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A NEW PRIMAL-DUAL INTERIOR POINT METHOD FOR SEMIDEFINITE OPTIMIZATION BASED ON A HYPERBOLIC KERNEL FUNCTION

ZHENXIANG WU[1], MINGWANG ZHANG[1,2,*]

[1]Department of Mathematics, Anhui Institute of Information Technology, Wuhu 241000, China
[2]College of Science, China Three Gorges University, Yichang 443002, China

Abstract. In primal-dual interior-point methods, kernel functions not only determine the search direction and measures the distance of the current iteration point from the μ-center but also affect the iteration complexity and practical computational efficiency of algorithms. In this paper, we propose a new primal-dual interior-point method for semidefinite optimization based on a hyperbolic kernel function. We derive the iteration bounds for both large-update and small-update methods, which are the best-known complexity results for such methods. We prove the efficiency of the new algorithm via numerical results.
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1. Introduction

Research on semidefinite optimization (SDO) has been particularly active in recent years, primarily because many practical problems in operations research, statistics, and combinatorial optimization can be transformed into SDO; see, e.g., [1, 36] and the references therein. Early work on SDO was given by Bellman and Fan [2] in 1963. However, due to the lack of effective algorithms, SDO did not attract attention for a long time. It was not until the early 1990s that people discovered that the effective algorithms such as the interior point methods (IPMs) for linear optimization (LO) could often be extended to the more general case of SDO. To the best of our knowledge, Nesterov and Nemirovskii [22] first showed that SDO can be solved by IPMs in 1994. Among different variants of IPMs, primal-dual IPMs are the most efficient from the computation point view. There are two types of the IPMs: Small-update methods that take small steps near the central path and large-update methods that take more aggressive steps. However, there is a gap between the theoretical behavior and practical computational efficiency

*Corresponding author.
E-mail address: zmwang@ctgu.edu.cn (M. Zhang).
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of the algorithm: the large-update IPMs have better practical performance than the small-update IPMs, but the theoretical result is relatively weaker [24]. For more related results about IPMs, we refer to [35, 37].

For LO, most primal-dual IPMs are based on the simple logarithmic barrier functions, with the complexity of large-update methods being $O(n\log\frac{n}{\varepsilon})$, where n is the size of the problem and ε is the accuracy parameter [29]. In order to improve the complexity of IPMs, Peng et al. [25] introduced the primal-dual IPM for LO based on a new class of barrier functions defined by the so-called self-regular kernel functions. The barrier function is not only used to determine the search direction but also measure the distance of the current iteration point to the μ-center. They significantly improved the theoretical complexity and obtained the best-known iteration bound for large-update IPM, specifically $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$. Bai et al. [3] further relaxed the conditions for the selection of kernel functions proposed by Peng et al. They achieved the best-known iteration bound for both large-update and small-update methods. Since then, a large number of studies on kernel function-based primal-dual IPMs for LO have emerged. El Ghami et al. [12] proposed a primal-dual IPM based on a trigonometric kernel function, and they established the worst-case iteration complexity as $O(n^{\frac{3}{4}}\log(\frac{n}{\varepsilon}))$. Later, Bouafia et al. [10] proposed a primal-dual IPM for LO based on a simple kernel function and obtained the iteration bound of $O(pn^{\frac{p+1}{2p}}\log(\frac{n}{\varepsilon}))$ for the large-update method. Then, Guerdouh et al. [13] proposed a primal-dual IPM for LO based on a hyperbolic kernel function and derived the iteration bound of $O(n^{\frac{3}{4}}\log(\frac{n}{\varepsilon}))$ for the large-update method in 2023. Very recently, Touil et al. [30] proposed a primal-dual IPM based on the hyperbolic kernel function for LO, achieving the best-known iteration bound for the large-update method. For more related papers about primal-dual IPM for LO, we refer to [4, 5, 14]. Several IPMs based on the kernel functions for LO have been successfully extended to SDO, due to their theoretical and practical efficiency. For example, Wang et al. [33] proposed primal-dual IPMs for SDO based on a simple non-self-regular kernel function, which was first introduced to LO in [4]. They derived the complexity bounds as $O(qn\log\frac{n}{\varepsilon})$ and $O(q^2\sqrt{n}\log\frac{n}{\varepsilon})$ for the large-update and small-update methods, respectively. Next, Qian et al. [28] proposed a new kernel function with simple algebraic expression and derived the iteration complexity as $O(n^{\frac{3}{4}}\log(\frac{n}{\varepsilon}))$ for large-update method. Later on, Kheirfam et al. [18] introduced a primal-dual IPM for SDO based on a new trigonometric kernel function, and they showed that the iteration bound were $O(n^{\frac{3}{4}}\log(\frac{n}{\varepsilon}))$ for large-update method. Recently, Abderrahim et al. [16] presented a primal-dual IPM for SDO based on logarithmic kernel functions, and derived the complexity bounds as $O(qsn^{\frac{sq+1}{2sq}}\log(\frac{n}{\varepsilon}))$ and $O(q^2s^2\sqrt{n}\log(\frac{n}{\varepsilon}))$ for the large-update and small-update methods, respectively. For additional research on primal-dual IPMs for SDO based on kernel functions, we refer to [7, 8, 9, 11, 19, 20, 26, 27, 34].

Recently, several researchers developed primal-dual IPMs based on hyperbolic kernel functions for SDO. Touil et al. [31] proposed primal-dual IPM based on the hyperbolic-logarithmic kernel function for SDO, and derived the iteration bound as $O(n^{\frac{2}{3}}\log\frac{n}{\varepsilon})$ for the large-update method. Subsequently, Touil et al. [32] proposed a primal-dual IPM for SDO based on a new hyperbolic kernel function, and derived the iteration bounds as $O(n^{\frac{3}{4}}\log\frac{n}{\varepsilon})$ and $O(\sqrt{n}\log\frac{n}{\varepsilon})$ for the large-update and small-update methods, respectively. Very recently, Bounibane et al. [6] proposed a primal-dual IPM based on a bi-barrier hyperbolic kernel function and obtained the best-known iteration bound. It is worth noting that although some IPMs based on a kernel function for SDO can be viewed as extensions of the IPMs for LO and have similar polynomial complexity results, in fact, obtaining a valid search direction and convergence results in the SDO case is more difficult than in the LO case.

Motivated by the previous research, in this paper, our main objective is to propose a primal-dual interior-point algorithm for SDO based on a new hyperbolic kernel function, which was first introduced in [15], and it was used to determine the iteration direction of the algorithm for LO. The change in the kernel function leads to differences between our algorithm and similar algorithms of SDO in the following aspects: barrier function, proximity measure, iteration direction, and convergence analysis. By establishing some new technical lemmas, we derive the iteration complexity bounds for both large-update and small-update methods, namely $O\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right)$ and $O\left(p\sqrt{n}\log\frac{n}{\varepsilon}\right)$, which are the best-known iteration bounds for such methods. In addition, via numerical experiments, we show that the practical performance of the algorithm is very promising.

Finally, we introduce the notations required for this paper. The set of all $(n\times n)$ matrices with real terms is represented by $R^{n\times n}$. The R^n denotes the set of n-dimensional vectors, the sets of n-dimensional nonnegative vectors and positive vectors are denoted as R^n_+ and R^n_{++}, respectively. Given $M\in R^{n\times n}$, M^T denotes the transpose of M. The set of all symmetric $n\times n$ real matrices is denoted by S^n. $S^n_+(S^n_{++})$ denotes the cone of positive semidefinite (positive definite) matrices in S^n. The inner product of matrix A and B is defined by $\langle A,B\rangle = \operatorname{tr}(A^TB) = \Sigma^n_{i,j=1}a_{ij}b_{ij}$. For $M\in S^n$, we denote $\lambda_i(M)(i=1,\cdots,n)$ as eigenvalues of M, and $\lambda_{\max}(M),\lambda_{\min}(M)$ as the largest eigenvalue and the smallest eigenvalue. In this paper, we adopt the Frobenius norm, i.e, $\|M\| = \langle M,M\rangle^{\frac{1}{2}} = \sqrt{\Sigma^n_{i=1}\lambda_i^2(M)}$. $M\succ 0(M\succeq 0)$ indicates that M is a positive definite (semidefinite) matrix.

2. Preliminaries

2.1. **Kernel function.** It is well known that the barrier function Ψ is determined by the univariate kernel function ψ. We define the kernel function as follows.

Definition 2.1. [3] The function $\psi:(0,+\infty)\to[0,+\infty)$ is a kernel function if ψ is twice differentiable and satisfies the following conditions:

(1) $\psi''(t)>0, \forall t>0$,
(2) $\psi'(1)=\psi(1)=0$,
(3) $\lim\limits_{t\to 0^+}\psi(t)=\lim\limits_{t\to+\infty}\psi(t)=+\infty$.

Using the concept of a matrix function (Horn and Johnson [17]), we are ready to show how a barrier function Ψ can be obtained from a kernel function ψ.

2.2. **Barrier function.** In this section, we introduce the concept of matrix functions and some primary results. This also serves as the necessary foundation for subsequent analysis.

Definition 2.2. [17] A matrix $M(t)$ is said to be a matrix function if each entry of $M(t)$ is a function of t, i.e., $M(t)=[M_{ij}(t)]$.

Observe that $\frac{d}{dt}(M(t)) = \left[\frac{d}{dt}M_{ij}(t)\right] = M'(t)$. Suppose that both matrix functions $M(t)$ and $N(t)$ are differentiable with respect to t. Then

$$\frac{d}{dt}(\mathrm{tr}(M(t))) = \mathrm{tr}\left[\frac{d}{dt}(M(t))\right] = \mathrm{tr}(M'(t));$$

$$\frac{d}{dt}(\mathrm{tr}(\psi(M(t)))) = \mathrm{tr}\left[\psi'(M(t))(M'(t))\right];$$

and

$$\frac{d}{dt}(M(t)N(t)) = \left[\frac{d}{dt}(M(t))\right]N(t) + M(t)\left[\frac{d}{dt}(N(t))\right] = M'(t)N(t) + M(t)N'(t).$$

Definition 2.3. [17] Suppose the matrix $V(V \succ 0)$ is diagonalizable and has the following eigen-decomposition

$$V = Q^T \mathrm{diag}(\lambda_1(V), \lambda_2(V), \cdots, \lambda_n(V))Q,$$

where Q is an orthogonal matrix with $Q^T = Q^{-1}$. Given a kernel function $\psi(t)$, the matrix function $\psi(V): S^n_{++} \to S^n$ is defined by

$$\psi(V) = Q^T \mathrm{diag}(\psi(\lambda_1(V)), \psi(\lambda_2(V)), \cdots, \psi(\lambda_n(V)))Q.$$

Similarly to the case in LO, we define barrier function $\Psi(V)$ as the trace of the matrix function $\psi(V)$, i.e., $\Psi(V) = \mathrm{tr}(\psi(V)) = \sum_{i=1}^n \psi((\lambda_i(V)))$.

In this paper, when we use the function $\psi(\cdot)$ and its derivative $\psi'(\cdot)$ and $\psi''(\cdot)$, they always represent a matrix function if the argument is a matrix; if the argument is in the real number set R, then they represent a general function from R to R.

2.3. **The central path.** We consider the standard form of SDO

$$(P)\begin{cases} \min\langle \mathrm{C}, \mathrm{X}\rangle \\ \mathscr{A}X = b, \\ X \in S^n_+, \end{cases}$$

where $b \in R^m, C \in S^n$, $\mathscr{A}$ is a linear operator from S^n to R^m defined by

$$\mathscr{A}X = (\langle A_1, X\rangle, \langle A_2, X\rangle, \cdots, \langle A_m, X\rangle)^T$$

with $A_i (i = 1, 2, \ldots, m) \in S^n$, and A_i are linearly independent. Its dual problem is given by

$$(D)\begin{cases} \max \quad b^T y \\ \mathscr{A}^* y + S = C, \\ S \in S^n_+, \end{cases}$$

where $\mathscr{A}^*$ is the adjoint of $\mathscr{A}$ defined from R^m to S^n by $\mathscr{A}^* y = \sum_{i=1}^m y_i A_i$.

The sets of strictly feasible solutions of (P) and (D) are

$$\mathscr{F}^0(P) = \{X \in S^n_{++} : \mathscr{A}X = b\},$$

$$\mathscr{F}^0(D) = \{(y, S) \in R^m \times S^n_{++} : \mathscr{A}^* y + S = C\},$$

respectively. In this paper, we assume that both problems (P) and (D) satisfy the interior point condition (IPC), i.e., $\mathscr{F}^0(P)\times\mathscr{F}^0(D)\neq\emptyset$. If the IPC holds, the optimal conditions for problems (P) and (D) are as follows:

$$\begin{cases} \mathscr{A}X=b, X\succ 0,\\ \mathscr{A}^*y+S=C, S\succ 0,\\ XS=0. \end{cases} \tag{2.1}$$

The third equation in problem (2.1) is called the complementarity condition for (P) and (D); the core idea of the primal-dual IPMs is to use the parameterized equation $XS=\mu I, \mu>0$ to replace the last equation in (2.1). Then the problem is transformed as follows:

$$\begin{cases} \mathscr{A}X=b, X\succ 0,\\ \mathscr{A}^*y+S=C, S\succ 0,\\ XS=\mu I, \mu>0. \end{cases} \tag{2.2}$$

Under the IPC, system (2.2) has a unique solution denoted by $(X(\mu),y(\mu),S(\mu))$, for each $\mu>0$. The set of all solutions $(X(\mu),y(\mu),S(\mu))$, with $\mu>0$, is known as the central path or central trajectory. If $\mu\to 0$, then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields an optimal solution for SDO.

Applying Newton's method to (2.2), we obtain the following system:

$$\begin{cases} \mathscr{A}\Delta X=0,\\ \mathscr{A}^*\Delta y+\Delta S=0,\\ X\Delta S+\Delta XS=\mu I-XS. \end{cases} \tag{2.3}$$

For SDO, a crucial observation is that ΔX in system (2.3) is not necessarily symmetric. To address this issue, several ways have been proposed for symmetrizing the third equation in the system (2.3), such that the resulting new system has a unique symmetric solution.

In this paper, we use the symmetrization scheme where the Nesterov-Todd (NT) direction was derived by Nesterov and Todd in [23]. The advantage is that the NT scaling technique transfers the primal variable X and the dual variable S into the same space: the so-called V-space. Define

$$P:=\left[X^{\frac{1}{2}}(X^{\frac{1}{2}}SX^{\frac{1}{2}})^{-\frac{1}{2}}X^{\frac{1}{2}}\right]^{-\frac{1}{2}}=\left[S^{-\frac{1}{2}}(S^{\frac{1}{2}}XS^{\frac{1}{2}})^{\frac{1}{2}}S^{-\frac{1}{2}}\right]^{-\frac{1}{2}}.$$

The matrix P can be used to scale X and S to the same matrix V,

$$V:=\frac{1}{\sqrt{\mu}}PXP=\frac{1}{\sqrt{\mu}}P^{-1}SP^{-1}.$$

It is noteworthy that matrices P and V are symmetric and positive definite. Furthermore, we have

$$V^2=\left(\frac{PXP}{\sqrt{\mu}}\right)\left(\frac{P^{-1}SP^{-1}}{\sqrt{\mu}}\right)=\frac{PXSP^{-1}}{\mu}.$$

Let us further define

$$\bar{A}_i=\frac{P^{-1}A_iP^{-1}}{\sqrt{\mu}}, \quad i=1,2,\cdots,m,$$

$$D_X:=\frac{P\Delta XP}{\sqrt{\mu}}, D_S:=\frac{P^{-1}\Delta SP^{-1}}{\sqrt{\mu}}. \tag{2.4}$$

Substitute $V, \bar{A}_i, D_X$ and D_S into equation (2.3), correspondingly the NT direction is the solution to (2.3), we obtain the following system:

$$\begin{cases} \bar{\mathscr{A}} D_X = 0, \\ \bar{\mathscr{A}}^* \Delta y + D_S = 0, \\ D_X + D_S = V^{-1} - V, \end{cases} \tag{2.5}$$

where $\bar{\mathscr{A}}^*$ is adjoint of $\bar{\mathscr{A}}$. The third equation in (2.5) is called the scaled centering equation.

Given the kernel function $\psi(t)$ in Definition 2.1, and associated matrix functions $\psi(V)$ and $\psi'(V)$ are defined in Definition 2.3. As [26], we replace $V^{-1} - V$ in the third equation in (2.5) with $-\psi'(V)$. Thus we consider the system

$$\begin{cases} \bar{\mathscr{A}} D_X = 0, \\ \bar{\mathscr{A}}^* \Delta y + D_S = 0, \\ D_X + D_S = -\psi'(V). \end{cases} \tag{2.6}$$

Due to the first two equations of the system (2.6), it is trivial to see that D_X and D_S are orthogonal, i.e., $\mathrm{tr}(D_X D_S) = \mathrm{tr}(D_S D_X) = 0$. Thus we have

$$D_X = D_S = 0_{n\times n} \Leftrightarrow \psi'(V) = 0_{n\times n} \Leftrightarrow V = I \Leftrightarrow \Psi(V) = 0 \Leftrightarrow XS = \mu I.$$

By taking a step along the search direction $(\Delta X, \Delta y, \Delta S)$, with the step-size α defined by some line search rules, the new iterate (X_+, y_+, S_+) is given by

$$X_+ = X + \alpha \Delta X; y_+ = y + \alpha \Delta y; S_+ = S + \alpha \Delta S. \tag{2.7}$$

From what has been discussed above, we know that if $(X, y, S) \neq (X(\mu), y(\mu), S(\mu))$, then $(\Delta X, \Delta y, \Delta S)$ is nonzero. In addition, the norm-based proximity measure is defined by

$$\delta(V) = \frac{1}{2}\|D_X + D_S\| = \frac{1}{2}\|\psi'(V)\| = \frac{1}{2}\sqrt{\mathrm{tr}(\psi'(V)^2)}.$$

2.4. **Hyperbolic kernel function and its properties.** In this section, we consider a hyperbolic kernel function, and present the relevant properties of this kernel function. For subsequent analysis, we give the following inequalities about kernel functions,

$$t\psi''(t) + \psi'(t) > 0, \forall t < 1; \tag{2.8}$$

$$t\psi''(t) - \psi'(t) > 0, \forall t > 1; \tag{2.9}$$

$$\psi'''(t) < 0, \forall t > 0; \tag{2.10}$$

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) > 0, \forall t < 1; \tag{2.11}$$

and

$$\psi''(t)\psi'(\beta t) - \beta \psi'(t)\psi''(\beta t) > 0, \forall t > 1, \beta > 1. \tag{2.12}$$

Definition 2.4. [3] Let $\psi : (0, +\infty) \to [0, +\infty)$ be a kernel function. If ψ satisfies (2.8), (2.10), (2.11) and (2.12), then $\psi(t)$ is an eligible kernel function.

Moreover, if $\psi(t)$ satisfies (2.9) and (2.10), then $\psi(t)$ satisfies (2.12) [3]. In this paper, we consider the following hyperbolic kernel function $\psi(t)$:

$$\psi(t) = \frac{t^2-1}{2} + \frac{a}{p}\coth^p(t) - \frac{\sinh^2(1)}{p}\coth(1), t > 0,$$

with $a = \frac{\sinh^2(1)}{\coth^{p-1}(1)}$ and $p \geq 2$. $\psi(t)$ was primarily proposed in [15] for solving LO. Next, we list $\psi'(t), \psi''(t)$ and $\psi'''(t)$ as shown below:

$$\psi'(t) = t - a\frac{\coth^{p-1}(t)}{\sinh^2(t)};$$

$$\psi''(t) = 1 + a\left(2\frac{\coth^p(t)}{\sinh^2(t)} + (p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)}\right) \geq 1; \tag{2.13}$$

and

$$\psi'''(t) = -a\left(4\frac{\coth^{p+1}(t)}{\sinh^2(t)} + (6p-4)\frac{\coth^{p-1}(t)}{\sinh^4(t)} + (p-1)(p-2)\frac{\coth^{p-3}(t)}{\sinh^6(t)}\right).$$

As can be seen from [15], $\psi(t)$ is an eligible kernel function.

Next, we cite the following lemmas, which provides several crucial properties for algorithmic analysis.

Lemma 2.5. [15] *For the kernel function* $\psi(t)$, $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\left[\psi'(t)\right]^2, t > 0$.

Lemma 2.6. [26] *Let the function* $\Psi(V) : S^n_{++} \to R$ *be defined above. For any* $X_1 \succ 0$ *and* $X_2 \succ 0$,

$$\Psi\left(\left(X_1^{\frac{1}{2}} X_2 X_1^{\frac{1}{2}}\right)^{\frac{1}{2}}\right) \leq \frac{1}{2}\left(\Psi(X_1) + \Psi(X_2)\right).$$

Furthermore, two inverse functions related to the kernel function have proved to be essential in the analysis of this algorithm and are defined below.

Lemma 2.7. *Let* $\sigma : [0, +\infty) \to [1, +\infty)$ *be the inverse function of* $\psi(t)$ *for* $t \geq 1$. *Then,*

$$1 + \sqrt{\frac{4s}{2p+5}} \leq \sigma(s) \leq 1 + \sqrt{2s}, s \geq 0.$$

Proof. Let $s = \psi(t), t \geq 1$, i.e., $\sigma(s) = t$. From Lemma 2.5, we have $\psi(t) \geq \frac{1}{2}(t-1)^2$, i.e., $t = \sigma(s) \leq 1 + \sqrt{2s}$. Using Taylor's development for $1 \leq \zeta \leq t$, we have

$$\psi(t) = \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi'''(\zeta)(t-1)^3.$$

From $\psi(1) = \psi'(1) = 0$, $\psi'''(t) < 0$ and $\psi''(1) = 1 + 2\coth(1) + \frac{2(p-1)}{\sinh(2)} < p + \frac{5}{2}$, we can obtain

$$\begin{aligned}
\psi(t) &= \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi'''(\zeta)(t-1)^3 \\
&\leq \frac{1}{2}\psi''(1)(t-1)^2 \\
&= \frac{1}{2}\left[1 + 2\coth(1) + \frac{2(p-1)}{\sinh(2)}\right](t-1)^2 \leq \frac{1}{4}(2p+5)(t-1)^2.
\end{aligned}$$

Hence, we may write $s \leq \frac{1}{4}(2p+5)(t-1)^2$, which implies $t \geq 1+\sqrt{\frac{4s}{2p+5}}$. This completes the proof. $\square$

Lemma 2.8. [15] *Let* $\rho : [0,+\infty) \to (0,1]$ *be the inverse function of* $-\frac{1}{2}\psi'(t)$ *for* $0 < t \leq 1$. *Then* $\tanh(1)\coth(\rho(z)) \leq (2z+1)^{\frac{1}{p+1}}$, $z > 0$.

During the course of the algorithm, the largest value of $\Psi(V)$ in Definition 2.3 occurs just after the updates of μ. Thus, it is important to estimate the effect of a μ-update on the value of $\Psi(V)$. The result is stated in the lemma below.

Lemma 2.9. [33] *Let* $\sigma : [0,+\infty) \to [1,+\infty)$ *be the inverse function of* $\psi(t)$ *for* $t \geq 1$. *If* $\beta = \frac{1}{\sqrt{1-\theta}} > 1, V \succ 0$, *then* $\Psi(\beta V) \leq n\psi\left(\beta\sigma\left(\frac{\Psi(V)}{n}\right)\right)$.

Lemma 2.10. *Let* $0 < \theta < 1, \tau > 0$, *and* $\beta = \frac{1}{\sqrt{1-\theta}}$. *If* $\Psi(V) \leq \tau$, *then*

$$\Psi(\beta V) \leq \frac{2p+5}{4(1-\theta)}\left(\sqrt{2\tau}+\theta\sqrt{n}\right)^2.$$

Proof. From $\beta = \frac{1}{\sqrt{1-\theta}} > 1$ and $\sigma\left(\frac{\Psi(V)}{n}\right) \geq 1$, we have $\frac{\sigma\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}} \geq 1$. Using Lemma 2.9, we have

$$\Psi(\beta V) \;\leq\; n\psi\left(\beta\sigma\left(\frac{\Psi(V)}{n}\right)\right) \leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi(V)}{n}\right)\right).$$

By Lemma 2.7, we obtain

$$\begin{aligned}
\Psi(\beta V) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\sigma\left(\frac{\Psi(V)}{n}\right)\right) \leq \frac{n}{4}(2p+5)\left(\left[\frac{\sigma\left(\frac{\Psi(V)}{n}\right)}{\sqrt{1-\theta}}\right]-1\right)^2 \\
&= \frac{n(2p+5)}{4(1-\theta)}\left(\sigma\left(\frac{\Psi(V)}{n}\right)-\sqrt{1-\theta}\right)^2 \leq \frac{n(2p+5)}{4(1-\theta)}\left(\left[1+\sqrt{\frac{2\Psi(V)}{n}}\right]-\sqrt{1-\theta}\right)^2 \\
&= \frac{n(2p+5)}{4(1-\theta)}\left(\left[1-\sqrt{1-\theta}\right]+\sqrt{\frac{2\Psi(V)}{n}}\right)^2 \\
&\leq \frac{n(2p+5)}{4(1-\theta)}\left(\sqrt{\frac{2\tau}{n}}+\theta\right)^2 = \frac{2p+5}{4(1-\theta)}\left(\sqrt{2\tau}+\theta\sqrt{n}\right)^2 := \Psi_0.
\end{aligned}$$

The last inequality is obtained from $1-\sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1+\theta}} \leq \theta$. This completes the proof. $\square$

Then Ψ_0 is an upper bound for $\Psi(\beta V)$ during the process of the algorithm.

Lemma 2.11. [20] *For* $V \succ 0$, $\delta(V) = \frac{1}{2}\|D_X+D_S\| = \frac{1}{2}\|\psi'(V)\| \geq \sqrt{\frac{\Psi(V)}{2}}$.

3. A NEW ALGORITHM OF IPM FOR SDO BASED ON KERNEL FUNCTIONS AND COMPLEXITY RESULT

In this section, we introduce our algorithm and analyze its complexity. The algorithm consists of two parts: the inner iteration and the outer iteration. If $\Psi(V) \leq \tau$, and an outer iteration starts, we decrease μ to $(1-\theta)\mu$ for $0<\theta<1$, which in general leads to an increase in the value of $\Psi(V)$. Then the algorithm starts the inner iterations if $\Psi(V)\geq\tau$, and the inner iterations will reduce the value of $\Psi(V)$. This iterative process is repeated until μ is small enough. In this process, we find the ε-solution of (P) and (D).

Algorithm 3.1. *(The Primal-dual Interior Point Algorithm for SDO)*
Input:
a threshold parameter $\tau>0$*;*
a accuracy parameter $\varepsilon>0$*;*
a fixed barrier update parameter $\theta\in(0,1)$*;*
(X^0,y^0,S^0) *satisfy the IPC and* $\mu^0=1$ *such that* $\Phi(X^0,S^0,\mu^0)=\Psi(V^0)\leq\tau$.
begin:
$X=:X^0; y:=y^0; S:=S^0; \mu=\mu^0$*;*
while: $n\mu\geq\varepsilon$ *do*
begin *(outer iteration)*

$$\mu:=(1-\theta)\mu;$$

while $\Phi(X,S,\mu)=\Psi(V)>\tau$ *do*
begin *(inner iteration)*
Solve system (2.6) *and use* (2.4) *to obtain* $(\Delta X,\Delta y,\Delta S)$*;*
choose a step-size α*;*

$$X:=X+\alpha\Delta X; y:=y+\alpha\Delta y; S:=S+\alpha\Delta S; V:=\frac{1}{\sqrt{\mu}}\left(PXSP^{-1}\right)^{\frac{1}{2}}.$$

end
end
end

The parameters θ,τ and the step-size α should be chosen to ensure that the number of iterations is as small as possible. If θ is a constant independent of dimension n of the problem, for instance $\theta=\frac{1}{2}$, then the algorithm is called a large-updated method. If θ depends on dimension n of the problem, such as $\theta=\frac{1}{\sqrt{n}}$, then the algorithm is called a small-update method.

From (2.4) and (2.7) for fixed μ, one has

$$X_+=X+\alpha\Delta X=X+\alpha\sqrt{\mu}P^{-1}D_XP^{-1}=\sqrt{\mu}P^{-1}(V+\alpha D_X)P^{-1},$$
$$S_+=S+\alpha\Delta S=S+\alpha\sqrt{\mu}PD_SP=\sqrt{\mu}P(V+\alpha D_S)P,$$

and V_+^2 is similar to the matrix $\frac{1}{\mu}X_+^{\frac{1}{2}}S_+X_+^{\frac{1}{2}}$, and thus the eigenvalues of V_+ are the same as those of the matrix $\left((V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}\right)^{\frac{1}{2}}$.

Define $f(\alpha):=\Psi(V_+)-\Psi(V)$, which denotes the decrease of the barrier function in each inner iteration. In this paper, we assume that the step-size α satisfies

$$V+\alpha D_X\succeq 0 \quad \text{and} \quad V+\alpha D_S\succeq 0.$$

From Lemma 2.6, we have

$$\Psi(V_+)=\Psi\left(\left((V+\alpha D_X)^{\frac{1}{2}}(V+\alpha D_S)(V+\alpha D_X)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)\leq\frac{1}{2}\left(\Psi(V+\alpha D_X)+\Psi(V+\alpha D_S)\right),$$

which implies

$$f(\alpha) \le f_1(\alpha) := \frac{1}{2}\left(\Psi(V+\alpha D_X)+\Psi(V+\alpha D_S)\right)-\Psi(V).$$

We thus have $f(0)=f_1(0)=0$. Taking the derivative of $f_1(\alpha)$ with respect to α, one has

$$f_1'(\alpha)=\frac{1}{2}\mathrm{tr}(\psi'(V+\alpha D_X)D_X+\psi'(V+\alpha D_S)D_S),$$

and its second derivative is $f_1''(\alpha)=\frac{1}{2}\mathrm{tr}(\psi''(V+\alpha D_X)D_X^2+\psi''(V+\alpha D_S)D_S^2)$. From third equality of (2.6), we have

$$f_1'(0)=\frac{1}{2}\mathrm{tr}(\psi'(V)D_X+\psi'(V)D_S)=\frac{1}{2}\mathrm{tr}(\psi'(V)(D_X+D_S))=-\frac{1}{2}\mathrm{tr}((\psi'(V))^2)=-2\delta(V)^2.$$

Below we use the short notation $\delta:=\delta(V)$, and we present the important lemmas from [33] as follows, which will be used to analyze the convergence of the algorithm.

Lemma 3.2. [33] *It holds that* $f_1''(\alpha)\le 2\delta^2\psi''(\lambda_{\min}(V)-2\alpha\delta)$.

Lemma 3.3. [33] *If step* α *satisfies* $\psi'(\lambda_{\min}(V))-\psi'(\lambda_{\min}(V)-2\alpha\delta)\le 2\delta$, *then* $f_1'(\alpha)\le 0$.

Lemma 3.4. [33] *Let* ρ *be defined in Lemma 2.8. Then the largest possible value of the step-size* α^* *satisfying* $\psi'(\lambda_{\min}(V))-\psi'(\lambda_{\min}(V)-2\alpha\delta)\le 2\delta$ *is given by* $\alpha^*:=\frac{\rho(\delta)-\rho(2\delta)}{2\delta}$.

Lemma 3.5. [33] *Let* ρ *and* α^* *be as defined in Lemma 3.4. Then* $\alpha^*\ge\frac{1}{\psi''(\rho(2\delta))}$.

Then we give a suitable step-size α for our algorithm.

Lemma 3.6. *If* α^* *is defined as Lemma 3.4, then* $\alpha^*\ge\frac{1}{4(p+1)(4\delta+1)^{\frac{p+2}{p+1}}+1}$.

Proof. Firstly, from (2.13), we have

$$\begin{aligned}\psi''(t) &= 1+a\left(2\frac{\coth^p(t)}{\sinh^2(t)}+(p-1)\frac{\coth^{p-2}(t)}{\sinh^4(t)}\right)\\ &= (p-1)\frac{\tanh^p(1)\sinh(2)\coth^{p-2}(t)}{2\sinh^4(\mathrm{t})}+\frac{1}{\sinh^2(\mathrm{t})}\tanh^p(1)\sinh(2)\coth^p(t)+1.\end{aligned}$$

Let $t=\rho(2\delta)\in(0,1]$. Using $\frac{1}{\sinh(\mathrm{t})}\le\coth(t)$, one has

$$\begin{aligned}\psi''(t) &\le \frac{p-1}{2}\tanh^p(1)\sinh(2)\coth^{p-2}(t)\coth^4(t)+\tanh^p(1)\sinh(2)\coth^p(t)\coth^2(t)+1\\ &\le \frac{p-1}{2}\tanh^p(1)\sinh(2)\coth^{p+2}(t)+\tanh^p(1)\sinh(2)\coth^{p+2}(t)+1.\end{aligned}$$

By $\frac{\sinh(2)}{\tanh^2(1)}=\frac{2\sinh(1)\cosh(1)}{\tanh^2(1)}=\frac{(e+e^{-1})^3}{2(e-e^{-1})}<8$, we obtain

$$\begin{aligned}\psi''(t) &\le \frac{\sinh(2)}{\tanh^2(1)}\frac{p+1}{2}(\tanh(1)\coth(t))^{p+2}+1\\ &\le 4(p+1)(\tanh(1)\coth(t))^{p+2}+1.\end{aligned}$$

Combining with the Lemmas 2.8 and 3.5 , we have

$$\begin{aligned}\alpha^* \geq \frac{1}{\psi''(\rho(2\delta))} &\geq \frac{1}{4(p+1)\,(\tanh(1)\coth(\rho(2\delta)))^{p+2}+1}\\ &\geq \frac{1}{4(p+1)\,(4\delta+1)^{\frac{p+2}{p+1}}+1},\end{aligned}$$

and the proof is complete. ☐

In the sequel, we use the notation

$$\bar{\alpha} := \frac{1}{4(p+1)\,(4\delta+1)^{\frac{p+2}{p+1}}+1},$$

and we use $\bar{\alpha}$ as the default step-size. It is obvious that $\alpha^* \geq \bar{\alpha}$.

Lemma 3.7. [33] *If the step-size $\alpha \leq \alpha^*$, then $f(\alpha) \leq -\alpha\delta^2$.*

By the following theorem, we can obtain the upper bound for decreasing value of the proximity in the inner iteration.

Theorem 3.8. *If $\Psi(V) \geq 1$ and $\bar{\alpha} = \frac{1}{4(p+1)(4\delta+1)^{\frac{p+2}{p+1}}+1}$, then*

$$f(\bar{\alpha}) \leq -\frac{[\Psi(V)]^{\frac{p}{2(p+1)}}}{64(p+2)}.$$

Proof. Using Lemmas 3.6 and 3.7, we derive that

$$f(\bar{\alpha}) \leq -\bar{\alpha}\delta^2 = \frac{-\delta^2}{4(p+1)\,(4\delta+1)^{\frac{p+2}{p+1}}+1} = \frac{-\delta^{\frac{p}{p+1}}}{4(p+1)(4+\frac{1}{\delta})^{\frac{p+2}{p+1}}+(\frac{1}{\delta})^{\frac{p+2}{p+1}}}.$$

Since $\Psi(V) \geq 1$, then from Lemma 2.11, we have $\delta \geq \sqrt{\frac{\Psi(V)}{2}} \geq \sqrt{\frac{1}{2}} > \frac{1}{2}$. Thus

$$\begin{aligned}f(\bar{\alpha}) &\leq \frac{-\delta^{\frac{p}{p+1}}}{4(p+1)(4+\frac{1}{\delta})^{\frac{p+2}{p+1}}+(\frac{1}{\delta})^{\frac{p+2}{p+1}}}\\ &\leq \frac{-\delta^{\frac{p}{p+1}}}{4(p+2)(4+\frac{1}{\delta})^{\frac{p+2}{p+1}}} \leq \frac{-\delta^{\frac{p}{p+1}}}{4(p+2)6^{\frac{4}{3}}}\\ &\leq -\frac{[\Psi(V)]^{\frac{p}{2(p+1)}}}{4(p+2)6^{\frac{4}{3}}2^{\frac{1}{2}}} \leq -\frac{[\Psi(V)]^{\frac{p}{2(p+1)}}}{64(p+2)}.\end{aligned}$$

The last inequality is obtained from $4(6^{\frac{4}{3}}2^{\frac{1}{2}}) < 64$. This completes the proof of the theorem. ☐

Now, we compute how many inner iterations are required to return to the situation where $\Psi(V) \leq \tau$. After the update of μ to $(1-\theta)\mu$, from Theorem 3.8, we have the decrease of Ψ is given by

$$\Psi_{k+1} \leq \Psi_k - \frac{[\Psi(V)]^{\frac{p}{2(p+1)}}}{64(p+2)}, \quad k = 0, 1, \cdots, K-1,$$

where K is the total number of inner iterations in an outer iteration.

Lemma 3.9. [26] *Let $t_0, t_1, \cdots, t_k$ be a sequence of positive numbers such that $t_{k+1} \le t_k - \beta t_k^{1-\gamma}$, $k = 0, 1, \cdots, K-1$, where $\beta > 0$ and $0 < \gamma \le 1$. Then $K \le \left\lfloor \frac{t_0^\gamma}{\beta\gamma} \right\rfloor$.*

Lemma 3.10. *If K denotes the number of inner iterations, then $K \le 128(p+1)\left[\Psi_0\right]^{\frac{p+2}{2(p+1)}}$.*

Proof. Let $t_k = \Psi_k$, $\beta = \frac{1}{64(p+2)}$, and $\gamma = \frac{p+2}{2(p+1)}$. Using Lemma 3.9, we have

$$K \le \frac{[\Psi_0]^\gamma}{\beta\gamma} = \frac{128(p+2)(p+1)\left[\Psi_0\right]^{\frac{p+2}{2(p+1)}}}{p+2} \le 128(p+1)\left[\Psi_0\right]^{\frac{p+2}{2(p+1)}}.$$

□

Theorem 3.11. *The total number of iterations required by the algorithm is at most*

$$\frac{K}{\theta}\log\left(\frac{n}{\varepsilon}\right) = 128(p+1)\left[\Psi_0\right]^{\frac{p+2}{2(p+1)}} \frac{\log(\frac{n}{\varepsilon})}{\theta}.$$

Proof. An upper bound for the total number of iterations is obtained by multiplying the upper bound K by the number of barrier parameter updates, where the number of barrier parameter updates is bounded by $\frac{1}{\theta}\log(\frac{n}{\varepsilon})$ [29]. The proof is complete here. □

Setting $\tau = O(1), \theta = \Theta(\frac{1}{\sqrt{n}})$, we conclude that the iteration bound of the small-update method is $O\left(p\sqrt{n}\log\frac{\mathrm{n}}{\varepsilon}\right)$. For large-update method with $\tau = O(n), \theta = \Theta(1)$, we have the iteration bound, namely

$$O\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right).$$

When $p = \frac{\log n}{2} - 1$, the best-current iteration bound, namely $O\left(\sqrt{n}(\log n)\log\frac{n}{\varepsilon}\right)$, is obtained.

4. Numerical experiments

In this section, we present some numerical results using MATLAB. Some kernel functions that we used in the experiments are as follows:

$$\psi_a(t) = \frac{t^2-1}{2} + \frac{(e-1)^{q+1}}{qe(e^{t-1})^q} - \frac{(e-1)}{qe};$$

$$\psi_b(t) = \frac{t^2-1}{2\sinh^2(1)} + \coth(t) - \coth(1);$$

$$\psi_c(t) = \frac{t^2-1}{2} + \tanh^2(1)(\coth(\mathrm{t}) - \log\mathrm{t}) - \tanh(1);$$

and

$$\psi_{\text{new}} = \frac{t^2-1}{2} + \frac{a}{p}\coth^p(t) - \frac{\sinh^2(1)}{p}\coth(1).$$

The first kernel function $\psi_a(t)$ was presented by Li et al. [21], and it is exactly the kernel function presented by Bai et al. [5] when $q = 1$. $\psi_b(t)$ and $\psi_c(t)$ are hyperbolic kernel functions in [32] and [13], respectively.

During the experiment, we set the threshold parameter $\tau = 2.5$, the accuracy parameter $\varepsilon = 10^{-7}$, and $p = q = 6, \theta = 0.6, \alpha = 0.5$, and " – " indicates that the runtime is too long, exceeding the upper limit, where " times " represents the runtime, and " iter " represents the number of iterations.

Example 4.1. $C=-I$ and define $A_k, k=1,2,\cdots,m$, are defined as

$$A_k(i,j)=\begin{cases}1, & \text{if} \quad i=j=k,\\ 1, & \text{if} \quad i=j \quad \text{and} \quad i=m+k,\\ 0, & \text{otherwise,}\end{cases}$$

and $b(i)=2, i=1,2,\cdots,m$. Given an initial point (X^0,y^0,S^0), we define the following

$$X^0(i,j)=\begin{cases}1.5, & \text{if} \quad i\le j,\\ 0.5, & \text{if} \quad i>j.\end{cases}$$

$y^0(i)=-2, i=1,2,\cdots,m$, $S^0=I$.

Example 4.2.

$$C(i,j)=\begin{cases}-1, & \text{if} \quad i=j\le m,\\ 0, & \text{otherwise.}\end{cases}$$

$A_k(k=1,2,\cdots,m)$ by

$$A_k(i,j)=\begin{cases}1, & \text{if} \quad i=j=k,\\ 1, & \text{if} \quad i=j \quad \text{and} \quad i=m+k,\\ 0, & \text{otherwise,}\end{cases}$$

and $b(i)=2, i=1,2,\cdots,m$. Given an initial point (X^0,y^0,S^0), we define the following

$$S^0(i,j)=\begin{cases}1, & \text{if} \quad i=j\le m,\\ 2, & \text{if} \quad i=j>m,\\ 0, & \text{otherwise,}\end{cases}$$

where $X^0=I$, $y^0(i)=-2, i=1,2,\cdots,m$.

During the experiment, the starting point (X^0,y^0,S^0) is feasible and the IPC is satisfied. The experimental results for Example 4.1 are shown in Table 1, and Table 2 displays the experimental results of Example 4.2.

From Tables 1 and 2, we may conclude a few remarks. The new algorithm based on ψ_{new} requires relatively fewer iterations, and the new algorithm also has a shorter running time, which indicates relatively higher efficiency. In most cases, our kernel function ψ_{new} has better numerical results than several other kernel functions. It can be seen from the experimental results that the new algorithm based on kernel function for SDO is feasible and this method is relatively efficient.

5. Conclusions

In this paper, we extended a primal-dual IPM based on the hyperbolic kernel function from LO to SDO. After that, we derived the iteration bounds of the large-update and small-update methods, which are $O\left(pn^{\frac{p+2}{2(p+1)}}\log\frac{n}{\varepsilon}\right)$ and $O\left(p\sqrt{n}\log\frac{n}{\varepsilon}\right)$, respectively. Moreover, some numerical results were represented, which indicate the efficiency of the proposed algorithm.

Some interesting topics for further works remain. The search direction adopted in this paper is based on NT-symmetrization scheme. It may be possible to design similar algorithms by using other symmetrization schemes and to obtain excellent polynomial complexity.

TABLE 1. Numerical results for Example 4.1

size(m,n)	result	ψ_a	ψ_b	ψ_c	ψ_{new}
$(5,10)$	iter	58	62	66	57
	times	2.23130e-1	1.9883e-1	2.48565e-1	2.12479e-1
	μ	1.759e-09	1.759e-09	1.759e-09	1.759e-09
$(10,20)$	iter	42	46	119	46
	times	4.18738e-1	3.7164e-1	6.54067e-1	4.20143e-1
	μ	2.815e-10	2.815e-10	2.815e-10	2.815e-10
$(15,30)$	iter	73	47	74	74
	times	6.96607e-1	6.90108e-1	8.49672e-1	6.76435e-1
	μ	1.126e-10	1.126e-10	1.126e-10	1.126e-10
$(20,40)$	iter	42	47	50	41
	times	1.275577	1.191475	1.150584	1.120332
	μ	1.126e-10	1.126e-10	1.126e-10	1.126e-10
$(25,50)$	iter	43	48	49	43
	times	1.945718	2.047980	2.026118	1.892404
	μ	4.504e-11	4.504e-11	4.504e-11	4.504e-11
$(50,100)$	iter	52	52	27	79
	times	8.536332	12.803630	8.553425	9.650027
	μ	1.801e-11	1.801e-11	1.801e-11	1.801e-11
$(100,200)$	iter	-	87	29	58
	times	-	114.571493	53.531214	78.695
	μ	-	2.882e-12	2.882e-12	2.882e-12

TABLE 2. Numerical results for Example 4.2

size(m,n)	result	ψ_a	ψ_b	ψ_c	ψ_{new}
$(5,10)$	iter	103	193	140	100
	times	2.89341e-1	3.18080e-1	3.98903e-1	2.14058e-1
	μ	1.759e-09	1.759e-09	1.759e-09	1.759e-09
$(10,20)$	iter	113	234	155	111
	times	4.18738e-1	7.30902e-1	5.9348e-1	3.01662e-1
	μ	2.815e-10	2.815e-10	2.815e-10	2.815e-10
$(15,30)$	iter	118	245	162	116
	times	7.74581e-1	1.106180	9.62593e-1	4.51815e-1
	μ	1.126e-10	1.126e-10	1.126e-10	1.126e-10
$(20,40)$	iter	118	245	162	116
	times	9.36336e-1	1.363724	1.062579	5.99587e-1
	μ	1.126e-10	1.126e-10	1.126e-10	1.126e-10
$(25,50)$	iter	123	256	169	121
	times	1.026126	1.724944	1.093114	8.67938e-1
	μ	4.504e-11	4.504e-11	4.504e-11	4.504e-11
$(50,100)$	iter	128	267	176	126
	times	6.278127	12.803630	7.757101	5.719420
	μ	1.801e-11	1.801e-11	1.801e-11	1.801e-11
$(100,200)$	iter	138	289	193	136
	times	106.977651	241.357710	145.837229	104.645898
	μ	2.882e-11	2.882e-11	2.882e-11	2.882e-11

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