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FIXED POINT THEOREMS FOR CRRSJ-TYPE AND ADMISSIBLE HYBRID CONTRACTIONS IN b-METRIC SPACES

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Abstract. In this paper, some fixed point theorems are proved in b-metric spaces for the map satisfying CRRSJ-type and admissible hybrid contractions. Some examples are also provided to validate of our main results.
Keywords. Admissible hybrid contraction; b-metric space; CRRSJ-type contraction; Fixed point.
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1. Introduction and Prelimnaries

Fixed points of nonlinear operators play an important role in studying various differential inclusion and equations; see, e.g., [12, 15, 17, 18] and the references therein. In 1837, Liouville [5] solved a differential equation by employing the method of successive approximation which implicitly brings a solution to a fixed point equation. In 1890, Picard [13] further developed the method of successive approximation for an initial value problem of differential equations. In 1992, Banach [2] obtained a celebrated fixed point theorem in the framework of a complete metric space. The metric concept has been generalised from variouis angles recently. One of the significant generalizations is the b-metric space.

In 1989, Bakhtin [1] introduced the notion of b-metric space as follows.

Definition 1.1. [1] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $b : X \times X \to [0, \infty)$ is called a b-metric if it satisfies the following properties, for each $x, y, z \in X$,

(1) $b(x,y) = 0 \iff x = y$;

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(2) $b(x,y) = b(y,x)$;
(3) $b(x,z) \leq s[b(x,y) + b(y,z)]$.

The pair (X,b) is called a b-metric space.

Definition 1.2. [1] Let (X,b) be a b-metric space. (X,b) is said to be complete if every Cauchy sequence is convergent in X. We say that $\{x_n\}$ is

(1) A Cauchy sequence if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n,m \geq N, b(x_n,x_m) < \varepsilon$.
(2) Convergent to $x \in X$ such that, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $m \geq N, b(x_m,x) < \varepsilon$.

In 2019, Mitrovic et al. [10] introduced the notion of hybrid contractions that integrate Reich-Rus-Ciric-type contractive and interpolative type mappings. In this paper, motivated by the work of Noorwaliand Yesilkaya [11] and Karpinar et al. [6, 7, 8, 9] that combine generalized CRRSJ type contractions and admissible hybrid contraction. We also consider examples to validate our results. The following tools, definitions and lemmas, are essential to our main fixed-point theorems.

Definition 1.3. A space is said to be ω-regular if $\{r_q\}$ is a sequence in X such that $\alpha(r_q,r_{q+1}) \geq 1$ for all $q \in \mathbb{N}$ and $r_q \to r \in X$ as $q \to \infty$, it holds that there exists a subsequence $\{r_{q(p)}\}$ of $\{r_q\}$ such that $w(r_{q(p)},r) \geq 1$ for all p.

Definition 1.4. [4] Let $s \geq 1$ be a constant. A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function with base $s \geq 1$ if

(1) φ is non-decreasing;
(2) $\lim_{n\to\infty}[\varphi^n(t)] = 0$ for all $t > 0$.

Clearly, if φ is a comparison function, then $\varphi(t) < t$ for each $t > 0$.

Lemma 1.5. [3] *If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a b-comparison function, then*

(1) $\sum_{l=0}^{\infty} s^l \varphi^l(z)$ converges for any $z \in \mathbb{R}^+$ (where $\mathbb{R}^+ = [0,\infty)$).
(2) The function $b_s : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $b_s(z) = \sum_{l=0}^{\infty} s^l \varphi^l(z)$, $z \in \mathbb{R}^+$, is increasing and continuous at 0.

Definition 1.6. [16] Let $\alpha : X \times X \to \mathbb{R}^+$ be a given function. A mapping $T : X \to X$ is said to be α-admissible if, for each $r,v \in X$, $\alpha(r,v) \geq 1 \Rightarrow \alpha(Tr,Tv) \geq 1$.

Popescu modified the definition of α-admissibility to ω-orbital admissibility as follows.

Definition 1.7. [14] Let $\omega : X \times X \to \mathbb{R}^+$ be a mapping and $X \neq \emptyset$. A map $T : X \to X$ is said to be ω-orbital admissible if, for every $r \in X$, $w(r,Tr) \geq 1 \Rightarrow w(Tr,T^2r) \geq 1$.

2. Main Results

We begin this section with the following definition.

Definition 2.1. Let(X,b,s) be a complete b-metric space and $\omega : X \times X \to [0,\infty)$ be a function. A map $T : X \to X$ is said to be a Ciric-Rus-Reich-Suzuki-Jaggi-Type generalized hybrid contraction if there exists $\psi \in \Psi$ such that

$$\frac{1}{2s} b(r,Tr) \leq b(r,v) \Rightarrow \omega(r,v) b(Tr,Tv) \leq \psi(J_T^a(r,v)),$$

for each $r, v \in X$, where $a \geq 0$ and $\sigma_i \geq 0, i = 1,2$ with $\sigma_1 + \sigma_2 = 1$ and

$$J_T^a(r,v) = \begin{cases} [\sigma_1(\frac{b(r,Tr).b(v,Tv)}{b(r,v)+1})^a + \sigma_2(b(r,v))^a]^{1/a}, \ for\ s > 0,\ \ r,\ v \in X,\ r \neq v, \\ (b(r,Tr))^{\sigma_1}(b(v,Tv))^{\sigma_2},\ for\ s = 0,\ \ r, v \in X \setminus Fix(T), \end{cases}$$

where $Fix(T) = \{x \in X\ : Tx = x\}$.

2.1. Fixed point theorem for CRRSJ type generalized hybrid contraction.

Theorem 2.2. *Let (X,b,s) be a complete b-metric space and ω-orbital admissible map $w(r_0,Tr_0) \geq 1$ for some $r_0 \in X$. Let $T\ : X \to X$ be a CRRSJ-type hybrid contraction satisfying one of the following conditions (1) (X,b,s) is ω-regular; (2) T is continuous; (3) T^2 is continuous and $\omega(r,Tr) \geq 1$, where $r \in Fix(T^2)$. Then, T has a fixed point.*

Proof. We take an iterative sequence $\{r_q\}$ such that $T^q(r_0) = r_q$ for $q = 0,1,2\ldots\ldots$ and $r_0 \in X$ with $\omega(r_0,Tr_0) \geq 1$. If $r_{q_0} = r_{q_{0+1}}$ for some integers q_0, then $r_{q_0} = Tr_{q_0}$. Suppose that $r_q \neq r_{q+1}$. As T is ω-orbital admissible, one sees that $\omega(r_0,Tr_0) = \omega(r_0,r_1) \geq 1$ implies that $\omega(r_1,Tr_1) = \omega(r_1,r_2) \geq 1$. Continuing this process, we have $\omega(r_q,r_{q+1}) \geq 1$. Here we have two conditions:

Condition 1. $a > 0$. Taking $r = r_{q-1}$ and $v = Tr_{q-1} = r_q$ yields

$$\begin{aligned} &\frac{1}{2s}b(r_{q-1},Tr_{q-1}) = \frac{1}{2s}b(r_{q-1},r_q) \leq b(r_{q-1},r_q) \\ &\Rightarrow \omega(r_{q-1},r_q)b(Tr_{q-1},Tr_q) \leq \psi(J_T^a(r_{q-1},r_q)), \end{aligned}$$

where

$$J_T^a(r_{q-1},Tr_{q-1}) = [\sigma_1(\frac{b(r_{q-1},r_q).b(r_q,r_{q+1})}{b(r_{q-1},r_q)+1})^a + \sigma_2(b(r_{q-1},r_q))^a]^{1/a}.$$

Thus

$$\begin{aligned} b(r_q,r_{q+1}) &\leq \psi(J_T^a(r_{q-1},r_q)) \\ &\leq \psi[\sigma_1(\frac{b(r_{q-1},r_q).b(r_q,r_{q+1})}{b(r_{q-1},r_q)+1})^a + \sigma_2(b(r_{q-1},r_q))^a]^{1/a} \\ &\leq \psi[\sigma_1(b(r_q,r_{q+1}))^a + \sigma_2(b\left(r_{q-1},r_q\right))^a]^{1/a}. \end{aligned}$$

Observe that ψ is nondecreasing. If $b(r_q,r_{q+1}) \geq b(r_{q-1},r_q)$, then

$$\begin{aligned} b(r_q,r_{q+1}) &\leq \psi[\sigma_1(b(r_q,r_{q+1}))^a + \sigma_2(b(r_{q-1},r_q))^a]^{1/a} \\ &< \psi[(\sigma_1+\sigma_2)(b(r_q,r_{q+1}))^a]^{1/a} \\ &< b(r_q,r_{q+1}), \end{aligned}$$

a contradiction. Thus, we obtain $b(r_q,r_{q+1}) < b(r_{q-1},r_q)$. It follows that

$$b(r_q,r_{q+1}) \leq \psi(b(r_{q-1},r_q)) < b(r_{q-1},r_q).$$

Similarly, one has $b(r_q,r_{q+1}) < \psi^q(b(r_0,r_1))$ for any $q \in \mathbb{N}$.

Now, we prove that $\{r_q\}$ is a Cauchy sequence in (X,b,s). Let $q,l\in\mathbb{N}$ such that $l>q$. Using the triangle inequality yields

$$\begin{aligned}b(r_q,r_l)&\leq sb(r_q,r_{q+1})+s^2b(r_{q+1},r_{q+2})+\cdots+s^{l-q}b(r_{l-1},r_l),\\&\leq s\psi^q(b(r_0,r_1))+s^2\psi^{q+1}(b(r_0,r_1))+\cdots+s^{l-q}\psi^{l-1}(b(r_0,r_1)),\\&\leq\frac{1}{s^{q-1}}\sum_{q=q}^{l-1}s^q\psi^q(b(r_0,r_1)).\end{aligned}$$

Using Lemma 1.4, one has that $\sum_{q=0}^{\infty}s^q\psi^q(b(r_0,r_1))$ is convergent. Thus $H_t=\sum_{q=0}^{t}s^q\psi^q(b(r_0,r_1))$ and $b(r_q,r_l)\leq\frac{1}{s^{q-1}}(H_{l-1}-H_{q-1})$. Taking $q,l\to\infty$, we obtain $b(r_q,r_l)\to 0$. Thus $\{r_q\}$ is a Cauchy sequence in X. Note that X is complete and that there exists $p\in X$ such that $\lim_{q\to\infty}b(r_q,p)\to 0$.

Next, we claim that p is a fixed point of T. If condition 1 holds, we have $\omega(r_q,p)\geq 1$, and we assert that

$$Either\ \frac{1}{2s}b(r_q,Tr_q)\leq b(r_q,p)\ or\ \frac{1}{2s}b(Tr_q,T(Tr_q))\leq b(Tr_q,p),$$

for every $q\in\mathbb{N}$. Note that $\{b(r_q,r_{q+1})\}$ is decreasing.

If $\frac{1}{2s}b(r_q,Tr_q)>b(r_q,p)$ and $\frac{1}{2s}b(Tr_q,T(Tr_q))>b(Tr_q,p)$, then

$$b(r_q,r_{q+1})\leq s(b(r_q,p)+b(p,Tr_q))<\frac{1}{2}b(r_q,r_{q+1})+\frac{1}{2}b(r_{q+1},r_{q+2})<b(r_q,r_{q+1}),$$

a contradiction. Therefore, for all $q\in\mathbb{N}$, either $\frac{1}{2s}b(r_q,Tr_q)\leq b(r_q,p)$ or $\frac{1}{2s}b(Tr_q,T(Tr_q))\leq b(Tr_q,p)$.

On the other hand, we have

$$b(r_{q+1},Tp)\leq\psi(J_T^a(r_q,p))<[\sigma_1(\frac{b(r_q,r_{q+1}).b(p,Tp)}{b(r_q,p)+1}))^a+\sigma_2(b(r_q,p))^a]^{1/a},$$

which implies that

$$\begin{aligned}b(r_{q+2},Tp)\leq\psi(J_T^a(Tr_q,p))&\leq\psi[\sigma_1(\frac{b(Tr_qT^2r_q).b(p,Tp)}{b(Tr_q,p)+1})^a+\sigma_2(b(Tr_q,p))^a]^{1/a}\\&<[\sigma_1(\frac{b(Tr_qT^2r_q).b(p,Tp)}{b(Tr_q,p)+1})^a+\sigma_2(b(Tr_q,p))^a]^{1/a}.\end{aligned}$$

By taking $q\to\infty$ in the equations above, one has p is a fixed point of T. If T ia continuous, one sees that $Tp=\lim_{q\to\infty}Tr_q=\lim_{q\to\infty}r_{q+1}=p$. For assumpition (3), we write $T^2p=\lim_{q\to\infty}T^2r_q=\lim_{q\to\infty}r_{q+2}=p$.

We now need to show that $Tp=p$. On the contrary let $p\neq Tp$. Then $\frac{1}{2s}b(Tp,T^2p)=\frac{1}{2s}b(Tp,p)\leq b(Tp,p)$. Note that

$$\begin{aligned}b(p,Tp)\leq\omega(Tp,p)b(Tp,p)&\leq\psi[\sigma_1(\frac{b(Tp,T^2p).b(p,Tp)}{b(Tp,p)+1})^a+\sigma_2(b(Tp,p))^a]^{1/a}\\&<[\sigma_1(\frac{b(Tp,T^2p).b(p,Tp)}{b(Tp,p)+1})^a+\sigma_2(b(Tp,p))^a]^{1/a}\\&=[(\sigma_1+\sigma_2)(b(p,Tp))^a]^{1/a}=b(p,Tp).\end{aligned}$$

a contradiction. Thus $p = Tp$.

Condition 2. Let $a = 0$. Note that

$$\begin{aligned}\frac{1}{2s}b(r_q, Tr_q) &\leq \omega(r_{q-1}, r_q)b(Tr_{q-1}, Tr_q) \leq \psi(J_T^a(r_{q-1}, r_q)) \\ &\leq \psi((b(r_{q-1}, r_q))_1^{\sigma}(b(r_q, r_{q+1}))^{\sigma_2}) \\ &< (b(r_{q-1}, r_q))_1^{\sigma}(b(r_q, r_{q+1}))^{\sigma_2}.\end{aligned}$$

Thus $(b(r_q, r_{q+1}))^{1-\sigma_2} < (b(r_{q-1}, r_q))^{\sigma_1}$. In view of $\sigma_1 + \sigma_2 = 1$, we see that $b(r_q, r_{q+1}) < b(r_{q-1}, r_q)$ for every $q \in \mathbb{N}$. It follows that $b(r_q, r_{q+1}) \leq \psi b(r_{q-1}, r_q)$. As Condition 1, we can show that $b(r_q, r_{q+1}) \leq \psi\ b(r_0, r_1)$. By using the same method as in the case of $a > 0$, we can say that $\{r_q\}$ is a Cauchy sequence in X, but X isg complete. Thus there exists $p \in X$ such that $\lim_{q\to\infty} b(r_q, p) = 0$.

Next, we show that $p = Tp$. Note that (X, b, s) is ω-regular. Thus $\omega(r_q, p) \geq 1$. Moreover, as in the proof of Condition 1, we see that either $\frac{1}{2s}b(r_q, Tr_q) \leq b(r_q, p)$ or $\frac{1}{2s}b(Tr_q, T(Tr_q)) \leq b(Tr_q, p)$, holds for each $q \in \mathbb{N}$. On the other hand, we have

$$\begin{aligned}b(r_{q+1}, Tp) &\leq \omega(r_q, p)b(Tr_q, Tp) \leq \psi(J_T^a(r_q, p) \\ &\leq \psi((b(r_q, r_{q+1}))^{\sigma_1}(b(p, Tp))^{\sigma_2} \\ &< (b(r_q, r_{q+1}))^{\sigma_1}(b(p, Tp))^{\sigma_2}\end{aligned}$$

and

$$\begin{aligned}b(r_{q+2}, Tp) &\leq \omega(r_{q+1}, p)b(T^2r_q, Tp) \leq \psi(J_T^a(Tr_q, p)) \\ &\leq \psi((b(r_{q+1}, r_{q+2}))^{\sigma_1}(b(p, Tp))^{\sigma_2} \\ &< (b(r_{q+1}, r_{q+2}))^{\sigma_1}(b(p, Tp))^{\sigma_2}.\end{aligned}$$

Letting $q \to \infty$, we conclude that $b(p, Tp) = 0$ and $p = Tp$. Also the continuity of T implies $p = Tp$. Now, assumption 3 leads to $T^2p = \lim_{q\to\infty} T^2r_q = \lim_{q\to\infty} r_{q+2} = p$. We next prove that $Tp = p$. On the contrary, we suppose that $p \neq Tp$. Then, $\frac{1}{2s}b(Tp, T^2p)) = \frac{1}{2s}b(Tp, p) \leq b(Tp, p)$. It follows that

$$\begin{aligned}b(p, Tp) &\leq \omega(Tp, p)b(T^2p, Tp) \leq \psi(J_T^a(Tp, p)) \\ &\leq \psi((b(Tp, T^2p))^{\sigma_1}(b(p, Tp))^{\sigma_2} \\ &< b(p, Tp),\end{aligned}$$

which is a contradiction to our assumption. Consequently, $p = Tp$. This completes the proof of the theorem. □

Now, we give an example to illustrate our theorem.

Example 2.3. Let $b : X \times X \to [0, \infty)$. Define $b(x, y) = \begin{cases} \max\{x, y\}, & x \neq y \\ 0, & x = y \end{cases}$ for every $x, y \in X$ with $s = 2$. Consider a function $\omega(x, y) = \begin{cases} 4, & if\ x, y \in [0, 1] \\ 1, & if\ x = 0, y = 2 \\ 0, & otherwise \end{cases}$ and the comparison function $\psi \in \Psi$ with $\psi(t) = t/4$.

Define a mapping $T\ : X \to X$ by $Tx = \begin{cases} \frac{1}{4}, & x \in [0,1], \\ \frac{x}{5}, & x \in (1,2]. \end{cases}$ Here T^2 is continuous, but T is not continuous, where $X = [0,2]$

(1) For $x, y \in [0,1]$, we have $b(Tx,Ty) = 0$.
(2) If $x = 0, y = 2$, then

$$\begin{aligned} \frac{1}{2s}b(0,T0) &= \frac{1}{16} < 2 = b(0,2) \\ &\Rightarrow \omega(0,2)b(T0,T2) = 0.4 < 0.503475 \\ &= \frac{1}{4}\sqrt{\frac{1}{2}(\frac{1}{6})^2 + \frac{1}{2}(2)^2} \\ &= \psi\sqrt[a]{\sigma_1(\frac{b(x,Tx).b(y,Ty)}{b(x,y)+1}) + \sigma_2(b(x,y))^a}. \end{aligned}$$

Letting $a = 2$ and $\sigma_1 = \sigma_2 = \frac{1}{2}$, we obtain that T is CRRSJ type contraction, which satisfies all the conditions of theorem. Hence $x = \frac{1}{4}$ is fixed point of T.

2.2. **The fixed point theorem for admissible hybrid contractions.** In this section, we prove a theorem for admissible hybrid contractions. We also give an example to illustrate it. First, we define admissible hybrid contractions.

Definition 2.4. Let (X,b,s) be a b-metric space, and let $\omega\ : X \times X \to [0,\infty)$ be a function. A map $T : X \to X$ is said to be a admissible hybrid contraction if there exists $\psi \in \Psi$ such that

$$\frac{1}{2s}b(r,Tr) \le b(r,v) \Rightarrow \omega(r,v)b(Tr,Tv) \le \psi(R_T^a(r,v)), \tag{2.1}$$

where $a \ge 0$ and $\lambda_i \ge 0, i = 1,2,3,4$, such that $\sum_{i=1}^4 \lambda_i = 1$ and

$$R_T^a(r,v) = \begin{cases} [\lambda_1 b^a(r,v) + \lambda_2 b^a(r,Tr) + \lambda_3 b^a(v,Tv) + \lambda_4(\frac{b(v,Tv).(1+b(r,Tr)}{b(r,v)+1})^a]^{1/a}, \\ for\ a > 0,\ r,v \in X, r \neq v \\ (b(r,v))^{\lambda_1}(b(r,Tr))^{\lambda_2}(b(v,Tv))^{\lambda_3}[\frac{b(v,Tv)(1+b(r,Tr))}{1+b(r,v)}]^{\lambda_4}, \\ for\ a = 0,\ r,v \in X \diagdown Fix(X). \end{cases}$$

Here $Fix(T) = \{x \in X\ : Tx = x\}$.

The concept of admissible hybrid contractions is inspired from the notion of interpolative contractions.

Theorem 2.5. *Let (X,b,s) be a complete b-metric space and ω-orbital admissible map $\omega(r_0,Tr_0) \ge 1$ for some $r_0 \in X$. Let $T : X \to X$ be an admissible hybrid contraction satisfying one of the following conditions:*
(h_1) (X,b,s) is ω-regular;
(h_2) T is continuous;
(h_3) T^2 is continuous and $w(r,Tr) \ge 1$, where $r \in Fix(T^2)$.
Then T has a fixed point.

Proof. We take an iterative sequence $\{r_q\}$ of points such that $T^q(r_0) = r_q$ for $q = 0,1,2\ldots\ldots$ and $r_0 \in X$ with $\omega(r_0,Tr_0) \geq 1$. If $r_{q_0} = r_{q_{0+1}}$ for some integers q_0, then $r_{q_0} = Tr_{q_0}$. Thus we suppose that $r_q \neq r_{q+1}$. As T is ω-orbital admissible, one has that $\omega(r_0,Tr_0) = \omega(r_0,r_1) \geq 1$, which implies that $\omega(r_1,Tr_1) = \omega(r_1,r_2) \geq 1$. Continuing this process, we arrive at $w(r_q,r_{q+1}) \geq 1$. Substituting $r = r_{q-1}$ and $v = Tr_{q-1} = r_q$ in equation (2.1), we have

$$\begin{aligned} \frac{1}{2s}b(r_{q-1},Tr_{q-1}) = \frac{1}{2s}b(r_{q-1},r_q) &\leq b(r_{q-1},r_q) \\ \Rightarrow \omega(r_{q-1},r_q)b(Tr_{q-1},Tr_q) &\leq \psi(R_T^a(r_{q-1},r_q)). \end{aligned}$$

Taking into account that T is ω-orbital admissible, we see that

$$b(r_q,r_{q+1}) \leq \omega(r_{q-1},r_q)b(Tr_{q-1},Tr_q) \leq \psi(R_T^a(r_{q-1},r_q)). \tag{2.2}$$

Here we have two conditions.

Condition 1: $a > 0$. Note that

$$R_T^a(r_{q-1},Tr_{q-1}) = [\lambda_1 b^a(r_{q-1},r_q) + \lambda_2 b^a(r_{q-1},r_q) + \lambda_3 b^a(r_q,r_{q+1}) + \lambda_4 b^a(r_q,r_{q+1})]^{1/a}.$$

From equation (2.1), we have

$$\begin{aligned} b(r_q,r_{q+1}) &\leq \psi(R_T^a(r_{q-1},r_q)), \\ &\leq \psi[\lambda_1 b^a(r_{q-1},r_q) + \lambda_2 b^a(r_{q-1},r_q) + \lambda_3 b^a(r_q,r_{q+1}) + \lambda_4 b^a(r_q,r_{q+1})]^{1/a} \\ &\leq \psi[(\lambda_1+\lambda_2)b^a(r_{q-1},r_q) + (\lambda_3+\lambda_4)b^a(r_q,r_{q+1})]^{1/a}. \end{aligned}$$

If $b(r_q,r_{q+1}) \geq b(r_{q-1},r_q)$, then

$$\begin{aligned} b(r_q,r_{q+1}) &\leq \psi[(\lambda_1+\lambda_2)b^a(r_q,r_{q+1}) + (\lambda_3+\lambda_4)b^a(r_q,r_{q+1})]^{1/a} \\ &< (\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{1/a} b(r_q,r_{q+1}), \\ &\leq b(r_q,r_{q+1}). \end{aligned}$$

a contradiction. Therefore, for every $q \in \mathbb{N}$, we have $b(r_q,r_{q+1}) < b(r_{q-1},r_q)$. Thus, we obtain

$$\begin{aligned} b(r_q,r_{q+1}) &\leq \psi[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{1/a} b(r_{q-1},r_q)] \\ &\leq \psi b(r_{q-1},r_q) < \psi^2 b(r_{q-2},r_{q-1}) \\ &\cdots \\ &< \psi^q(b(r_0,r_1)), \end{aligned}$$

for any $q \in \mathbb{N}$. Now, we claim that $\{r_q\}$ is a Cauchy sequence in (X,b,s). Let $q,l \in \mathbb{N}$ be such that $l > q$. Using the triangle inequality yields that

$$\begin{aligned} b(r_q,r_l) &\leq sb(r_q,r_{q+1}) + s^2 b(r_{q+1},r_{q+2}) + \cdots + s^{l-q} b(r_{l-1},r_l) \\ &\leq s\psi^q(b(r_0,r_1)) + s^2\psi^{q+1}(b(r_0,r_1)) + \cdots + s^{l-q}\psi^{l-1}(b(r_0,r_1)) \\ &\leq \frac{1}{s^{q-1}}(s^q\psi^q(b(r_0,r_1)) + s^{q+1}\psi^{q+1}(b(r_0,r_1)) + \cdots + s^{l-1}\psi^{l-1}(b(r_0,r_1))) \\ &= \frac{1}{s^{q-1}}\sum_{q=q}^{l-1} s^q\psi^q(b(r_0,r_1)). \end{aligned}$$

By using Lemma 1.5, one obtains that $\sum_{q=0}^{\infty} s^q \psi^q(b(r_0,r_1))$ is convergent. Here

$$H_t = \sum_{q=0}^{t} s^q \psi^q(b(r_0,r_1)).$$

It follows that $b(r_q,r_l) \leq \frac{1}{s^{q-1}}(H_{l-1} - H_{q-1})$. Taking $q,l \to \infty$, we obtain $b(r_q,r_l) \to 0$. Thus $\{r_q\}$ is a Cauchy sequence in X. Since X is complete, one sees that there exists $p \in X$ such that $\lim_{q\to\infty} b(r_q,p) \to 0$.

Now, we show that p is a fixed point of T. If (h_1) holds, then $\omega(r_q,p) \geq 1$ and we assert that

$$either\ \frac{1}{2s} b(r_q,Tr_q) \leq b(r_q,p)\ or\ \frac{1}{2s} b(Tr_q,T(Tr_q)) \leq b(Tr_q,p),$$

for every $q \in \mathbb{N}$. If $\frac{1}{2s} b(r_q,Tr_q) > b(r_q,p)$ and $\frac{1}{2s} b(Tr_q,T(Tr_q)) > b(Tr_q,p)$, then we find by using the condition of b-metric spaces that $\{b(r_q,r_{q+1})\}$ is decreasing. We write

$$\begin{aligned} b(r_q,r_{q+1}) = b(r_q,Tr_q) &\leq s(b(r_q,p) + b(p,Tr_q)) \\ &< \frac{1}{2} b(r_q,Tr_q) + \frac{1}{2} b(Tr_q,T(Tr_q)) \\ &= \frac{1}{2} b(r_q,r_{q+1}) + \frac{1}{2} b(r_{q+1},r_{q+2}) \\ &< b(r_q,r_{q+1}). \end{aligned}$$

which is a contradiction. Therefore, for all $q \in \mathbb{N}$, either $\frac{1}{2s} b(r_q,Tr_q) \leq b(r_q,p)$ or $\frac{1}{2s} b(Tr_q,T(Tr_q))$ $\leq b(Tr_q,p)$. Note that

$$\begin{aligned} & b(r_{q+1},Tp) \leq \psi(R_T^a(r_q,p)) \\ & \leq \psi[\lambda_1 b^a(r_q,p) + \lambda_2 b^a(r_q,r_{q+1}) + \lambda_3 b^a(p,Tp) + \lambda_4 (\frac{b(p,Tp).(1+b(r_q,r_{q+1})}{(1+b(r_q,p)})^a]^{1/a} \\ & < (\lambda_3 + \lambda_4)^{1/a} b(p,Tp), \end{aligned}$$

which yields that

$$\begin{aligned} & b(r_{q+2},Tp) \leq \psi(R_T^a\left(Tr_q,p\right)) \\ & \leq \psi[\lambda_1 b^a(r_{q+1},p) + \lambda_2 b^a(r_{q+1},r_{q+2})\ + \lambda_3 b^a(p,Tp) + \lambda_4 (\frac{b(p,Tp).(1+b(r_{q+1},r_{q+2})}{(1+b(r_{q+1},p)}^a]^{1/a} \\ & < (\lambda_3 + \lambda_4)^{1/a} b\left(p,Tp\right). \end{aligned}$$

Letting $q \to \infty$ in the two equations above, we have

$$b(p,Tp) < (\lambda_3 + \lambda_4)^{1/a} b(p,Tp) \leq b(p,Tp),$$

a contradiction. Therefore, $b(p,Tp) = 0$, that is, $p = Tp$. If assumption (h_2) holds, that is, T is continuous, then $Tp = \lim_{q\to\infty} Tr_q = \lim_{q\to\infty} r_{q+1} = p$. In case that (h_3) holds, we write

$$T^2 p = \lim_{q\to\infty} T^2 r_q = \lim_{q\to\infty} r_{q+2} = p.$$

We need to show that $Tp = p$. On the contrary let $p \neq Tp$. From equation (2.1), we have

$$\begin{aligned}
& b(p,Tp) \leq \omega(Tp,p)b(Tp,p) \\
& \leq \psi\big[\lambda_1 b^a(Tp,p)+\lambda_2 b^a(Tp,T^2p)+\lambda_3 b^a(p,Tp)+\lambda_4(\frac{b(p,Tp).(1+b(Tp,T^2p)}{(1+b(Tp,p)})^a\big]^{1/a} \\
& < \psi[\lambda_1 b^a(Tp,p)+\lambda_2 b^a(Tp,p)+\lambda_3 b^a(Tp,p)+\lambda_4 b^a(Tp,p)]^{1/a} \\
& < [(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^{1/a} b^a(Tp,p)] \\
& \leq b(p,Tp),
\end{aligned}$$

a contradiction. Thus $p = Tp$.

Condition 2. $a = 0$. Taking$r = r_q$ and $v = Tr_q$ in equation (2.2), we have

$$\begin{aligned}
b(r_q,r_{q+1}) & \leq \omega(r_{q-1},r_q)b(Tr_{q-1},Tr_q) \leq \psi(R_T^a(r_{q-1},r_q)) \\
& \leq \psi[(b(r_{q-1},r_q))^{\lambda_1}(b(r_{q-1},r_q))^{\lambda_2}(b(r_q,r_{q+1}))^{\lambda_3}[\frac{b(r_q,r_{q+1})(1+b(r_{q-1},r_q))}{1+b(r_{q-1},r_q)}]^{\lambda_4}].
\end{aligned}$$

If $b(r_q,r_{q+1}) \geq b(r_{q-1},r_q)$, then $b(r_q,r_{q+1}) \leq \psi(b(r_q,r_{q+1}))^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} < b(r_q,r_{q+1})$, a contradiction. Thus, we have $b(r_q,r_{q+1}) < b(r_{q-1},r_q)$. On the other hand, we have

$$b(r_q,r_{q+1}) \leq \psi(b(r_{q-1},r_q))^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \leq \psi(b(r_{q-1},r_q)) < (b(r_{q-1},r_q)).$$

Similarly, we obtain $(b(r_q,r_{q+1})) \leq \psi^q(b(r_0,r_1)$. By using the same method as the case of $a > 0$, we can say that $\{r_q\}$ is a Cauchy sequence in X. Since X is complete, one sees that there exists $p \in X$ such that $\lim_{q\to\infty} b(r_q,p) = 0$.

Next, we claim that $p = Tp$. Note that (X,b,s) is w-regular and $\omega(r_q,r_{q+1}) \geq 1$ for each $q \in \mathbb{N}$. We obtain $\omega(r_q,p) \geq 1$. Moreover, as in the proof of Condition 1, we see that either $\frac{1}{2s}b(r_q,Tr_q) \leq b(r_q,p)$ or $\frac{1}{2s}b(Tr_q,T(Tr_q)) \leq b(Tr_q,p)$, holds for each $q \in \mathbb{N}$. Thus

$$\begin{aligned}
b(r_{q+1},Tp) & \leq \omega(r_q,p)b(Tr_q,Tp) \\
& \leq \psi(R_T^a(r_q,p)) \\
& \leq \psi[(b(r_q,p))^{\lambda_1}(b(r_q,r_{q+1}))^{\lambda_2}(b(p,Tp))^{\lambda_3}[\frac{b(p,Tp)(1+b(r_q,r_{q+1}))}{1+b(r_q,p)}]^{\lambda_4}].
\end{aligned}$$

Note that

$$\begin{aligned}
& b(r_{q+2},Tp) \leq \omega(r_{q+1},p)b(T^2r_q,Tp) \\
& \leq \ \psi(J_T^a(Tr_q,p)) \\
& \leq \psi(b(r_{q+1},p))^{\lambda_1}(b(r_{q+1},r_{q+2}))^{\lambda_2}(b(p,Tp))^{\lambda_3}[\frac{b(p,Tp)(1+b(r_{q+1},r_{q+2}))}{1+b(r_{q+1},p)}]^{\lambda_4}.
\end{aligned}$$

Letting $q \to \infty$, we conclude that $b(p,Tp) = 0$ and $p = Tp$. Now, the continuity of T implies $p = Tp$. Therefore, (h_3) leads to $T^2p = \lim_{q\to\infty} T^2r_q = \lim_{q\to\infty} r_{q+2} = p$. We next prove that

$Tp = p$. On the contrary, let us suppose that $p \neq Tp$. Using equation (2.1), we find that

$$\begin{aligned}
b(p,Tp &\leq \omega(Tp,p)b(T^2p,Tp) \leq \psi(R_T^a(Tp,p)) \\
&\leq \psi[(b(Tp,p))^{\lambda_1}(b(Tp,T^2p))^{\lambda_2}(b(p,Tp))^{\lambda_3}[\frac{b(p,Tp)(1+b(Tp,T^2p))}{1+b(Tp,p)}]^{\lambda_4}] \\
&\leq \psi[(b(Tp,p))^{\lambda_1}(b(Tp,p))^{\lambda_2}(b(p,Tp))^{\lambda_3}[\frac{b(p,Tp)(1+b(Tp,p))}{1+b(Tp,p)}]^{\lambda_4}] \\
&\leq \psi[(b(Tp,p))]^{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\
&< b(p,Tp),
\end{aligned}$$

which is a contradiction to our assumption. Consequently, $p = Tp$. This completes the proof of the theorem. ☐

Example 2.6. Let $X = [0,2]$, $b\ : X \times X \to [0,\infty)$ be the usual metric, $b(x,y) = |x-y|$ for all $x,y \in X$,and the mapping $T : X \to X$ be defined by $T(x) = \begin{cases} \frac{2}{3}, & x \in [0,1], \\ \frac{x}{2}, & x \in (1,2]. \end{cases}$ Consider a function

$$\omega(x,y) = \begin{cases} 2, & x,y \in [0,1], \\ 1, & x = 0, y = 2, \\ 0, & otherwise, \end{cases}$$

and the comparison function $\psi \in \Psi$ with $\psi(t) = \frac{t}{2}$. Here $T^2(x) = \frac{2}{3}$ is continuous, but T is not continuous, where $X = [0,2]$. We choose $a = 2$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$. Then

(1) For $x,y \in [0,1]$, $b(Tx,Ty) = 0$.
(2) If $x = 0, y = 2$, then

$$\begin{aligned}
\omega(0,2)b(T0,T2) = \frac{1}{3} < \frac{1}{2}\sqrt{\frac{466}{324}} &= \frac{1}{2}\sqrt{\frac{1}{4}(4+\frac{4}{9}+1+\frac{25}{81}} \\
&= \frac{1}{2}[\frac{1}{4}b^2(0,2)+\frac{1}{4}b^2(0,T0)+\frac{1}{4}b^2(2,T2)+\frac{1}{4}(\frac{b(2,T2)(1+b(0,T0)t)}{1+b(0,2)})^2]^{\frac{1}{2}}.
\end{aligned}$$

Therefore, equation (2.1) is satisfied. Thus, taking $a = 2$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/4$, we obtain T is an admissible hybrid contraction which satisfies all the conditions of theorem. Hence $x = \frac{2}{3}$ is a fixed point of T.

3. Conclusion

In this paper, we established two fixed point results in the setting of CRRJ type and admissible hybrid contractions in the setting of b-metric space. Both the results were supported by examples. It will be an open problem to generalise the results in the more generalised metric and metric like spaces such as generalised metric space, b-metric space, bipolar metric spaces, and so on.

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