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ON THE REAL PARTS OF A MATRIX AND ITS ALUTHGE TRANSFORM

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Abstract. Our goal in this paper is to discuss possible bounds and relations for the real part of a complex matrix. In particular, we present several relations between the real parts of the matrix and its Aluthge transform. We show that if T is an $n\times n$ complex matrix, then

$$\|\Re T\| \le \frac{1}{2}\max\left\{\left\||T|+\Re\widetilde{T}\right\|,\left\||T|-\Re\widetilde{T}\right\|\right\} \le \frac{1}{2}\|\,|T^*|+|T|\,\|,$$

where $\|\cdot\|, |\cdot|, \Re(\cdot), \widetilde{T}$ denote the usual operator norm, the absolute value, the real part, and the Aluthge transform, respectively. We also present numerical radius bounds, a spectral radius identity, and an arithmetic-geometric mean inequality for the real part and the numerical radius. In particular, we show that if A, B are $n\times n$ complex matrices, then

$$\|\Re(AB)\| = \frac{1}{2}\rho\left(\begin{bmatrix} BA & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & (BA)^* \end{bmatrix}\right),$$

where $\rho(\cdot)$ is the spectral radius, and $t>0$ is arbitrary.

Keywords. Generalized Aluthge transform; Operator matrix; Numerical radius; Norm inequality.
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1. Introduction

By $\mathscr{M}_n$, we denote the algebra of all $n\times n$ complex matrices, with identity I and zero element O. Given $T\in\mathscr{M}_n$, the Cartesian decomposition of T is defined as $T=\Re T+\mathrm{i}\Im T$, where $\Re T=\frac{T+T^*}{2}$ and $\Im T=\frac{T-T^*}{2\mathrm{i}}$ are the real and imaginary parts of T, respectively. Here, T^* is the conjugate transpose of T. If $\Im T=O$, then T is said to be Hermitian. The usual operator norm $\|\cdot\|$ is a matrix norm defined on $\mathscr{M}_n$ by $\|T\|=\sup_{\|x\|=1}\|Tx\|$. It follows immediately from the properties of norms that $\|T\|\le\|\Re T\|+\|\Im T\|$ for any $T\in\mathscr{M}_n$. Also, using the property $\|T\|=\|T^*\|$, we can easily see the validity of the inequality $\|\Re T\|\le\|T\|$. Investigating

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possible relations for the real part of $T \in \mathscr{M}_n$ is an important topic in operator theory due to the significance that could be reached. For example, numerical radius bounds can be found by using properties of the real part due to the formula [23]

$$\omega(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re\left(e^{\mathrm{i}\theta} T\right) \right\|,$$

where $\omega(\cdot)$ is the numerical radius, defined by $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. Also, the real part is the key idea behind the theory of accretive matrices, which are defined by those matrices $T \in \mathscr{M}_n$ for which $\Re T \geq O$. In this context, we say that $T \geq O$ if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$. Among the most basic and powerful properties of the real part of the product, we have the following result, which can be found in [7, Proposition IX.1.2].

Lemma 1.1. *Let $A, B \in \mathscr{M}_n$. If AB is Hermitian, then $\|AB\| \leq \|\Re(BA)\|$.*

Being Hermitian seems to be a harsh restriction. We say that T is normal when it commutes with T^*. In dealing with normal operators on Hilbert spaces, hyponormal operators acquired attention. While studying hyponormal operators, Aluthge [4] defined the Aluthge transform of T by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of T. Later, in [8], the weighted Aluthge transform was defined by $\widetilde{T}_r = |T|^r U |T|^{1-r}$, for $0 \leq r \leq 1$. These transforms received the attention of numerous researchers due to their effectiveness in finding new bounds or sharper bounds than the existing ones. We refer the reader to [3, 8, 11, 23] as a sample of references for this transformation.

By sub-multiplicativity of $\|\cdot\|$, it follows immediately that $\|\widetilde{T}_r\| \leq \|T\|, 0 \leq r \leq 1$. By manipulating the Aluthge transform and the polar decomposition of T, we can present refinements of the basic relation $\|\Re T\| \leq \|T\|$, where intermediate terms including the Aluthge transform will be found. For example, we show that if $A \in \mathscr{M}_n$, then

$$\|\Re A\| \leq \frac{1}{4} \left\| t^2 |A|^{2(1-r)} + \frac{1}{t^2} |A|^{2r} \right\| + \frac{1}{2} \left\| \Re \widetilde{A}_r \right\|, t > 0, 0 \leq r \leq 1.$$

Our discussion will lead to a new identity that relates the norm of the real part of the product with the spectral radius ρ. More precisely, we show that

$$\|\Re(AB)\| = \frac{1}{2} \rho \left(\begin{bmatrix} BA & \frac{1}{t^2} |B^*|^2 \\ t^2 |A|^2 & (BA)^* \end{bmatrix} \right), \ t > 0.$$

Another line of findings in this article is some equality conditions. Finding sufficient conditions that convert certain well-known inequalities into identities has been an interesting topic in the literature recently. We show that if $\Re \widetilde{T} = O$, then

$$\|T\| = \frac{1}{2} \max\left\{ \| \, |T^*| + |T| + 2\Re T \, \|, \| \, |T^*| + |T| - 2\Re T \, \| \right\},$$

as a new contribution to this line of research. Other results that treat the numerical radius of the product are presented too and will be compared with the existing literature. Before proceeding to the main results, we list some lemmas, which are needed in our analysis.

Lemma 1.2. [12] *Let $A, B, C, D \in \mathscr{M}_n$. Then* $\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|$.

The following is the arithmetic-geometric mean inequality [7, Corollary IX.4.4].

Lemma 1.3. *Let $A,B\in\mathscr{M}_n$. Then* $\|AB\|\le\frac{1}{2}\|A^*A+BB^*\|$.

Notice that Lemma 1.3 immediately implies $\|\Re(AB)\|\le\frac{1}{2}\|A^*A+BB^*\|$. At this point, it is interesting to ask whether this latter inequality is true without the norms. That is, if $A,B\in\mathscr{M}_n$, do we have

$$\Re(AB)\le\frac{1}{2}(A^*A+BB^*)?$$

In this context, if X,Y are Hermitian, we say that $X\le Y$ if $\langle Xx,x\rangle\le\langle Yx,x\rangle$ for all $x\in\mathbb{C}^n$. This is usually called Löwener partial ordering. This partial order is stronger than the norm order. In other words, if $X\le Y$, then $\|X\|\le\|Y\|$, but the converse is not true, in general. We refer the reader to [19] for some studies in this direction.

Lemma 1.4. [6, Corollary 1] *Let $A,B\in\mathscr{M}_n$. Then* $\left\|\begin{bmatrix}A & B\\ B & A\end{bmatrix}\right\|=\max\{\|A+B\|,\|A-B\|\}$.

In the next lemma, we use the notation $\sharp$ to denote the geometric mean. Recall that if A,B are positive definite, then the geometric mean of A,B is defined by

$$A\sharp B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}.$$

If $A,B\ge O$, then $A\sharp B=\lim_{\varepsilon\downarrow 0}(A+\varepsilon I)\sharp(B+\varepsilon I)$. The first part of this lemma can be found in [5, Theorem 3.4], and the second part in [22, Theorem 1].

Lemma 1.5. *Let $A,B\in\mathscr{M}_n$. If* $\begin{bmatrix}A & C\\ C^* & B\end{bmatrix}\ge O$, *then*

(i) *If C is Hermitian, then $\pm C\le A\sharp B$.*
(ii) $2\|C\|\le\left\|\begin{bmatrix}A & C\\ C^* & B\end{bmatrix}\right\|$.

We notice that if $\begin{bmatrix}A & C\\ C^* & B\end{bmatrix}\ge O$, then $A,B\ge O$, necessarily.

Lemma 1.6. *Let $T\in\mathscr{M}_n$ be with polar decomposition $T=U|T|$. Then*

(i) [9, Theorem 3, p. 54] *$U^*U=UU^*=I$. That is, U is unitary.*
(ii) [9, Theorem 4, p. 58] $|T^*|^q=U|T|^qU^*$ *for any* $q>0$.

An important property of the spectral radius $\rho(\cdot)$ is a commutativity property, which asserts that:

Lemma 1.7. *Let $A,B\in\mathscr{M}_n$. Then $\rho(AB)=\rho(BA)$.*

Lemma 1.8. [3, Corollary 2.4] *Let $A,B\in\mathscr{M}_n$. Then*

$$\rho(A+B)\le\frac{1}{2}\omega(A)+\omega(B)+\sqrt{(\omega(A)-\omega(B))^2+4\min\{\|AB\|\,\|BA\|\}}.$$

2. Main Results for the Real Part

In this section, we present our main results regarding the real parts of a matrix and its Aluthge transform, with comparisons to the existing literature, when applicable.

Theorem 2.1. *Let $A \in \mathscr{M}_n$ and let $0 \le r, s \le 1$. Then, for any $t > 0$,*

$$\begin{aligned}
&\|\Re A\| \\
&\le \frac{1}{4}\left(\left\|\Re\widetilde{A}_r\right\| + \left\|\Re\widetilde{A}_s\right\| + \sqrt{\left(\left\|\Re\widetilde{A}_r\right\| - \left\|\Re\widetilde{A}_s\right\|\right)^2 + \left\|\frac{1}{t^2}|A|^{r+s} + t^2|A|^{2-(r+s)}\right\|^2}\right).
\end{aligned}$$

Proof. Let $A = U|A|$ be the polar decomposition of A. Let $S = \begin{bmatrix} tU|A|^{1-r} & \frac{1}{t}|A|^s \\ O & O \end{bmatrix}$ and $T = \begin{bmatrix} \frac{1}{t}|A|^r & O \\ t|A|^{1-s}U^* & O \end{bmatrix}$. Then $ST = T^*S^* = \begin{bmatrix} 2\Re A & O \\ O & O \end{bmatrix}$, which shows that ST is Hermitian. Thus, we can use Lemma 1.1 to obtain

$$\begin{aligned}
&2\|\Re A\| \\
&\le \left\|\Re\left(\begin{bmatrix} \frac{1}{t}|A|^r & O \\ t|A|^{1-s}U^* & O \end{bmatrix}\begin{bmatrix} tU|A|^{1-r} & \frac{1}{t}|A|^s \\ O & O \end{bmatrix}\right)\right\| \\
&\le \left\|\Re\left(\begin{bmatrix} |A|^r U|A|^{1-r} & \frac{1}{t^2}|A|^{r+s} \\ t^2|A|^{2-(r+s)} & |A|^{1-s}U^*|A|^s \end{bmatrix}\right)\right\| \\
&= \left\|\begin{bmatrix} \Re\widetilde{A}_r & \frac{1}{2}\left(\frac{1}{t^2}|A|^{r+s} + t^2|A|^{2-(r+s)}\right) \\ \frac{1}{2}\left(\frac{1}{t^2}|A|^{r+s} + t^2|A|^{2-(r+s)}\right) & \Re\widetilde{A}_s \end{bmatrix}\right\| \\
&\le \left\|\begin{bmatrix} \left\|\Re\widetilde{A}_r\right\| & \frac{1}{2}\left\|\frac{1}{t^2}|A|^{r+s} + t^2|A|^{2-(r+s)}\right\| \\ \frac{1}{2}\left\|\frac{1}{t^2}|A|^{r+s} + t^2|A|^{2-(r+s)}\right\| & \left\|\Re\widetilde{A}_s\right\| \end{bmatrix}\right\| \\
&= \frac{1}{2}\left(\left\|\Re\widetilde{A}_r\right\| + \left\|\Re\widetilde{A}_s\right\| + \sqrt{\left(\left\|\Re\widetilde{A}_r\right\| - \left\|\Re\widetilde{A}_s\right\|\right)^2 + \left\|\frac{1}{t^2}|A|^{r+s} + t^2|A|^{2-(r+s)}\right\|^2}\right),
\end{aligned}$$

where Lemma 1.2 is used to find the second inequality, and the eigenvalues of the treated matrix is calculated in the last step. This completes the proof. ∎

Corollary 2.2. *Let $A \in \mathscr{M}_n$ and let $0 \le r, s \le 1$. Then*

$$\|\Re A\| \le \frac{1}{4}\left(\left\|\Re\widetilde{A}_r\right\| + \left\|\Re\widetilde{A}_s\right\| + \sqrt{\left(\left\|\Re\widetilde{A}_r\right\| - \left\|\Re\widetilde{A}_s\right\|\right)^2 + 4\|A\|^2}\right).$$

Proof. By using the triangle inequality in Theorem 2.1, we obtain

$$\begin{aligned}
&\|\Re A\| \\
&\le \frac{1}{4}\left(\left\|\Re\widetilde{A}_r\right\| + \left\|\Re\widetilde{A}_s\right\| + \sqrt{\left(\left\|\Re\widetilde{A}_r\right\| - \left\|\Re\widetilde{A}_s\right\|\right)^2 + \left(\frac{1}{t^2}\left\||A|^{r+s}\right\| + t^2\left\||A|^{2-(r+s)}\right\|\right)^2}\right) \\
&= \frac{1}{4}\left(\left\|\Re\widetilde{A}_r\right\| + \left\|\Re\widetilde{A}_s\right\| + \sqrt{\left(\left\|\Re\widetilde{A}_r\right\| - \left\|\Re\widetilde{A}_s\right\|\right)^2 + \left(\frac{1}{t^2}\|A\|^{r+s} + t^2\|A\|^{2-(r+s)}\right)^2}\right).
\end{aligned}$$

The result follows by finding the minimum value of the above bound when $t > 0$. ∎

Remark 2.3. Letting $r=s$ in Corollary 2.2, we obtain

$$\|\Re A\| \le \frac{1}{2}\left(\|\Re\widetilde{A}_r\| + \|A\|\right) \le \frac{1}{2}\left(\|\widetilde{A}_r\| + \|A\|\right) \le \|A\|.$$

This provides a refinement of the basic inequality $\|\Re A\| \le \|A\|$.

Letting $r=s$ in Theorem 2.1 implies

$$\|\Re A\| \le \frac{1}{4}\left\| t^2|A|^{2(1-r)} + \frac{1}{t^2}|A|^{2r} \right\| + \frac{1}{2}\left\|\Re\widetilde{A}_r\right\|, t>0, 0\le r\le 1.$$

This latter inequality can also be deduced from a more general result, as follows.

Corollary 2.4. *Let $A,B \in \mathscr{M}_n$ and let $0 \le r \le 1$. Then*

$$\|A+B^*\| \le \frac{1}{2}\sqrt{\left\||B|^{2r}+|B|^{2(1-r)}\right\| \left\||A|^{2r}+|A|^{2(1-r)}\right\|} + \max\{\|A\|,\|B\|\}.$$

Proof. Let $A=U|A|$ and $B=V|B|$ be the polar decompositions of A and B, respectively. Let $T=\begin{bmatrix} O & A \\ B & O \end{bmatrix}$. It can be easily verified that

$$\Re T = \frac{1}{2}\begin{bmatrix} O & A+B^* \\ A^*+B & O \end{bmatrix}, \widetilde{T}_r = \begin{bmatrix} O & |B|^r U|A|^{1-r} \\ |A|^r V|B|^{1-r} & \end{bmatrix}.$$

Applying the first inequality in Remark 2.3 on T, we have

$$\begin{aligned}
&\|\Re T\| \\
&= \frac{1}{2}\|A+B^*\| \\
&\le \frac{1}{4}\left\|\begin{bmatrix} O & |B|^r U|A|^{1-r} + |B|^{1-r}V^*|A|^r \\ |A|^r V|B|^{1-r} + |A|^{1-r}U^*|B|^r & O \end{bmatrix}\right\| + \frac{1}{2}\max\{\|A\|,\|B\|\} \\
&= \frac{1}{4}\left\||B|^r U|A|^{1-r} + |B|^{1-r}V^*|A|^r\right\| + \frac{1}{2}\max\{\|A\|,\|B\|\}.
\end{aligned}$$

That is,

$$\|A+B^*\| \le \frac{1}{2}\left\||B|^r U|A|^{1-r} + |B|^{1-r}V^*|A|^r\right\| + \max\{\|A\|,\|B\|\}.$$

We know that [10, Theorem 11]

$$\|T_1^*T_2 + T_3^*T_4\| \le \sqrt{\left\||T_1|^2+|T_3|^2\right\| \left\||T_2|^2+|T_4|^2\right\|}.$$

Letting $T_1^*=|B|^r$, $T_2=U|A|^{1-r}$, $T_3^*=|B|^{1-r}V^*$, and $T_4=|A|^r$, we have

$$\|A+B^*\| \le \frac{1}{2}\sqrt{\left\||B|^{2r}+|B|^{2(1-r)}\right\| \left\||A|^{2r}+|A|^{2(1-r)}\right\|} + \max\{\|A\|,\|B\|\}$$

as expected. ∎

Theorem 2.5. *Let $A\in\mathscr{M}_n$ with polar decomposition $A=U|A|$, and let $0\le r,s\le 1$. Then, for any $t>0$,*

$$\begin{aligned}&\|\Re A\|\\&\le\frac{1}{8}\left(\left\|t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r}\right\|+\left\|\frac{1}{t^2}|A|^{2s}+t^2|A|^{2(1-s)}\right\|\right.\\&\quad\left.+\sqrt{\left(\left\|t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r}\right\|-\left\|\frac{1}{t^2}|A|^{2s}+t^2|A|^{2(1-s)}\right\|\right)^2+4\left\||A|^{1-r}U^*|A|^s+|A|^rU|A|^{1-s}\right\|^2}\right).\end{aligned}$$

Proof. Let $A=U\,|A|$ be the polar decomposition of A. Then

$$\begin{aligned}&2\,\|\Re A\|\\&=\left\|\begin{bmatrix}U\,|A|+|A|\,U^* & O\\ O & O\end{bmatrix}\right\|\\&=\left\|\begin{bmatrix}tU|A|^{1-r} & \frac{1}{t}|A|^s\\ O & O\end{bmatrix}\begin{bmatrix}\frac{1}{t}|A|^r & O\\ t|A|^{1-s}U^* & O\end{bmatrix}\right\|\\&\le\frac{1}{2}\left\|\begin{bmatrix}t^2|A|^{2(1-r)} & |A|^{1-r}U^*|A|^s\\ |A|^sU|A|^{1-r} & \frac{1}{t^2}|A|^{2s}\end{bmatrix}+\begin{bmatrix}\frac{1}{t^2}|A|^{2r} & |A|^rU|A|^{1-s}\\ |A|^{1-s}U^*|A|^r & t^2|A|^{2(1-s)}\end{bmatrix}\right\|\\&=\frac{1}{2}\left\|\begin{bmatrix}t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r} & |A|^{1-r}U^*|A|^s+|A|^rU|A|^{1-s}\\ |A|^sU|A|^{1-r}+|A|^{1-s}U^*|A|^r & \frac{1}{t^2}|A|^{2s}+t^2|A|^{2(1-s)}\end{bmatrix}\right\|\\&\le\frac{1}{2}\left\|\begin{bmatrix}\left\|t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r}\right\| & \left\||A|^{1-r}U^*|A|^s+|A|^rU|A|^{1-s}\right\|\\ \left\||A|^sU|A|^{1-r}+|A|^{1-s}U^*|A|^r\right\| & \left\|\frac{1}{t^2}|A|^{2s}+t^2|A|^{2(1-s)}\right\|\end{bmatrix}\right\|,\end{aligned}$$

where the first inequality follows from Lemma 1.3, while the second inequality follows from Lemma 1.2. Calculating the eigenvalues of the later matrix, we deduce

$$\begin{aligned}&\|\Re A\|\\&\le\frac{1}{8}\left(\left\|t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r}\right\|+\left\|\frac{1}{t^2}|A|^{2s}+t^2|A|^{2(1-s)}\right\|\right.\\&\quad\left.+\sqrt{\left(\left\|t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r}\right\|-\left\|\frac{1}{t^2}|A|^{2s}+t^2|A|^{2(1-s)}\right\|\right)^2+4\left\||A|^{1-r}U^*|A|^s+|A|^rU|A|^{1-s}\right\|^2}\right),\end{aligned}$$

as required. □

Letting $r=s$ in Theorem 2.5, we have the following bound, a consequence of Theorem 2.1.

Corollary 2.6. *Let $A\in\mathscr{M}_n$ and let $0\le r\le 1$. Then, for any $t>0$,*

$$\|\Re A\|\le\frac{1}{4}\left\|t^2|A|^{2(1-r)}+\frac{1}{t^2}|A|^{2r}\right\|+\frac{1}{2}\left\|\Re\widetilde{A}_r\right\|.$$

Due to the role of block matrices in obtaining norm bounds, we present the following result, which shows the positivity of certain operator matrices involving the real part of the product. In conclusion of this, we present an arithmetic-geometric mean inequality for the real part.

Proposition 2.7. *Let $A,B\in\mathscr{M}_n$. Then*

(i)
$$\begin{bmatrix} |A^*|^2+|B|^2 & 2\Re(AB) \\ 2\Re(AB) & |A^*|^2+|B|^2 \end{bmatrix}, \begin{bmatrix} |A|^2+|B^*|^2 & 2\Re(BA) \\ 2\Re(BA) & |A|^2+|B^*|^2 \end{bmatrix} \geq O.$$
(ii)
$$\begin{aligned} \max\left\{\|A^*+B\|^2, \|A^*-B\|^2\right\} &= \left\| \begin{bmatrix} |A^*|^2+|B|^2 & 2\Re(AB) \\ 2\Re(AB) & |A^*|^2+|B|^2 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} |A|^2+|B^*|^2 & 2\Re(BA) \\ 2\Re(BA) & |A|^2+|B^*|^2 \end{bmatrix} \right\|. \end{aligned}$$

Proof. (i) The proof follows from the following
$$\begin{bmatrix} |A^*|^2+|B|^2 & 2\Re(AB) \\ 2\Re(AB) & |A^*|^2+|B|^2 \end{bmatrix} = \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix}^* \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix} \geq O,$$
and
$$\begin{bmatrix} |A|^2+|B^*|^2 & 2\Re(BA) \\ 2\Re(BA) & |A|^2+|B^*|^2 \end{bmatrix} = \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix} \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix}^* \geq O.$$
(ii) Using the property $\|XX^*\| = \|X^*X\|$, we have
$$\begin{aligned} \left\| \begin{bmatrix} |A^*|^2+|B|^2 & 2\Re(AB) \\ 2\Re(AB) & |B|^2+|A^*|^2 \end{bmatrix} \right\| &= \left\| \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix}^* \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix} \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix}^* \right\| \\ &= \left\| \begin{bmatrix} |A|^2+|B^*|^2 & 2\Re(BA) \\ 2\Re(BA) & |B^*|^2+|A|^2 \end{bmatrix} \right\|. \end{aligned}$$
Consequently,
$$\left\| \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix}^* \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix} \right\| = \left\| \begin{bmatrix} A^* & B \\ B & A^* \end{bmatrix} \right\|^2 = \max\left\{\|A^*+B\|^2, \|A^*-B\|^2\right\}.$$
This completes the proof. □

At this point, we can present a new arithmetic-geometric mean inequality for the real part. We refer to the discussion that followed Lemma 1.3 to see the significance of this coming result.

Corollary 2.8. *Let $A, B \in \mathscr{M}_n$. Then $2\Re(AB) \leq |A^*|^2 + |B|^2$.*

Proof. From Proposition 2.7, we have
$$\begin{bmatrix} |A^*|^2+|B|^2 & 2\Re(AB) \\ 2\Re(AB) & |A^*|^2+|B|^2 \end{bmatrix} \geq O.$$
Then the desired conclusion follows immediately from Lemma 1.5, part (i). □

Remark 2.9. It was demonstrated in [18, Remark 3.5] that if $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq O$, then $\|\Re C\| \leq \frac{1}{2}\|A+B\|$. Applying this result on $\begin{bmatrix} |A^*|^2+|B|^2 & 2\Re(AB) \\ 2\Re(AB) & |A^*|^2+|B|^2 \end{bmatrix}$ implies $\|\Re(AB)\| \leq \frac{1}{2}\left\||A^*|^2+|B|^2\right\|$. Consequently, Corollary 2.8 provides a significance improvement of this inequality.

Remark 2.10. We notice that Corollary 2.8 also follows from the observation $(A-B^*)(A-B^*)^* \geq O$.

Remark 2.11. It follows from Proposition 2.7 that

$$\max\{\|\Re(BA)\|,\|\Re(AB)\|\} \leq \frac{1}{4}\max\left\{\|A^*+B\|^2,\|A^*-B\|^2\right\},$$

noting part (ii) of Lemma 1.5. So,

$$\begin{aligned}
&\frac{1}{2}\left(\|\Re(BA\pm AB)\| + |\,\|\Re(BA)\| - \|\Re(AB)\|\,|\right)\\
&=\frac{1}{2}\left(\|\Re(BA)\pm\Re(AB)\| + |\,\|\Re(BA)\| - \|\Re(AB)\|\,|\right)\\
&\leq\frac{1}{2}\left(\|\Re(BA)\|+\|\Re(AB)\| + |\,\|\Re(BA)\| - \|\Re(AB)\|\,|\right)\\
&=\max\{\|\Re(BA)\|,\|\Re(AB)\|\}\\
&\leq\frac{1}{4}\max\left\{\|A^*+B\|^2,\|A^*-B\|^2\right\}.
\end{aligned}$$

That is,

$$\|\Re(BA\pm AB)\| + |\,\|\Re(BA)\| - \|\Re(AB)\|\,| \leq \frac{1}{2}\max\left\{\|A^*+B\|^2,\|A^*-B\|^2\right\}.$$

The quantity $\|\,|T^*|+|T|\,\|$ appeared in numerous works dealing with norm and numerical radius bounds, as seen in [14]. We observe that $\frac{1}{2}\|\,|T^*|+|T|\,\| \leq \|T\|$. The following is a reversed version of this observation. Also, an interesting identity involving the Aluthge transform is obtained.

Corollary 2.12. *Let $T\in\mathscr{M}_n$. Then, for any $0\leq r\leq 1$,*

$$\begin{aligned}
\|T\| &\leq \frac{1}{2}\max\left\{\left\||T^*|^{2(1-r)}+|T|^{2r}+2\Re T\right\|,\left\||T^*|^{2(1-r)}+|T|^{2r}-2\Re T\right\|\right\}\\
&=\frac{1}{2}\max\left\{\left\||T|^{2(1-r)}+|T|^{2r}+2\Re\widetilde{T}_r\right\|,\left\||T|^{2(1-r)}+|T|^{2r}-2\Re\widetilde{T}_r\right\|\right\}.
\end{aligned}$$

Proof. Let $T=U|T|$ be the polar decomposition of T, and let $A=U|T|^{1-r}$ and $B=|T|^r$. We notice that

$$\begin{aligned}
|A^*|^2 &= U|T|^{2(1-r)}U^* = |T^*|^{2(1-r)}, |B|^2=|T|^{2r}, AB=T\\
|A|^2 &= |T|^{2(1-r)}, |B^*|^2=|T|^{2r}, BA=\widetilde{T}_r.
\end{aligned}$$

As a direct consequence of Proposition 2.7, we have

$$\begin{aligned}
&\left\|\begin{bmatrix}|A^*|^2+|B|^2 & 2\Re(AB)\\ 2\Re(AB) & |A^*|^2+|B|^2\end{bmatrix}\right\|\\
&=\left\|\begin{bmatrix}|T|^{2r} & T^*\\ T & |T^*|^{2(1-r)}\end{bmatrix}+\begin{bmatrix}|T^*|^{2(1-r)} & T\\ T^* & |T|^{2r}\end{bmatrix}\right\| \qquad (2.1)\\
&=\left\|\begin{bmatrix}|T^*|^{2(1-r)}+|T|^{2r} & 2\Re T\\ 2\Re T & |T^*|^{2(1-r)}+|T|^{2r}\end{bmatrix}\right\|.
\end{aligned}$$

Using Lemma 1.4, we have

$$\begin{aligned}&\left\|\begin{bmatrix}|A^*|^2+|B|^2 & 2\Re(AB)\\ 2\Re(AB) & |A^*|^2+|B|^2\end{bmatrix}\right\|\\&=\max\left\{\left\||T^*|^{2(1-r)}+|T|^{2r}+2\Re T\right\|,\left\||T^*|^{2(1-r)}+|T|^{2r}-2\Re T\right\|\right\}.\end{aligned}\tag{2.2}$$

On the other hand,

$$\begin{aligned}&\left\|\begin{bmatrix}|A|^2+|B^*|^2 & 2\Re(BA)\\ 2\Re(BA) & |A|^2+|B^*|^2\end{bmatrix}\right\|\\&=\left\|\begin{bmatrix}|T|^{2(1-r)}+|T|^{2r} & 2\Re\widetilde{T}_r\\ 2\Re\widetilde{T}_r & |T|^{2(1-r)}+|T|^{2r}\end{bmatrix}\right\|\\&=\max\left\{\left\||T|^{2(1-r)}+|T|^{2r}+2\Re\widetilde{T}_r\right\|,\left\||T|^{2(1-r)}+|T|^{2r}-2\Re\widetilde{T}_r\right\|\right\},\end{aligned}\tag{2.3}$$

by Lemma 1.4 again. Notice that (see the proof of Theorem 1 in [15])

$$\begin{bmatrix}|T|^{2r} & T^*\\ T & |T^*|^{2(1-r)}\end{bmatrix},\begin{bmatrix}|T^*|^{2(1-r)} & T\\ T^* & |T|^{2r}\end{bmatrix}\geq O.$$

We know that if $A,B\geq O$, then $\max\{\|A\|,\|B\|\}\leq\|A+B\|$ (see, e.g., [13, Corollary 1]). Consequently,

$$\max\left\{\left\|\begin{bmatrix}|T|^{2r} & T^*\\ T & |T^*|^{2(1-r)}\end{bmatrix}\right\|,\left\|\begin{bmatrix}|T^*|^{2(1-r)} & T\\ T^* & |T|^{2r}\end{bmatrix}\right\|\right\}\leq\left\|\begin{bmatrix}|T|^{2r} & T^*\\ T & |T^*|^{2(1-r)}\end{bmatrix}+\begin{bmatrix}|T^*|^{2(1-r)} & T\\ T^* & |T|^{2r}\end{bmatrix}\right\|.$$

On the other hand, by Lemma 1.5 part (ii), we have

$$2\|T\|\leq\left\|\begin{bmatrix}|T|^{2r} & T^*\\ T & |T^*|^{2(1-r)}\end{bmatrix}\right\|,\left\|\begin{bmatrix}|T^*|^{2(1-r)} & T\\ T^* & |T|^{2r}\end{bmatrix}\right\|.$$

So, using this, together with (2.1), (2.2), and (2.3) implies

$$\begin{aligned}\|T\|&\leq\frac{1}{2}\max\left\{\left\||T^*|^{2(1-r)}+|T|^{2r}+2\Re T\right\|,\left\||T^*|^{2(1-r)}+|T|^{2r}-2\Re T\right\|\right\}\\&=\frac{1}{2}\max\left\{\left\||T|^{2(1-r)}+|T|^{2r}+2\Re\widetilde{T}_r\right\|,\left\||T|^{2(1-r)}+|T|^{2r}-2\Re\widetilde{T}_r\right\|\right\},\end{aligned}$$

as required. □

Corollary 2.13. *Let $T\in\mathscr{M}_n$. Then, for any $0\leq r\leq 1$,*

$$\begin{aligned}\|\Re T\|&\leq\frac{1}{4}\max\left\{\left\||T^*|^{2(1-r)}+|T|^{2r}+2\Re T\right\|,\left\||T^*|^{2(1-r)}+|T|^{2r}-2\Re T\right\|\right\}\\&=\frac{1}{4}\max\left\{\left\||T|^{2(1-r)}+|T|^{2r}+2\Re\widetilde{T}_r\right\|,\left\||T|^{2(1-r)}+|T|^{2r}-2\Re\widetilde{T}_r\right\|\right\}.\end{aligned}$$

In particular,

$$\begin{aligned}\|\Re T\|&\leq\frac{1}{4}\max\{\||T^*|+|T|+2\Re T\|,\||T^*|+|T|-2\Re T\|\}\\&=\frac{1}{2}\max\left\{\left\||T|+\Re\widetilde{T}\right\|,\left\||T|-\Re\widetilde{T}\right\|\right\}.\end{aligned}\tag{2.4}$$

Proof. We have by (2.1), (2.2) and (2.3) the desired result immediately. □

The following equality condition is an interesting application of Corollary 2.12. Recall that, for any $T \in \mathscr{M}_n$, we have $\frac{1}{2}\| \, |T| + |T^*| \, \| \leq \|T\|$. The following corollary provides sufficient conditions that make this inequality equality.

Corollary 2.14.

(i) *If* $\Re T = O$, *then* $\|T\| = \frac{1}{2}\| \, |T^*| + |T| \, \|$ *and*

$$\|T\| = \max\left\{\left\||T| + \Re\widetilde{T}\right\|, \left\||T| - \Re\widetilde{T}\right\|\right\}.$$

(ii) *If* $\Re\widetilde{T} = O$, *then* $\|T\| = \frac{1}{2}\| \, |T^*| + |T| \, \|$ *and*

$$\|T\| = \frac{1}{2}\max\left\{\| \, |T^*| + |T| + 2\Re T \, \|, \| \, |T^*| + |T| - 2\Re T \, \|\right\}.$$

Remark 2.15. We know that if $\begin{bmatrix} A & C \\ C^* & B \end{bmatrix} \geq O$, then $\left\|\begin{bmatrix} A & C \\ C^* & B \end{bmatrix}\right\| \leq \|A\| + \|B\|$. Thus, using part (ii) from Proposition 2.7,

$$\max\left\{\|A^* + B\|^2, \|A^* - B\|^2\right\} \leq 2\min\left\{\left\||A^*|^2 + |B|^2\right\|, \left\||A|^2 + |B^*|^2\right\|\right\}.$$

Let $T = U|T|$ be the polar decomposition of T. Letting $A = U|T|^{1-r}$ and $B = |T|^r$. Then

$$\begin{aligned} &\max\left\{\left\||T|^{2(1-r)} + |T|^{2r} + 2\Re\widetilde{T}_r\right\|, \left\||T|^{2(1-r)} + |T|^{2r} - 2\Re\widetilde{T}_r\right\|\right\} \\ &\leq 2\min\left\{\left\||T^*|^{2(1-r)} + |T|^{2r}\right\|, \left\||T|^{2(1-r)} + |T|^{2r}\right\|\right\}. \end{aligned}$$

In particular,

$$\max\left\{\left\||T| + \Re\widetilde{T}\right\|, \left\||T| - \Re\widetilde{T}\right\|\right\} \leq \| \, |T^*| + |T| \, \|, \tag{2.5}$$

since $\| \, |T^*| + |T| \, \| \leq 2\|T\|$.

Remark 2.16. By (2.4) and (2.5), we infer that

$$\|\Re T\| \leq \frac{1}{2}\max\left\{\left\||T| + \Re\widetilde{T}\right\|, \left\||T| - \Re\widetilde{T}\right\|\right\} \leq \frac{1}{2}\| \, |T^*| + |T| \, \|.$$

3. Applications

In this section, we present some numerical radius inequalities and identities. These results will follow mainly from our discussion in the previous section. We begin with the following simple observations.

Remark 3.1. If we replace A by $e^{i\theta}A$ in the first inequality in Remark 2.3, we obtain

$$\left\|\Re\left(e^{i\theta}A\right)\right\| \leq \frac{1}{2}\left(\left\|\Re\left(e^{i\theta}\widetilde{A}_r\right)\right\| + \|A\|\right),$$

which follows from the fact that $\widetilde{(e^{i\theta}A_r)} = e^{i\theta}\widetilde{A}_r$. Taking the supremum over θ, we get

$$\omega(A) \leq \frac{1}{2}\left(\omega\left(\widetilde{A}_r\right) + \|A\|\right), 0 \leq r \leq 1,$$

a result that was shown in [3, Theorem 3.2], with a different technique.

Remark 3.2. Replacing A by $e^{\mathrm{i}\theta}A$ in Corollary 2.6, and arguing like Remark 3.1, we show

$$\omega(A) \leq \frac{1}{4}\left\|t^2|A|^{2(1-r)} + \frac{1}{t^2}|A|^{2r}\right\| + \frac{1}{2}\omega\left(\widetilde{A}_r\right),$$

where $0 \leq r \leq 1$ and $t > 0$.

Remark 3.3. If we replace T by $e^{\mathrm{i}\theta}T$ in Remark 2.16, then taking the supremum over real θ, we reach

$$\omega(T) \leq \frac{1}{2}\max\left\{\sup_{\theta\in\mathbb{R}}\left\||T| + \Re\left(e^{\mathrm{i}\theta}\widetilde{T}\right)\right\|, \sup_{\theta\in\mathbb{R}}\left\||T| - \Re\left(e^{\mathrm{i}\theta}\widetilde{T}\right)\right\|\right\} \leq \frac{1}{2}\|\,|T| + |T^*|\,\|.$$

This provides a refinement of the celebrated result [14]

$$\omega(T) \leq \frac{1}{2}\|\,|T| + |T^*|\,\|.$$

Remark 3.4. If we replace A by $e^{\mathrm{i}\theta}A$ in Remark 2.11, and we use the identity [11, (4.6)]

$$\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|X + e^{\mathrm{i}\theta}Y^*\right\| = \omega\left(\begin{bmatrix} O & X \\ Y & O \end{bmatrix}\right),$$

we obtain

$$\max\left\{\omega(AB), \omega(BA)\right\} \leq \omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right).$$

This result was recently proved in [16, Theorem 2.1] by using a different technique.

In the above remarks, we used the well-known identity $\omega(A) = \sup_{\theta\in\mathbb{R}}\left\|\Re\left(e^{\mathrm{i}\theta}A\right)\right\|$. This identity was used frequently in the literature as an alternative to the original definition of the numerical radius, as one can see in [1, 11, 17, 23] to mention a few. The following theorem gives an identity for the numerical radius of a block matrix.

Theorem 3.5. *Let $A, B \in \mathscr{M}_n$. Then*

$$\begin{aligned} 2\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) &= \sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} |A^*|^2 + |B|^2 & 2\Re\left(e^{\mathrm{i}\theta}AB\right) \\ 2\Re\left(e^{\mathrm{i}\theta}AB\right) & |A^*|^2 + |B|^2 \end{bmatrix}\right\|^{\frac{1}{2}} \\ &= \sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} |A|^2 + |B^*|^2 & 2\Re\left(e^{\mathrm{i}\theta}BA\right) \\ 2\Re\left(e^{\mathrm{i}\theta}BA\right) & |A|^2 + |B^*|^2 \end{bmatrix}\right\|^{\frac{1}{2}}. \end{aligned}$$

Proof. From ([11, (4.6)]), we know that

$$\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|X + e^{\mathrm{i}\theta}Y^*\right\| = \omega\left(\begin{bmatrix} O & X \\ Y & O \end{bmatrix}\right).$$

Also, we clearly have $\sup_{\theta\in\mathbb{R}}\|X+e^{\mathrm{i}\theta}Y^*\|=\sup_{\theta\in\mathbb{R}}\|X-e^{\mathrm{i}\theta}Y^*\|$. So, by part (ii) of Proposition 2.7, we infer that

$$\begin{aligned}2\omega\left(\begin{bmatrix}O & A\\ B & O\end{bmatrix}\right)&=2\omega\left(\begin{bmatrix}O & A^*\\ B^* & O\end{bmatrix}\right)\\
&=\max\left\{\sup_{\theta\in\mathbb{R}}\left\|A^*+e^{\mathrm{i}\theta}B\right\|,\sup_{\theta\in\mathbb{R}}\left\|A^*-e^{\mathrm{i}\theta}B\right\|\right\}\\
&=\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix}|A^*|^2+|B|^2 & 2\Re\left(e^{\mathrm{i}\theta}AB\right)\\ 2\Re\left(e^{\mathrm{i}\theta}AB\right) & |A^*|^2+|B|^2\end{bmatrix}\right\|^{\frac{1}{2}}\\
&=\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix}|A|^2+|B^*|^2 & 2\Re\left(e^{\mathrm{i}\theta}BA\right)\\ 2\Re\left(e^{\mathrm{i}\theta}BA\right) & |A|^2+|B^*|^2\end{bmatrix}\right\|^{\frac{1}{2}}.\end{aligned}$$

Here we used the following facts $\omega(X)=\omega(X^*)$ and

$$\omega\left(\begin{bmatrix}O & X\\ Y & O\end{bmatrix}\right)=\omega\left(\begin{bmatrix}O & Y\\ X & O\end{bmatrix}\right).$$

This completes the proof. $\square$

Theorem 3.6. *Let $A,B\in\mathscr{M}_n$. Then*

$$\omega^2\left(\begin{bmatrix}O & A\\ B & O\end{bmatrix}\right)\le\frac{1}{2}\min\left\{\omega\left(\begin{bmatrix}O & |A|^2+|B^*|^2\\ 2BA & O\end{bmatrix}\right),\omega\left(\begin{bmatrix}O & |A^*|^2+|B|^2\\ 2AB & O\end{bmatrix}\right)\right\}.$$

Proof. It follows from Theorem 3.5 and Lemma 1.4 that

$$\begin{aligned}&4\omega^2\left(\begin{bmatrix}O & A\\ B & O\end{bmatrix}\right)\\
&=\max\left\{\sup_{\theta\in\mathbb{R}}\left\||A|^2+|B^*|^2+2\Re\left(e^{\mathrm{i}\theta}BA\right)\right\|,\sup_{\theta\in\mathbb{R}}\left\||A|^2+|B^*|^2-2\Re\left(e^{\mathrm{i}\theta}BA\right)\right\|\right\}.\end{aligned}$$

We have

$$\begin{aligned}&4\omega^2\left(\begin{bmatrix}O & A\\ B & O\end{bmatrix}\right)\\
&=\max\left\{\sup_{\theta\in\mathbb{R}}\left\||A|^2+|B^*|^2+2\Re\left(e^{\mathrm{i}\theta}BA\right)\right\|,\sup_{\theta\in\mathbb{R}}\left\||A|^2+|B^*|^2-2\Re\left(e^{\mathrm{i}\theta}BA\right)\right\|\right\}\\
&\le\max\left\{\sup_{\theta\in\mathbb{R}}\left\||A|^2+|B^*|^2+2e^{\mathrm{i}\theta}BA\right\|,\sup_{\theta\in\mathbb{R}}\left\||A|^2+|B^*|^2-2e^{\mathrm{i}\theta}BA\right\|\right\}\\
&=2\max\left\{\omega\left(\begin{bmatrix}O & |A|^2+|B^*|^2\\ 2A^*B^* & O\end{bmatrix}\right),\omega\left(\begin{bmatrix}O & |A|^2+|B^*|^2\\ -2A^*B^* & O\end{bmatrix}\right)\right\}\\
&=2\omega\left(\begin{bmatrix}O & |A|^2+|B^*|^2\\ 2A^*B^* & O\end{bmatrix}\right)\\
&=2\omega\left(\begin{bmatrix}O & |A|^2+|B^*|^2\\ 2BA & O\end{bmatrix}\right),\end{aligned}$$

that is,

$$\omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \le \frac{1}{2}\,\omega\left(\begin{bmatrix} O & |A|^2+|B^*|^2 \\ 2BA & O \end{bmatrix}\right). \tag{3.1}$$

On the other hand,

$$\begin{aligned} &4\omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \\ &= \max\left\{\sup_{\theta\in\mathbb{R}}\left\| |A^*|^2+|B|^2+2\Re\left(e^{\mathrm{i}\theta}AB\right)\right\|, \sup_{\theta\in\mathbb{R}}\left\| |A^*|^2+|B|^2-2\Re\left(e^{\mathrm{i}\theta}AB\right)\right\|\right\} \end{aligned}$$

which implies

$$\omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \le \frac{1}{2}\,\omega\left(\begin{bmatrix} O & |A^*|^2+|B|^2 \\ 2AB & O \end{bmatrix}\right). \tag{3.2}$$

Combining two inequalities (3.1) and (3.2) gives the desired result. □

Remark 3.7. Notice that (see Remark 3.4)

$$\max\{\omega(AB), \omega(BA)\} \le \omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right).$$

Therefore,

$$\begin{aligned} \max\{\omega(AB), \omega(BA)\} &\le \omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \\ &\le \frac{1}{2}\min\left\{\omega\left(\begin{bmatrix} O & |A|^2+|B^*|^2 \\ 2BA & O \end{bmatrix}\right), \omega\left(\begin{bmatrix} O & |A^*|^2+|B|^2 \\ 2AB & O \end{bmatrix}\right)\right\}. \end{aligned}$$

In particular,

$$\omega(T) \le \omega^2\left(\begin{bmatrix} O & U|T|^{\frac{1}{2}} \\ |T|^{\frac{1}{2}} & O \end{bmatrix}\right) \le \min\left\{\omega\left(\begin{bmatrix} O & |T| \\ \widetilde{T} & O \end{bmatrix}\right), \omega\left(\begin{bmatrix} O & \frac{|T^*|+|T|}{2} \\ T & O \end{bmatrix}\right)\right\}. \tag{3.3}$$

Remark 3.8. By (3.3), if $\widetilde{T} = O$, then

$$\omega^2\left(\begin{bmatrix} O & U|T|^{\frac{1}{2}} \\ |T|^{\frac{1}{2}} & O \end{bmatrix}\right) = \frac{1}{2}\|T\|.$$

Indeed,

$$\frac{1}{2}\|T\| \le \omega(T) \le \omega^2\left(\begin{bmatrix} O & U|T|^{\frac{1}{2}} \\ |T|^{\frac{1}{2}} & O \end{bmatrix}\right) \le \omega\left(\begin{bmatrix} O & |T| \\ O & O \end{bmatrix}\right) = \frac{1}{2}\|T\|,$$

where the last equality is obtained from [2, Corollary 3].

A discussion related to the possible versions of the arithmetic-geometric mean inequality for the numerical radius can be found in [20, 21]. It is interesting that Theorem 3.6 can be used to present some related arithmetic-geometric mean inequalities for ω, as follows. We notice the validity of the relation $\omega(AB) \le \frac{1}{2}\|\,|A|^2+|B^*|^2\|$, which is due to the fact that $w(T) \le \|T\|$ and Lemma 1.3.

Corollary 3.9. *Let $A, B \in \mathscr{M}_n$. Then*

$$\omega(AB) \le \frac{1}{2}\min\left\{\omega\left(\begin{bmatrix} O & |A|^2+|B^*|^2 \\ 2BA & O \end{bmatrix}\right), \omega\left(\begin{bmatrix} O & |A^*|^2+|B|^2 \\ 2AB & O \end{bmatrix}\right)\right\}.$$

Proof. The result follows immediately by Theorem 3.6 and Remark 3.4. □

Next, we show another bound for the numerical radius. First, a new identity for the spectral radius is shown in terms of the norm of the real part. Then we use this to present a bound for the numerical radius.

In the proof of the following result, we use the fact that $\|X\| = \rho(X)$ for Hermitian X.

Theorem 3.10. *Let $A, B \in \mathscr{M}_n$ and let $0 \le r \le 1$. Then for any $t > 0$,*

$$\|\Re(AB)\| = \frac{1}{2}\rho\left(\begin{bmatrix} BA & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & (BA)^* \end{bmatrix}\right).$$

In particular,

$$\|\Re T\| = \frac{1}{2}\rho\left(\begin{bmatrix} \widetilde{T}_r & \frac{1}{t^2}|T|^{2r} \\ t^2|T|^{2(1-r)} & \left(\widetilde{T}_r\right)^* \end{bmatrix}\right).$$

Proof. We have

$$\begin{aligned}
\|\Re(AB)\| &= \frac{1}{2}\left\|\begin{bmatrix} AB + B^*A^* & O \\ O & O \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} A & B^* \\ O & O \end{bmatrix}\begin{bmatrix} B & O \\ A^* & O \end{bmatrix}\right\| \\
&= \frac{1}{2}\left\|\begin{bmatrix} tA & \frac{1}{t}B^* \\ O & O \end{bmatrix}\begin{bmatrix} \frac{1}{t}B & O \\ tA^* & O \end{bmatrix}\right\| \\
&= \frac{1}{2}\rho\left(\begin{bmatrix} tA & \frac{1}{t}B^* \\ O & O \end{bmatrix}\begin{bmatrix} \frac{1}{t}B & O \\ tA^* & O \end{bmatrix}\right) \\
&= \frac{1}{2}\rho\left(\begin{bmatrix} \frac{1}{t}B & O \\ tA^* & O \end{bmatrix}\begin{bmatrix} tA & \frac{1}{t}B^* \\ O & O \end{bmatrix}\right) \\
&= \frac{1}{2}\rho\left(\begin{bmatrix} BA & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & (BA)^* \end{bmatrix}\right)
\end{aligned}$$

as required.

To get the second equality, we suppose that $T = U|T|$ is the polar decomposition of T. Letting $A = U|T|^{1-r}$ and $B = |T|^r$ in the first equality, we reach the expected result. □

Now Theorem 3.10 is used to show a bound for the real part, which we use next to obtain the desired numerical radius bounds.

Proposition 3.11. *Let $A, B \in \mathscr{M}_n$ and let $t > 0$. Then*

$$\begin{aligned}
\|\Re(AB)\| \le{}& \frac{1}{4}\left(\omega(BA) + \frac{1}{2}\left\|t^2|A|^2 + \frac{1}{t^2}|B^*|^2\right\|\right) \\
&+ \frac{1}{4}\sqrt{\left(\omega(BA) - \frac{1}{2}\left\|t^2|A|^2 + \frac{1}{t^2}|B^*|^2\right\|\right)^2 + 4\max\left\{\frac{1}{t^2}\left\|BA|B^*|^2\right\|, t^2\left\|(BA)^*|A|^2\right\|\right\}}.
\end{aligned}$$

Proof. Using Theorem 3.10, then Lemma 1.8, we have

$$\begin{aligned}
&\|\Re(AB)\| \\
&= \frac{1}{2}\rho\left(\begin{bmatrix} BA & O \\ O & (BA)^* \end{bmatrix} + \begin{bmatrix} O & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & O \end{bmatrix}\right) \\
&\le \frac{1}{4}\left(\omega\left(\begin{bmatrix} BA & O \\ O & (BA)^* \end{bmatrix}\right) + \omega\left(\begin{bmatrix} O & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & O \end{bmatrix}\right)\right) \\
&\quad + \frac{1}{4}\left(\left(\omega\left(\begin{bmatrix} BA & O \\ O & (BA)^* \end{bmatrix}\right) - \omega\left(\begin{bmatrix} O & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & O \end{bmatrix}\right)\right)^2\right. \\
&\quad\quad \left. + 4\min\left\{\left\|\begin{bmatrix} BA & O \\ O & (BA)^* \end{bmatrix}\begin{bmatrix} O & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & O \end{bmatrix}\right\|, \left\|\begin{bmatrix} O & \frac{1}{t^2}|B^*|^2 \\ t^2|A|^2 & O \end{bmatrix}\begin{bmatrix} BA & O \\ O & (BA)^* \end{bmatrix}\right\|\right\}\right)^{\frac{1}{2}} \\
&= \frac{1}{4}\left(\omega(BA) + \frac{1}{2}\left\|t^2|A|^2 + \frac{1}{t^2}|B^*|^2\right\|\right) \\
&\quad + \frac{1}{4}\sqrt{\left(\omega(BA) - \frac{1}{2}\left\|t^2|A|^2 + \frac{1}{t^2}|B^*|^2\right\|\right)^2 + 4\max\left\{\frac{1}{t^2}\left\|BA|B^*|^2\right\|, t^2\left\|(BA)^*|A|^2\right\|\right\}},
\end{aligned}$$

which completes the proof. □

Replacing A by $e^{\mathrm{i}\theta}$ in Proposition 3.11 yields the following bound for the numerical radius of the product.

Corollary 3.12. *Let $A, B \in \mathscr{M}_n$, and let $t > 0$. Then*

$$\begin{aligned}
\omega(AB) \le{}& \frac{1}{4}\left(\omega(BA) + \frac{1}{2}\left\|t^2|A|^2 + \frac{1}{t^2}|B^*|^2\right\|\right) \\
&+ \frac{1}{4}\sqrt{\left(\omega(BA) - \frac{1}{2}\left\|t^2|A|^2 + \frac{1}{t^2}|B^*|^2\right\|\right)^2 + 4\max\left\{\frac{1}{t^2}\left\|BA|B^*|^2\right\|, t^2\left\||A|^2BA\right\|\right\}}.
\end{aligned}$$

Now we have the following bound for $\omega(T)$ in terms of the Aluthge transform.

Corollary 3.13. *Let $T \in \mathscr{M}_n$. If $t > 0$ and $0 \le r \le 1$, then*

$$\begin{aligned}
\omega(T) \le{}& \frac{1}{4}\left(\omega\left(\widetilde{T}_r\right) + \frac{1}{2}\left\|t^2|T|^{2(1-r)} + \frac{1}{t^2}|T|^{2r}\right\|\right) \\
&+ \frac{1}{4}\sqrt{\left(\omega\left(\widetilde{T}_r\right) - \frac{1}{2}\left\|t^2|T|^{2(1-r)} + \frac{1}{t^2}|T|^{2r}\right\|\right)^2 + 4\max\left\{\frac{1}{t^2}\left\|\widetilde{T}_r|T|^{2r}\right\|, t^2\left\||T|^{2(1-r)}\widetilde{T}_r\right\|\right\}}.
\end{aligned}$$

In particular,

$$\begin{aligned}
\omega(T) \le{}& \frac{1}{4}\left(\omega\left(\widetilde{T}\right) + \frac{1}{2}\left\|t^2|T| + \frac{1}{t^2}|T|\right\|\right) \\
&+ \frac{1}{4}\sqrt{\left(\omega\left(\widetilde{T}\right) - \frac{1}{2}\left\|t^2|T| + \frac{1}{t^2}|T|\right\|\right)^2 + 4\max\left\{\frac{1}{t^2}\left\|\widetilde{T}|T|\right\|, t^2\left\||T|\widetilde{T}\right\|\right\}},
\end{aligned}$$

and

$$\omega(T) \le \frac{1}{4}\left(\omega\left(\widetilde{T}\right) + \|T\| + \sqrt{\left(\omega\left(\widetilde{T}\right) - \|T\|\right)^2 + 4\max\left\{\left\|\widetilde{T}\,|T|\right\|, \left\||T|\,\widetilde{T}\right\|\right\}}\right).$$

Proof. Assume T is invertible, and let $T = U|T|$ be the polar decomposition of T. In Corollary 3.12, let $A = U|T|^{r-1}$ and $B = |T|^r$. Then direct calculations show the first desired result. To obtain the second desired result, let $r = \frac{1}{2}$ in the first obtained inequality. Letting $t = 1$ in the second inequality completes the proof. □

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