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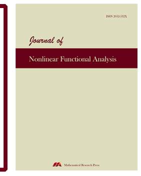

ON EXTENDED GRONWALL-BELLMAN-BIHARI TYPE INTEGRAL INEQUALITIES VIA (k, ψ)-HILFER PROPORTIONAL FRACTIONAL OPERATORS AND THEIR APPLICATIONS

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Abstract. In the framework of the (k, ψ)-Hilfer proportional fractional operator ((k, ψ)-HPFO), we investigate a new type of integral inequalities of Gronwall-Bellman-Bihari. In addition to covering current results in fractional calculus, these results provide a generalized framework that connects them. The integral inequalities are a helpful tool for examining how solutions to the nonlinear (k, ψ)-Hilfer proportional fractional differential equations behave, especially in terms of their stability, uniqueness, and boundedness. The framework produces additional generalizations and reproduces a number of known results as special cases by choosing appropriate parameter values and kernel functions. Several examples are given to demonstrate how our method works for fractional-order systems with memory-dependent dynamics.

Keywords. Convex functions; Gronwall inequality; Integral inequalities; (k, ψ)-Hilfer proportional fractional derivative operator; Mittag-Leffler function.

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1. Introduction

Fractional calculus generalizes traditional calculus by enabling the differentiation and integration of arbitrary, including non-integer, orders. Fractional calculus is proven to be an appropriate mathematical foundation for understanding complex systems in a variety of disciplines, including engineering science, biology, and physics. Several fractional operators were designed to fulfill the diverse requirements of theoretical research and practical applications. Recently,

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authors designed new fractional operators to satisfy specific needs in modeling memory and heredity qualities of various materials and processes, such as Hadamard [1, 2], Katugampola [3, 4], Hilfer derivative [5], and so on. Several fractional operators were developed into more generalized forms, especially the Hilfer operator, which is an outstanding example. The Hilfer operator is a generalized fractional derivative incorporating an interpolation parameter to bridge the gap between the $\mathbb{RL}$ and Caputo derivatives, which was initially revealed by Hilfer. Subsequently, researchers expanded on the concept of the Hilfer operator and developed its generalized form, including ψ-Hilfer [6], (k,ψ)-Hilfer [7], and (k,ψ)-HPFO [8]. Moreover, ψ-Hilfer is a generalized operator, which can be reduced to classical Hilfer fractional derivative and the $\mathbb{RL}$ or Caputo fractional derivatives, depending on the values of its parameters. The (k,ψ)-Hilfer is a more advanced generalization than ψ-Hilfer, while the (k,ψ)-HPFO furthers and encompasses the (k,ψ)–Hilfer fractional operator that incorporates a proportional factor. Both fractional operators can be reduced to classical Hilfer derivative, the (k,ψ)-$\mathbb{RL}$ derivative, and the (k,ψ)-Caputo derivative. These operators were widely applied in various types of fractional differential equations; see, e.g., [9, 10, 11, 12, 13].

Since most differential and integral equations cannot be solved exactly in closed forms, researchers employed integral inequalities as a powerful fundamental tool to investigate and establish various qualitative properties of the solutions to such equations. Several famous inequalities were often used to bound solutions, such as Gronwall's, Bellman's, Bihari's, and so on. Indeed, Gronwall's inequality was first investigated in [14]. In addition, Bellman's inequality, which is an extension of Gronwall's inequality, was first introduced in [15]. Later, Bihari's inequality, which is an extension of Bellman's inequality to the fully nonlinear case, was disclosed in [16]. In recent years, numerous studies investigated differential and integral equations involving integral inequalities. For instance, in 2018. Nisar *et al.* [17] derived specific forms of Gronwall–type inequalities within the framework of k-$\mathbb{RL}$ and k-Hadamard fractional operators. In 2019, Alzabut *et al.* [18] investigated Gronwall-Bellman's inequality in the context of the $\mathbb{RL}$ and Caputo proportional fractional derivatives. In 2020, Foukrach and Meftah [19] examined nonlinear Gronwall-Bellman-Bihari's inequalities for the k-$\mathbb{RL}$-fractional integral operator. After a span of three years, their investigation extended to the inequalities of Gronwall-Bellman-Bihari type involving the ψ-Hilfer fractional derivative, as documented in [20]. We also refer to [21, 22, 23] for some recent results.

The literature review [8, 19, 20] motivates us to fill the gap in the boundary of this study area, as the (k,ψ)-HPFO associated with the Gronwall-Bellman-Bihari's type has not yet been examined. Moreover, this operator leverages the flexibility of three parameters k,β,ρ, and a function ψ. We aim to analyze some novel extensions of integral inequalities of Gronwall-Bellman-Bihari's type via the (k,ψ)-HPFO of the proposed problem which have the general form

$$\begin{cases} {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau)=f\left(\tau,u(\tau)\right), \quad \alpha/k\in(n-1,n], \quad \tau\in(a,b], \quad 0\leq a<b<\infty, \\ \lim\limits_{\tau\to a^{+}} {}_{k}\mathfrak{D}^{n-i,\rho;\psi}\left({}_{a,k}\mathscr{I}^{nk-\gamma,\rho;\psi}u(\tau)\right)=c_j, \quad c_j\in\mathbb{R}, \quad j=1,\ldots,n, \quad n=1,2,\ldots, \end{cases}$$

where $f\in\mathscr{C}([a,b]\times\mathbb{R},\mathbb{R})$, ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}$ denotes (k,ψ)-HPFO of order α and type β with $\beta\in[0,1]$, $\rho\in(0,1]$, $k\in\mathbb{R}^{+}$, ${}_{k}\mathfrak{D}^{n-i,\rho;\psi}$ denotes the (k,ψ)-proportional derivative operator ((k,ψ)-PDO) of order $n-i$, $i=1,\ 2,\ \ldots,\ n$, and ${}_{a,k}\mathscr{I}^{nk-\gamma,\rho;\psi}$ denotes the (k,ψ)-$\mathbb{RL}$-proportional fractional integral operator ((k,ψ)-$\mathbb{RL}$-PFIO) of order $nk>\gamma$.

The subsequent portions of this work are organized below. Section 2 presents essential definitions and lemmas associated with the (k,ψ)-HPFO. The desired results are established in Section 3 by focusing on generalized Gronwall-Bellman-Bihari's type integral inequalities within the framework of (k,ψ)-HPFO. Section 4 demonstrates the applicability of our results through various nonlinear proportional problems. Finally, the concluding remarks are presented in Section 5

2. Basic Definitions and Lemmas

This section presents some basic definitions and lemmas. We provide the following symbol for simple calculation in this work

$$ {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)=\exp\left(\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(s))\right)(\psi(\tau)-\psi(s))^{\frac{\alpha}{k}-1}. $$

Definition 2.1 ([8, 24].)**.** Let $\alpha,k\in\mathbb{R}^+$, $\rho\in(0,1]$, and $u\in L^1([a,b],\mathbb{R})$, where $0\le a<b<\infty$. Then, the (k,ψ)-$\mathbb{RL}$-PFIO of α of u is defined as

$$ {}_{a,k}\mathscr{I}^{\alpha,\rho;\psi}u(\tau)=\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^{\tau}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)u(s)\psi'(s)ds, $$

where $\Gamma_k(u)=\int_0^\infty s^{u-1}e^{-s^k/k}ds$, $u\in\mathbb{C}$, $Re(u)>0$, and $\Gamma(u)=\Gamma_k(u)$ as $k\to1$, $\Gamma_k(z)=k^{z/k-1}\Gamma(z/k)$, $\Gamma_k(z+k)=z\Gamma_k(z)$, and $\Gamma_k(k)=1$.

Definition 2.2 ([8])**.** Let $\alpha,k\in\mathbb{R}^+$, $\rho\in(0,1]$, $u\in\mathscr{C}([a,b],\mathbb{R})$, $\psi(\tau)\in\mathscr{C}^n([a,b],\mathbb{R})$, $\psi'(\tau)\neq0$, and $n=1,2,\ldots$, with $n=\lfloor\alpha/k\rfloor+1$. Then, the (k,ψ)-$\mathbb{RL}$-proportional fractional derivative operator ((k,ψ)-$\mathbb{RL}$-PFDO) of α of u is defined as

$$ {}^{\mathbb{RL}}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau)=\frac{\rho^{-\frac{nk-\alpha}{k}}{}_k\mathfrak{D}^{n,\rho;\psi}}{k\Gamma_k(nk-\alpha)}\int_a^{\tau}{}^{\rho}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(\tau,s)u(s)\psi'(\tau)ds={}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathscr{I}^{nk-\alpha,\rho;\psi}u(\tau)\right), $$

where ${}_k\mathfrak{D}^{n,\rho;\psi}=\underbrace{{}_k\mathfrak{D}^{\rho;\psi}\,{}_k\mathfrak{D}^{\rho;\psi}\cdots{}_k\mathfrak{D}^{\rho;\psi}}_{n\text{ times}}$, and ${}_k\mathfrak{D}^{1,\rho;\psi}u(\tau)={}_k\mathfrak{D}^{\rho;\psi}u(\tau)=(1-\rho)u(\tau)+k\rho\frac{u'(\tau)}{\psi'(\tau)}$.

Definition 2.3 ([8])**.** Let $\alpha,k\in\mathbb{R}^+$, $\rho\in(0,1]$, $u\in\mathscr{C}^n([a,b],\mathbb{R})$, $\psi(\tau)\in\mathscr{C}^n([a,b],\mathbb{R})$, $\psi'(\tau)\neq0$, and $n=1,2,\ldots$, with $n=\lfloor\alpha/k\rfloor+1$. Hence, the (k,ψ)-Caputo proportional fractional derivative operator ((k,ψ)-Caputo-PFDO) of α of u is defined as

$$ \begin{aligned} {}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau) &= \frac{1}{\rho^{\frac{nk-\alpha}{k}}k\Gamma_k(nk-\alpha)}\int_a^{\tau}{}^{\rho}_{k}\Psi_{\psi}^{\frac{nk-\alpha}{k}-1}(\tau,s)\left({}_k\mathfrak{D}^{n,\rho;\psi}u(s)\right)\psi'(s)ds \\ &= {}_{a,k}\mathscr{I}^{nk-\alpha,\rho;\psi}\left({}_k\mathfrak{D}^{n,\rho;\psi}u(\tau)\right). \end{aligned} $$

Definition 2.4 ([8])**.** Let $\alpha,k\in\mathbb{R}^+$, $\rho\in(0,1]$, $\beta\in[0,1]$, $u\in\mathscr{C}^n([a,b],\mathbb{R})$, $\psi(\tau)\in\mathscr{C}^n([a,b],\mathbb{R})$, $\psi'(\tau)\neq0$, and $n=1,2,\ldots$, with $n=\lfloor\alpha/k\rfloor+1$. Hence, the (k,ψ)-HPFDO of α and β of u is defined as

$$ \begin{aligned} {}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau) &= {}_{a,k}\mathscr{I}^{\beta(nk-\alpha),\rho;\psi}\left({}_k\mathfrak{D}^{n,\rho;\psi}\left({}_{a,k}\mathscr{I}^{(1-\beta)(nk-\alpha),\rho;\psi}u(\tau)\right)\right) \\ &= \begin{cases} {}^{\mathbb{RL}}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau), & \text{if } \beta=0, \\ {}^{C}_{a,k}\mathfrak{D}^{\alpha,\rho;\psi}u(\tau), & \text{if } \beta=1. \end{cases} \end{aligned} $$

Lemma 2.5 ([8])**.** *Assume $\alpha,\ \delta \in \mathbb{R}^+ \cup \{0\}$, $k,\ \gamma \in \mathbb{R}^+$, $\rho \in (0,1]$, $\omega \in \mathbb{R}$, and $\omega/k > -1$. Hence,*

(i) ${}_{a,k}\mathscr{I}^{\alpha,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(\tau,a)\right] = \frac{\Gamma_k(\omega){}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega+\alpha}{k}-1}(\tau,a)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega+\alpha)}$.

(ii) ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega}{k}-1}(\tau,a)\right] = \frac{\rho^{\frac{\alpha}{k}}\Gamma_k(\omega){}^{\rho}_{k}\Psi_{\psi}^{\frac{\omega-\alpha}{k}-1}(\tau,a)}{\Gamma_k(\omega-\alpha)}$. *In particular, $m = 0,\dots,n-1$ with $n = \lfloor \omega/k \rfloor + 1$, we achieve* ${}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}\left[{}^{\rho}_{k}\Psi_{\psi}^{m}(\tau,a)\right] = 0$.

(iii) ${}_{a,k}\mathscr{I}^{\alpha,\rho;\psi}\left({}_{a,k}\mathscr{I}^{\delta,\rho;\psi}u(\tau)\right) = {}_{a,k}\mathscr{I}^{\delta,\rho;\psi}\left({}_{a,k}\mathscr{I}^{\alpha,\rho;\psi}u(\tau)\right) = {}_{a,k}\mathscr{I}^{\delta+\alpha,\rho;\psi}u(\tau)$.

(iv) ${}^{H}_{a,k}\mathfrak{D}^{\omega,\beta,\rho;\psi}\left({}_{a,k}\mathscr{I}^{\gamma,\rho;\psi}u(\tau)\right) = {}_{a,k}\mathscr{I}^{\gamma-\omega,\rho;\psi}u(\tau)$, *where $n = \lfloor \omega/k \rfloor + 1$, and $\gamma > nk$.*

(v) ${}_{a,k}\mathscr{I}^{\alpha,\rho;\psi}\left({}^{H}_{a,k}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau)\right) = u(\tau) - \sum_{i=1}^{n}\frac{{}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-i}(\tau,a)}{\rho^{\frac{\gamma-ki}{k}}\Gamma_k(\gamma+k-ki)}\left[{}_{k}\mathfrak{D}^{n-i,\rho;\psi}\left({}_{a,k}\mathscr{I}^{nk-\gamma,\rho;\psi}u(a^+)\right)\right]$, *with $\gamma = \alpha + \beta(nk-\alpha)$.*

Definition 2.6 ([25])**.** If $\varphi(u+v) \le \varphi(u) + \varphi(v)$ for any $u,\ v \ge 0$, then $\varphi(u)$ is known as sub-additive.

Definition 2.7 ([26])**.** If $\varphi(uv) \le \varphi(u)\varphi(v)$, for any $u,\ v \ge 0$, then $\varphi(u)$ is known as sub-multiplicative.

Definition 2.8 ([27])**.** The function $\varphi : [0,+\infty) \to [0,+\infty)$ is said to be belong to a class Ω if (i) $\varphi(u) > 0$ is continuous and non-decreasing for $u \ge 0$; (ii) $(1/p)\varphi(u) \le \varphi(u/p)$, for any $p > 0$.

Lemma 2.9. ([28]). *Let $a,\ p,\ q \ge 0$ with $p \ge q$, and $p \ne 0$. Hence, for all $\varepsilon > 0$,*

$$a^{\frac{q}{p}} \le \left(\frac{q}{p}\right)\varepsilon^{\frac{q-p}{p}}a + \left(\frac{p-q}{p}\right)\varepsilon^{\frac{q}{p}}.$$

3. Main Results

This section analyzes a variety of extended integral inequalities of Gronwall-Bellman-Bihari's type in frame of (k,ψ)-HPFO.

Theorem 3.1. *Assume that two functions ϕ and u are locally integrable and non-negative defined on $[a,b)$, where $0 \le a < b < \infty$, and a function η is a non-negative, non-decreasing, and continuous on $[a,b)$ so that η is bounded on $[a,b)$, i.e., $|\eta(\tau)| \le \mathscr{M}$ for any $\tau \in [a,b)$, ψ is positive monotone increasing function on $(a,T]$, with a continuous derivative $\psi'(\tau)$ on (a,b), and given arbitrary constants $p,\ q,\ \varepsilon,\ \alpha \in \mathbb{R}^+$ such that $p \ge q$. If*

$$u^p(\tau) \le \phi(\tau) + \frac{\eta(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)u^q(s)\psi'(s)ds, \tag{3.1}$$

then

$$u(\tau) \le \left(\phi_1(\tau) + \sum_{i=1}^{\infty}\frac{(\eta_1(\tau))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\phi_1(s)\psi'(s)ds\right)^{\frac{1}{p}}, \tag{3.2}$$

where

$$\phi_1(\tau) := \phi(\tau) + \frac{(p-q)\varepsilon^{\frac{q}{p}}}{p\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(\tau,a)\eta(\tau) \quad and \quad \eta_1(\tau) := \frac{q\varepsilon^{\frac{q-p}{p}}}{p}\eta(\tau). \tag{3.3}$$

Proof. Let $v(\tau)$ be a function defined by

$$v(\tau)=\phi(\tau)+\frac{\eta(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)u^q(s)\psi'(s)ds. \tag{3.4}$$

In view of (3.1) and (3.4), one has $u(\tau)\le v^{\frac{1}{p}}(\tau)$, which together with (3.4) and Lemma 2.5 with the property (i) in Lemma 2.9 yields

$$\begin{aligned} v(\tau) &\le \phi(\tau)+\frac{\eta(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)[v(s)]^{\frac{q}{p}}ds \\ &\le \phi(\tau)+\frac{\eta(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)\left[\left(\frac{q}{p}\right)\varepsilon^{\frac{q-p}{p}}v(s)+\left(\frac{p-q}{p}\right)\varepsilon^{\frac{q}{p}}\right]ds \\ &= \phi_1(\tau)+\frac{\eta_1(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)v(s)ds, \end{aligned} \tag{3.5}$$

where ϕ_1 and η_1 are provided by (3.3). Next, we define a function $\mathscr{B}:[a,b)\to\mathbb{R}^+$, that is

$$\mathscr{B}v(\tau):=\frac{\eta_1(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)v(s)ds. \tag{3.6}$$

By using (3.5) and (3.6), we obtain $v(\tau)\le\phi_1(\tau)+\mathscr{B}v(\tau)$, which implies that

$$v(\tau)\le\phi_1(\tau)+\mathscr{B}\phi_1(\tau)+\mathscr{B}^2v(\tau)\le\phi_1(\tau)+\mathscr{B}\phi_1(\tau)+\mathscr{B}^2\phi_1(\tau)+\mathscr{B}^3v(\tau).$$

Utilizing the iterative procedure, one has

$$v(\tau)\le\sum_{i=0}^{n-1}\mathscr{B}^i\phi_1(\tau)+\mathscr{B}^nv(\tau),\quad n\ge 1. \tag{3.7}$$

For any $\tau\in[a,T)$ and $n\ge 1$, we obtain

$$\mathscr{B}^nv(\tau)\le\frac{(\eta_1(\tau))^n}{\rho^{\frac{n\alpha}{k}}k\Gamma_k(n\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{n\alpha}{k}-1}(\tau,s)\psi'(s)v(s)ds. \tag{3.8}$$

For $n=1$, it is easy to see that inequality (3.8) is satisfied. Next, let inequality (3.8) hold for any $n\le m$. We show that it holds for any $n=m+1$. Using the non-decreasing property of η_1, i.e., $\eta_1(s)\le\eta_1(\tau)$, and the Dirichlet's formula, we have

$$\mathscr{B}^{m+1}v(\tau) \tag{3.9}$$

$$\begin{aligned} &= \frac{\eta_1(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)[\mathscr{B}^mv(s)]ds \\ &\le \frac{(\eta_1(\tau))^{m+1}}{\rho^{\frac{(m+1)\alpha}{k}}k^2\Gamma_k(\alpha)\Gamma_k(m\alpha)}\int_a^\tau\left[\int_a^s {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s){}_k^\rho\Psi_\psi^{\frac{m\alpha}{k}-1}(s,r)\psi'(r)v(r)dr\right]\psi'(s)ds \\ &\le \frac{(\eta_1(\tau))^{m+1}}{\rho^{\frac{(m+1)\alpha}{k}}k^2\Gamma_k(\alpha)\Gamma_k(m\alpha)}\int_a^\tau\exp\left(\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(r))\right) \\ &\quad\times\left[\int_r^\tau(\psi(\tau)-\psi(s))^{\frac{\alpha}{k}-1}(\psi(s)-\psi(r))^{\frac{m\alpha}{k}-1}\psi'(s)ds\right]\psi'(r)v(r)dr. \end{aligned} \tag{3.10}$$

Changing variable $\psi(s)-\psi(r)=z(\psi(\tau)-\psi(r))$, $a\le r\le s\le\tau$ with the k-beta function, one has

$$\begin{aligned}&\int_r^\tau \big(\psi(\tau)-\psi(s)\big)^{\frac{\alpha}{k}-1}\big(\psi(s)-\psi(r)\big)^{\frac{m\alpha}{k}-1}\psi'(s)ds\\ =\;& \big(\psi(\tau)-\psi(r)\big)^{\frac{(m+1)\alpha}{k}-1}\frac{k\Gamma_k(\alpha)\Gamma_k(m\alpha)}{\Gamma_k((m+1)\alpha)}.\end{aligned}\tag{3.11}$$

Inserting (3.11) into (3.9) yields

$$\begin{aligned}\mathscr{B}^{m+1}v(\tau) \;\le\;& \frac{(\eta_1(\tau))^{m+1}}{\rho^{\frac{(m+1)\alpha}{k}}k^2\Gamma_k(\alpha)\Gamma_k(m\alpha)}\int_a^\tau e^{\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(r))}\\ &\times\left[\big(\psi(\tau)-\psi(r)\big)^{\frac{(m+1)\alpha}{k}-1}\frac{k\Gamma_k(\alpha)\Gamma_k(m\alpha)}{\Gamma_k((m+1)\alpha)}\right]\psi'(r)v(r)dr\\ \le\;& \frac{(\eta_1(\tau))^{m+1}}{\rho^{\frac{(m+1)\alpha}{k}}k\Gamma_k((m+1)\alpha)}\int_a^\tau {}_k^{\rho}\Psi_\psi^{\frac{(m+1)\alpha}{k}-1}(\tau,r)\psi'(r)v(r)dr.\end{aligned}$$

Thus one has (3.7). Since v is locally integrable and non-negative on the interval $[a,b)$, we obtain that v is integrable on $[a,\tau]$. Also, a function v is bounded on $[a,\tau]$, and there exists a constant $\mathscr{L}>0$ for which $|v(\tau)|\le\mathscr{L}$ with $|\eta(\tau)|\le\mathscr{M}$ for any $\tau\in[a,b]$. Then,

$$\begin{aligned}\mathscr{B}^n v(\tau) \;\le\;& \left(\frac{q\varepsilon^{\frac{q-p}{p}}}{p}\mathscr{M}\right)^n\frac{\mathscr{L}}{\rho^{\frac{n\alpha}{k}}k\Gamma_k(n\alpha)}\int_a^\tau {}_k^{\rho}\Psi_\psi^{\frac{n\alpha}{k}-1}(\tau,s)\psi'(s)ds\\ \le\;& \left(\frac{q\varepsilon^{\frac{q-p}{p}}}{p}\mathscr{M}\right)^n\frac{\mathscr{L}}{\rho^{\frac{n\alpha}{k}}\Gamma_k(n\alpha+k)}{}_k^{\rho}\Psi_\psi^{\frac{n\alpha}{k}}(\tau,a).\end{aligned}\tag{3.12}$$

Since $0<\exp\left(\frac{\rho-1}{k\rho}(\psi(\tau)-\psi(a))\right)\le1$ and $(\psi(\tau)-\psi(a))^{\frac{n\alpha}{k}}\le(\psi(\tau))^{\frac{n\alpha}{k}}$, for any $\tau\in[a,b]$, then

$$\mathscr{B}^n v(\tau)\le\frac{\mathscr{L}}{\Gamma_k(n\alpha+k)}\left(\frac{q\varepsilon^{\frac{q-p}{p}}\mathscr{M}\big(\psi(\tau)\big)^{\frac{\alpha}{k}}}{\rho^{\frac{\alpha}{k}}p}\right)^n.\tag{3.13}$$

Applying Stirling's formula to (3.13), i.e. $n!\sim\sqrt{2\pi n}(n/e)^n$, one has

$$\frac{\mathscr{L}}{\Gamma_k(n\alpha+k)}\left(\frac{q\varepsilon^{\frac{q-p}{p}}\mathscr{M}\big(\psi(\tau)\big)^{\frac{\alpha}{k}}}{p\rho^{\frac{\alpha}{k}}}\right)^n\approx\mathscr{L}\left(\frac{k}{2\pi\alpha}\right)^{\frac12}\frac{\Theta^n}{n^{\frac{n\alpha}{k}+\frac12}},$$

where $\Theta:=\frac{q\varepsilon^{\frac{q-p}{p}}\mathscr{M}}{p}\left(\frac{e\psi(\tau)}{\rho\alpha}\right)^{\frac{\alpha}{k}}$. Using the fact $p\ge q>0$ yields that

$$\lim_{n\to\infty}\left[\mathscr{L}\left(\frac{k}{2\pi\alpha}\right)^{\frac12}\frac{\Theta^n}{n^{\frac{n\alpha}{k}+\frac12}}\right]=0.$$

Therefore,

$$v(\tau)\le\phi_1(\tau)+\sum_{i=1}^{\infty}\frac{(\eta_1(\tau))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^\tau {}_k^{\rho}\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)\phi_1(s)ds.$$

Thus inequality (3.2) is obtained by taking $u^p(\tau)\le v(\tau)$. This completes the proof. □

Remark 3.2. Under all the conditions of Theorem 3.1, we conclude the following assertions

(i) Theorem 3.1 is reduced to [29, Theorem 1] if we set $\psi(\tau)=\tau$, $a=0$, and $p=q=\rho=k=1$. In addition, if $f(\tau)\equiv b$, then Theorem 3.1 is reduced to [29, Corollary 1].

(ii) Theorem 3.1 is reduced to [17, Theorem 2.1] if we set $\psi(\tau)=\tau$, $f(\tau)\equiv\kappa\chi(\tau)$, $a=0$, and $p=q=\rho=k=1$. In addition, if we set $\psi(\tau)=\ln\tau$ and $a=1$, then Theorem 3.1 is deduced to [17, Theorem 2.4].

(iii) Theorem 3.1 is reduced to [30, Theorem 1.4.1] if we set $\psi(\tau)=\tau$, $f(\tau)\equiv b\in\mathbb{R}$, $a=0$, and $p=q=\rho=k=1$.

(iv) Theorem 3.1 is deduced to [19, Theorem 3.1] if we set $\psi(\tau)=\tau$, $a=0$, $\alpha=\lambda/r$, and $\rho=k=1$.

(v) Theorem 3.1 is deduced to [19, Theorem 3.1] if we set $\psi(\tau)=\tau$, $a=0$, and $\rho=k=1$.

Corollary 3.3. *Let all the conditions in Theorem 3.1 hold. Assume that $\phi(\tau)$ is a non-decreasing function on $[a,b)$ and inequality (3.1) is true. Then,*

$$u(\tau)\le\left\{\phi_1(\tau)\mathbb{E}_{k,\alpha,k}\left(\rho^{-\frac{\alpha}{k}}\eta_1(\tau)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(\tau,a)\right)\right\}^{\frac{1}{p}},\tag{3.14}$$

where $\phi_1(\tau)$ and $\eta_1(\tau)$ are given by (3.3) and $\mathbb{E}_{k,\alpha,k}(\cdot)$ is the k-Mittag-Leffler function provided by

$$\mathbb{E}_{k,\alpha,k}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(n\alpha+k)},\quad z\in\mathbb{R},\quad\alpha\in\mathbb{R}^+,\quad k>0.\tag{3.15}$$

Proof. Applying the inequality (3.2) in Theorem 3.1, one has

$$u(\tau)\le\left\{\phi_1(\tau)+\sum_{i=1}^{\infty}\frac{(\eta_1(\tau))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^{\tau}{}^{\rho}_{k}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)\phi_1(s)ds\right\}^{\frac{1}{p}}.\tag{3.16}$$

Since $\phi_1(\tau)$ is a non-decreasing for each $\tau\in[a,b]$, $0\le a<b<\infty$, applying the property (i) in Lemma 2.9, one has

$$\begin{aligned}
u(\tau)&\le\left\{\phi_1(\tau)\left(1+\sum_{i=1}^{\infty}\frac{(\eta_1(\tau))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^{\tau}{}^{\rho}_{k}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)ds\right)\right\}^{\frac{1}{p}}\\
&\le\left\{\phi_1(\tau)\left(1+\sum_{i=1}^{\infty}\frac{(\eta_1(\tau))^i}{\rho^{\frac{i\alpha}{k}}\Gamma_k(i\alpha+k)}{}^{\rho}_{k}\Psi_{\psi}^{\frac{i\alpha}{k}}(\tau,a)\right)\right\}^{\frac{1}{p}}\\
&\le\left\{\phi_1(\tau)\left(1+\sum_{i=1}^{\infty}\frac{\left[\rho^{-\frac{\alpha}{k}}\eta_1(\tau)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(\tau,a)\right]^i}{\Gamma_k(i\alpha+k)}\right)\right\}^{\frac{1}{p}}\\
&=\left\{\phi_1(\tau)\sum_{i=0}^{\infty}\frac{\left[\rho^{-\frac{\alpha}{k}}\eta_1(\tau)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(\tau,a)\right]^i}{\Gamma_k(i\alpha+k)}\right\}^{\frac{1}{p}}\\
&=\left\{\phi_1(\tau)\mathbb{E}_{k,\alpha,k}\left(\rho^{-\frac{\alpha}{k}}\eta_1(\tau)\,{}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}}(\tau,a)\right)\right\}^{\frac{1}{p}}.
\end{aligned}$$

Hence, inequality (3.14) is obtained. □

Remark 3.4. Under all the assumptions of Corollary 3.3, we have the following statements.

(*i*) If we set $\psi(\tau)=\tau$, $a=0$, $\alpha=\lambda$, and $\rho=k=1$, then Corollary 3.3 is reduced to [29, Corollary 2].

(*ii*) If we set $\psi(\tau)=\tau$, $f(\tau)\equiv\kappa\chi(\tau)$, $a=0$, and $p=q=\rho=k=1$, then Corollary 3.3 is reduced to [17, Corollary 2.3].

(*iii*) If we set $\psi(\tau)=\tau$, $a=0$, $\alpha=\lambda/r$, and $\rho=k=1$, then Corollary 3.3 is reduced to [19, Corollary 3.5].

(*iv*) If we set $\psi(\tau)=\tau$, $a=0$, and $\rho=k=1$, then Theorem 3.3 is reduced to [19, Theorem 3.6].

Theorem 3.5. *Assume that two functions ϕ and u are locally integrable and non-negative defined on $[a,b)$, $0\le a<b<\infty$, and suppose that η is a non-negative, non-decreasing and continuous on $[a,T)$ such that η is bounded on $[a,b)$, i.e. $|\eta(\tau)|\le\mathscr{M}$ for any $\tau\in[a,b)$, ψ is positive monotone increasing on $(a,b]$, with a continuous derivative $\psi'(\tau)$ on (a,b), and $z\in\Omega$ is a sub-additive and convex. If*

$$u(\tau)\le\phi(\tau)+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)\eta(s)z(u(s))ds, \tag{3.17}$$

then

$$u(\tau)\le z^{-1}\left\{z(\phi(\tau))+\sum_{i=1}^{\infty}\frac{((\tau-a)z(\mathscr{M}))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z(\phi(s))ds\right\},$$

for any $\tau\in[a,t_0]$, where z^{-1} is the inverse function of z, $[a,t_0]$ is sub-interval, which is

$$\left\{z(\phi(\tau))+\sum_{i=1}^{\infty}\frac{((\tau-a)z(\mathscr{M}))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z(\phi(s))ds\right\}\in Dom\, z^{-1}.$$

Proof. Taking z into (3.17), we arrive at

$$z(u(\tau))\le z\left(\phi(\tau)+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)\eta(s)z(u(s))ds\right).$$

Applying the continuity of z and the property of sub-additive [25], we have

$$z(u(\tau))\le z(\phi(\tau))+z\left(\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)\eta(s)z(u(s))ds\right).$$

Using Jensen's inequality with $z\in\Omega$, we obtain

$$\begin{aligned}
z(u(\tau)) &\le z(\phi(\tau))+\frac{\tau-a}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau z\left({}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)\eta(s)z(u(s))\right)ds\\
&\le z(\phi(\tau))+\frac{\tau-a}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(\eta(s))z(u(s))ds\\
&\le z(\phi(\tau))+\frac{(\tau-a)z(\mathscr{M})}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(u(s))ds.
\end{aligned} \tag{3.18}$$

Substituting $v(\tau)=z(u(\tau))$ and $p=q=1$ into (3.18), we conclude that

$$v(\tau)\le z(\phi(\tau))+\frac{(\tau-a)z(\mathscr{M})}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(u(s))ds.$$

Note that

$$v(\tau) \leq z(\phi(\tau)) + \sum_{i=1}^{\infty} \frac{((\tau-a)z(\mathscr{M}))^i}{\rho^{\frac{i\alpha}{k}} k\Gamma_k(i\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z(\phi(s))ds.$$

For any $\tau \in [a,\tau_0)$, we have

$$u(\tau) \leq z^{-1}\left\{z(\phi(\tau)) + \sum_{i=1}^{\infty} \frac{((\tau-a)z(\mathscr{M}))^i}{\rho^{\frac{i\alpha}{k}} k\Gamma_k(i\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z(\phi(s))ds\right\}.$$

This completes the proof. □

Remark 3.6. Under all the conditions of Theorem 3.5, we have the following conclusions

(i) Theorem 3.5 is reduced to [19, Theorem 3.8] if we set $\psi(\tau)=\tau$, $a=0$, $\alpha=\lambda/r$, and $\rho=k=1$.

(ii) Theorem 3.5 is reduced to [19, Theorem 3.10] if we set $\psi(\tau)=\tau$, $a=0$, and $\rho=k=1$.

Corollary 3.7. *Let all the conditions of Theorem 3.5 and (3.17) be true. Then,*

$$u(\tau) \leq \phi(\tau) + \sum_{i=1}^{\infty} \frac{(z(1)\eta(\tau))^{i-1}}{\rho^{\frac{i\alpha}{k}} k\Gamma_k(i\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z(1)\eta(s)\phi(s)ds.$$

Proof. Since $z \in \Omega$, we find by inequality (3.17) that

$$u(\tau) \leq \phi(\tau) + \frac{1}{\rho^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)\eta(s)u(s)z(1)ds. \tag{3.19}$$

Multiplying $z(1)\eta(\tau)$ into (3.19) with $z(1)\eta(\tau)u(\tau)=v(\tau)$, we have

$$v(\tau) \leq z(1)\eta(\tau)\phi(\tau) + \frac{z(1)\eta(\tau)}{\rho^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)v(s)ds.$$

Setting $p=q=1$ in Theorem 3.1 yields

$$v(\tau) \leq z(1)\eta(\tau)\left(\phi(\tau) + \sum_{i=1}^{\infty} \frac{(z(1)\eta(\tau))^{i-1}}{\rho^{\frac{i\alpha}{k}} k\Gamma_k(i\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z(1)\eta(s)\phi(s)ds\right). \tag{3.20}$$

Substituting $z(1)\eta(\tau)u(\tau)=v(\tau)$ and multiplying $(z(1)\eta(\tau))^{-1}$ to (3.20), we obtain the desired conclusion immediately. □

Remark 3.8. Under the assumptions of Corollary 3.7, we conclude the following assertions

(i) Corollary 3.7 is reduced to [19, Theorem 3.9] if we set $\psi(\tau)=\tau$, $a=0$, $\alpha=\lambda/r$, and $\rho=k=1$.

(ii) Corollary 3.7 is reduced to [19, Theorem 3.12] if we set $\psi(\tau)=\tau$, $a=0$, and $\rho=k=1$.

Theorem 3.9. *Assume that two functions ϕ and u are locally integrable and non-negative defined on $[a,b)$, $0\leq a<b<\infty$, and suppose that g is non-negative, non-decreasing, continuous, and bounded on $[a,b)$, i.e. $|g(\tau)|\leq\mathscr{N}$ for all $\tau\in[a,b)$, ψ is positive monotone increasing on $(a,T]$, having a continuous derivative $\psi'(\tau)$ on (a,b), and $z\in\Omega$ is a sub-multiplicative and convex such that $z(0)=0$ and $z(u)>0$ on $[a,b)$. If*

$$u(\tau) \leq \phi(\tau) + \frac{g(\tau)}{\rho^{\frac{\alpha}{k}} k\Gamma_k(\alpha)} \int_a^\tau {}_k^\rho\Psi_\psi^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(u(s))ds, \tag{3.21}$$

then

$$u(\tau) \le g(\tau)z^{-1}\left\{z\left(\frac{\phi(\tau)}{g(\tau)}\right)+\sum_{i=1}^{\infty}\frac{((\tau-a)z(z(\mathcal{N}))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z\left(\frac{\phi(s)}{g(s)}\right)ds\right\}, \tag{3.22}$$

for any $\tau\in[a,t_0]$, *where* z^{-1} *is the inverse of* z *and* $[a,t_0]$ *is the sub-interval, which is*

$$\left\{z\left(\frac{\phi(\tau)}{g(\tau)}\right)+\sum_{i=1}^{\infty}\frac{((\tau-a)z(z(\mathcal{N}))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z\left(\frac{\phi(s)}{g(s)}\right)ds\right\}\in Dom\, z^{-1}.$$

Proof. Since$g$ is non-decreasing, continuous, and bounded on $[a,b)$, we obtain by inequality (3.21) that

$$\frac{u(\tau)}{g(\tau)}\le\frac{\phi(\tau)}{g(\tau)}+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(u(s))ds. \tag{3.23}$$

Putting $u(\tau)=g(\tau)v(\tau)$ into (3.23) with the sub-multiplicative property yields

$$\begin{aligned} v(\tau) &\le \frac{\phi(\tau)}{g(\tau)}+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(g(s)v(s))ds \\ &\le \frac{\phi(\tau)}{g(\tau)}+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(g(s))z(v(s))ds. \end{aligned}$$

Setting $\eta(s)=z(g(s))$ in Theorem 3.5, one has

$$v(\tau)\le z^{-1}\left\{z\left(\frac{\phi(\tau)}{g(\tau)}\right)+\sum_{i=1}^{\infty}\frac{((\tau-a)z(z(\mathcal{N}))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z\left(\frac{\phi(s)}{g(s)}\right)ds\right\}.$$

Letting $u(\tau)=g(\tau)v(\tau)$, we obtain the desired conclusion immediately. □

Remark 3.10. Theorem 3.9 is reduced to [19, Theorem 3.14] if $\psi(\tau)=\tau$, $a=0$, and $\rho=k=1$.

4. Some Applications

Example 4.1. Consider the following nonlinear integral equation under the (k,ψ)-PFIO:

$$u(\tau)=\tau+\frac{\sqrt{\tau}}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_0^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)u(s)ds, \quad \tau\in[0,1). \tag{4.1}$$

From the nonlinear integral equation (4.1) with parameters $p=q=1$, $\phi(\tau)=\tau$, and $\eta(\tau)=\sqrt{\tau}$, we proceed to show how the main results can be applied. This is divided into four categories and is shown in Figure 1 (Fig. 1a-Fig.1d).

(i) Set $\psi(\tau)=\tau$, $\alpha=1.2$, $k=0.8$, $\rho=1$, and $\varepsilon=1$. By invoking Theorem 3.1, one has

$$u(\tau)\le\tau+\sum_{i=1}^{\infty}\frac{(\sqrt{\tau})^i}{0.8\Gamma_{0.8}(1.2i)}\int_0^{\tau}(\tau-s)^{1.5i-1}sds.$$

Moreover, by Corollary 3.3, dynamic (k,ψ)-proportional fractional integral equation (4.1) has the estimate upper bounded: $u(\tau)\le\tau\mathbb{E}_{0.8,1.2,0.8}(\tau^2)$. The graphical representation for estimation of $u(\tau)$ for (4.1) is shown in Fig. 1a.

(ii) Set $\psi(\tau)=1-e^{-2\tau}$, $\alpha=1.4$, $k=0.6$, $\rho=0.8$, and $\varepsilon=0.4$. Applying Theorem 3.1, one has

$$\begin{aligned} u(\tau) &\leq \tau+\sum_{i=1}^{\infty}\frac{(\sqrt{\tau})^{i}}{0.6(0.8)^{2.33i}\Gamma_{0.6}(1.4i)} \\ &\quad\times\int_{0}^{\tau}2e^{-2s}e^{0.4167e^{-2\tau}-0.4167e^{-2s}}(-e^{-2\tau}+e^{-2s})^{2.33i-1}sds. \end{aligned}$$

Moreover, by Corollary 3.3, dynamic (k,ψ)-proportional fractional integral equation (4.1) has the following estimate upper bounded:

$$u(\tau)\leq\tau\mathbb{E}_{0.6,1.4,0.6}\left(1.68315\sqrt{\tau}e^{-0.41667+0.41667e^{-2\tau}}(1-e^{-2\tau})^{2.3333}\right).$$

The graphical representation for estimation of $u(\tau)$ in (4.1) is shown in Fig. 1b.

(iii) Set $\psi(\tau)=\tan\tau$, $\alpha=0.25$, $k=0.1$, $\rho=0.6$, and $\varepsilon=1.4$. Applying Theorem 3.1, one has

$$\begin{aligned} u(\tau) &\leq \tau+\sum_{i=1}^{\infty}\frac{(\sqrt{\tau})^{i}}{0.1(0.6)^{2.5i}\Gamma_{0.1}(0.25i)} \\ &\quad\times\int_{0}^{\tau}\sec^{2}se^{-6.67\tan\tau+6.67\tan s}(\tan\tau-\tan s)^{2.5i-1}sds. \end{aligned}$$

Moreover, by Corollary 3.3, dynamic (k,ψ)-proportional fractional integral equation (4.1) has the following estimate upper bounded:

$$u(\tau)\leq\tau\mathbb{E}_{0.1,0.25,0.1}\left(3.58610\sqrt{\tau}e^{-6.66667\tan\tau}\tan^{2.5}\tau\right).$$

The graphical representation for estimation of $u(\tau)$ in (4.1) is shown in Fig. 1c.

(iv) Set $\psi(\tau)=\ln(\tau+e)$, $\alpha=0.7$, $k=1.15$, $\rho=0.4$, and $\varepsilon=0.01$. Applying Theorem 3.1, one has

$$\begin{aligned} u(\tau) &\leq \tau+\sum_{i=1}^{\infty}\frac{(\sqrt{\tau})^{i}}{1.15(0.4)^{0.61i}\Gamma_{1.15}(0.7i)} \\ &\quad\times\int_{0}^{\tau}\left(\frac{s}{s+e}\right)e^{\left(\ln\left(\frac{\tau+e}{s+e}\right)\right)^{-1.31}}\left(\ln\left(\frac{\tau+e}{s+e}\right)\right)^{0.61i-1}ds. \end{aligned}$$

Moreover, by Corollary 3.3, dynamic (k,ψ)-proportional fractional integral equation (4.1) has the following estimate upper bounded:

$$u(\tau)\leq\tau\mathbb{E}_{1.15,0.7,1.15}\left(1.75\sqrt{\tau}e^{-1.31(\ln(\tau+e)-1)}(\ln(\tau+e)-1)^{0.61}\right).$$

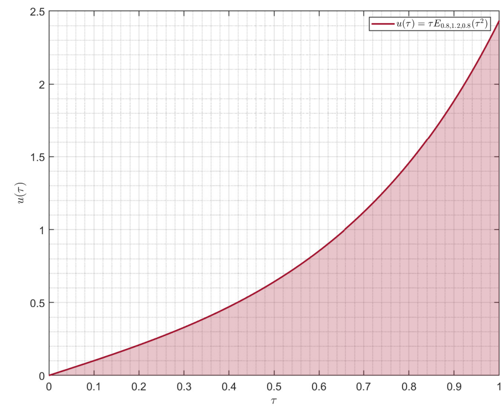
The graphical representation for estimation of $u(\tau)$ in (4.1) is shown in Fig. 1d.

Example 4.2. The dynamic integral equation governed by the (k,ψ)-PFIO can be represented:

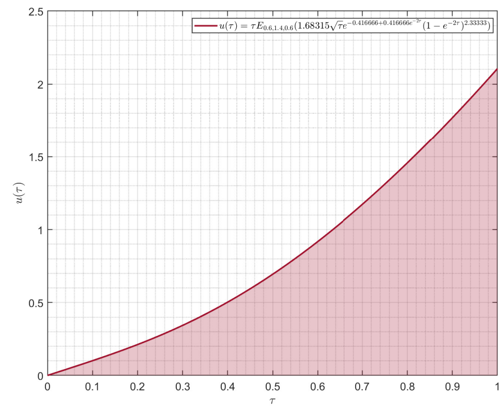
$$u^{p}(\tau)=\phi(\tau)+\frac{\eta(\tau)}{\rho^{\frac{\alpha}{k}}k\Gamma_{k}(\alpha)}\int_{a}^{\tau}{}_{k}^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)u^{q}(s)ds,\quad \tau\in[0,1). \tag{4.2}$$

Setting $\eta(\tau)=\tau$ and $\psi(\tau)=\tau$, the upper approximation of $u(\tau)$ for (4.2) is obtained that

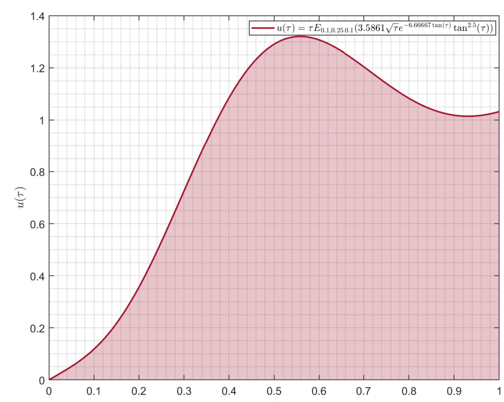
$$u^{p}(\tau)=\tau+\frac{\tau}{\rho^{\frac{\alpha}{k}}k\Gamma_{k}(\alpha)}\int_{a}^{\tau}{}_{k}^{\rho}\Psi_{\tau}^{\frac{\alpha}{k}-1}(\tau,s)u^{q}(s)ds,\quad \tau\in[0,1). \tag{4.3}$$



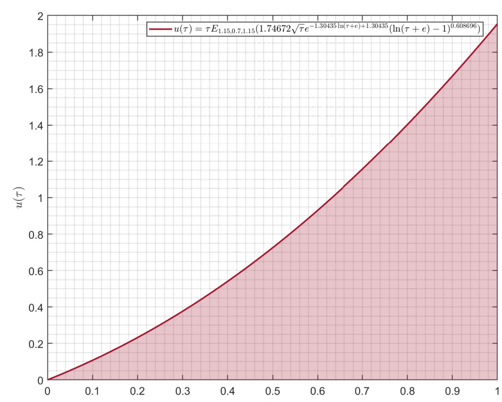
(A) $\psi(\tau)=\tau$



(B) $\psi(\tau)=1-e^{-2\tau}$



(C) $\psi(\tau)=\tan\tau$



(D) $\psi(\tau)=\ln(e+\tau)$

FIGURE 1. Graphical representation of $u(\tau)$ for Example 4.1 via various functions $\psi(\tau)=\tau$, $1-e^{-2\tau}$, $\tan\tau$, and $\ln(\tau+e)$.

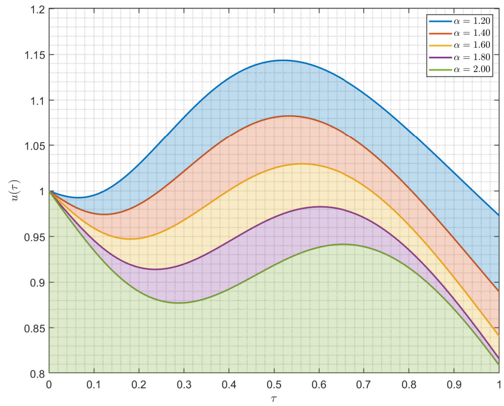
Using Corollary 3.3 in inequality (4.3), we obtain the following estimate

$$u(\tau)\leq\left(\left(\tau+\frac{(p-q)\varepsilon^{\frac{q}{p}}\tau}{p\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{}_k^\rho\Psi_\tau^{\frac{\alpha}{k}}(\tau,a)\right)\mathbb{E}_{k,\alpha,k}\left(\frac{q\varepsilon^{\frac{q-p}{p}}\tau}{p\rho^{\frac{\alpha}{k}}}{}_k^\rho\Psi_\psi^{\frac{\alpha}{k}}(\tau,a)\right)\right)^{\frac{1}{p}}. \tag{4.4}$$

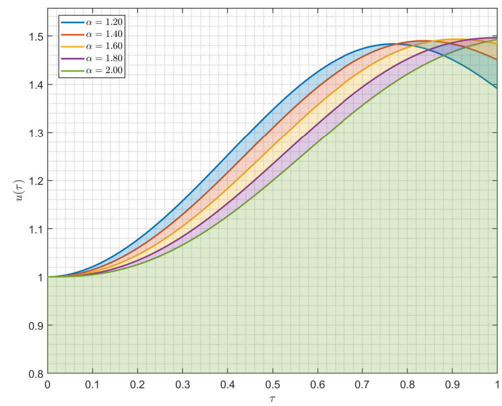
The graphical results of the inequality (4.4) via $\alpha=1.20$, 1.40, 1.60, 1.80, 2.00 with various functions $\phi(\tau)$, and constants k, ρ, p, q, ε are illustrated in Figure 2 (Fig. 2a-Fig. 2d).

Example 4.3. Consider the nonlinear initial value problem governed by the (k,ψ)-HPFDO:

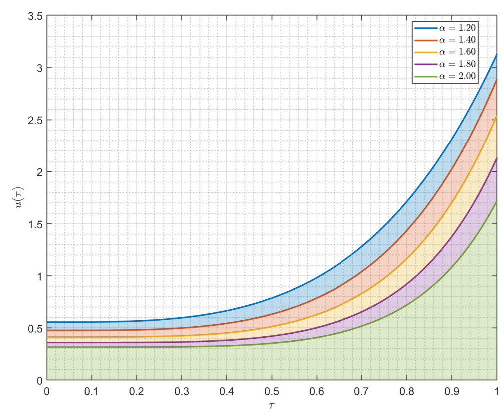
$$\begin{cases} {}_{a,k}^{H}\mathfrak{D}^{\alpha,\beta,\rho;\psi}u(\tau)=f(\tau,u(\tau)), \quad \alpha/k\in(n-1,n],\ \beta\in[0,1],\ \rho\in(0,1],\ \tau\in(a,b], \\ \lim\limits_{\tau\to a^+}{}_k\mathfrak{D}^{n-i,\rho;\psi}\left({}_{a,k}\mathscr{I}^{nk-\gamma,\rho;\psi}u(\tau)\right)=c_j, \quad c_j\in\mathbb{R},\ j=1,2,\ldots,n,\ n\in\mathbb{N},\ k>0. \end{cases} \tag{4.5}$$



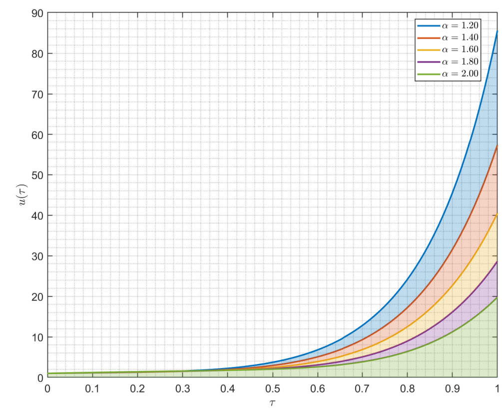
(A) $\phi(\tau)=\frac{1}{\alpha^{\tau}}$, $k=0.60$, $\rho=0.25$, $p=3$, $q=2$, $\varepsilon=1$.



(B) $\phi(\tau)=\frac{\alpha}{\tau^2+36}$, $k=1.20$, $\rho=0.45$, $p=2$, $q=1$, $\varepsilon=0.25$.



(C) $\phi(\tau)=\ln\left(\frac{2\sqrt{\tau}+\alpha}{\alpha\sqrt{\tau}+1}\right)$, $k=0.45$, $\rho=0.65$, $p=1.50$, $q=1$, $\varepsilon=1.25$.



(D) $\phi(\tau)=1+\sin(\alpha\tau+1)$, $k=0.35$, $\rho=0.85$, $p=3.50$, $q=1.50$, $\varepsilon=0.01$.

FIGURE 2. Graphical representation for the upper bound estimation of $u(\tau)$ in Example 4.2 via various functions $\psi(\tau)$, $\rho\in(0,1]$, k, p, q, $\varepsilon>0$.

which equivalent integral equation:

$$u(\tau)=\sum_{i=1}^{n}\frac{c_j\,{}_k^{\rho}\Psi_{\psi}^{\frac{\gamma}{k}-i}(\tau,a)}{\rho^{\frac{\gamma-ki}{k}}\Gamma_k(\gamma+k(1-i))}+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)f(s,u(s))\,ds. \tag{4.6}$$

If $0<\alpha\leq 1$, then ${}_{\rho}\mathscr{I}_{a^+}^{k-\gamma;\psi}u(\tau)=\mathscr{A}_1$. Note that equation (4.6) can be written as

$$u(\tau)=\frac{\mathscr{A}_1\,{}_k^{\rho}\Psi_{\psi}^{\frac{\gamma}{k}-1}(\tau,a)}{\rho^{\frac{\gamma}{k}-1}\Gamma_k(\gamma)}+\frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^{\tau}{}_k^{\rho}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)f(s,u(s))\,ds. \tag{4.7}$$

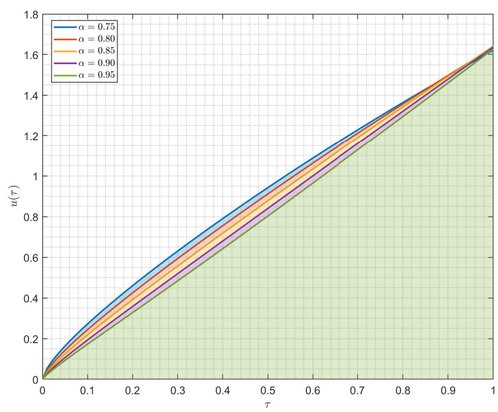
Applying Theorem 3.1 with $f(\tau,u(\tau)) = \sqrt{u(\tau)}$ and $\mathscr{A}_1 = 0$, one sees that

$$u(\tau) \leq \frac{\sqrt{\varepsilon}\,{}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(\tau,a)}{2\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} + \sum_{i=1}^{\infty}\frac{1}{(2\sqrt{\varepsilon}\rho^{\frac{\alpha}{k}})^{i}k\Gamma_k(i\alpha)}\int_a^{\tau}{}^{\rho}_{k}\Psi^{\frac{i\alpha}{k}-1}_{\psi}(\tau,s)\psi'(s)\left(\frac{\sqrt{\varepsilon}\,{}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(s,a)}{2\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\right)ds.$$

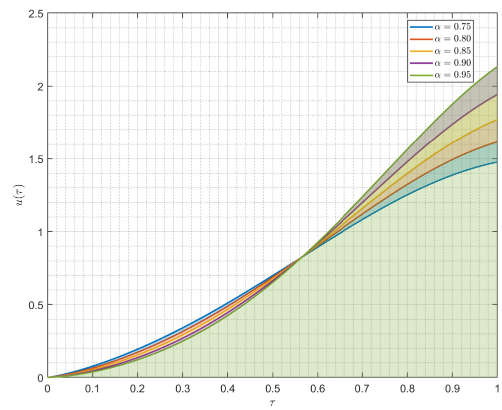
Hence, by Corollary 3.3, we have

$$u(\tau) \leq \frac{\sqrt{\varepsilon}\,{}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(\tau,a)}{2\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\mathbb{E}_{k,\alpha,k}\left(\frac{{}^{\rho}_{k}\Psi^{\frac{\alpha}{k}}_{\psi}(\tau,a)}{2\sqrt{\varepsilon}\rho^{\frac{\alpha}{k}}}\right).$$

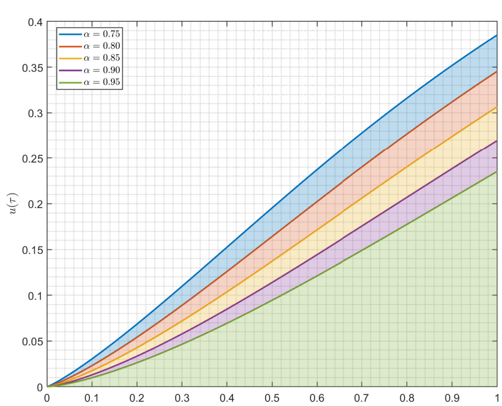
The estimated solution $u(\tau)$ of (4.7) via $\alpha = 0.75, 0.80, 0.85, 0.90, 0.95$ with various functions $\psi(\tau)$ and the constants $\rho > 0$, is presented in Figure 3 (Fig. 3a - Fig. 3d).



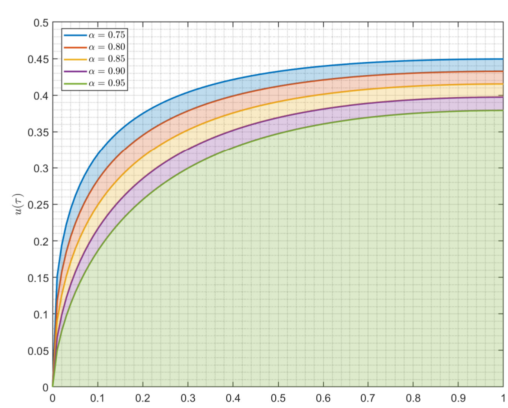
(A) $\psi(\tau) = \sqrt{\tau}$, $k = 0.55$, $\rho = 0.80$, $\varepsilon = 0.90$.



(B) $\psi(\tau) = \frac{e^{\alpha\tau}}{\alpha}$, $k = 0.60$, $\rho = 0.70$, $\varepsilon = 1.00$.



(C) $\psi(\tau) = \ln(\tau + 2\alpha)$, $k = 0.65$, $\rho = 0.60$, $\varepsilon = 1.10$.



(D) $\psi(\tau) = \frac{\sin\sqrt{\tau^{\alpha}}}{\cosh\sqrt{\tau^{\alpha}+1}}$, $k = 0.70$, $\rho = 0.50$, $\varepsilon = 1.20$.

FIGURE 3. Graphical representation for the estimated solution $u(\tau)$ in Example 4.3 via various functions $\psi(\tau)$, with parameters $\rho \in (0,1]$, k, $\varepsilon > 0$.

In addition, if one selects $f(\tau,u(\tau)) = z(u(\tau))$ into (4.7) where z is a positive, then

$$u(\tau) = \frac{\mathscr{A}_1\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-1}(\tau,a)}{\rho^{\frac{\gamma}{k}-1}\Gamma_k(\gamma)} + \frac{1}{\rho^{\frac{\alpha}{k}}k\Gamma_k(\alpha)}\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{\alpha}{k}-1}(\tau,s)\psi'(s)z(u(\tau))ds. \tag{4.8}$$

Applying Theorem 3.5 with $\eta(\tau)=1$, (4.8) can be calculated as

$$\begin{aligned} u(\tau) \ \le\ & z^{-1}\Bigg\{z\left(\frac{\mathscr{A}_1\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-1}(\tau,a)}{\rho^{\frac{\gamma}{k}-1}\Gamma_k(\gamma)}\right) + \sum_{i=1}^{\infty}\frac{((\tau-a)z(1))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)} \\ & \times\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z\left(\frac{\mathscr{A}_1\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-1}(s,a)}{\rho^{\frac{\gamma}{k}-1}\Gamma_k(\gamma)}\right)ds\Bigg\}. \end{aligned}$$

Furthermore, if we set $g(\tau)=1$ in Theorem 3.9, then

$$\begin{aligned} u(\tau) \ \le\ & z^{-1}\Bigg\{z\left(\frac{\mathscr{A}_1\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-1}(\tau,a)}{\rho^{\frac{\gamma}{k}-1}\Gamma_k(\gamma)}\right) + \sum_{i=1}^{\infty}\frac{((\tau-a)z(z(1)))^i}{\rho^{\frac{i\alpha}{k}}k\Gamma_k(i\alpha)} \\ & \times\int_a^\tau {}^{\rho}_{k}\Psi_{\psi}^{\frac{i\alpha}{k}-1}(\tau,s)\psi'(s)z\left(\frac{\mathscr{A}_1\, {}^{\rho}_{k}\Psi_{\psi}^{\frac{\gamma}{k}-1}(s,a)}{\rho^{\frac{\gamma}{k}-1}\Gamma_k(\gamma)}\right)ds\Bigg\}. \end{aligned}$$

5. Conclusion

In this paper, we formulated and analyzed a new class of extended Gronwall-Bellman-Bihari's type integral inequalities in the framework of the (k,ψ)-HPFO. The derived results provide a broader perspective that unifies and generalizes several celebrated integral inequalities. By appropriate selections of parameters and functional forms, our results yield particular cases which were previously studied and also offer some extensions beyond those earlier works. Furthermore, the theoretical results presented in this paper contribute to the qualitative analysis of the nonlinear differential equations under the (k,ψ)-HPFDO, especially in investigating solution behaviors and establishing bounds. Several illustrative examples were provided to support the main theorems. This main results obtained in this paper can be viewed as useful tools in the ongoing study of fractional systems and their dynamic properties.

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