+\sum_{k\in S}(i'_k-i_k)p_{k-1}=m+\sum_{k=0}^{\infty}(i'_k-i_k)p_{k-1}.$$

So, for $K \geq 0$,

$$\begin{aligned} m' =& m + \sum_{k=0}^{\infty} (i'_k - i_k) p_{k-1} \\ =& r_K + m_K p_K + \sum_{k=K+1}^{\infty} (i'_k - i_k) p_{k-1} + \sum_{k=0}^{K} i'_k p_{k-1} \quad \text{(by (4.4))} \\ \in& r_K + p_K \mathbb{Z} + i'_K p_{K-1} + [0, p_{K-1}) \cap \mathbb{Z} \end{aligned}$$

where the last statement holds since $i'_k < n_k$ for all $k$ and then the sum $\sum_{k=0}^{K-1} i'_k p_{k-1} < p_{K-1}$. For $K \geq 0$, we set

$$m'_K = m_K + \sum_{k=K+1}^{\infty} (i'_k - i_k) \frac{p_{k-1}}{p_K}. \tag{4.5}$$

So

$$m' = r_K + m'_K p_K + \sum_{k=0}^{K} i'_k p_{k-1} \in r_K + m'_K p_K + i'_K p_{K-1} + [0, p_{K-1}) \cap \mathbb{Z}. \tag{4.6}$$

We claim that $x_m = x_{m'}$ and $y_m = y_{m'}$. Let $\hat{K} = \max S$ and $x^{(0)} = \sigma^{r_{\hat{K}+1} + m_{\hat{K}+1} p_{\hat{K}+1} + i_{\hat{K}+1} p_{\hat{K}}} x \in Y_{\hat{K}}$. So $i_k = i'_k$ for $k > \hat{K}$. Since $x^{(0)} \in Y_{\hat{K}}$, by (ii) of Lemma 4.3, we have

$$(\sigma^{i_{\hat{K}} p_{\hat{K}-1}} x^{(0)})|_{[0, p_{\hat{K}-1})} = (\sigma^{i'_{\hat{K}} p_{\hat{K}-1}} x^{(0)})|_{[0, p_{\hat{K}-1})}. \tag{4.7}$$

Let $x^{(1)} = \sigma^{i'_{\hat{K}} p_{\hat{K}-1}} x^{(0)} \in Y_{\hat{K}-1}$. Again, by (ii) of Lemma 4.3, we have

$$(\sigma^{i_{\hat{K}-1} p_{\hat{K}-2}} x^{(1)})|_{[0, p_{\hat{K}-2})} = (\sigma^{i'_{\hat{K}-1} p_{\hat{K}-2}} x^{(1)})|_{[0, p_{\hat{K}-2})}. \tag{4.8}$$

So

$$\begin{aligned} & (\sigma^{i_{\hat{K}} p_{\hat{K}-1} + i_{\hat{K}-1} p_{\hat{K}-2}} x^{(0)})|_{[0, p_{\hat{K}-2})} \\ =& (\sigma^{i_{\hat{K}} p_{\hat{K}-1}} x^{(0)})|_{[i_{\hat{K}-1} p_{\hat{K}-2}, (i_{\hat{K}-1}+1) p_{\hat{K}-2})} \\ =& x^{(1)}|_{[i_{\hat{K}-1} p_{\hat{K}-2}, (i_{\hat{K}-1}+1) p_{\hat{K}-2})} \quad \text{(by (4.7))} \\ =& (\sigma^{i_{\hat{K}-1} p_{\hat{K}-2}} x^{(1)})|_{[0, p_{\hat{K}-2})} \\ =& (\sigma^{i'_{\hat{K}} p_{\hat{K}-1} + i'_{\hat{K}-1} p_{\hat{K}-2}} x^{(0)})|_{[0, p_{\hat{K}-2})} \quad \text{(by (4.8))}. \end{aligned}$$

Similarly, for $1 \leq k \leq \hat{K}$, we have $x^{(k)} = \sigma^{i'_{\hat{K}-k+1} p_{\hat{K}-k}} x^{(k-1)} \in Y_{\hat{K}-k}$ and

$$(\sigma^{\sum_{k'=\hat{K}-k}^{\hat{K}} i_{k'} p_{k'-1}} x^{(k)})|_{[0, p_{\hat{K}-k-1})} = (\sigma^{\sum_{k'=\hat{K}-k}^{\hat{K}} i'_{k'} p_{k'-1}} x^{(k)})|_{[0, p_{\hat{K}-k-1})}.$$

Therefore, we have $(\sigma^{\sum_{k'=0}^{\hat{K}} i_{k'} p_{k'-1}} x^{(0)})_0 = (\sigma^{\sum_{k'=0}^{\hat{K}} i'_{k'} p_{k'-1}} x^{(0)})_0$, that is, $x_m = x_{m'}$. Replacing $x$ by $y$, we also have $y_m = y_{m'}$ by the same arguments. Now we have found $m'$ instead of $m$ such that $x_{m'} \neq *$, $x_{m'} \neq y_{m'}$ and for all $k \geq 0$, $m' \in r_k + m'_k p_k + i'_k p_{k-1} + [0, p_{k-1})$, where $i'_k \in C_k$. So $m' - (r_k + m'_k p_k) \in J_k$ for all $k \geq 0$, noticing that for any $k \geq k' \geq 0$,

$$m' - (r_k + m'_k p_k) = \sum_{k''=0}^{k} i'_{k''} p_{k''-1} \in p_{k'} \mathbb{Z} + i'_{k'} p_{k'-1} + [0, p_{k'-1}) \cap \mathbb{Z}.$$

Finally, we show that points $x$ and $y$ contain the $\mathbb{Z}$-orbit of $x_{m'}$ and $y_{m'}$ at the same coordinate, respectively. To show this similarity, fix $z = (z_i)_{i \in \mathbb{Z}} \in \{x, y\}$. For $k \geq 0$, let $\alpha_k = \phi_k^{-1}(m' - (r_k +$

$m'_k p_k)$). Since $\sigma^{r_k+m'_k p_k} z \in Y_k$, by (i) of Lemma 3.6, we have $z_{r_k+m'_k p_k+\phi_k(n)} = T_*^n z_{r_k+m'_k p_k}$ for all $n \in I_k$. In particular,

$$z_{m'} = z_{r_k+m'_k p_k+\phi_k(\alpha_k)} = T_*^{\alpha_k} z_{r_k+m'_k p_k}$$

So for all $k \geq 0$,

$$z_{r_k+m'_k p_k+\phi_k(n+\alpha_k)} = T_*^n z_{m'} \text{ for all } n \in I_k - \alpha_k. \tag{4.9}$$

We next show that, for any integer $N \geq 1$, there exists $k \geq 0$ such that

$$I_k - \alpha_k \supset (-N,N) \cap \mathbb{Z}. \tag{4.10}$$

Fix integer $N \geq 1$ and choose $k > \max S$ such that $\# I_{k-1} > N$. Since $k > \max S$, we have $i'_k = i_k \in C_k \setminus \{\tau_k^{-1}(1), \tau_k^{-1}(b_k-2)\}$. So $\tau_k(i'_k) \in \{2,3,\ldots,b_k-3\}$ and n $f_k(i'_k) \in \{-(b_k-4),\ldots,-2,-1\}$. By (ii) of Lemma 3.6, $\alpha_k \in f_k(i'_k)b'_{k-1} + I_{k-1}$ since $m' - (r_k + m'_k p_k) \in i'_k p_{k-1} + J_{k-1}$. Since $I_k = f_k(B_k)b'_{k-1} + I_{k-1}$ and $f_k(B_k) = \{-(b_k-3),\ldots,-2,-1,0\}$, we have

$$I_k - \alpha_k \supset I_{k-1} \cup (b'_{k-1} + I_{k-1}) = (-b'_{k-1}, b'_{k-1}] \cap \mathbb{Z}.$$

By $b'_{k-1} = \# I_{k-1} > N$, (4.10) is proved.

Now we have proved that, for any $z \in \{x,y\}$, (4.9) holds, and for any $N \geq 1$, there exists $k \geq 0$ such that (4.10) holds. Therefore if $y_{m'} = *$, $(x,y)$ is not two-sided asymptotic by $x_{m'} \neq *$. In the case of $y_{m'} \neq *$, since all the proper pairs of $(X,T)$ are not two-sided asymptotic, so is $(x,y)$. □

By Proposition 4.5, the existence of homeomorphisms with positive topological entropy such that all the proper pairs of $(X,T)$ are not two-sided asymptotic implies the existence of completely Li–Yorke chaotic homeomorphisms with positive topological entropy. So by Theorem 4.4, we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* By Theorem 4.4, we choose a t.d.s. $(X,T)$ such that $h_{\text{top}}(X,T) > 0$ and all the proper pairs of $X$ are not two-sided asymptotic. And then choose a decreasing sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ such that $0 < \varepsilon_k < 1/2$ for $k \geq 0$, and $\prod_{k=0}^{\infty}(1-2\varepsilon_k) > 0$. Let $(Y,\sigma)$ be defined as (3.4) in Section 3. By Proposition 3.7, $h_{\text{top}}(Y,\sigma) > 0$. By Proposition 3.5 and Proposition 4.5, all the proper pairs are either positively or negatively scrambled pairs, which ends the proof. □

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