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EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR A CAPUTO FRACTIONAL SYSTEM DEPENDING ON TWO PARAMETERS

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Abstract. This paper studies the existence and uniqueness of positive solutions for a Caputo fractional system involving with two parameters. By using a fixed point result for two increasing operators in ordered Banach spaces, some results on the existence and uniqueness of positive solutions depending on two parameters are obtained. By taking any initial point in a special set, we obtain a sequence which approximates the unique solution. Finally, a concrete example is present to validate the main conclusion.
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1. INTRODUCTION

In past decades, the theory of fractional calculus, which finds wide applications in engineering, biology, and physics, was intensively investigated by numerous authors. In particular, the existence of solutions or positive solutions for various fractional problems is now under the academic spotlight. For some recent results on the fractional equations or systems and their applications, we refer to [1]-[16] and the references therein. There are various fractional derivatives, and the Caputo fractional derivative is one of important fractional derivatives. Indeed, this derivative has been used to construct differential equations or inclusions in various fields including physics, engineering, electrochemistry, biology mathematics, and so on; see [17, 18, 19, 20, 21] and the references therein.

Recently, many authors investigated various Caputo fractional differential equations or systems. For example, in [21], Ma and Cui investigated the following problem involving with

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Caputo derivative

$$\begin{cases} {}^cD^{\sigma}p(\tau)+\mu\zeta(\tau,p(\tau))=0, \quad \tau\in[0,1],\\ p(0)=p''(0)=0,\\ p(1)=\int_0^1 p(\tau)dA(\tau), \end{cases}$$

where ${}^cD^{\sigma}$ is the Caputo fractional derivative (CFD for short) operator of order $\sigma\in(2,3)$, and $\zeta:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous, and $\mu>0$ is a parameter. By applying the Guo-Krasnoselskii fixed point theorem, the existence and non-existence results for positive solutions to this problem were proved.

In [22], Li and Chen considered a new Caputo fractional system

$$\begin{cases} {}^cD^{\theta_1}_{0_+}p(\tau)+\zeta_1(\tau,p(\tau),q(\tau))=a_1(\tau),\tau\in[0,1],\\ {}^cD^{\theta_2}_{0_+}q(\tau)+\zeta_2(\tau,p(\tau),q(\tau))=a_2(\tau),\tau\in[0,1],\\ p(0)=p''(0)=0,p(1)=\int_0^1 p(\tau)dA_1(\tau),\\ q(0)=q''(0)=0,q(1)=\int_0^1 q(\tau)dA_2(\tau), \end{cases}$$

where $\theta_i\in(2,3),i=1,2$; ${}^cD^{\theta_i}_{0_+},i=1,2$ are the CFDs; $\zeta_1,\zeta_2:[0,1]\times(-\infty,+\infty)\times(-\infty,+\infty)\to(-\infty,+\infty)$ are continuous; $a_1,a_2:[0,1]\to[0,+\infty)$ also are continuous; two functions A_1,A_2 are bounded variation with positive measure, and $B_i=\int_0^1\tau dA_i(\tau)<1,i=1,2$. This system includes two Caputo fractional equations and Riemann-Stieltjes integral conditions which is a new form. To obtain the existence and uniqueness of positive solutions for this Caputo system, they investigated a fixed-point method: fixed point theorem of increasing φ-(h,e)-concave operator by Zhai and Wang [23].

In [24], a Riemann-Liouville fractional system was studied, and this system involves p-Laplacian operators and two parameters as the follows:

$$\begin{cases} D^{\alpha_1}_{0^+}(\varphi_{p_1}(D^{\beta_1}_{0^+}p(\tau)))+\lambda\zeta(\tau,p(\tau),q(\tau))=0,\tau\in(0,1),\\ D^{\alpha_2}_{0^+}(\varphi_{p_2}(D^{\beta_2}_{0^+}q(\tau)))+\mu\chi(\tau,p(\tau),q(\tau))=0,\tau\in(0,1),\\ p(0)=p(1)=p'(0)=p'(1)=0,D^{\beta_1}_{0^+}p(0)=0,D^{\beta_1}_{0^+}p(1)=c_1D^{\beta_1}_{0^+}p(\tau_1),\\ q(0)=q(1)=q'(0)=q'(1)=0,D^{\beta_2}_{0^+}q(0)=0,D^{\beta_2}_{0^+}q(1)=c_2D^{\beta_2}_{0^+}q(\tau_2), \end{cases}$$

where $\alpha_i\in(1,2]$, $\beta_i\in(3,4]$, $D^{\alpha_i}_{0^+}$, $D^{\beta_i}_{0^+}$ are the Riemann-Liouville derivatives, $\varphi_{p_i}(s)=\mid s\mid^{p_i-2}$, $p_i>1$, $\varphi^{-1}_{p_i}=\varphi_{q_i}$, $\frac{1}{p_i}+\frac{1}{q_i}=1$, $\tau_i\in(0,1)$, $c_i\in(0,\tau_i^{\frac{(1-\alpha_i)}{(p_i-1)}}),i=1,2$, $\zeta,\chi:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$, and $\lambda,\mu>0$. The solutions of this system depend on two parameters. By utilizing a fixed point theorem of monotone operators in partial order spaces, the existence and uniqueness of positive solutions was established.

Inspired by [21, 24], we investigate a different fractional system involving with CFDs:

$$\begin{cases} -{}^cD^{\theta_1}p(t)=\lambda\zeta_1(\tau,p(\tau),q(\tau)),\tau\in[0,1],\\ -{}^cD^{\theta_2}q(\tau)=\mu\zeta_2(\tau,p(\tau),q(\tau)),\tau\in[0,1],\\ p(0)=p''(0)=0,p(1)=\int_0^1 p(\tau)dA_1(\tau),\\ q(0)=q''(0)=0,q(1)=\int_0^1 q(t)dA_2(\tau), \end{cases} \tag{1.1}$$

Where ${}^cD^{\theta_i}$ is the CFDs; $\zeta_i : [0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous; $\theta_i\in(2,3)$; A_i is bounded variation with positive measure, $B_i=\int_0^1 \tau dA_i(\tau)<1, i=1,2$; $\lambda>0$ and $\mu>0$ are parameters. We study the existence and uniqueness of positive solutions.

In [25], Li and Chen studied the case of system (1.1) when $\lambda=\mu=1$, and by using fixed-point index theory, they established the existence results of multiple positive solutions. Compared with [25], the main aim of this paper is to demonstrate the existence and uniqueness of positive solutions for the system (1.1) involving with CFDs and two parameters by the method in [26]. The rest paper is divided into several sections. Section 2 lists some necessary definitions and preliminary facts about fractional calculus. Section 3 is our main results. The existence and uniqueness of solutions for (1.1) is given, and a concrete example is present to validate the main conclusion in this section.

2. Preliminaries

This section gives some needed concepts and lemmas for the further discussion.

Let $p\in C^n[0,+\infty)$. The CFD of order $\theta>0$ is defined as (see [1])

$$ {}^cD^{\theta}p(\tau)=\frac{1}{\Gamma(n-\theta)}\int_0^{\tau}(\tau-s)^{n-\theta-1}p^{(n)}(s)ds, n-1<\theta<n. $$

Lemma 2.1. [21] *Let $q\in C^n[0,1]$ and $\theta_1,\theta_2\in(2,3)$. Then the solution p of the following equation involving with CFD*

$$ \begin{cases} {}^cD^{\theta_i}p(\tau)+q(\tau)=0, & \tau\in[0,1],\\ p(0)=p''(0)=0,\\ p(1)=\int_0^1 p(\tau)dA_i(\tau), \end{cases} $$

can be shown by the integral form $p(\tau)=\int_0^1 G_i(\tau,s)q(s)ds$, where

$$ G_i(\tau,s)=\frac{1}{\Gamma(\theta_i)}\begin{cases} \frac{\tau}{1-B_i}[(1-s)^{\theta_i-1}-\int_s^1(\tau-s)^{\theta_i-1}dA_i(\tau)]-(\tau-s)^{\theta_i-1}, & 0\le s\le\tau\le1,\\ \frac{\tau}{1-B_i}[(1-s)^{\theta_i-1}-\int_s^1(\tau-s)^{\theta_i-1}dA_i(\tau)], & 0\le\tau\le s\le1, \end{cases} \tag{2.1} $$

and $B_i=\int_0^1\tau dA_i(\tau)<1, i=1,2$.

Lemma 2.2. [22] *The Green's function $G_i(\tau,s)$, $i=1,2$, satisfies:*

(i) $G_i(\tau,s)\ge0$ and $G_i(\tau,s)$ is continuous for $\tau,s\in[0,1]$;

(ii)

$$ \frac{\tau(1-s)^{\theta_i-1}\int_0^1(\tau-\tau^{\theta_i-1})dA_i(\tau)}{\Gamma(\theta_i)(1-B_i)}\le G_i(\tau,s)\le\frac{\tau(1-s)^{\theta_i-1}}{\Gamma(\theta_i)(1-B_i)}, \tau,s\in[0,1]. $$

Now, $(E,\|\cdot\|_E)$ is a real Banach space, θ is the zero element in E, $P\subset E$ is a cone, and it induces a partial order "$\le$". For $p,q\in E$ with $\theta\le p\le q$, if $\exists N>0$ satisfies $\|p\|_E\le N\|q\|_E$, then P is called normal. Take $h_0>\theta$, define a set $P_{h_0}=\{p\in E\mid \lambda_0h_0\le p\le\mu_0h_0,\lambda_0,\mu_0>0\}$. Obviously, $P_{h_0}\subset P$. Let a vector $h_0=(h_0^{(1)},h_0^{(2)})$, and $h_0^{(1)},h_0^{(2)}\in P$ with $h_0^{(1)},h_0^{(2)}\ne\theta$. Then $h_0\in\widetilde{P}:=P\times P$. Obviously, P is normal$\Rightarrow\widetilde{P}$ is normal.

Lemma 2.3. [26] $\widetilde{P}_{h_0}=\{(p,q):p\in P_{h_0^{(1)}},q\in P_{h_0^{(2)}}\}$ *and* $\widetilde{P}_{h_0}=P_{h_0^{(1)}}\times P_{h_0^{(2)}}$.

Lemma 2.4. [26] *Let E be a Banach space, P be a normal cone and $h_0 = (h_0^{(1)}, h_0^{(2)}) \in P \times P$ with $h_0^{(1)}, h_0^{(2)} \neq \theta$. Let operators $L_1, L_2 : P \times P \to P$ be increasing and satisfy*

(C_1) For $\forall p, q \in P$, there exist $\varphi_1, \varphi_2 : (0,1) \to (0,1)$ such that

$$L_1(\iota p, \iota q) \geq \varphi_1(\iota) L_1(p,q), L_2(\iota p, \iota q) \geq \varphi_2(\iota) L_2(p,q), p, q \in P,$$

where $\varphi_i(\iota) > \iota, \iota \in (0,1), i = 1,2$;

(C_2) There exists $(e_1, e_2) \in \widetilde{P}_{h_0}$, such that $L_1(e_1, e_2) \in P_{h_0^{(1)}}$, $L_2(e_1, e_2) \in P_{h_0^{(2)}}$.

Then

(1) $L_1 : P_{h_0^{(1)}} \times P_{h_0^{(2)}} \longrightarrow P_{h_0^{(1)}}$, $L_2 : P_{h_0^{(1)}} \times P_{h_0^{(2)}} \longrightarrow P_{h_0^{(2)}}$, and $\exists p_1, q_1 \in P_{h_0^{(1)}}, p_2, q_2 \in P_{h_0^{(2)}}$, $\rho \in (0,1)$, such that $\rho(q_1, q_2) \leq (p_1, p_2) \leq (q_1, q_2)$, $p_1 \leq L_1(p_1, p_2) \leq q_1$, $p_2 \leq L_2(p_1, p_2) \leq q_2$;

(2) for $\forall \lambda, \mu > 0$, the equation $(p,q) = (\lambda L_1(p,q), \mu L_2(p,q))$ has a solution $(p^*_{\lambda,\mu}, q^*_{\lambda,\mu})$ and it is unique in $\widetilde{P}_{h_0}$.

For any fixed point $(p_0, q_0) \in \widetilde{P}_{h_0}$, If $(p_n, q_n) = (\lambda L_1(p_{n-1}, q_{n-1}), \mu L_2(p_{n-1}, q_{n-1})), n = 1, 2, \cdots,$ then $\|p_n - p^*_{\lambda,\mu}\| \to 0$ and $\|q_n - q^*_{\lambda,\mu}\| \to 0$ as $n \to \infty$.

3. Main Results

This section presents our main results in a Banach space $E = C[0,1]$ with its norm

$$\|p\| = \sup\{| p(\tau) | : \tau \in [0,1]\}.$$

Let $\|(p,q)\|_E = \max\{\|p\|, \|q\|\}$, for $(p,q) \in E \times E$. Then $(E \times E, \|(\cdot,\cdot)\|_E)$ is also a Banach space. Let $P = \{p \in E | p(\tau) \geq 0, \tau \in [0,1]\}$, a cone in E. Then

$$\widetilde{P} = \{(p,q) \in E \times E | p(\tau), q(\tau) \geq 0, \tau \in [0,1]\},$$

also is a cone. Clearly, $\widetilde{P} = P \times P \subset E \times E$ is normal. Naturally,

$$(p_1, q_1) \leq (p_2, q_2) \Longleftrightarrow p_1(\tau) \leq p_2(\tau) \,\text{and}\, q_1(\tau) \leq q_2(\tau), \tau \in [0,1].$$

Applying Lemma 2.1, the positive solution of (1.1) has the following forms

$$\begin{cases} p(\tau) = \lambda \int_0^1 G_1(\tau,s) \zeta_1(s, p(s), q(s)) ds, \\ q(\tau) = \mu \int_0^1 G_2(\tau,s) \zeta_2(s, p(s), q(s)) ds. \end{cases}$$

Theorem 3.1. *Let $\theta_i \in (2,3)$, $h_0^{(1)}(\tau) = \tau$, $h_0^{(2)}(\tau) = \tau$, and $\tau \in [0,1]$. Assume*

(H_1) $\zeta_1, \zeta_2 : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ are continuous, $\zeta_1(\tau,0,0) \not\equiv 0$, $\zeta_2(\tau,0,0) \not\equiv 0$, $\tau \in [0,1]$;

(H_2) ζ_1, ζ_2 are increasing about the second and third variables, i.e., $\zeta_1(\tau, p_1, q_1) \leq \zeta_1(\tau, p_2, q_2)$, $\zeta_2(\tau, p_1, q_1) \leq \zeta_2(\tau, p_2, q_2)$ for $\tau \in [0,1]$, $0 \leq p_1 \leq p_2, 0 \leq q_1 \leq q_2$;

(H_3) for $\forall \iota \in (0,1)$, there exists $\varphi_i(\iota) \in (0,1), i = 1,2$ such that $\varphi_i(\iota) > \iota$ and $\zeta_1(\tau, \iota p, \iota q) \geq \varphi_1(\iota) \zeta_1(\tau, p, q)$, $\zeta_2(\tau, \iota p, \iota q) \geq \varphi_2(\iota) \zeta_2(\tau, p, q)$, for $\tau \in [0,1]$, $p, q \in [0,+\infty)$;

(H_4) $0 < R = \int_0^1 (\tau - \tau^{\theta_i - 1}) dA_i(\tau) < 1, i = 1,2$, where function A_i is bounded variation.

Then,

(1) There exist $p_1, q_1 \in P_{h_0^{(1)}}$, $p_2, q_2 \in P_{h_0^{(2)}}$, $\rho \in (0,1)$ such that $\rho(q_1, q_2) \leq (p_1, p_2) \leq (q_1, q_2)$ and

$$p_1(\tau) \leq \int_0^1 G_1(\tau,s) \zeta_1(s, p_1(s), q_1(s)) ds \leq q_1(\tau), \tau \in [0,1],$$

$$p_2(\tau) \leq \int_0^1 G_2(\tau,s)\zeta_2(s,p_2(s),q_2(s))ds \leq q_2(\tau), \tau \in [0,1];$$

(2) Let $\lambda,\mu > 0$ be fixed. Then (1.1) has a unique solution $(p^*_{\lambda,\mu}, q^*_{\lambda,\mu}) \in \widetilde{P}_{h_0}$, where $h_0(\tau) = (\tau,\tau), \tau \in [0,1]$;

(3) For any $(p_0,q_0) \in \widetilde{P}_{h_0}$, set

$$p_{n+1}(\tau) = \lambda \int_0^1 G_1(\tau,s)\zeta_1(s,p_n(s),q_n(s))ds, n = 0,1,2\ldots,$$

$$q_{n+1}(\tau) = \mu \int_0^1 G_2(\tau,s)\zeta_2(s,p_n(s),q_n(s))ds, n = 0,1,2\ldots.$$

Then

$$\|p_n - p^*_{\lambda,\mu}\| \to 0, \|q_n - q^*_{\lambda,\mu}\| \to 0, n \to \infty.$$

Proof. Define operators $L_1 : P \times P \to E$, $L_2 : P \times P \to E$, and $L : P \times P \to E \times E$ by

$$L_1(p,q)(\tau) = \int_0^1 G_1(\tau,s)\zeta_1(s,p(s),q(s))ds, \tau \in [0,1],$$

$$L_2(p,q)(\tau) = \int_0^1 G_2(\tau,s)\zeta_2(s,p(s),q(s))ds, \tau \in [0,1],$$

and

$$L(p,q) = (\lambda L_1(p,q), \mu L_2(p,q)), \forall (p,q) \in P \times P,$$

where $G_1(\tau,s)$ and $G_2(\tau,s)$ are from (2.1). Obviously, $L_1 : \widetilde{P} \longrightarrow P$, $L_2 : \widetilde{P} \longrightarrow P$, and $L : \widetilde{P} \longrightarrow \widetilde{P}$. So, one can see that, if $(p,q) \in \widetilde{P}$ is a solution to (1.1), then (p,q) is a fixed point of operator L.

Firstly, we show that conclusion (1) is true.

For for any $p_i, q_i \in P$, $i = 1,2$, and $p_1 \leq p_2$ and $q_1 \leq q_2$, we have $p_1(\tau) \leq p_2(\tau)$, $q_1(\tau) \leq q_2(\tau)$, $\tau \in [0,1]$. By using (H_2) and Lemma 2.2, we have

$$\begin{aligned} L_1(p_1,q_1)(\tau) &= \int_0^1 G_1(\tau,s)\zeta_1(s,p_1(s),q_1(s))ds \\ &\leq \int_0^1 G_1(\tau,s)\zeta_1(s,p_2(s),q_2(s))ds = L_1(p_2,q_2)(\tau), \end{aligned}$$

and

$$\begin{aligned} L_2(p_1,q_1)(\tau) &= \int_0^1 G_2(\tau,s)\zeta_2(s,p_1(s),q_1(s))ds \\ &\leq \int_0^1 G_2(\tau,s)\zeta_2(s,p_2(s),q_2(s))ds = L_2(p_2,q_2)(\tau), \end{aligned}$$

Thus $L_1(p_1,q_1) \leq L_1(p_2,q_2)$ and $L_2(p_1,q_1) \leq L_2(p_2,q_2)$.

We show that L_1 and L_2 satisfy (C_1) of Lemma 2.4. For $\iota \in (0,1)$ and $\forall p,q \in P$, one has

$$\begin{aligned} L_1(\iota p, \iota q)(\tau) &= \int_0^1 G_1(\tau,s)\zeta_1(s,\iota p(s),\iota q(s))ds \\ &\geq \varphi_1(\iota)\int_0^1 G_1(\tau,s)\zeta_1(s,p(s),q(s))ds \\ &= \varphi_1(\iota)L_1(p,q)(t), \end{aligned}$$

and

$$\begin{aligned} L_2(\iota p,\iota q)(\tau) &= \int_0^1 G_2(\tau,s)\zeta_2(s,\iota p(s),\iota q(s))ds \\ &\geq \varphi_2(\iota)\int_0^1 G_2(\tau,s)\zeta_2(s,p(s),q(s))ds \\ &= \varphi_2(\iota)L_2(p,q)(\tau), \end{aligned}$$

that is, $L_1(\iota p,\iota q)\geq\varphi_1(\iota)L_1(p,q)$ and $L_2(\iota p,\iota q)\geq\varphi_2(\iota)L_2(p,q)$ for all $p,q\in P$, $\iota\in(0,1)$. Let $h_0=(h_0^{(1)},h_0^{(2)})$, $h_0^{(1)}(\tau)=\tau$, and $h_0^{(2)}(\tau)=\tau,\tau\in[0,1]$. Then $(h_0^{(1)},h_0^{(2)})\in\widetilde{P}_{h_0}$. In addition, by using (H_2) and Lemma 2.2, we can obtain

$$\begin{aligned} L_1(h_0^{(1)},h_0^{(2)})(\tau) &= \int_0^1 G_1(\tau,s)\zeta_1(s,\tau,\tau)ds \\ &\geq \int_0^1 \frac{\tau(1-s)^{\theta_1-1}\int_0^1(\tau-\tau^{\theta_1-1})dA_1(\tau)}{\Gamma(\theta_1)(1-B_1)}\zeta_1(s,0,0)ds \\ &\geq \frac{\tau}{\Gamma(\theta_1)(1-B_1)}\int_0^1 R(1-s)^{\theta_1-1}\zeta_1(s,0,0)ds \\ &= \frac{h_0^{(1)}(\tau)}{\Gamma(\theta_1)(1-B_1)}\int_0^1 R(1-s)^{\theta_1-1}\zeta_1(s,0,0)ds, \end{aligned}$$

and

$$\begin{aligned} L_1(h_0^{(1)},h_0^{(2)})(\tau) &= \int_0^1 G_1(\tau,s)\zeta_1(s,\tau,\tau)ds \\ &\leq \int_0^1 \frac{\tau(1-s)^{\theta_1-1}}{\Gamma(\theta_1)(1-B_1)}\zeta_1(s,1,1)ds \\ &= \frac{\tau}{\Gamma(\theta_1)(1-B_1)}\int_0^1 (1-s)^{\theta_1-1}\zeta_1(s,1,1)ds \\ &= \frac{h_0^{(1)}(\tau)}{\Gamma(\theta_1)(1-B_1)}\int_0^1 (1-s)^{\theta_1-1}\zeta_1(s,1,1)ds. \end{aligned}$$

Let

$$l_1=\frac{1}{\Gamma(\theta_1)(1-B_1)}\int_0^1 R(1-s)^{\theta_1-1}\zeta_1(s,0,0)ds,$$

and

$$l_2=\frac{1}{\Gamma(\theta_1)(1-B_1)}\int_0^1 (1-s)^{\theta_1-1}\zeta_1(s,1,1)ds.$$

From conditions (H_1), (H_2), and (H_4), we have

$$R(1-t)^{\theta_1-1}\zeta_1(\tau,0,0)\not\equiv 0$$

and

$$(1-\tau)^{\theta_1-1}\zeta_1(\tau,1,1)\not\equiv 0.$$

Thus $l_1,l_2>0$ with $l_1\leq l_2$, and $l_1h_0^{(1)}(\tau)\leq L_1(h_0^{(1)},h_0^{(2)})(\tau)\leq l_2h_0^{(1)}(\tau)$, $\tau\in[0,1]$, that is, $L_1(h_0^{(1)},h_0^{(2)})\in P_{h_0^{(1)}}$. Similarly, we can obtain $L_2(h_0^{(1)},h_0^{(2)})\in P_{h_0^{(2)}}$. Using Lemma 2.4, we obtain that conclusion (1) is true.

Secondly, from the above process, we see that $(p,q)=(\lambda L_1(p,q),\mu L_2(p,q))$ has a unique solution$(u^*_{\lambda,\mu},v^*_{\lambda,\mu})\in \widetilde{P}_{h_0}$, where $\lambda,\mu>0$. Thus one obtains $(p^*_{\lambda,\mu},q^*_{\lambda,\mu})=(\lambda L_1(p^*_{\lambda,\mu},q^*_{\lambda,\mu}),$ $\mu L_2(p^*_{\lambda,\mu},q^*_{\lambda,\mu}))$. System (1.1) has a positive solution $(p^*_{\lambda,\mu},q^*_{\lambda,\mu})$ and it is unique in $\widetilde{P}_{h_0}$. That is, conclusion (2) is true.

Finally, for $(p_0,q_0)\in \widetilde{P}_{h_0}$, putting

$$p_{n+1}(\tau)=\lambda\int_0^1 G_1(\tau,s)\zeta_1(s,p_n(s),q_n(s))ds, n=0,1,2\cdots,$$

and

$$q_{n+1}(\tau)=\mu\int_0^1 G_2(\tau,s)\zeta_2(s,p_n(s),q_n(s))ds, n=0,1,2\cdots,$$

we have

$$\lim_{n\to\infty}\|p_n-p^*_{\lambda,\mu}\|=\lim_{n\to\infty}\|q_n-q^*_{\lambda,\mu}\|=0.$$

That is, conclusion (3) is true. □

If $\lambda=\mu=1$, we have from Theorem 3.1 the following result.

Corollary 3.2. *Let $(H_1)-(H_4)$ be satisfied. Then the conclusion (1) of Theorem 3.1 is true and*

(i) For following system

$$\begin{cases} -^cD^{\theta_1}p(\tau)=\zeta_1(\tau,p(\tau),q(\tau)),\tau\in[0,1],\\ -^cD^{\theta_2}q(\tau)=\zeta_2(\tau,p(\tau),q(\tau)),\tau\in[0,1],\\ p(0)=p''(0)=0,p(1)=\int_0^1 p(\tau)dA_1(\tau),\\ q(0)=q''(0)=0,q(1)=\int_0^1 q(\tau)dA_2(\tau),\end{cases}$$

there exists a solution (p^*,q^*) and the solution is unique in $\widetilde{P}_{h_0}$, where $h_0(\tau)=(\tau,\tau),\tau\in[0,1]$;

(ii) For any $(p_0,q_0)\in\widetilde{P}_{h_0}$, if

$$p_{n+1}(\tau)=\int_0^1 G_1(\tau,s)\zeta_1(s,p_n(s),q_n(s))ds, n=0,1,2\dots$$

and

$$q_{n+1}(\tau)=\int_0^1 G_2(\tau,s)\zeta_2(s,p_n(s),q_n(s))ds, n=0,1,2\dots,$$

then

$$\|p_n-p^*\|\to 0,\|q_n-q^*\|\to 0,n\to\infty.$$

Example 3.3. Consider the following system:

$$\begin{cases} -^cD^{\frac{5}{2}}p(\tau)=\lambda(p^{\frac{1}{2}}+q^{\frac{1}{2}}+\tau^2),\tau\in[0,1],\\ -^cD^{\frac{5}{2}}q(\tau)=\mu(p^{\frac{1}{3}}+q^{\frac{1}{3}}+2\tau^3),\tau\in[0,1],\\ p(0)=p''(0)=0,p(1)=\int_0^1\frac{1}{2}p(\tau)d\tau,\\ q(0)=q''(0)=0,q(1)=\int_0^1\frac{1}{2}q(\tau)d\tau,\end{cases} \tag{3.1}$$

where $\zeta_1(\tau,p,q)=p^{\frac{1}{2}}+q^{\frac{1}{2}}+\tau^2$ and $\zeta_2(\tau,p,q)=p^{\frac{1}{3}}+q^{\frac{1}{3}}+2\tau^3$. Take $\theta_1=\theta_2=\frac{5}{2}$, $A_1(\tau)=A_2(\tau)=\frac{1}{2}\tau$, and $B_1=B_2=\frac{1}{4}$, $\lambda,\mu>0$. Obviously, $\zeta_1,\zeta_2:[0,1]\times[0,+\infty)\times[0,+\infty)\longrightarrow[0,+\infty)$, $\zeta_1(\tau,0,0)=\tau^2\not\equiv 0$, and $\zeta_2(\tau,0,0)=2\tau^3\not\equiv 0,\tau\in[0,1]$. Since $p^{\frac{1}{2}},p^{\frac{1}{3}},q^{\frac{1}{2}}$, and $q^{\frac{1}{3}}$ are increasing in $[0,+\infty)$, it shows that $\zeta_1(\tau,p,q)$ and $\zeta_2(\tau,p,q)$ are increasing about the second and third variables in $[0,+\infty)$.

In addition, setting $\varphi_1(\iota)=\iota^{\frac{1}{2}}$ and $\varphi_2(\iota)=\iota^{\frac{1}{3}}$, $\iota\in(0,1)$, one has $\varphi_1(\iota),\varphi_2(\iota)\in(0,1)$, $\varphi_1(\iota)=\iota^{\frac{1}{2}}>\iota,\varphi_2(\iota)=\iota^{\frac{1}{3}}>\iota$,

$$\zeta_1(\tau,\iota p,\iota q)=(\iota p)^{\frac{1}{2}}+(\iota q)^{\frac{1}{2}}+\tau^2\geq\iota^{\frac{1}{2}}(p^{\frac{1}{2}}+q^{\frac{1}{2}})+\iota^{\frac{1}{2}}\tau^2=\iota^{\frac{1}{2}}\zeta_1(\tau,p,q)=\varphi_1(\iota)\zeta_1(\tau,p,q),$$

and

$$\zeta_2(\tau,\iota p,\iota q)=(\iota p)^{\frac{1}{3}}+(\iota q)^{\frac{1}{3}}+2\tau^3\geq\iota^{\frac{1}{3}}(p^{\frac{1}{3}}+q^{\frac{1}{3}})+\iota^{\frac{1}{3}}2\tau^3=\varphi_2(\iota)\zeta_2(\tau,p,q),$$

where $\tau\in[0,1]$, $p,q\in[0,+\infty)$.

For (3.1), one sees by Theorem 3.1 that there exists a solution $(p^*_{\lambda,\mu},q^*_{\lambda,\mu})$ and it is unique in $\widetilde{P}_{h_0}$. Let $(p_0,q_0)\in\widetilde{P}_{h_0}$ and $h_0(\tau)=(\tau,\tau)$, for $\tau\in[0,1]$. Setting

$$p_{n+1}=\lambda\int_0^1 G_1(\tau,s)((p_n(s))^{\frac{1}{2}}+(q_n(s))^{\frac{1}{2}}+s^2)ds, n=0,1,2\ldots$$

and

$$q_{n+1}=\mu\int_0^1 G_2(\tau,s)((p_n(s))^{\frac{1}{3}}+(q_n(s))^{\frac{1}{3}}+2s^3)ds, n=0,1,2\ldots,$$

where

$$G_1(\tau,s)=G_2(\tau,s)=\frac{1}{\Gamma(\frac{5}{2})}\begin{cases}\frac{4}{3}\tau[(1-s)^{\frac{3}{2}}-\int_s^1\frac{1}{2}(\tau-s)^{\frac{3}{2}}d\tau]-(\tau-s)^{\frac{3}{2}}, & 0\leq s\leq\tau\leq 1,\\ \frac{4}{3}\tau[(1-s)^{\frac{3}{2}}-\int_s^1\frac{1}{2}(\tau-s)^{\frac{3}{2}}d\tau], & 0\leq\tau\leq s\leq 1,\end{cases}$$

one has

$$\lim_{n\to\infty}\|p_n-p^*_{\lambda,\mu}\|=\lim_{n\to\infty}\|q_n-q^*_{\lambda,\mu}\|=0.$$

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REFERENCES

[1] J. Jiang, H. Wang, Existence and uniqueness of solutions for a fractional differential equation with multi-point boundary value problems, J. Appl. Anal. Comput, 9 (2019) 2156-2168.

[2] M. Li, J. Wang, Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations, Appl. Math. Comput. 324 (2018) 254-265.

[3] D. Hertz, Optimal robust sparse solutions for linear equations via mixed linear (fractional) programming and rank study of complex matrices and their real counterpart, Commun. Optim. Theory 2024 (2024) 31.

[4] D. Luo, Q. Zhu, Z. Luo, A novel result on averaging principle of stochastic Hilfer-type fractional system involving non-Lipschitz coefficients, Appl. Math. Lett. 122 (2021) 107549.

[5] S. Bai, Y. Song, D.D. Repovš, Existence and multiplicity of solutions for critical Kirchhoff-Choquard equations involving the fractional p-Laplacian on the Heisenberg group, J. Nonlinear Var. Anal. 8 (2024) 143-166.

[6] B. Ahmad, S.K. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, Fract. Calc. Appl. Anal. 17 (2014) 348-360.

[7] L. Sadek, The methods of fractional backward differentiation formulas for solving two-term fractional differential Sylvester matrix equations, Appl. Set-Valued Anal. Optim. 6 (2024) 137-155.

[8] W. Yang, Positive solutions for singular Hadamard fractional differential system with four-point coupled boundary conditions. J. Appl. Math. Comput. 49(2015) 357-381.

[9] W. Wang, Unique positive solutions for boundary value problem of p-Laplacian fractional differential equation with a sign-changed nonlinearity, Nonlinear Anal. Model. Control, 27 (2022) 1110-1128.
[10] B. Ahmad, R. Luca, Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions, Appl. Math. Comput. 339 (2018) 516-534.
[11] A. Cabada, G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, J. Math. Anal. Appl, 389 (2012) 403-411.
[12] A. Ghanmi, S. Horrigue, Existence of positive solutions for a coupled system of nonlinear fractional differential equations, Ukr. Math. J. 71 (2019) 39-49.
[13] J. Wang, M. Feckan, Y. Zhou, A survey on impulsive fractional differential equations, Fract. Calc. Appl. Anal. 19 (2016) 806-831.
[14] J. Wang, M. Feckan, Y. Zhou, Fractional order differential dwitched dystems with coupled nonlocal initial and impulsive conditions, Bull. Sci. Math, 141 (2017) 727-746.
[15] J. Wang, X. Li, A uniformed method to Ulam-Hyers stability for some linear fractional equations, Mediterr. J. Math. 13 (2016) 625-635.
[16] Z. Shuang, J. Xiao, Weighted p-Laplacian parabolic equation and (p, α, β)-spectrum, Ann. Math. Sci. Appl. 9 (2024) 405-437.
[17] Agarwal R, O'regan D, Hristova S, Sun N. Stability of Caputo fractional differential equations by Lyapunov functions, Appl Math, 2015, 60:653–76.
[18] L. Li, J. Liu, A generalized definition of Caputo derivatives and its application to fractional ODEs, SIAM J Math Anal, 2018,50:2867–900.
[19] M. Kirane, B. T. Torebek, A Lyapunov-type inequality for a fractional boundary value problem with Caputo-Fabrizio derivative, J. Math. Inequ, 2018, 12(4): 1005-1012.
[20] D. Luo, T. Abdeljawad, Z. Luo, Ulam-Hyers stability results for a novel nonlinear nabla Caputo fractional variable-order difference system, Turk. J. Math, 2021, 45: 456-470.
[21] W. Ma, Y. Cui, The eigenvalue problem for Caputo type fractional differential equation with Riemann-Stieltjes integral boundary conditions, J. Funct. Spaces, 2018, 2176809.
[22] H. Li, Y. Chen, Existence and uniqueness of positive solutions for a new system of fractional differential equations, Discrete Dyn. Nat. Soc, 2020, 2617272.
[23] C. Zhai, L. Wang, $\varphi-(h,e)$-concave operators and applications, J. Math. Anal. Appl. 454(2) (2017) 571-584.
[24] C. Yang, X. Zhu, Positive solutions depending on parameters for a nonlinear fractional system with p-Laplacian operators, Adv. Math. Phys, 2020, 9563791.
[25] H. Li, Y. Chen, Multiple positive solutions for a system of nonlinear Caputo-type fractional differential equations, J. Funct. Spaces, 2020, 2437530.
[26] C. Zhai, J. Ren, Some properties of sets, fixed point theorems in ordered product spaces and applications to a nonlinear system of fractional differential equations, Topol. Methods Nonl. Anal, 2017, 49(2): 625-645.