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THE EXISTENCE OF RADIAL POSITIVE SOLUTIONS OF A CLASS OF QUASI-LINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we study the weighted embedding theorem in Orlicz-Sobolev spaces, and we obtain the existence of nontrivial solution of the following equation

$$\begin{cases} -\Delta_\Phi u = |x|^\alpha |u|^{q-2}u, \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases}$$

where $B \subset \mathbb{R}^N$ $(N \geq 3)$ is the unique ball, $\alpha > 0$ is a constant, $\phi \in C^1(0, +\infty)$ and $2 < q < \infty$. If the nonlinear term is sub-linear, by Clark's theorem, we obtain the existence of infinity many solutions of the equation.

Keywords. Quasilinear elliptic equation; Variational method; Weighted nonlinearity.
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1. Introduction

In this paper, we consider the following quasi-linear elliptic equation:

$$\begin{cases} -\Delta_\Phi u \equiv div(\phi(|\nabla u|)\nabla u) = |x|^\alpha |u|^{q-2}u, \text{ in } B, \\ u = 0 \text{ on } \partial B, \end{cases} \tag{1.1}$$

where $B \subset \mathbb{R}^N$ $(N \geq 3)$ is the unique ball, $2 < q < \infty$, $\alpha > 0$ is a constant, and $\phi \in C^1(0, +\infty)$. Assume that Φ is a continuous function defined by

$$\Phi(t) = \int_0^t \phi(s) s ds.$$

Without loss of generality, we may assume $\Phi(1) = 1$. In this paper, we assume that ϕ satisfies the following conditions:

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(ϕ_1) : $\lim_{t\to 0}\phi(t)=0$, $\lim_{t\to\infty}\phi(t)=\infty$, $(\phi(t)t)'>0$, $\phi(t)>0$, for $t>0$;

(ϕ_2) : there exist $l,m\in(1,N)$ such that $l\le\frac{\phi(t)t^2}{\Phi(t)}\le m$, for $t>0$.

For any $p\ge 1$ and $\Omega\subset\mathbb{R}^N$, we define

$$L^p(\Omega;|x|^\alpha)=\left\{u\Big|\int_\Omega |x|^\alpha|u|^p dx<\infty\right\}.$$

Then it is easy to check that $L^p(\Omega;|x|^\alpha)$ is a Banach space with the norm

$$||u||_{L^p(\Omega,|x|^\alpha)}=\left(\int_\Omega |x|^\alpha|u|^p dx\right)^{1/p}.$$

In recent years, many authors considered the following weighted elliptic equation

$$-\Delta u=|x|^\alpha u^p,\ x\in\Omega,\ u>0,\ x\in\Omega,\ u=0,\ x\in\partial\Omega, \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ is a bound domain and p is subcritical exponent. (1.2) is called Hénon type elliptic equation and it was considered by Hénon [9] in the context of astrophysics. In [11], Ni studied the existence of equation (1.2), and proved a embedding theorem. By mountain pass lemma, he obtained the existence of a nontrivial solution. In particularly, Ni observed that the presence of weighted term $|x|^\alpha$ leads to a new critical exponents for the nonexistence of classical positive solutions of (1.2), i.e., $2^*_\alpha=\dfrac{2N+2\alpha}{N-2}$. If $\alpha>0$, 2^*_α is greater than the classical Sobolev critical exponent. After Ni's work, researchers focus their interest on this type equations. For example, in [12], Phan and Souplet considered the Hénon type Liouville theorem, and they proved that

Theorem A. Let $\alpha>0$, $p>1$, and $N=3$. If $p<2^*_\alpha$, then equation (1.2) has no positive and bounded solutions in $\mathbb{R}^N$.

Recently, Li and Zhang [10] improved Phan-Souplet's work to the Hénon-Lane-Emden system; see [10] and the references therein, however there is no results about $N>3$. Since the weighted $|x|^\alpha,\alpha>0$, the moving plan method cannot be used. Smets, Willem and Su [13] studied the symmetry-breaking results of (1.2) via the rescaling method and ground state energy estimate, and they also studied the asymptotic as $p\to 2^*$ and 2. For the symmetric results of elliptic systems, we refer to [2]. In [14, 15], Su and Tian considered the weighted $p-Laplace$ equation (1.2). Following Ni's ideas, they proved a embedding theorem and considered the existence of the equations. Su and Tian also considered the bifurcation and sublinear problems of the equations. In [4, 6], the authors studied the general quasilinear elliptic equation

$$-\Delta_\Phi u=f(x,u),\ x\in\Omega,\ u=0,\ x\in\partial\Omega, \tag{1.3}$$

where Ω is a bound smooth domain, $\Phi(u)$ is a continuous function satisfying some conditions. The authors considered the problem in Orlicz-Sobolev spaces. By using classical variational method and minimax theorems, they obtained the existence of nontrivial solutions of (1.3). They also studied the classical subcritical and critical Sobolev exponents.

Motivated by these results, we consider the general quasilinear elliptic equation with the Hénon type weighted term. By proving a new embedding theorem, we have the following result.

Theorem 1.1. *Let ϕ satisfy the conditions $(\phi_1),(\phi_2)$ and $q>m$. Then (1.1) has a positive solution.*

In [16], Wang considered the effect of concave nonlinearities for the solution structure of nonlinear boundary value problems. By a result of Clark [3], Wang proved that the elliptic problem has infinitely many nontrivial solutions. It is known that Clark's theorem is useful tool to study sub-linear problems. In this paper, we consider the sub-linear quasilinear elliptic problem and prove the following result.

Theorem 1.2. *Assume* $m < q < \dfrac{m(\alpha+N)(l-1)}{(m-1)(N-1)}$ *and the sequence* $\{u_n\} \subset W^1_{0,r}L_\Phi(B)$ *satisfies* $\mathscr{J}(u_n) \to c$ *and* $\mathscr{J}'(u_n) \to 0$ *as* $n \to +\infty$. *Then there exists* $u \in W^1_{0,r}L_\Phi(B)$ *such that* $\mathscr{J}(u) = c$, $\mathscr{J}'(u) = 0$.

The paper is organized as follows. In Section 2, we list some preliminary results. In Section 3, we prove a embedding theorem of $\Phi - Laplace$ operator. Finally, in Section 4, we prove some lemmas and then give the proof to our main results.

2. Orlicz-Sobolev Spaces

In this section, we recall some useful knowledge for Orlicz-Sobolev spaces and give some inequalities on Φ. The reader can refer [1, 7] for more details.

By Condition (ϕ_1) and the definition of Φ, Φ is a N-function. The complementary of Φ is defined by $\widetilde{\Phi}(s) = \max_{t\geq 0}(st - \Phi(t))$ for $s \geq 0$. It is easy to see that Φ and $\widetilde{\Phi}$ are complementary $N-$functions and satisfy Δ_2-condition. Suppose that Ω is a subset of $\mathbb{R}^N$. Under assumptions (ϕ_1) and (ϕ_2), the Orlicz Space $L_\Phi(\Omega)$ contains all measurable functions $u : \Omega \to \mathbb{R}$ such that $\int_\Omega \Phi(|u(x)|)dx < \infty$, and the Luxemburg norm on $L_\Phi(\Omega)$ is defined by

$$||u||_{\Phi,\Omega} = \inf\left\{\lambda > 0 \Big| \int_\Omega \Phi\left(\frac{u(x)}{\lambda}\right)dx \leq 1\right\}.$$

The corresponding Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ is defined by

$$W^1L_\Phi(\Omega) = \left\{u \in L_\Phi(\Omega) \Big| \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), i = 1,2,\cdots,N\right\},$$

and the norm on $W^1L_\Phi(\Omega)$ is defined by

$$||u||_{1,\Phi,\Omega} = ||u||_{\Phi,\Omega} + \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{\Phi,\Omega}.$$

The spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are reflexive Banach spaces (see [1, Theorem 8.20 and Theorem 8.31]). The space $W^1_0L_\Phi(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. It follows from Poincaré inequality [8] for the $\Phi-$laplacian operator that $||u||_{\Phi,\Omega} \leq C||\nabla u||_{\Phi,\Omega}$, for $u \in W^1_0L_\Phi(\Omega)$ and some $C > 0$. Hence $||u||_\Omega \equiv ||\nabla u||_{\Phi,\Omega}$ is a norm on $W^1_0L_\Phi(\Omega)$ and equivalent to $||u||_{1,\Phi,\Omega}$. In the following of this paper, we use $||\cdot||_\Omega$ as the norm of space $W^1_0L_\Phi(\Omega)$. Set

$$C^\infty_{0,r}(\Omega) = \left\{u \in C_0^\infty(\Omega) \big| u \text{ is a radially symmetry function}\right\}.$$

The completion of $C^\infty_{0,r}(\Omega)$ under the norm $||\cdot||_\Omega$ is denoted by $W^1_{0,r}L_\Phi(\Omega)$. Since Φ and $\widetilde{\Phi}$ are complementary $N-$functions, the following generalize Hölder inequality (see [1]) holds:

$$\left|\int_\Omega u(x)v(x)dx\right| \leq 2\,\|u\|_{\Phi,\Omega}\,\|v\|_{\widetilde{\Phi},\Omega}, \text{ for any } u \in L_\Phi(\Omega), v \in L_{\widetilde{\Phi}}(\Omega). \tag{2.1}$$

Next, we recall some inequalities on Φ.

Lemma 2.1. [7] *Let* $\zeta_0(t)=\min\{t^l,t^m\}$ *and* $\zeta_1(t)=\max\{t^l,t^m\}$, $t\ge 0$. *Then*

$$\zeta_0(t)\Phi(\rho)\le\Phi(\rho t)\le\zeta_1(t)\Phi(\rho)\ \textit{for}\ \rho,t\ge 0,$$
$$\zeta_0(||u||_{\Phi,\Omega})\le\int_\Omega\Phi(|u|)dx\le\zeta_1(||u||_{\Phi,\Omega})\ \textit{for}\ u\in L_\Phi(\Omega).$$

Lemma 2.2. [7] *Let* $\zeta_2(t)=\min\{t^{l/(l-1)},t^{m/(m-1)}\}$ *and* $\zeta_3(t)=\max\{t^{l/(l-1)},t^{m/(m-1)}\}$, $t\ge 0$. *Then*

$$\zeta_2(e)\widetilde{\Phi}(\rho)\le\widetilde{\Phi}(\rho t)\le\zeta_3(t)\widetilde{\Phi}(\rho)\ \textit{for}\ \rho,t\ge 0,$$
$$\zeta_2(||u||_{\widetilde{\Phi},\Omega})\le\int_\Omega\widetilde{\Phi}(|u|)dx\le\zeta_3(||u||_{\widetilde{\Phi},\Omega})\ \textit{for}\ u\in L_{\widetilde{\Phi}}(\Omega).$$

Lemma 2.3. *Let* $\Omega=\left\{y\in\mathbb{R}^N\Big|0<|x|\le|y|\le 1\right\}$. *Then there exists* $C=C(l,m,N,\widetilde{\Phi})>0$ *such that* $\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\le C\,|x|^{\frac{(l-N)(m-1)}{m(l-1)}}$.

Proof. If $\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\ge 1$, then it follows from Lemma 2.2 that

$$\zeta_2\left(\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\right)=\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}^{m/(m-1)}\le\int_\Omega\widetilde{\Phi}\left(|y|^{1-N}\right)dy. \tag{2.2}$$

Notice that $|y|^{1-N}\ge 1$ for $y\in\Omega$. In view of Lemma 2.2, one has

$$\begin{aligned}\int_\Omega\widetilde{\Phi}\left(|y|^{1-N}\right)dy&\le\int_\Omega\widetilde{\Phi}(1)\,|y|^{(1-N)\frac{l}{l-1}}dy=\widetilde{\Phi}(1)\,\omega_N\int_{|x|}^1 r^{(1-N)\frac{l}{l-1}}r^{N-1}dr\\&=\widetilde{\Phi}(1)\,\omega_N\tfrac{l-1}{N-l}\left(|x|^{\frac{l-N}{l-1}}-1\right)\le\widetilde{\Phi}(1)\,\omega_N\tfrac{l-1}{N-l}|x|^{\frac{l-N}{l-1}},\end{aligned} \tag{2.3}$$

where ω_N is the surface area of the unit ball of $\mathbb{R}^N$. Then it follows from (2.2) and (2.3) that

$$\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\le\left(\widetilde{\Phi}(1)\,\omega_N\frac{l-1}{N-l}\right)^{\frac{m-1}{m}}|x|^{\frac{(l-N)(m-1)}{m(l-1)}}. \tag{2.4}$$

If $\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}<1$, one has

$$\zeta_2\left(\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\right)=\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}^{l/(l-1)}\le\int_\Omega\widetilde{\Phi}\left(|y|^{1-N}\right)dy.$$

Computing similarly as in (2.4), one obtains

$$\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\le\left(\widetilde{\Phi}(1)\,\omega_N\frac{l-1}{N-l}\right)^{\frac{l-1}{l}}|x|^{\frac{l-N}{l}}.$$

Note that $|x|\le 1$ and $1<l\le m<N$. Thus

$$\left\||y|^{1-N}\right\|_{\widetilde{\Phi},\Omega}\le C|x|^{\frac{(l-N)(m-1)}{m(l-1)}},$$

with $C=\max\left\{\left(\widetilde{\Phi}(1)\,\omega_N\frac{l-1}{N-l}\right)^{\frac{m-1}{m}},\left(\widetilde{\Phi}(1)\,\omega_N\frac{l-1}{N-l}\right)^{\frac{l-1}{l}}\right\}$. This ends the proof. □

3. Embedding Results

Lemma 3.1. *Let $u \in W^1_{0,r}L_\Phi(B)$. Then there exists $C = C(l,m,N,\widetilde{\Phi}) > 0$ such that $|u(x)| \leq C\|u\|_B |x|^{\frac{(l-N)(m-1)}{m(l-1)}}$.*

Proof. Notice that $u(1) = 0$ and $u(1) - u(|x|) = \int_{|x|}^1 u'(t)dt$. It follows that

$$|u(|x|)| \leq \int_{|x|}^1 |u'(t)|dt = \frac{1}{\omega_N}\int_{\mathbb{S}}\int_{|x|}^1 |u'(t)|t^{1-N}t^{N-1}\omega(\theta)dtd\theta = \frac{1}{\omega_N}\int_\Omega |\nabla u(y)|\,|y|^{1-N}dy,$$

where $\Omega = \left\{y \in \mathbb{R}^N \middle| 0 < |x| \leq |y| \leq 1\right\}$ and $\mathbb{S}$ is the unit sphere in $\mathbb{R}^N$. Using (2.1) and Lemma 2.3, one obtains

$$\int_\Omega |\nabla u(y)|\,|y|^{1-N}dy \leq 2\|u\|_\Omega \left\||y|^{1-N}\right\|_{\tilde{\Phi},\Omega} \leq C\|u\|_B |x|^{\frac{(l-N)(m-1)}{m(l-1)}}.$$

This completes the proof. □

Theorem 3.2. *Let $q < \frac{(\alpha+N)m(l-1)}{(m-1)(N-l)}$. Then the embedding $W^1_{0,r}L_\Phi(B) \hookrightarrow L^q(B,|x|^\alpha)$ is continuous and compact.*

Proof. Using Lemma 3.1, one has that, for any $q < \frac{(\alpha+N)m(l-1)}{(m-1)(N-l)}$ and $u \in W^1_{0,r}L_\Phi(B)$,

$$\int_B |x|^\alpha |u|^q dx \leq C^q \int_B |x|^\alpha \|u\|_B^q |x|^{\frac{q(l-N)(m-1)}{m(l-1)}} dx = C^q C(\alpha,N,q,l,m)\|u\|_B^q \tag{3.1}$$

with $C(\alpha,N,q,l,m) = \frac{\omega_N}{\alpha+N+\frac{q(l-N)(m-1)}{m(l-1)}} > 0$. Hence

$$\|u\|_{L^q(B,|x|^\alpha)} \leq CC^{1/q}(\alpha,N,q,l,m)\|u\|_B.$$

This means that the embedding $W^1_{0,r}L_\Phi(B) \hookrightarrow L^q(B,|x|^\alpha)$ is continuous.

Next, we show that the embedding is compact. Notice that $W^1_0L_\Phi(B)$ is compactly embedded in $L^1(B)$ (see [1, Theorem 8.35]). Hence, for any $0 < \beta < 1$,

$$\int_B |x|^\alpha |u|^q dx = \int_B |x|^\alpha |u|^{q-\beta}|u|^\beta dx \leq ||u||^\beta_{L^1(B)}\left(\int_B |x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-\beta}{1-\beta}}dx\right)^{1-\beta}.$$

Since $q < \frac{(\alpha+N)m(l-1)}{(m-1)(N-l)}$, there exists a constant $\beta > 0$ small enough such that $\frac{q}{1-\beta} < \frac{(\alpha+N)m(l-1)}{(m-1)(N-l)}$. It follows that

$$\frac{q-\beta}{1-\beta} < \frac{(\frac{\alpha}{1-\beta}+N)m(l-1)}{(m-1)(N-l)}.$$

By (3.1), one sees that there exists a constant $C = C(\alpha,\beta,N,l,m,q) > 0$ such that

$$\int_B |x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-\beta}{1-\beta}}dx \leq C(\alpha,\beta,N,l,m,q)\|u\|_B^{\frac{q-\beta}{1-\beta}}.$$

Then

$$\int_B |x|^\alpha |u|^q dx \leq C^{1-\beta}(\alpha,\beta,N,l,m,q)||u||^\beta_{L^1(B)}\|u\|_B^{q-\beta}.$$

It follows that

$$\|u\|_{L^q(B,|x|^\alpha)} \leq C^{\frac{1-\beta}{q}}(\alpha,\beta,N,l,m,q)||u||^{\frac{\beta}{q}}_{L^1(B)}\|u\|_B^{\frac{q-\beta}{q}}.$$

Then we conclude that $W_0^1 L_\Phi(B) \hookrightarrow L^q(B,|x|^\alpha)$ is compact. □

4. The Existence of the Solutions

In this section, we prove the existence of solutions for (1.1). For any $u \in W_0^1 L_\Phi(B)$, we define

$$J(u) = \int_B \Phi(|\nabla u|)dx - \frac{1}{q}\int_B |x|^\alpha |u|^q dx, \text{ and}$$
$$J_1(u) = \int_B \Phi(|\nabla u|)dx,\ J_2(u) = \frac{1}{q}\int_B |x|^\alpha |u|^q dx.$$

Then $J(u), J_1(u)$, and $J_2(u)$ are well defined and are of C^1. For $u, v \in W_0^1 L_\Phi(B)$, one has

$$\langle J'(u), v\rangle = \int_B \phi(|\nabla u|)\nabla u\cdot\nabla v dx - \int_B |x|^\alpha |u|^{q-2} uv dx,$$
$$\langle J_1'(u), v\rangle = \int_B \phi(|\nabla u|)\nabla u\cdot\nabla v dx, \text{ and}$$
$$\langle J_2'(u), v\rangle = \int_B |x|^\alpha |u|^{q-2} uv dx.$$

Hence, the weak solution of (1.1) is the critical point of J.

Lemma 4.1. *Let $m < q < \frac{(\alpha+N)m(l-1)}{(m-1)(N-l)}$. Suppose that the sequence $\{u_n\} \subset W_{0,r}^1 L_\Phi(B)$ satisfies the following condition $J(u_n) \to c$, $J'(u_n) \to 0$ as $n \to \infty$. Then there exists $u \in W_{0,r}^1 L_\Phi(B)$ such that $J(u) = c$ and $J'(u) = 0$.*

Proof. First, we show that $\sup_n ||u_n||_B < \infty$. Since $J(u_n) \to c$, there exists a positive constat d such that $\sup_n J(u_n) \le d$. By condition (ϕ_2) and Lemma 2.1, we have

$$\begin{aligned} J(u_n) - \frac{1}{q}\langle J'(u_n), u_n\rangle &= \int_B \Phi(|\nabla u_n|) - \frac{1}{q}\phi(|\nabla u_n|)|\nabla u_n|^2 dx \\ &\ge (1-\frac{m}{q})\int_B \Phi(|\nabla u_n|)dx \ge (1-\frac{m}{q})\zeta_0(||u_n||_B). \end{aligned} \tag{4.1}$$

On the other hand,

$$J(u_n) - \frac{1}{q}\langle J'(u_n), u_n\rangle \le d + ||u_n||_B. \tag{4.2}$$

It follows from (4.1) and (4.2) that $||u_n||_B$ is bounded.

Next, we show that c is a critical values of J. Since $W_0^1 L_\Phi(B)$ is a reflexive space, we can assume that $\{u_n\}$ converges weakly in $W_0^1 L_\Phi(B)$. Since $W_{0,r}^1 L_\Phi(B) \hookrightarrow L^q(B,|x|^\alpha)$ compactly, we can assume that $u_n \to u$ in $L^q(B,|x|^\alpha)$ as $n \to \infty$. Since J_2 is C^1 in $L^q(B,|x|^\alpha)$, then $\lim_{n\to\infty} J_2(u_n) = J_2(u)$ and $\lim_{n\to\infty} J_2'(u_n) = J_2'(u)$. Because $\lim_{n\to\infty} J'(u_n) = 0$, one has that

$$\lim_{n\to\infty} J_1'(u_n) = \lim_{n\to\infty}(J'(u_n) + J_2'(u_n)) = J_2'(u). \tag{4.3}$$

Since Φ is convex and C^1, then, for any $s, t > 0$, $\Phi(s) \le \Phi(t) + \Phi'(t)(t-s)$. It follows that

$$J_1(u_n) \le J_1(u) + \langle J_1'(u_n), u_n - u\rangle. \tag{4.4}$$

It follows from (4.3) and (4.4) that

$$\begin{aligned} \limsup_{n\to\infty} J_1(u_n) &\le J_1(u) + \limsup_{n\to\infty}\langle J_1'(u_n), u_n - u\rangle \\ &= J_1(u) + \limsup_{n\to\infty}\big(\langle J_1'(u_n) - J_2(u), u_n - u\rangle + \langle J_2(u), u_n - u\rangle\big) = J_1(u). \end{aligned}$$

Because J_1 is convex, J_1 is weakly lower semi-continuous. Hence $\liminf_{n\to\infty} J_1(u_n) \geq J_1(u)$. Therefore $\lim_{n\to\infty} J_1(u_n) = J_1(u)$, which together with (4.5) yields that $\lim\limits_{n\to\infty} J(u_n) = J(u)$. By condition (ϕ_1), one sees that $\phi(s)s$ is monotone increasing. It follows that, for $v \in W_0^1 L_\Phi(B)$,

$$\langle J_1'(v) - J_1'(u_n), v - u_n\rangle \geq 0.$$

Thus

$$\lim_{n\to\infty} \langle J_1'(v) - J_1'(u_n), v - u_n\rangle = \langle J_1'(v) - J_2'(u), v - u\rangle \geq 0.$$

Letting $v = u + th$ with $h \in W_0^1 L_\Phi(B), t > 0$, one obtains $\langle J_1'(u+th) - J_2'(u), th\rangle \geq 0$. It follows that, for any $h \in W_0^1 L_\Phi(B)$,

$$\lim_{t\to 0} \langle J_1'(u+th) - J_2'(u), h\rangle = \langle J_1'(u) - J_2'(u), h\rangle \geq 0.$$

Therefore $J'(u) = J_2'(u) - J_1'(u) = 0$. This ends the proof. □

The Proof of Theorem 1.1. By Lemma 2.1 and Theorem 3.2, we have

$$J(u) \geq \zeta_0(||u||_B) - \frac{C^q}{q}||u||_B^q. \tag{4.5}$$

Then there exists $r > 0$ such that $b \equiv \inf_{||u||_B = r} J(u) > 0 = J(0)$. Let $u \in W_0^1 L_\Phi(B)$ with $u > 0$ on B. We have that, for $t > 0$,

$$J(tu) = \int_B \Phi(t|\nabla u|)dx - \frac{t^q}{q}\int_B |x|^\alpha u^q dx \leq \zeta_1(t)\int_B \Phi(|\nabla u|)dx - \frac{t^q}{q}\int_B |x|^\alpha u^q dx.$$

Hence there exists a $e = tu$ such that $||e||_B > r$ and $J(e) < 0$. By the mountain pass theorem and Lemma 4.1 , J has a positive critical value and equation (1.1) has a nontrivial solution.

Definition 4.2. ([14]) Let X be a Banach space and X^* be the dual space of X. The operator $A : X \to X^*$ is said to satisfy the (S^+) condition if, for any $u_n \subset X$ such that $u_n \rightharpoonup u$ weakly and $\limsup\limits_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, $u_n \to u$ strongly.

Definition 4.3. ([14]) Let X be a Banach space. A convex functional $\Psi : X \to \mathbb{R}$ is said to be uniformly convex $E \subset X$ if, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\Psi\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Psi(u) + \frac{1}{2}\Psi(v) - \delta(\varepsilon), \text{ for } u, v \in E, ||u - v|| \geq \varepsilon.$$

Ψ is said to be local uniform convex if Ψ is uniform convex on each ball of X.

Lemma 4.4. *([5]) Assume that $\Psi : X \to \mathbb{R}$ is a C^1 locally uniformly convex functional and is locally bounded. Then $\Psi' : X \to X^*$ satisfies the (S^+) condition.*

Proposition 4.5. *([16]) Let $J \in C^1(X, \mathbb{R})$, where X is a Banach space. Assume that J satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If, for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X_k and $\rho_k > 0$ such that*

$$\sup_{u \in X_k \cap S_{\rho_k}} J(u) < 0, \tag{4.6}$$

where $S_{\rho_k} = \{u \in X \mid ||u|| = \rho\}$, then J has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to 0$.

In the following part, we suppose that ϕ satisfies the following hypothesis

(ϕ_3) : Φ is uniformly convex on $\mathbb{R}^+$.

The Proof of Theorem 1.2. We use Proposition 4.5 to prove this result. Hence, we first verify that J satisfies the conditions in Proposition 4.5. From the definition of J, it is easy to see that J is even, and $J(0) = 0$. For $u \in W^1_{0,r}L_\Phi(B)$ and $||u||_B > 1$, one finds by (4.5)

$$J(u) \geq ||u||^l_B - \frac{C^q}{q}||u||^q_B.$$

In view of $q < l$, one has $J(u) \to +\infty$ as $||u||_B \to +\infty$. Hence J is bounded from below. Next, we show that J satisfies (PS) condition. Suppose that $\{u_n\} \subset W^1_{0,r}L_\Phi(B)$ satisfies

$$\{J(u_n)\} \text{ is bounded, and } J'(u_n) \to 0, \text{ as } n \to \infty.$$

Note that $\{u_n\}$ is bounded in $W^1_0L_\Phi(B)$. Since $W^1_0L_\Phi(B)$ is reflexivity, by Theorem 3.2, up to a subsequence if necessary, we can assume that there exists $u \in W^1_{0,r}L_\Phi(B)$ such that

$$u_n \rightharpoonup u \text{ in } W^1_0L_\Phi(B), \text{ and } u_n \to u \text{ in } L^q(B, |x|^\alpha), \text{ as } n \to \infty.$$

Hence

$$\langle J'_1(u_n), u_n - u\rangle = \langle J'(u_n), u_n - u\rangle + \langle J'_2(u_n), u_n - u\rangle \to 0, \text{ as } n \to \infty.$$

By condition (ϕ_3), one sees that J_1 is uniformly convex on $W^1_{0,r}L_\Phi(B)$. Hence J'_1 satisfies the (S^+) property. Therefore $u_n \to u$ in $W^1_{0,r}L_\Phi(B)$, as $n \to \infty$. It means that J satisfies (PS) condition. At last, we need to verify that J satisfies (4.6). For any $k \in \mathbb{N}$, we can choose k independent smooth functions $\phi_i \in C^\infty_{0,r}(B)(i = 1, 2, \cdots, k)$. Set $X_k = \{\phi_1, \phi_2, \cdots, \phi_k\}$. Then, for $\rho_k > 0$ small enough and $u \in X_k \cap S_{\rho_k}$, by Lemma 2.1,

$$J(u) \leq \zeta_1(||\nabla u||_B) - \frac{1}{q}\int_B |x|^\alpha u^q dx = ||\nabla u||^m_B - \frac{1}{q}\int_B |x|^\alpha u^q dx.$$

Since the norms on finite dimensional X_k are equivalent and $m > l > q$, we have $\sup_{u \in X_k \cap S_{\rho_k}} J(u) < 0$. Using Proposition 4.5, we obtain that J has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$. This ends the proof.

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