



SOLVABILITY OF NONLINEAR INTEGRAL EQUATIONS VIA COMMON FIXED POINTS OF GENERALIZED (α, φ, ψ) -MEIR-KEELER HYBRID CONTRACTIVE MAPPINGS IN PARTIAL b -METRIC SPACES

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Abstract. In this paper, we first introduce the concepts of generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I and II via simulation functions by triangular (B, α) -admissible and composite α -admissible mapping pairs. Next, we study the sufficient conditions for the existence of unique common fixed point theorems for such mappings in the setting of complete partial b -metric spaces. Then we also provide two examples to show the applicability and validity of our results. Finally, we present an application to the existence of solutions to an integral equation by means of one of our results.

Keywords. Common fixed point; Generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings; Partial b -metric; Weakly contractive mapping.

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1. INTRODUCTION

The Banach fixed point theorem [6], which is known as Banach contraction principle, is a rewarding result in fixed point theory. It has widespread applications in both pure and applied mathematics and been extended in various different directions. One of the most popular and interesting topics among them is the study of new classes of spaces and their fundamental properties.

In 2014, Shukla [24] introduced the concept of partial b -metric spaces and proved some fixed point theorems of contractive mappings in partial b -metric space. After that, some authors have researched on the fixed point theorems of various new type of contractive conditions in partial b -metric space. Pant et al. [21] obtained a fixed point theorem which is the first Meir-Keeler type solution. In [25], Choudhury and Chaitali proved a Meir-Keeler type coupled fixed point results in metric spaces. Aoyama and Toyoda [4] provided characterizations of a Meir-Keeler type mapping and a fixed point theorem for the mapping in a metric space endowed with

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a transitive relation. Aydi et al. [5] considered some Nemytzki-Edelstein-Meir-Keeler type results and Debnath [9] presented some new set-valued Meir-Keeler, Geraghty and Edelstein type fixed point theorems in b -metric spaces. Azam and Shagari [2] proved a Meir-Keeler type common fixed point theorem for soft set-valued map and point-to-point mapping. The authors [12, 14] introduced α -Meir-Keeler and generalized α -Meir-Keeler contractions on Branciari b -metric spaces. Gholamian and Khanehgir [15] introduced the notion of generalized Meir-Keeler contraction mappings in the setting of b -metric-like spaces.

In 2022, Karapinar et al. [13] reviewed Meir-Keeler contraction mappings results on various abstract spaces. Karapinar et al. [3] presented a fixed point theorem for generalized (α, ψ) -Meir-Keeler type contractions in the setting of generalized b -metric spaces. Mamud and Tola [17] introduced the notion of generalized (α, φ) -Meir-Keeler hybrid contractive mappings of type I and II via simulation function and established fixed point theorems for such mappings in the setting of complete b -metric spaces. Jain et al. [10] introduced $\xi - (\alpha, \beta)$ -contractive mappings in b -metric spaces. In [7], Cvetković discussed the relation between F -contractions and Meir-Keeler contractions in complete metric space. Salvador Romaguera and Pedro Tirado [22] studied quasi-metric versions of the famous Meir-Keeler fixed point theorem from which they deduced quasi-metric generalizations of Boyd-Wong's fixed point theorem. Saiedinezhad [23] established the Boyd-Wong type and Meir-Keeler type contractions in new generalized b -metric spaces. For recent development on fixed point theory, we refer to [1, 8, 11, 16, 18, 19, 20].

In this paper, motivated and inspired by [17, Theorems 4] and [20, Theorem 26], using triangular (B, α) -admissible mapping and composite α -admissible mapping pairs, we introduce the notions of generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I and II via simulation functions and studied the sufficient conditions for the existence of unique common fixed point theorems for such mappings in partial b -metric spaces. Two examples are provided to support our results. Finally, we give an application to the existence of solutions to an integral equation by means of one of our results.

2. PRELIMINARIES

Firstly, we recall some definitions and lemmas in partial b -metric spaces.

Definition 2.1. ([24]) Let $\mathbb{G}$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $p : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ is said to be a partial b -metric if, for all $\kappa, \tau, v \in \mathbb{G}$, $s \geq 1$, the following conditions are satisfied:

- (i) $p(\kappa, v) \geq 0$, $\kappa = \tau$ if and only if $p(\kappa, \tau) = p(\kappa, \kappa) = p(\tau, \tau)$;
- (ii) $p(\kappa, \kappa) \leq p(\kappa, \tau)$;
- (iii) $p(\kappa, \tau) = p(\tau, \kappa)$;
- (iv) $p(\kappa, \tau) \leq s(p(\kappa, v) + p(\tau, v)) - p(v, v)$.

The pair $(\mathbb{G}, p)$ is called a partial b -metric space. It is clear that the class of partial b -metric spaces is larger than that of partial metric spaces.

Remark 2.2. Every partial b -metric p defines b -metric p^s , where $p^s(\kappa, \tau) = 2p(\kappa, \tau) - p(\kappa, \kappa) - p(\tau, \tau)$.

Definition 2.3. ([19]) Let $(\mathbb{G}, p)$ be a partial b -metric space with parameter $s \geq 1$. Then a sequence $\{\kappa_n\}$ in $\mathbb{G}$ is said to be:

- (i) convergent if and only if there exists $\kappa \in \mathbb{G}$ such that $\lim_{n \rightarrow \infty} p(\kappa_n, \kappa) = p(\kappa, \kappa)$;

(ii) a Cauchy sequence if and only if $\lim_{n,m \rightarrow \infty} p(\kappa_n, \kappa_m)$ exists (and is finite).

A partial b -metric space is called complete if each Cauchy sequence in this space is convergent to a point $\kappa \in \mathbb{G}$ such that $\lim_{n,m \rightarrow \infty} p(\kappa_n, \kappa_m) = \lim_{n \rightarrow \infty} p(\kappa_n, \kappa) = p(\kappa, \kappa)$.

Lemma 2.4. ([19]) *A sequence $\{\kappa_n\}$ is a Cauchy sequence in a partial b -metric space $(\mathbb{G}, p)$ if and only if it is a b -Cauchy sequence in the b -metric space $(\mathbb{G}, p^s)$.*

Lemma 2.5. ([19]) *A partial b -metric space $(\mathbb{G}, p)$ is p_b -complete if and only if the b -metric space $(\mathbb{G}, p^s)$ is b -complete. Moreover, $\lim_{n,m \rightarrow \infty} p^s(\kappa_n, \kappa_m) = 0$ if and only if*

$$\lim_{n,m \rightarrow \infty} p(\kappa_n, \kappa_m) = \lim_{n \rightarrow \infty} p(\kappa_n, \kappa) = p(\kappa, \kappa).$$

Definition 2.6. Let A and B be two self-mappings on a nonempty set $\mathbb{G}$. Let $C(A, B)$ denote the set of all coincidence points of A and B . If $w = A\kappa = B\kappa$, for some $\kappa \in \mathbb{G}$, then κ is said to be the coincidence point of A and B , where w is called the point of coincidence of A and B .

Definition 2.7. ([16]) Let f and g be two self-mappings defined on a nonempty set $\mathbb{G}$. Then f and g is said to be weakly compatible if they commute at every coincidence point, that is, $fg\kappa = gf\kappa$ for every $\kappa \in C(f, g)$.

We also need the following lemmas to obtain our main results.

Lemma 2.8. ([19]) *Let $(\mathbb{G}, p)$ be a partial b -metric space with parameter $s \geq 1$. Assume that $\{\kappa_n\}$ and $\{\tau_n\}$ are convergent to κ and τ , respectively. Then,*

$$\begin{aligned} & \frac{1}{s^2} p(\kappa, \tau) - \frac{1}{s} p(\kappa, \kappa) - p(\tau, \tau) \\ & \leq \liminf_{n \rightarrow \infty} p(\kappa_n, \tau_n) \leq \limsup_{n \rightarrow \infty} p(\kappa_n, \tau_n) \leq sp(\kappa, \kappa) + s^2 p(\tau, \tau) + s^2 p(\kappa, \tau). \end{aligned}$$

In particular, if $p(\kappa, \tau) = 0$, then we have $\lim_{n \rightarrow \infty} p(\kappa_n, \tau_n) = 0$. Moreover, for each $v \in \mathbb{G}$,

$$\frac{1}{s} p(\kappa, v) - p(\kappa, \kappa) \leq \liminf_{n \rightarrow \infty} p(\kappa_n, v) \leq \limsup_{n \rightarrow \infty} p(\kappa_n, v) \leq sp(\kappa, v) + sp(\kappa, \kappa).$$

Furthermore, if $p(\kappa, \kappa) = 0$, then $\frac{1}{s} p(\kappa, v) \leq \liminf_{n \rightarrow \infty} p(\kappa_n, v) \leq \limsup_{n \rightarrow \infty} p(\kappa_n, v) \leq sp(\kappa, v)$.

Lemma 2.9. ([8]) *Let $(\mathbb{G}, p)$ be a partial b -metric space with parameter $s \geq 1$ and $\{\kappa_n\}$ be a sequence in $\mathbb{G}$ such that $\{p(\kappa_n, \kappa_{n+1})\}$ is non-increasing and $\lim_{n \rightarrow \infty} p(\kappa_n, \kappa_{n+1}) = 0$. If $\{\kappa_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k > k$ and four following sequences:*

$$p(\kappa_{2m_k}, \kappa_{2n_k}), p(\kappa_{2m_k}, \kappa_{2n_k+1}), p(\kappa_{2m_k-1}, \kappa_{2n_k}), p(\kappa_{2m_k-1}, \kappa_{2n_k+1})$$

satisfy

$$\begin{aligned} \varepsilon & \leq \liminf_{k \rightarrow \infty} p(\kappa_{2m_k}, \kappa_{2n_k}) \leq \limsup_{k \rightarrow \infty} p(\kappa_{2m_k}, \kappa_{2n_k}) \leq s\varepsilon, \\ \frac{\varepsilon}{s} & \leq \liminf_{k \rightarrow \infty} p(\kappa_{2m_k}, \kappa_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} p(\kappa_{2m_k}, \kappa_{2n_k+1}) \leq s^2\varepsilon, \\ \frac{\varepsilon}{s} & \leq \liminf_{k \rightarrow \infty} p(\kappa_{2m_k-1}, \kappa_{2n_k}) \leq \limsup_{k \rightarrow \infty} p(\kappa_{2m_k-1}, \kappa_{2n_k}) \leq s\varepsilon, \end{aligned}$$

and

$$\frac{\varepsilon}{s^2} \leq \liminf_{k \rightarrow \infty} p(\kappa_{2m_k-1}, \kappa_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} p(\kappa_{2m_k-1}, \kappa_{2n_k+1}) \leq s^2\varepsilon.$$

3. MAIN RESULTS

In this section, we establish some results for the existence of common fixed points of generalized weakly contractive mappings in the setting of complete partial b -metric spaces.

Let Φ denote the class of the functions $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (1) φ is non-decreasing,
- (2) φ is continuous,
- (3) $\varphi(t) = 0$ if and only if $t = 0$.

Let Ψ denote the class of the functions $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the following conditions:

- (1) $\psi(t) \leq \varphi(t)$ for all $t \in \mathbb{R}^+$,
- (2) ψ is upper semi-continuous.

Let Ξ denote the class of the functions $\xi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the following conditions:

- (1) $\xi(0, 0) = 0$,
- (2) $\xi(t, s) < s - t$ for all $t, s > 0$.

Let $(\mathbb{G}, p)$ be a partial b -metric space with $s \geq 1$ and A, B be two self-mappings on $\mathbb{G}$. We define a mapping $M : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^+$ by

$$M(\kappa, \tau) = \max\left\{p(B\kappa, B\tau), p(A\kappa, B\kappa), p(A\tau, B\tau), \frac{p(A\tau, B\kappa) + p(A\kappa, B\tau)}{2s}, \right. \\ \left. \frac{p(A\kappa, B\kappa)p(A\tau, B\tau)}{1 + p(A\kappa, A\tau)}, \frac{p(A\kappa, B\kappa)p(A\tau, B\tau)}{1 + p(B\kappa, B\tau)}, \right. \\ \left. \frac{1 + p(A\kappa, B\tau) + p(B\kappa, A\tau)}{1 + sp(A\kappa, B\kappa) + sp(A\tau, B\tau)}p(A\kappa, B\kappa)\right\}.$$

Let $p : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^+$ be a mapping. The following conditions are used in this section:

- (P_M^1) $\kappa \neq \tau$ and $p(A\kappa, B\kappa) \leq p(B\kappa, B\tau)$ imply $p(\kappa, \tau) \leq M(\kappa, \tau)$;
- (P_M^2) $\kappa_{2m_k} \neq \kappa_{2n_k+1}$ and $\lim_{k \rightarrow \infty} \sup p(A\kappa_{2m_k}, B\kappa_{2m_k}) \leq \lim_{k \rightarrow \infty} \sup p(B\kappa_{2m_k}, B\kappa_{2n_k+1})$ imply $p(\kappa_{2m_k}, \kappa_{2n_k+1}) \leq M(\kappa_{2m_k}, \kappa_{2n_k+1})$.

Definition 3.1. Let $(\mathbb{G}, p)$ be a partial b -metric space with parameter $s \geq 1$, and let $A, B : \mathbb{G} \rightarrow \mathbb{G}$, $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ be given mappings. The mapping A is said to be triangular (B, α) -admissible if, for all $\kappa, \tau, \nu \in \mathbb{G}$, the following conditions are satisfied:

- (1) $\alpha(B\kappa, B\tau) \geq s^2$ implies $\alpha(A\kappa, A\tau) \geq s^2$;
- (2) $\alpha(\kappa, \tau) = \alpha(\tau, \kappa)$;
- (3) $\alpha(\kappa, \tau) \geq s^2$ and $\alpha(\tau, \nu) \geq s^2$ imply $\alpha(\kappa, \nu) \geq s^2$.

Remark 3.2. For $p(\kappa, \kappa) = 0, \forall \kappa \in \mathbb{G}$, the definition reduces to the definition of a triangular (B, α) -admissible mapping in a partial metric space.

Let $(\mathbb{G}, p)$ be a p_b -complete partial b -metric space with parameter $s \geq 1$ and let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$. Then,

- (a) If $\{A\kappa_{n-1}\}$ is a sequence in $\mathbb{G}$ such that $A\kappa_{n-1} \rightarrow z$ as $n \rightarrow \infty$, then there exists a subsequence $\{A\kappa_{n_i-1}\}$ of $\{A\kappa_{n-1}\}$ with $\alpha(A\kappa_{n_i-1}, z) \geq s^2$ for all $i \in \mathbb{N}$.
- (b) If $\{A\kappa_{n-1}\}$ is a sequence in $\mathbb{G}$ such that $A\kappa_{n-1} \rightarrow z$ as $n \rightarrow \infty$, then there exists a subsequence $\{A\kappa_{n_k-1}\}$ of $\{A\kappa_{n-1}\}$ with $\alpha(A\kappa_{n_k-1}, Az) \geq s^2$ for all $k \in \mathbb{N}$.
- (c) For all $z, w \in C(A, B)$, $\alpha(z, w) \geq s^2$.

Definition 3.3. Let A, B be two self-mappings on a partial b -metric space $(\mathbb{G}, d)$ and $\xi \in \Xi$. Suppose that $p : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^+$ is a function that satisfies (P_M^1) and (P_M^2) . Then A, B are said to

be the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I if the following conditions are satisfied:

- (i) for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\kappa \neq \tau$ and $p(\kappa, \tau) < \varepsilon + \delta(\varepsilon)$ imply $p(A\kappa, A\tau) \leq \frac{\varepsilon}{s}$;
- (ii) $\kappa \neq \tau$ and $p(\kappa, \tau) > 0$ imply $\xi(\varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)), \psi(p(\kappa, \tau))) \geq 0$, where $\varphi \in \Phi$, $\psi \in \Psi$, $\xi \in \Xi$ and $q > 1$ is a constant.

Remark 3.4. If A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I, $p(A\kappa, A\tau) > 0$, then $\varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)) < \psi(p(\kappa, \tau)) \leq \varphi(p(\kappa, \tau)) \leq \varphi(M(\kappa, \tau))$.

Proof. If $p(\kappa, \tau) = 0$, from Definition 3.3, we have $p(\kappa, \tau) < \varepsilon + \delta(\varepsilon)$ for any $\varepsilon > 0$, which implies $0 \leq p(A\kappa, A\tau) \leq \frac{\varepsilon}{s}$. But $\varepsilon > 0$ is arbitrary. Thus $p(A\kappa, A\tau) = 0$. We consider $p(\kappa, \tau) > 0$, and $p(A\kappa, A\tau) > 0$. In this case, according to the nature of the function ξ and Definition 3.3, we have

$$\begin{aligned} 0 &\leq \xi(\varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)), \psi(p(\kappa, \tau))) \\ &< \psi(p(\kappa, \tau)) - \varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)). \end{aligned}$$

This completes the proof. $\square$

Theorem 3.5. Let $(\mathbb{G}, p)$ be a complete partial b -metric space with parameter $s \geq 1$ and let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$, $A, B : \mathbb{G} \rightarrow \mathbb{G}$ be given mappings and $A(\mathbb{G}) \subseteq B(\mathbb{G})$. Suppose $\varphi \in \Phi$, $\psi \in \Psi$, $\xi \in \Xi$ and $q > 1$. If the following conditions are satisfied:

- (i) A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I,
 - (ii) A is a triangular (B, α) -admissible mapping,
 - (iii) there is $\kappa_0 \in \mathbb{G}$ with $\alpha(A\kappa_0, B\kappa_0) \geq s^2$,
 - (iv) properties (a), (b) and (c) are satisfied,
 - (v) $B(\mathbb{G})$ is complete and (A, B) is weakly compatible,
- then A, B have a unique common fixed point.

Proof. By (iii), there exists $\kappa_0 \in \mathbb{G}$ such that $\alpha(A\kappa_0, B\kappa_0) \geq s^2$. Define two sequences $\{\kappa_n\}$ and $\{\tau_n\}$ in $\mathbb{G}$ by $\tau_n = A\kappa_n = B\kappa_{n+1}$ for all $n \in \mathbb{N}$. If $p(\tau_n, \tau_{n+1}) = 0$ which implies $\tau_n = \tau_{n+1}$ for some $n \in \mathbb{N}$, then $\tau_n = B\kappa_{n+1} = A\kappa_{n+1} = \tau_{n+1}$ and it is easy to see that A and B have a point of coincidence. Without loss of generality, we assume that $p(\tau_n, \tau_{n+1}) > 0$ for all $n \in \mathbb{N}$. By the condition (ii), we have

$$\begin{aligned} \alpha(B\kappa_0, B\kappa_1) &= \alpha(B\kappa_0, A\kappa_0) \geq s^2, \\ \alpha(B\kappa_1, B\kappa_2) &= \alpha(A\kappa_0, A\kappa_1) \geq s^2, \end{aligned}$$

and

$$\alpha(B\kappa_2, B\kappa_3) = \alpha(A\kappa_1, A\kappa_2) \geq s^2.$$

Therefore, by induction, we obtain $\alpha(B\kappa_{n+1}, B\kappa_{n+2}) = \alpha(\tau_n, \tau_{n+1}) \geq s^2$ for all $n \in \mathbb{N}$. It follows from Definition 3.1 that $\alpha(B\kappa_{n+1}, B\kappa_{m+1}) = \alpha(\tau_n, \tau_m) \geq s^2$ for all $n, m \in \mathbb{N}$.

Now, we show that $\{p(\tau_n, \tau_{n+1})\}$ is monotone decreasing. Since

$$p(\tau_{n+1}, \tau_n) = p(A\kappa_{n+1}, B\kappa_{n+1}) = p(B\kappa_{n+1}, B\kappa_{n+2}),$$

we have

$$p(\kappa_{n+1}, \kappa_{n+2}) \leq M(\kappa_{n+1}, \kappa_{n+2})$$

by (P_M^1) , where

$$\begin{aligned}
& M(\kappa_{n+1}, \kappa_{n+2}) \\
&= \max\{p(B\kappa_{n+1}, B\kappa_{n+2}), p(A\kappa_{n+1}, B\kappa_{n+1}), p(A\kappa_{n+2}, B\kappa_{n+2}), \\
&\quad \frac{p(A\kappa_{n+2}, B\kappa_{n+1}) + p(A\kappa_{n+1}, B\kappa_{n+2})}{2s}, \\
&\quad \frac{p(A\kappa_{n+1}, B\kappa_{n+1})p(A\kappa_{n+2}, B\kappa_{n+2})}{1 + p(A\kappa_{n+1}, A\kappa_{n+2})}, \frac{p(A\kappa_{n+1}, B\kappa_{n+1})p(A\kappa_{n+2}, B\kappa_{n+2})}{1 + p(B\kappa_{n+1}, B\kappa_{n+2})}, \\
&\quad \frac{1 + p(A\kappa_{n+1}, B\kappa_{n+2}) + p(B\kappa_{n+1}, A\kappa_{n+2})}{1 + sp(A\kappa_{n+1}, B\kappa_{n+1}) + sp(A\kappa_{n+2}, B\kappa_{n+2})} p(A\kappa_{n+1}, B\kappa_{n+1})\}, \\
&= \max\{p(\tau_n, \tau_{n+1}), p(\tau_{n+1}, \tau_n), p(\tau_{n+2}, \tau_{n+1}), \frac{p(\tau_{n+2}, \tau_n) + p(\tau_{n+1}, \tau_{n+1})}{2s}, \\
&\quad \frac{p(\tau_{n+1}, \tau_n)p(\tau_{n+2}, \tau_{n+1})}{1 + p(\tau_{n+1}, \tau_{n+2})}, \frac{p(\tau_{n+1}, \tau_n)p(\tau_{n+2}, \tau_{n+1})}{1 + p(\tau_{n+1}, \tau_n)}, \\
&\quad \frac{1 + p(\tau_{n+1}, \tau_{n+1}) + p(\tau_n, \tau_{n+2})}{1 + sp(\tau_{n+1}, \tau_n) + sp(\tau_{n+2}, \tau_{n+1})} p(\tau_{n+1}, \tau_n)\} = \max\{p(\tau_n, \tau_{n+1}), p(\tau_{n+1}, \tau_{n+2})\}.
\end{aligned}$$

By (i) and Remark 3.4, we see that

$$\begin{aligned}
\varphi(p(\tau_{n+1}, \tau_{n+2})) &= \varphi(p(A\kappa_{n+1}, A\kappa_{n+2})) \\
&\leq \varphi((\alpha(B\kappa_{n+1}, B\kappa_{n+2}) + s^{-q})p(A\kappa_{n+1}, A\kappa_{n+2})) \\
&< \psi(p(\kappa_{n+1}, \kappa_{n+2})) \leq \varphi(M(\kappa_{n+1}, \kappa_{n+2})).
\end{aligned} \tag{3.1}$$

If $M(\kappa_{n+1}, \kappa_{n+1}) = p(\tau_{n+1}, \tau_{n+2})$, we find by (3.1) that $\varphi(p(\tau_{n+1}, \tau_{n+2})) \leq \varphi(p(\tau_{n+1}, \tau_{n+2}))$, which yields a contradiction. So, $M(\kappa_{n+1}, \kappa_{n+2}) = p(\tau_n, \tau_{n+1})$. We deduce

$$\varphi(p(\tau_{n+1}, \tau_{n+2})) < \varphi(p(\tau_n, \tau_{n+1})).$$

Since φ is non-decreasing, we see that $p(\tau_{n+1}, \tau_{n+2}) < p(\tau_{n+1}, \tau_n)$. Therefore, $\{p(\tau_n, \tau_{n+1})\}$ is a decreasing sequence, and $\lim_{n \rightarrow \infty} p(\tau_n, \tau_{n+1}) = p \geq 0$. Let $\varepsilon = p > 0$. One can choose a N . When $n > N$, there exists $p = \varepsilon < p(\tau_n, \tau_{n+1}) < \varepsilon + \delta(\varepsilon)$, and it follows that $p(\kappa_{n+1}, \kappa_{n+2}) \leq M(\kappa_{n+1}, \kappa_{n+2}) = p(\tau_n, \tau_{n+1}) < \varepsilon + \delta(\varepsilon)$, which implies $p(A\kappa_{n+1}, A\kappa_{n+2}) = p(\tau_{n+1}, \tau_{n+2}) \leq \frac{\varepsilon}{s}$, that is, $\varepsilon < p(\tau_{n+1}, \tau_{n+2}) \leq \frac{\varepsilon}{s}$. This is a contradiction. It follows that $\lim_{n \rightarrow \infty} p(\tau_n, \tau_{n+1}) = p = 0$.

We next prove that $\{\tau_n\}$ is a Cauchy sequence in the partial b -metric space $(\mathbb{G}, p)$. Suppose that is not true. Then, using Lemma 2.9, we see that exists $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k > k$ and the four following sequences

$$p(\tau_{2m_k}, \tau_{2n_k}), p(\tau_{2m_k}, \tau_{2n_k+1}), p(\tau_{2m_k-1}, \tau_{2n_k}), p(\tau_{2m_k-1}, \tau_{2n_k+1})$$

satisfy

$$\begin{aligned}
\varepsilon &\leq \liminf_{k \rightarrow \infty} p(\tau_{2m_k}, \tau_{2n_k}) \leq \limsup_{k \rightarrow \infty} p(\tau_{2m_k}, \tau_{2n_k}) \leq s\varepsilon, \\
\frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} p(\tau_{2m_k}, \tau_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} p(\tau_{2m_k}, \tau_{2n_k+1}) \leq s^2\varepsilon, \\
\frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} p(\tau_{2m_k-1}, \tau_{2n_k}) \leq \limsup_{k \rightarrow \infty} p(\tau_{2m_k-1}, \tau_{2n_k}) \leq s\varepsilon,
\end{aligned}$$

and

$$\frac{\varepsilon}{s^2} \leq \liminf_{k \rightarrow \infty} p(\tau_{2m_k-1}, \tau_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} p(\tau_{2m_k-1}, \tau_{2n_k+1}) \leq s^2\varepsilon.$$

Let $\kappa = \kappa_{2m_k}$, $\tau = \kappa_{2n_k+1}$ and $\kappa \neq \tau$. One can deduce that $p(A\kappa_{2m_k}, B\kappa_{2m_k}) = p(\tau_{2m_k}, \tau_{2m_k-1})$ and $p(B\kappa_{2m_k}, B\kappa_{2n_k+1}) = p(\tau_{2m_k-1}, \tau_{2n_k})$. It follows that

$$\limsup_{k \rightarrow \infty} p(A\kappa_{2m_k}, B\kappa_{2m_k}) \leq \limsup_{k \rightarrow \infty} p(B\kappa_{2m_k}, B\kappa_{2n_k+1})$$

which implies $p(\kappa_{2m_k}, \kappa_{2n_k+1}) \leq M(\kappa_{2m_k}, \kappa_{2n_k+1})$ by (P_M^2) , where

$$\begin{aligned} M(\kappa_{2m_k}, \kappa_{2n_k+1}) &= \max\{p(B\kappa_{2m_k}, B\kappa_{2n_k+1}), p(A\kappa_{2m_k}, B\kappa_{2m_k}), p(A\kappa_{2n_k+1}, B\kappa_{2n_k+1}), \\ &\quad \frac{p(A\kappa_{2n_k+1}, B\kappa_{2m_k}) + p(A\kappa_{2m_k}, B\kappa_{2n_k+1})}{2s}, \\ &\quad \frac{p(A\kappa_{2m_k}, B\kappa_{2m_k})p(A\kappa_{2n_k+1}, B\kappa_{2n_k+1})}{1 + p(A\kappa_{2m_k}, A\kappa_{2n_k+1})}, \\ &\quad \frac{p(A\kappa_{2m_k}, B\kappa_{2m_k})p(A\kappa_{2n_k+1}, B\kappa_{2n_k+1})}{1 + p(B\kappa_{2m_k}, B\kappa_{2n_k+1})}, \\ &\quad \frac{1 + p(A\kappa_{2m_k}, B\kappa_{2n_k+1}) + p(B\kappa_{2m_k}, A\kappa_{2n_k+1})}{1 + sp(A\kappa_{2m_k}, B\kappa_{2m_k}) + sp(A\kappa_{2n_k+1}, B\kappa_{2n_k+1})} p(A\kappa_{2m_k}, B\kappa_{2m_k})\}, \\ &= \max\{p(\tau_{2m_k-1}, \tau_{2n_k}), p(\tau_{2m_k}, \tau_{2m_k-1}), \\ &\quad p(\tau_{2n_k+1}, \tau_{2n_k}), \frac{p(\tau_{2n_k+1}, \tau_{2m_k-1}) + p(\tau_{2m_k}, \tau_{2n_k})}{2s}, \\ &\quad \frac{p(\tau_{2m_k}, \tau_{2m_k-1})p(\tau_{2n_k+1}, \tau_{2n_k})}{1 + p(\tau_{2m_k}, \tau_{2n_k+1})}, \frac{p(\tau_{2m_k}, \tau_{2m_k-1})p(\tau_{2n_k+1}, \tau_{2n_k})}{1 + p(\tau_{2m_k-1}, \tau_{2n_k})}, \\ &\quad \frac{1 + p(\tau_{2m_k}, \tau_{2n_k}) + p(\tau_{2m_k-1}, \tau_{2n_k+1})}{1 + sp(\tau_{2m_k}, \tau_{2m_k-1}) + sp(\tau_{2n_k+1}, \tau_{2n_k})} p(\tau_{2m_k}, \tau_{2m_k-1})\}. \end{aligned}$$

Taking the upper limit as $k \rightarrow \infty$, one can obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varphi(M(\kappa_{2m_k}, \kappa_{2n_k+1})) &= \varphi(\limsup_{k \rightarrow \infty} M(\kappa_{2m_k}, \kappa_{2n_k+1})) \\ &\leq \varphi(\max\{s\varepsilon, 0, 0, \frac{s^2\varepsilon + s\varepsilon}{2s}, 0, 0, 0\}) = \varphi(s\varepsilon). \end{aligned}$$

In view of Remark 3.4 and $\alpha(B\kappa_{2m_k}, B\kappa_{2n_k+1}) = \alpha(\tau_{2m_k-1}, \tau_{2n_k}) \geq s^2$, we arrive at

$$\begin{aligned} &\varphi((s^2 + s^{-q})p(\tau_{2m_k}, \tau_{2n_k+1})) \\ &= \varphi((s^2 + s^{-q})p(A\kappa_{2m_k}, A\kappa_{2n_k+1})) \\ &\leq \varphi((\alpha(B\kappa_{2m_k}, B\kappa_{2n_k+1}) + s^{-q})p(A\kappa_{2m_k}, A\kappa_{2n_k+1})) \\ &< \psi(p(\kappa_{2m_k}, \kappa_{2n_k+1})) \leq \varphi(M(\kappa_{2m_k}, \kappa_{2n_k+1})), \end{aligned}$$

where

$$\varphi(s\varepsilon + s^{-q-1}\varepsilon) \leq \limsup_{k \rightarrow \infty} \varphi((s^2 + s^{-q})p(\tau_{2m_k}, \tau_{2n_k+1})) = \varphi(\limsup_{k \rightarrow \infty} (s^2 + s^{-q})p(\tau_{2m_k}, \tau_{2n_k+1})).$$

Letting $k \rightarrow \infty$, we obtain $\varphi(s\varepsilon + s^{-q-1}\varepsilon) \leq \varphi(s\varepsilon)$, which is impossible. This shows that $\{\tau_{2n}\}$ is a Cauchy sequence and hence $\{\tau_n\}$ is a Cauchy sequence both in $(\mathbb{G}, p)$ and in $(\mathbb{G}, p^s)$. Since $(\mathbb{G}, p)$ is complete if and only if $(\mathbb{G}, p^s)$ is b -complete, and there exists $z \in \mathbb{G}$ such that $\lim_{n, m \rightarrow \infty} p(\tau_n, \tau_m) = \lim_{n \rightarrow \infty} p(\tau_n, z) = p(z, z)$, or equivalently $\lim_{n, m \rightarrow \infty} p^s(\tau_n, \tau_m) = 0$. Moreover, in view of $p^s(\tau_n, \tau_m) = 2p(\tau_n, \tau_m) - p(\tau_n, \tau_n) - p(\tau_m, \tau_m)$ and $\lim_{n \rightarrow \infty} p(\tau_n, \tau_{n+1}) = 0$, we have $\lim_{n, m \rightarrow \infty} p(\tau_n, \tau_m) = 0$. It follows that $\lim_{n, m \rightarrow \infty} p(\tau_n, \tau_m) = \lim_{n \rightarrow \infty} p(\tau_n, z) = p(z, z) = 0$.

Since $\tau_n \rightarrow z$, $\{A\kappa_n\}$ and $\{B\kappa_{n+1}\}$ converge to z . Assume that $B(\mathbb{G})$ is complete. Then there exists $u \in \mathbb{G}$ such that $z = Bu$. We next show that $z = Au$. On the contrary, we assume that $p(z, Au) > 0$. First of all, from condition (a), we can choose a subsequence $\{\kappa_{n_i-1}\}$ of $\{\kappa_{n-1}\}$ such that $\{A\kappa_{n_i-1}\} \rightarrow z$ with $\alpha(B\kappa_{n_i}, z) = \alpha(A\kappa_{n_i-1}, z) \geq s^2$. Letting $\kappa = \kappa_{n_i}$, $\tau = u$ and $\kappa \neq \tau$, we have $\lim_{i \rightarrow \infty} \sup p(A\kappa_{n_i}, B\kappa_{n_i}) = 0 = \lim_{i \rightarrow \infty} \sup p(B\kappa_{n_i}, Bu)$, which implies $p(\kappa_{n_i}, u) \leq M(\kappa_{n_i}, u)$ by (P_M^2) , where

$$\begin{aligned}
M(\kappa_{n_i}, u) &= \max\{p(B\kappa_{n_i}, Bu), p(A\kappa_{n_i}, B\kappa_{n_i}), p(Au, Bu), \frac{p(Au, B\kappa_{n_i}) + p(A\kappa_{n_i}, Bu)}{2s}, \\
&\quad \frac{p(A\kappa_{n_i}, B\kappa_{n_i})p(Au, Bu)}{1 + p(A\kappa_{n_i}, Au)}, \frac{p(A\kappa_{n_i}, B\kappa_{n_i})p(Au, Bu)}{1 + p(B\kappa_{n_i}, Bu)}, \\
&\quad \frac{1 + p(A\kappa_{n_i}, Bu) + p(B\kappa_{n_i}, Au)}{1 + sp(A\kappa_{n_i}, B\kappa_{n_i}) + sp(Au, Bu)}p(A\kappa_{n_i}, B\kappa_{n_i})\}, \\
&= \max\{p(\tau_{n_i-1}, z), p(\tau_{n_i}, \tau_{n_i-1}), p(Au, z), \frac{p(Au, \tau_{n_i-1}) + p(\tau_{n_i}, z)}{2s}, \\
&\quad \frac{p(\tau_{n_i}, \tau_{n_i-1})p(Au, z)}{1 + p(\tau_{n_i}, Au)}, \frac{p(\tau_{n_i}, \tau_{n_i-1})p(Au, z)}{1 + p(\tau_{n_i-1}, z)}, \\
&\quad \frac{1 + p(\tau_{n_i}, z) + p(\tau_{n_i-1}, Au)}{1 + sp(\tau_{n_i}, \tau_{n_i-1}) + sp(Au, z)}p(\tau_{n_i}, \tau_{n_i-1})\}.
\end{aligned} \tag{3.2}$$

According to Remark 3.4, we have

$$\begin{aligned}
\varphi((s^2 + s^{-q})p(A\kappa_{n_i}, Au)) &\leq \varphi((\alpha(B\kappa_{n_i}, Bu) + s^{-q})p(A\kappa_{n_i}, Au)) \\
&< \psi(p(\kappa_{n_i}, u)) \leq \varphi(M(\kappa_{n_i}, u)).
\end{aligned} \tag{3.3}$$

Taking the upper limit as $i \rightarrow \infty$ in (3.2) and (3.3), we obtain

$$\begin{aligned}
\limsup_{i \rightarrow \infty} \varphi(M(\kappa_{n_i}, u)) &= \varphi(\limsup_{i \rightarrow \infty} M(\kappa_{n_i}, u)) \\
&\leq \varphi(\max\{0, 0, p(Au, z), \frac{sp(Au, z)}{2s}, 0, 0, 0\}) = \varphi(p(Au, z)),
\end{aligned}$$

$$\begin{aligned}
\varphi((s + s^{-q-1})p(z, Au)) &\leq \limsup_{i \rightarrow \infty} \varphi((s^2 + s^{-q})p(A\kappa_{n_i}, Au)) \\
&= \varphi(\limsup_{i \rightarrow \infty} (s^2 + s^{-q})p(A\kappa_{n_i}, Au)),
\end{aligned}$$

and

$$\varphi((s + s^{-q-1})p(Au, z)) \leq \varphi(p(Au, z)),$$

which is a contradiction. Therefore, $p(z, Au) = 0$ implies $Au = z = Bu$.

Since (A, B) is weakly compatible, one has $Az = ABu = BAu = Bz$. Next, we prove that $Az = z$. On the contrary, we assume that $p(z, Az) > 0$. Similarly, by condition (b), we can also obtain a subsequence $\{\kappa_{n_k-1}\}$ of $\{\kappa_{n-1}\}$ such that $\{A\kappa_{n_k-1}\} \rightarrow z$ with $\alpha(B\kappa_{n_k}, Bz) = \alpha(A\kappa_{n_k-1}, Az) \geq s^2$. Letting $\kappa = \kappa_{n_k}$, $\tau = z$ and $\kappa \neq \tau$, one can deduce

$$\limsup_{k \rightarrow \infty} p(A\kappa_{n_k}, B\kappa_{n_k}) = 0 \leq \limsup_{k \rightarrow \infty} p(B\kappa_{n_k}, Bz).$$

It follows that $p(\kappa_{n_k}, z) \leq M(\kappa_{n_k}, z)$ by (P_M^2) , where

$$\begin{aligned}
 M(\kappa_{n_k}, z) &= \max\{p(B\kappa_{n_k}, Bz), p(A\kappa_{n_k}, B\kappa_{n_k}), p(Az, Bz), \frac{p(Az, B\kappa_{n_k}) + p(A\kappa_{n_k}, Bz)}{2s}, \\
 &\quad \frac{p(A\kappa_{n_k}, B\kappa_{n_k})p(Az, Bz)}{1 + p(A\kappa_{n_k}, Az)}, \frac{p(A\kappa_{n_k}, B\kappa_{n_k})p(Az, Bz)}{1 + p(B\kappa_{n_k}, Bz)}, \\
 &\quad \frac{1 + p(A\kappa_{n_k}, Bz) + p(B\kappa_{n_k}, Az)}{1 + sp(A\kappa_{n_k}, B\kappa_{n_k}) + sp(Az, Bz)} p(A\kappa_{n_k}, B\kappa_{n_k})\}, \\
 &= \max\{p(\tau_{n_k-1}, Az), p(\tau_{n_k}, \tau_{n_k-1}), p(Az, Az), \frac{p(Az, \tau_{n_k-1}) + p(\tau_{n_k}, Az)}{2s}, \\
 &\quad \frac{p(\tau_{n_k}, \tau_{n_k-1})p(Az, Az)}{1 + p(\tau_{n_k}, Az)}, \frac{p(\tau_{n_k}, \tau_{n_k-1})p(Az, Az)}{1 + p(\tau_{n_k-1}, Bz)}, \\
 &\quad \frac{1 + p(\tau_{n_k}, Az) + p(\tau_{n_k-1}, Az)}{1 + sp(\tau_{n_k}, \tau_{n_k-1}) + sp(Az, Az)} p(\tau_{n_k}, \tau_{n_k-1})\}.
 \end{aligned} \tag{3.4}$$

According to Remark 3.4 again, we have

$$\begin{aligned}
 \varphi((s^2 + s^{-q})p(A\kappa_{n_k}, Az)) &\leq \varphi((\alpha(B\kappa_{n_k}, Bz) + s^{-q})p(A\kappa_{n_k}, Az)) \\
 &< \psi(p(\kappa_{n_k}, z)) \leq \varphi(M(\kappa_{n_k}, z)).
 \end{aligned} \tag{3.5}$$

Letting $k \rightarrow \infty$ in (3.4) and (3.5), we obtain

$$\varphi((s + s^{-q-1})p(z, Az)) \leq \limsup_{k \rightarrow \infty} \varphi((s^2 + s^{-q})p(A\kappa_{n_k}, Az)),$$

$$\begin{aligned}
 \limsup_{k \rightarrow \infty} \varphi(M(\kappa_{n_k}, z)) &= \varphi(\limsup_{k \rightarrow \infty} M(\kappa_{n_k}, z)) \\
 &\leq \varphi(\max\{sp(Az, z), 0, p(Az, Az), \frac{sp(Az, z) + sp(z, Az)}{2s}, 0, 0, 0\}) \\
 &= \varphi(sp(Az, z)),
 \end{aligned}$$

and

$$\varphi((s + s^{-q-1})p(Az, z)) \leq \varphi(sp(Az, z)),$$

a contradiction. Hence, $p(z, Az) = 0$ and this implies $Az = z = Bz$. That is, A, B have common fixed point z .

Suppose on the contrary that there exists another common fixed point w of A and B . By condition (c), we can also get $\alpha(w, z) \geq s^2$. Letting $\kappa = z$, $\tau = w$ and $\kappa \neq \tau$, we have $p(Az, Bz) = 0 \leq p(Az, Aw) = p(z, w)$, which implies $p(z, w) \leq M(z, w)$ by (P_M^1) , where $M(z, w) = p(z, w)$. Meanwhile, in light of Remark 3.4, we obtain

$$\begin{aligned}
 \varphi((s^2 + s^{-q})p(z, w)) &= \varphi((s^2 + s^{-q})p(Az, Aw)) \\
 &\leq \varphi((\alpha(Bz, Bw) + s^{-q})p(Az, Aw)) \\
 &< \psi(p(z, w)) \leq \varphi(M(z, w)) \\
 &< \varphi(p(z, w)),
 \end{aligned}$$

which is a contradiction. Therefore, $p(z, w) = 0$ implies $z = w$, and A, B have a unique common fixed point. $\square$

Example 3.6. Let $\mathbb{G} = [0, 2]$, $p(\kappa, \tau) = \max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2$, $s = 2$ and $q > 1$. Define mappings A, B by

$$A\kappa = \begin{cases} \frac{\kappa}{16}, & \kappa \in [0, 1] \\ \frac{\kappa}{8}, & \kappa \in (1, 2] \end{cases}, B\kappa = \begin{cases} \frac{\kappa}{2}, & \kappa \in [0, 1] \\ \frac{\kappa}{4} + \frac{1}{4}, & \kappa \in (1, 2] \end{cases}.$$

Since $AB0 = BA0 = 0$, (A, B) is weakly compatible. It is clear that $A(\mathbb{G}) \subseteq B(\mathbb{G})$ and $B(\mathbb{G})$ is complete. Define mappings $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ and $\alpha(\kappa, \tau) = \alpha(\tau, \kappa)$ by

$$\alpha(\kappa, \tau) = \begin{cases} s^2, & \kappa, \tau \in [0, \frac{1}{2}] \\ 0, & \text{otherwise} \end{cases},$$

$\varphi(t) = \psi(t) = t$, and

$$p(\kappa, \tau) = \max\{p(B\kappa, B\tau), p(A\tau, B\tau), \frac{p(A\tau, B\kappa) + p(A\kappa, B\tau)}{2s}, \frac{p(A\kappa, B\kappa)p(A\tau, B\tau)}{1 + p(A\kappa, A\tau)}, \frac{1 + p(A\kappa, B\tau) + p(B\kappa, A\tau)}{1 + sp(A\kappa, B\kappa) + sp(A\tau, B\tau)}p(A\kappa, B\kappa)\},$$

First, we note that $p(\kappa, \tau)$ satisfies condition (P_M^1) and $p(\kappa, \tau) > 0$ for all $\kappa \neq \tau$. For $\kappa, \tau \in \mathbb{G}$ such that $\alpha(B\kappa, B\tau) \geq s^2$, we see that $B\kappa, B\tau \in [0, \frac{1}{2}]$, which implies that $\kappa, \tau \in [0, 1]$. It follows that $\alpha(A\kappa, A\tau) \geq s^2$, since $A\kappa, B\tau \in [0, \frac{1}{2}]$, that is, A is a triangular (B, α) -admissible mapping. For $\kappa, \tau \in [0, 1]$, we choose $\delta(\varepsilon) = 15\varepsilon$. Then

$$\begin{aligned} 32p(A\kappa, A\tau) &\leq \frac{32}{128} \max\{\kappa^2, \tau^2\} = \frac{1}{4} \max\{\kappa^2, \tau^2\} \\ &\leq p(B\kappa, B\tau) \leq p(\kappa, \tau) < \varepsilon + 15\varepsilon, \end{aligned}$$

which implies $p(A\kappa, A\tau) < \frac{\varepsilon}{2}$. For $\kappa \in [0, 1]$, $\tau \in (1, 2]$, letting $\delta(\varepsilon) = \frac{1}{2}\varepsilon$, we have

$$3p(A\kappa, A\tau) \leq \frac{6}{64} \tau^2 \leq \frac{\tau^2}{16} + \frac{\tau}{8} + \frac{1}{16} \leq p(B\kappa, B\tau) \leq p(\kappa, \tau) < \varepsilon + \frac{1}{2}\varepsilon,$$

which implies $p(A\kappa, A\tau) < \frac{\varepsilon}{2}$. For $\kappa \in (1, 2]$, $\tau \in [0, 1]$, taking $\delta(\varepsilon) = \frac{1}{2}\varepsilon$, we deduce

$$3p(A\kappa, A\tau) \leq \frac{6}{64} \kappa^2 \leq \frac{\kappa^2}{16} + \frac{\kappa}{8} + \frac{1}{16} \leq p(B\kappa, B\tau) \leq p(\kappa, \tau) < \varepsilon + \frac{1}{2}\varepsilon,$$

which implies $p(A\kappa, A\tau) < \frac{\varepsilon}{2}$. For $\kappa, \tau \in (1, 2]$, setting $\delta(\varepsilon) = \frac{1}{2}\varepsilon$, one can obtain

$$\begin{aligned} 3p(A\kappa, A\tau) &\leq \frac{6}{64} \max\{\kappa^2, \tau^2\} \leq \max\{(\frac{\kappa}{4} + \frac{\kappa}{4})^2, (\frac{\tau}{4} + \frac{\kappa}{4})^2\} \\ &\leq p(B\kappa, B\tau) \leq p(\kappa, \tau) < \varepsilon + \frac{1}{2}\varepsilon, \end{aligned}$$

which implies $p(A\kappa, A\tau) < \frac{\varepsilon}{2}$. Then condition (i) of Definition 3.3 is met about Theorem 3.5. Let $\xi \in \Xi$ be given by $\xi(t, s) = \frac{2s}{3} - t$. Now we consider the following cases. For $\kappa, \tau \in [0, 1]$, $\kappa \neq \tau$, we have

$$p(A\kappa, A\tau) = \frac{1}{256} \max\{\kappa^2, \tau^2\} + \frac{1}{256} (\kappa - \tau)^2,$$

and

$$p(B\kappa, B\tau) = \frac{1}{4} \max\{\kappa^2, \tau^2\} + \frac{1}{4} (\kappa - \tau)^2.$$

Thus

$$\begin{aligned}
& \frac{2}{3}\psi(p(\kappa, \tau)) - \varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)) \\
& \geq \frac{2}{3}\psi(p(B\kappa, B\tau)) - \varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)) \\
& = \frac{1}{6}(\max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2) - (\frac{1}{64} + 2^{-q-8})(\max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2) \geq 0.
\end{aligned}$$

For other cases, we obtain $\alpha(B\kappa, B\tau) = 0$, which implies $\varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)) \leq 2^{-q-5} \max\{\kappa^2, \tau^2\}$. It follows that

$$\begin{aligned}
& \frac{2}{3}\varphi(p(\kappa, \tau)) - \varphi((\alpha(B\kappa, B\tau) + s^{-q})p(A\kappa, A\tau)) \\
& \geq \frac{7}{96} \max\{\kappa^2, \tau^2\} - 2^{-q-5}(\max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2) \geq 0.
\end{aligned}$$

To sum up, all conditions of Theorem 3.5 are satisfied. It is obvious that 0 is the unique common fixed point of A, B .

Definition 3.7. Let A, B be two self-mappings on a partial b -metric space $(\mathbb{G}, p)$ and $\xi \in \Xi$. Suppose that $p : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^+$ is a function that satisfies (P_M^1) and (P_M^2) . Then A, B are said to be the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I^* if the following conditions are satisfied:

- (i) for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\kappa \neq \tau$ and $p(\kappa, \tau) < \varepsilon + \delta(\varepsilon)$ imply $p(A\kappa, A\tau) \leq \frac{\varepsilon}{s}$;
- (ii) $\kappa \neq \tau$ and $p(\kappa, \tau) > 0$ imply $\xi(\varphi((\alpha(B\kappa, B\tau))p(A\kappa, A\tau)), \psi(p(\kappa, \tau))) \geq 0$.

Theorem 3.8. Let $(\mathbb{G}, p)$ be a complete partial b -metric space with parameter $s \geq 1$ and let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$, $A, B : \mathbb{G} \rightarrow \mathbb{G}$ be given mappings and $A(\mathbb{G}) \subseteq B(\mathbb{G})$. Suppose $\varphi \in \Phi$, $\psi \in \Psi$ and $\xi \in \Xi$. If the following conditions are satisfied:

- (i) A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I^* ,
- (ii) A is a triangular (B, α) -admissible mapping,
- (iii) there is $\kappa_0 \in \mathbb{G}$ with $\alpha(A\kappa_0, B\kappa_0) \geq s^2$,
- (iv) properties (a), (b) and (c) are satisfied,
- (v) $B(\mathbb{G})$ is complete and (A, B) is weakly compatible,

then A, B have a unique common fixed point.

Proof. Since the proof of Theorem 3.8 is similar to that of Theorem 3.5, we omit it. $\square$

If $B = I_{\mathbb{G}}$ and $p(\kappa, \kappa) = 0$, for all $\kappa \in \mathbb{G}$ in Theorem 3.8, we have a corollary.

Corollary 3.9. Let $(\mathbb{G}, p)$ be a complete b -metric space with parameter $s \geq 1$. Let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ and $A : \mathbb{G} \rightarrow \mathbb{G}$ be given mappings. Suppose $\xi \in \Xi$. If the following conditions are satisfied:

- (i) A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I^* , where the inequality becomes to

$$\xi(\alpha(\kappa, \tau)\varphi(p(A\kappa, A\tau)), \psi(p(\kappa, \tau))) \geq 0,$$

- (ii) A is a triangular α -admissible mapping,
- (iii) there is $\kappa_0 \in \mathbb{G}$ with $\alpha(\kappa_0, A\kappa_0) \geq s^2$,

(iv) A is continuous,
 (v) properties (a), (b) and (c) (for all $z, w \in \text{Fix}(A)$, we have the condition of $\alpha(z, w) \geq s^2$) are satisfied,
 then A has a unique fixed point.

Let $p : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}^+$ be a mapping. The following conditions are used in the following part:
 $(\mathfrak{P}_{\mathfrak{M}}^1)$ $\kappa \neq \tau$ and $p(\kappa, B\kappa) \leq p(\kappa, \tau)$ imply $p(\kappa, \tau) \leq \mathfrak{M}(\kappa, \tau)$;
 $(\mathfrak{P}_{\mathfrak{M}}^2)$ $\kappa_{2m_k} \neq \kappa_{2n_k+1}$ and $\lim_{k \rightarrow \infty} \sup p(\kappa_{2n_k+1}, B\kappa_{2n_k+1}) \leq \lim_{k \rightarrow \infty} \sup p(\kappa_{2n_k+1}, \kappa_{2m_k})$ imply $p(\kappa_{2n_k+1}, \kappa_{2m_k}) \leq \mathfrak{M}(\kappa_{2n_k+1}, \kappa_{2m_k})$.

$$\begin{aligned} \mathfrak{M}(\kappa, \tau) = & \max\{p(\tau, A\tau), p(\kappa, B\kappa), p(\kappa, \tau), \frac{p(\tau, B\kappa) + p(\kappa, A\tau)}{2s}, \\ & \frac{1 + p(\tau, A\tau)}{1 + p(A\tau, B\kappa)} p(\kappa, B\kappa), \frac{1 + p(\kappa, B\kappa)}{1 + p(\kappa, \tau)} p(\tau, A\tau), \\ & \frac{1 + p(\kappa, A\tau) + p(\tau, B\kappa)}{1 + sp(\kappa, \tau) + sp(A\tau, B\kappa)} p(\kappa, B\kappa)\}. \end{aligned}$$

Definition 3.10. Let $(\mathbb{G}, p)$ be a partial b -metric space with parameter $s \geq 1$. Let $A, B : \mathbb{G} \rightarrow \mathbb{G}$ and $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ be given mappings. The pair (A, B) is said to be composite α -admissible if, for all $\kappa, \tau, v \in \mathbb{G}$, the following conditions are satisfied:

- (1) $\alpha(B\kappa, A\tau) \geq s^5$ implies $\alpha(AB\kappa, BA\tau) \geq s^5$;
- (2) $\alpha(\kappa, \tau) = \alpha(\tau, \kappa)$;
- (3) $\alpha(\kappa, \tau) \geq s^5$ and $\alpha(\tau, v) \geq s^5$ imply $\alpha(\kappa, v) \geq s^5$.

Let $(\mathbb{G}, p)$ be a p_b -complete partial b -metric space with parameter $s \geq 1$ and let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$. Then,

- (a') if $\{B\kappa_{2n+1}\}$ is a sequence in $\mathbb{G}$ such that $B\kappa_{2n+1} \rightarrow z$ as $n \rightarrow \infty$, then there exists a subsequence $\{B\kappa_{2n_i+1}\}$ of $\{B\kappa_{2n+1}\}$ with $\alpha(B\kappa_{2n_i+1}, Az) \geq s^5$ for all $i \in \mathbb{N}$.
- (b') for all $z, w \in C(A, B)$, we have $\alpha(z, w) \geq s^5$.

Definition 3.11. Let A, B be two self-mappings on a partial b -metric space $(\mathbb{G}, p)$ and $\xi \in \Xi$. Suppose that $p : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}^+$ is a function that satisfies $(\mathfrak{P}_{\mathfrak{M}}^1)$ and $(\mathfrak{P}_{\mathfrak{M}}^2)$. Then A, B are said to be the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type II if the following conditions are satisfied:

- (i) for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\kappa \neq \tau$ and $p(\kappa, \tau) < \varepsilon + \delta(\varepsilon)$ imply $p(B\kappa, A\tau) \leq \frac{\varepsilon}{s}$;
- (ii) $\kappa \neq \tau$ and $p(\kappa, \tau) > 0$ imply $\xi(\varphi((\alpha(B\kappa, A\tau) + s^{-q})p(B\kappa, A\tau)), \psi(p(\kappa, \tau))) \geq 0$.

Remark 3.12. If A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type II, $p(B\kappa, A\tau) > 0$, then

$$\varphi((\alpha(B\kappa, A\tau) + s^{-q})p(B\kappa, A\tau)) < \psi(p(\kappa, \tau)) \leq \varphi(p(\kappa, \tau)) \leq \varphi(\mathfrak{M}(\kappa, \tau)).$$

Proof. The proof is similar to that of Remark 3.4. □

Theorem 3.13. Let $(\mathbb{G}, p)$ be a complete partial b -metric space with parameter $s \geq 1$. Let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ and $A, B : \mathbb{G} \rightarrow \mathbb{G}$ be given mappings. Suppose $\varphi \in \Phi$, $\psi \in \Psi$, $\xi \in \Xi$, and $q > 1$. If the following conditions are satisfied:

- (i) A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type II,

- (ii) (A, B) is composite α -admissible,
- (iii) there is $\kappa_0 \in \mathbb{G}$ with $\alpha(A\kappa_0, BA\kappa_0) \geq s^5$,
- (iv) properties (a') and (b') are satisfied,
- (v) $p(A\kappa, B\kappa) \leq p(\kappa, A\kappa)$ for all $\kappa \in \mathbb{G}$,

then A, B have a unique common fixed point.

Proof. It follows from condition (iii) that there exists $\kappa_0 \in \mathbb{G}$ such that $\alpha(A\kappa_0, BA\kappa_0) \geq s^5$. Define the sequence $\{\kappa_n\}$ in $\mathbb{G}$ by $\kappa_{2n+1} = A\kappa_{2n}$, $\kappa_{2n+2} = B\kappa_{2n+1}$ for all $n \in \mathbb{N}$. By the condition (ii), we obtain

$$\begin{aligned}\alpha(B\kappa_1, A\kappa_0) &= \alpha(A\kappa_0, BA\kappa_0) \geq s^5, \\ \alpha(B\kappa_1, A\kappa_2) &= \alpha(AB\kappa_1, BA\kappa_0) \geq s^5, \\ \alpha(B\kappa_3, A\kappa_2) &= \alpha(BA\kappa_2, AB\kappa_1) \geq s^5, \\ &\dots\dots\dots\end{aligned}$$

By induction, one can obtain $\alpha(B\kappa_{2n+1}, A\kappa_{2n}) = \alpha(\kappa_{2n+1}, \kappa_{2n+2}) \geq s^5$ for all $n \in \mathbb{N}$. In view of Definition 8, we have $\alpha(B\kappa_{2m+1}, A\kappa_{2n}) = \alpha(A\kappa_{2n}, B\kappa_{2m+1}) = \alpha(\kappa_{2n+1}, \kappa_{2m+2}) \geq s^5$ for all $n, m \in \mathbb{N}$. If $p(\kappa_{2n+1}, \kappa_{2n+2}) = 0$ for some $n \in \mathbb{N}$, we deduce that $\kappa_{2n+1} = \kappa_{2n+2}$. It follows that we assert that $\kappa_{2n+2} = \kappa_{2n+3}$.

On the contrary, we assume that

$$p(\kappa_{2n+2}, \kappa_{2n+3}) = p(A\kappa_{2n+2}, B\kappa_{2n+1}) > 0.$$

According to $p(\kappa_{2n+1}, B\kappa_{2n+1}) = p(\kappa_{2n+1}, \kappa_{2n+2})$, we have $p(\kappa_{2n+1}, \kappa_{2n+2}) \leq \mathfrak{M}(\kappa_{2n+1}, \kappa_{2n+2})$ by $(\mathfrak{P}_{\mathfrak{M}}^1)$, where

$$\begin{aligned}&\mathfrak{M}(\kappa_{2n+1}, \kappa_{2n+2}) \\ &= \max\{p(\kappa_{2n+2}, A\kappa_{2n+2}), p(\kappa_{2n+1}, B\kappa_{2n+1}), p(\kappa_{2n+1}, \kappa_{2n+2}), \\ &\quad \frac{p(\kappa_{2n+2}, B\kappa_{2n+1}) + p(\kappa_{2n+1}, A\kappa_{2n+2})}{2s}, \\ &\quad \frac{1 + p(\kappa_{2n+2}, A\kappa_{2n+2})}{1 + p(A\kappa_{2n+2}, B\kappa_{2n+1})} p(\kappa_{2n+1}, B\kappa_{2n+1}), \\ &\quad \frac{1 + p(\kappa_{2n+1}, B\kappa_{2n+1})}{1 + p(\kappa_{2n+1}, \kappa_{2n+2})} p(\kappa_{2n+2}, A\kappa_{2n+2}), \\ &\quad \frac{1 + p(\kappa_{2n+1}, A\kappa_{2n+2}) + p(\kappa_{2n+2}, B\kappa_{2n+1})}{1 + sp(\kappa_{2n+1}, \kappa_{2n+2}) + sp(A\kappa_{2n+2}, B\kappa_{2n+1})} p(\kappa_{2n+1}, B\kappa_{2n+1})\}, \\ &= \max\{p(\kappa_{2n+2}, \kappa_{2n+3}), p(\kappa_{2n+1}, \kappa_{2n+2}), p(\kappa_{2n+1}, \kappa_{2n+2}), \\ &\quad \frac{p(\kappa_{2n+2}, \kappa_{2n+2}) + p(\kappa_{2n+1}, \kappa_{2n+3})}{2s}, \\ &\quad \frac{1 + p(\kappa_{2n+2}, \kappa_{2n+3})}{1 + p(\kappa_{2n+3}, \kappa_{2n+2})} p(\kappa_{2n+1}, \kappa_{2n+2}), \frac{1 + p(\kappa_{2n+1}, \kappa_{2n+2})}{1 + p(\kappa_{2n+1}, \kappa_{2n+2})} p(\kappa_{2n+2}, \kappa_{2n+3}), \\ &\quad \frac{1 + p(\kappa_{2n+1}, \kappa_{2n+3}) + p(\kappa_{2n+2}, \kappa_{2n+2})}{1 + sp(\kappa_{2n+1}, \kappa_{2n+2}) + sp(\kappa_{2n+3}, \kappa_{2n+2})} p(\kappa_{2n+1}, \kappa_{2n+2})\} \\ &= p(\kappa_{2n+2}, \kappa_{2n+3}).\end{aligned}$$

In light of (i) and Remark 3.12, we deduce

$$\begin{aligned}
\varphi(p(\kappa_{2n+2}, \kappa_{2n+3})) &= \varphi(p(B\kappa_{2n+1}, A\kappa_{2n+2})) \\
&\leq \varphi((\alpha(B\kappa_{2n+1}, A\kappa_{2n+2}) + s^{-q})p(B\kappa_{2n+1}, A\kappa_{2n+2})) \\
&< \psi(p(\kappa_{2n+1}, \kappa_{2n+2})) \\
&\leq \varphi(\mathfrak{M}(\kappa_{2n+1}, \kappa_{2n+2})) \\
&= \varphi(p(\kappa_{2n+2}, \kappa_{2n+3})),
\end{aligned}$$

which gives a contradiction. So, $p(\kappa_{2n+2}, \kappa_{2n+3}) = 0$ which implies $\kappa_{2n+3} = \kappa_{2n+2} = \kappa_{2n+1}$, that is, A and B have a point of coincidence.

Without loss of generality, we assume that $p(\kappa_n, \kappa_{n+1}) > 0$ and $n = \text{even}$. Now, we show that the sequence $\{p(\tau_{2n}, \tau_{2n+1})\}$ is monotone decreasing. Since $p(\kappa_{2n-1}, B\kappa_{2n-1}) = p(\kappa_{2n-1}, \kappa_{2n})$, we have $p(\kappa_{2n-1}, \kappa_{2n}) \leq \mathfrak{M}(\kappa_{2n-1}, \kappa_{2n})$ by $(\mathfrak{P}_{\mathfrak{M}}^1)$, where

$$\begin{aligned}
\mathfrak{M}(\kappa_{2n-1}, \kappa_{2n}) &= \max\{p(\kappa_{2n}, A\kappa_{2n}), p(\kappa_{2n-1}, B\kappa_{2n-1}), p(\kappa_{2n-1}, \kappa_{2n}), \\
&\quad \frac{p(\kappa_{2n}, B\kappa_{2n-1}) + p(\kappa_{2n-1}, A\kappa_{2n})}{2s}, \\
&\quad \frac{1 + p(\kappa_{2n}, A\kappa_{2n})}{1 + p(A\kappa_{2n}, B\kappa_{2n-1})}p(\kappa_{2n-1}, B\kappa_{2n-1}), \\
&\quad \frac{1 + p(\kappa_{2n-1}, B\kappa_{2n-1})}{1 + p(\kappa_{2n-1}, \kappa_{2n})}p(\kappa_{2n}, A\kappa_{2n}), \\
&\quad \frac{1 + p(\kappa_{2n-1}, A\kappa_{2n}) + p(\kappa_{2n}, B\kappa_{2n-1})}{1 + sp(\kappa_{2n-1}, \kappa_{2n}) + sp(A\kappa_{2n}, B\kappa_{2n-1})}p(\kappa_{2n-1}, B\kappa_{2n-1})\} \\
&= \max\{p(\kappa_{2n}, \kappa_{2n+1}), p(\kappa_{2n-1}, \kappa_{2n}), p(\kappa_{2n-1}, \kappa_{2n}), \\
&\quad \frac{p(\kappa_{2n}, \kappa_{2n}) + p(\kappa_{2n-1}, \kappa_{2n+1})}{2s}, \\
&\quad \frac{1 + p(\kappa_{2n}, \kappa_{2n+1})}{1 + p(\kappa_{2n+1}, \kappa_{2n})}p(\kappa_{2n-1}, \kappa_{2n}), \frac{1 + p(\kappa_{2n-1}, \kappa_{2n+1})}{1 + p(\kappa_{2n-1}, \kappa_{2n})}p(\kappa_{2n}, \kappa_{2n+1}), \\
&\quad \frac{1 + p(\kappa_{2n-1}, \kappa_{2n+1}) + p(\kappa_{2n}, \kappa_{2n})}{1 + sp(\kappa_{2n-1}, \kappa_{2n}) + sp(\kappa_{2n+1}, \kappa_{2n})}p(\kappa_{2n-1}, \kappa_{2n})\} \\
&= \max\{p(\kappa_{2n}, \kappa_{2n+1}), p(\kappa_{2n-1}, \kappa_{2n})\}.
\end{aligned}$$

According to condition (i) and Remark 3.12 above, we obtain

$$\begin{aligned}
\varphi(p(\kappa_{2n}, \kappa_{2n+1})) &= \varphi(p(B\kappa_{2n-1}, A\kappa_{2n})) \\
&\leq \varphi((\alpha(B\kappa_{2n-1}, A\kappa_{2n}) + s^{-q})p(B\kappa_{2n-1}, A\kappa_{2n})) \\
&< \psi(p(\kappa_{2n-1}, \kappa_{2n})) \leq \varphi(\mathfrak{M}(\kappa_{2n-1}, \kappa_{2n})).
\end{aligned} \tag{3.6}$$

Since $\mathfrak{M}(\kappa_{2n-1}, \kappa_{2n}) = \max\{p(\kappa_{2n}, \kappa_{2n+1}), p(\kappa_{2n-1}, \kappa_{2n})\}$, if $\mathfrak{M}(\kappa_{2n-1}, \kappa_{2n}) = p(\kappa_{2n}, \kappa_{2n+1})$, we find by (3.6) that $\varphi(p(\kappa_{2n}, \kappa_{2n+1})) < \varphi(p(\kappa_{2n}, \kappa_{2n+1}))$, a contradiction. Thus, $\mathfrak{M}(\kappa_{2n-1}, \kappa_{2n}) = p(\kappa_{2n-1}, \kappa_{2n})$, and $\varphi(p(\kappa_{2n}, \kappa_{2n+1})) < \varphi(p(\kappa_{2n-1}, \kappa_{2n}))$. It follows that

$$p(\kappa_{2n}, \kappa_{2n+1}) < p(\kappa_{2n-1}, \kappa_{2n}).$$

Similarly, one obtains $p(\kappa_{2n+1}, \kappa_{2n+2}) < p(\kappa_{2n}, \kappa_{2n+1})$. Hence, $\{p(\kappa_n, \kappa_{n+1})\}$ is a decreasing sequence. Set $\lim_{n \rightarrow \infty} p(\kappa_n, \kappa_{n+1}) = p^* \geq 0$.

Put $\varepsilon = p^* > 0$. It follows that there exists N such that $p^* = \varepsilon < p(\kappa_n, \kappa_{n+1}) < \varepsilon + \delta(\varepsilon)$, when $n > N$. Hence, $p(\kappa_{2n-1}, \kappa_{2n}) \leq \mathfrak{M}(\kappa_{2n-1}, \kappa_{2n}) = p(\kappa_{2n-1}, \kappa_{2n}) < \varepsilon + \delta(\varepsilon)$, which implies $p(A\kappa_{2n}, B\kappa_{2n+1}) = p(\kappa_{2n+1}, \kappa_{2n+2}) \leq \frac{\varepsilon}{s}$. Consequently, $\varepsilon < p(\kappa_{2n}, \kappa_{2n+1}) \leq \frac{\varepsilon}{s}$, which is impossible. Therefore, $\lim_{n \rightarrow \infty} p(\kappa_n, \kappa_{n+1}) = p^* = 0$.

Next, we present that $\{\kappa_n\}$ is a Cauchy in $(\mathbb{G}, p)$. Suppose that is not true. By Lemma 2.9, we obtain that there is $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $m_k > n_k > k$ and

$$\frac{\varepsilon}{s} \leq \liminf_{k \rightarrow \infty} p(\kappa_{2m_k}, \kappa_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} p(\kappa_{2m_k}, \kappa_{2n_k+1}) \leq s^2 \varepsilon.$$

Taking $\tau = \kappa_{2m_k}$, $\kappa = \kappa_{2n_k+1}$ and $\kappa \neq \tau$, we have $p(\kappa_{2n_k+1}, B\kappa_{2n_k+1}) = p(\kappa_{2n_k+1}, \kappa_{2n_k+2})$ and $\lim_{k \rightarrow \infty} \sup p(\kappa_{2n_k+1}, B\kappa_{2n_k+1}) \leq \lim_{k \rightarrow \infty} \sup p(\kappa_{2n_k+1}, \kappa_{2m_k})$ which implies $p(\kappa_{2n_k+1}, \kappa_{2m_k}) \leq \mathfrak{M}(\kappa_{2n_k+1}, \kappa_{2m_k})$, where

$$\begin{aligned} \mathfrak{M}(\kappa_{2n_k+1}, \kappa_{2m_k}) &= \max\{p(\kappa_{2m_k}, A\kappa_{2m_k}), p(\kappa_{2n_k+1}, B\kappa_{2n_k+1}), p(\kappa_{2n_k+1}, \kappa_{2m_k}), \\ &\quad \frac{p(\kappa_{2m_k}, B\kappa_{2n_k+1}) + p(\kappa_{2n_k+1}, A\kappa_{2m_k})}{2s}, \\ &\quad \frac{1 + p(\kappa_{2m_k}, A\kappa_{2m_k})}{1 + p(A\kappa_{2m_k}, B\kappa_{2n_k+1})} p(\kappa_{2n_k+1}, B\kappa_{2n_k+1}), \\ &\quad \frac{1 + p(\kappa_{2n_k+1}, B\kappa_{2n_k+1})}{1 + p(\kappa_{2n_k+1}, \kappa_{2m_k})} p(\kappa_{2m_k}, A\kappa_{2m_k}), \\ &\quad \frac{1 + p(\kappa_{2n_k+1}, A\kappa_{2m_k}) + p(\kappa_{2m_k}, B\kappa_{2n_k+1})}{1 + sp(\kappa_{2n_k+1}, \kappa_{2m_k}) + sp(A\kappa_{2m_k}, B\kappa_{2n_k+1})} p(\kappa_{2n_k+1}, B\kappa_{2n_k+1})\} \\ &= \max\{p(\kappa_{2m_k}, \kappa_{2m_k+1}), p(\kappa_{2n_k+1}, \kappa_{2n_k+2}), p(\kappa_{2n_k+1}, \kappa_{2m_k}), \\ &\quad \frac{p(\kappa_{2m_k}, \kappa_{2n_k+2}) + p(\kappa_{2n_k+1}, \kappa_{2m_k+1})}{2s}, \\ &\quad \frac{1 + p(\kappa_{2m_k}, \kappa_{2m_k+1})}{1 + p(\kappa_{2m_k+1}, \kappa_{2n_k+2})} p(\kappa_{2n_k+1}, \kappa_{2n_k+2}), \\ &\quad \frac{1 + p(\kappa_{2n_k+1}, \kappa_{2n_k+2})}{1 + p(\kappa_{2n_k+1}, \kappa_{2m_k})} p(\kappa_{2m_k}, \kappa_{2m_k+1}), \\ &\quad \frac{1 + p(\kappa_{2n_k+1}, \kappa_{2m_k+1}) + p(\kappa_{2m_k}, \kappa_{2n_k+2})}{1 + sp(\kappa_{2n_k+1}, \kappa_{2m_k}) + sp(\kappa_{2m_k+1}, \kappa_{2n_k+2})} p(\kappa_{2n_k+1}, \kappa_{2n_k+2})\} \end{aligned}$$

Taking the upper limit as $k \rightarrow \infty$, we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \varphi(\mathfrak{M}(\kappa_{2n_k+1}, \kappa_{2m_k})) &= \varphi(\limsup_{k \rightarrow \infty} \mathfrak{M}(\kappa_{2n_k+1}, \kappa_{2m_k})) \\ &\leq \varphi(\max\{0, 0, s^2 \varepsilon, s^2 \varepsilon, 0, 0, 0\}) = \varphi(s^2 \varepsilon). \end{aligned}$$

In view of Remark 3.12 and $\alpha(B\kappa_{2n_k+1}, A\kappa_{2m_k}) = \alpha(\kappa_{2m_k+1}, \kappa_{2n_k+2}) \geq s^5$, we obtain

$$\begin{aligned} &\varphi((s^5 + s^{-q})p(\kappa_{2m_k+1}, \kappa_{2n_k+2})) \\ &= \varphi((s^5 + s^{-q})p(B\kappa_{2n_k+1}, A\kappa_{2m_k})) \\ &\leq \varphi((\alpha(B\kappa_{2n_k+1}, A\kappa_{2m_k}) + s^{-q})p(B\kappa_{2n_k+1}, A\kappa_{2m_k})) \\ &< \psi(p(\kappa_{2n_k+1}, \kappa_{2m_k})) \leq \varphi(\mathfrak{M}(\kappa_{2n_k+1}, \kappa_{2m_k})), \end{aligned}$$

where

$$\begin{aligned}\varphi(s^2\varepsilon + s^{-q-3}\varepsilon) &\leq \limsup_{k \rightarrow \infty} \varphi((s^5 + s^{-q})p(\tau_{2m_k+1}, \tau_{2n_k+2})) \\ &= \varphi(\limsup_{k \rightarrow \infty} (s^5 + s^{-q})p(\tau_{2m_k+1}, \tau_{2n_k+2})).\end{aligned}$$

It follows that $\varphi(s^2\varepsilon + s^{-q-3}\varepsilon) \leq \varphi(s^2\varepsilon)$, which is a contradiction. Thus, $\{\kappa_{2n}\}$ is a Cauchy sequence and hence $\{\kappa_n\}$ is a Cauchy sequence both in $(\mathbb{G}, p)$ and in $(\mathbb{G}, p^s)$.

It is the same as the proof of Theorem 3.5, we also obtain $\lim_{n,m \rightarrow \infty} p(\kappa_n, \kappa_m) = \lim_{n \rightarrow \infty} p(\kappa_n, z) = p(z, z) = 0$. Since $\kappa_n \rightarrow z$, $\{A\kappa_{2n}\}$ and $\{B\kappa_{2n+1}\}$ converge to z .

We now prove that $z = Az$. On the contrary, we assume that $p(z, Au) > 0$. From the condition (iv), we can get a subsequence $\{\kappa_{2n_i+1}\}$ of $\{\kappa_{2n+1}\}$ such that $\{B\kappa_{2n_i+1}\} \rightarrow z$ with $\alpha(B\kappa_{2n_i+1}, Az) \geq s^5$. Letting $\kappa = \kappa_{2n_i+1}$, $\tau = z$ and $\kappa \neq \tau$, we obtain $\lim_{i \rightarrow \infty} \sup p(\kappa_{2n_i+1}, B\kappa_{2n_i+1}) = 0 = \lim_{i \rightarrow \infty} \sup p(\kappa_{2n_i+1}, z)$ and which imply $p(\kappa_{2n_i+1}, z) \leq \mathfrak{M}(\kappa_{2n_i+1}, z)$ by $(\mathfrak{P}_{\mathfrak{M}}^2)$, where

$$\begin{aligned}\mathfrak{M}(\kappa_{2n_i+1}, z) &= \max\{p(z, Az), p(\kappa_{2n_i+1}, B\kappa_{2n_i+1}), p(\kappa_{2n_i+1}, z), \\ &\quad \frac{p(z, B\kappa_{2n_i+1}) + p(\kappa_{2n_i+1}, Az)}{2s}, \\ &\quad \frac{1 + p(z, Az)}{1 + p(Az, B\kappa_{2n_i+1})} p(\kappa_{2n_i+1}, B\kappa_{2n_i+1}), \frac{1 + p(\kappa_{2n_i+1}, B\kappa_{2n_i+1})}{1 + p(\kappa_{2n_i+1}, z)} p(z, Az), \\ &\quad \frac{1 + p(\kappa_{2n_i+1}, Az) + p(z, B\kappa_{2n_i+1})}{1 + sp(\kappa_{2n_i+1}, z) + sp(Az, B\kappa_{2n_i+1})} p(\kappa_{2n_i+1}, B\kappa_{2n_i+1})\} \\ &= \max\{p(z, Az), p(\kappa_{2n_i+1}, \kappa_{2n_i+2}), p(\kappa_{2n_i+1}, z), \frac{p(z, \kappa_{2n_i+2}) + p(\kappa_{2n_i+1}, Az)}{2s}, \\ &\quad \frac{1 + p(z, Az)}{1 + p(Az, \kappa_{2n_i+2})} p(\kappa_{2n_i+1}, \kappa_{2n_i+2}), \frac{1 + p(\kappa_{2n_i+1}, \kappa_{2n_i+2})}{1 + p(\kappa_{2n_i+1}, z)} p(z, Az), \\ &\quad \frac{1 + p(\kappa_{2n_i+1}, Az) + p(z, \kappa_{2n_i+2})}{1 + sp(\kappa_{2n_i+1}, z) + sp(Az, \kappa_{2n_i+2})} p(\kappa_{2n_i+1}, \kappa_{2n_i+2})\}.\end{aligned}\tag{3.7}$$

It follows from Remark 3.12 that

$$\begin{aligned}\varphi((s^4 + s^{-q-1})p(z, Az)) &\leq \varphi((\alpha(B\kappa_{2n_i+1}, Az) + s^{-q})p(B\kappa_{2n_i+1}, Az)) \\ &< \psi(p(\kappa_{2n_i+1}, z)) \leq \varphi(\mathfrak{M}(\kappa_{2n_i+1}, z)).\end{aligned}$$

Taking the upper limit as $i \rightarrow \infty$ in (3.7), one can deduce

$$\begin{aligned}\limsup_{i \rightarrow \infty} \varphi(\mathfrak{M}(\kappa_{2n_i+1}, z)) &= \varphi(\limsup_{i \rightarrow \infty} \mathfrak{M}(\kappa_{2n_i+1}, z)) \\ &\leq \varphi(\max\{p(Az, z), 0, 0, \frac{p(Az, z)}{2}, 0, p(Az, z), 0\}) \\ &= \varphi(p(Az, z)),\end{aligned}$$

$$\varphi((s^4 + s^{-q-1})p(Az, z)) \leq \varphi(p(Az, z)),$$

which is a contradiction. Thus, $p(Az, z) = 0$ and which implies $Az = z$. By the condition (v), we have $p(z, Bz) = p(Az, Bz) \leq p(z, Az) = 0$. Therefore, $Az = z = Bz$, and A, B have a common fixed point z .

Suppose that w is another common fixed point of A and B . The condition (c) ensures $\alpha(w, z) \geq s^5$. Taking $\kappa = z$, $\tau = w$ and $\kappa \neq \tau$, we have $p(z, Bz) = 0 \leq p(z, w)$, which implies $p(z, w) \leq \mathfrak{M}(z, w)$ by $(\mathfrak{P}_{\mathfrak{M}}^1)$, where $\mathfrak{M}(z, w) = p(z, w)$. Moreover, in view of Remark 3.12, we obtain

$$\begin{aligned} \varphi((s^5 + s^{-q})p(z, w)) &= \varphi((s^5 + s^{-q})p(Bw, Az)) \\ &\leq \varphi((\alpha(Bw, Az) + s^{-q})p(Bw, Az)) \\ &< \psi(p(z, w)) \\ &\leq \varphi(\mathfrak{M}(z, w)) \\ &< \varphi(p(z, w)), \end{aligned}$$

a contradiction. It follows that $p(z, w) = 0$, which implies $z = w$, that is, A, B have a unique common fixed point. This completes the proof. $\square$

Example 3.14. Let $\mathbb{G} = [0, 2]$, $p(\kappa, \tau) = \max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2$, $s = 2$, and $q > 1$. Define mappings A, B by

$$A\kappa = \begin{cases} \frac{\kappa}{12}, & \kappa \in [0, 1] \\ \frac{\kappa}{6}, & \kappa \in (1, 2] \end{cases}, B\kappa = \begin{cases} \frac{\kappa}{14}, & \kappa \in [0, 1] \\ \frac{1}{8}, & \kappa \in (1, 2] \end{cases}.$$

We have

$$AB\kappa = \begin{cases} \frac{\kappa}{168}, & \kappa \in [0, 1] \\ \frac{1}{96}, & \kappa \in (1, 2] \end{cases}, BA\kappa = \begin{cases} \frac{\kappa}{168}, & \kappa \in [0, 1] \\ \frac{\kappa}{84}, & \kappa \in (1, 2] \end{cases}.$$

It is easy to show that $p(A\kappa, B\kappa) \leq p(\kappa, A\kappa)$ for all $\kappa \in [0, 2]$. Define mappings $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ and $\alpha(\kappa, \tau) = \alpha(\tau, \kappa)$ by

$$\alpha(\kappa, \tau) = \begin{cases} s^5, & \kappa, \tau \in [0, \frac{1}{12}] \\ 0, & \text{otherwise} \end{cases},$$

and

$$\varphi(t) = t, \psi(t) = \frac{3t}{4},$$

$$\begin{aligned} p(\kappa, \tau) &= \max\{p(\tau, A\tau), p(\kappa, \tau), \frac{p(\tau, B\kappa) + p(\kappa, A\tau)}{2s}, \frac{1 + p(\tau, A\tau)}{1 + p(A\tau, B\kappa)}p(\kappa, B\kappa), \\ &\quad \frac{1 + p(\kappa, A\tau) + p(\tau, B\kappa)}{1 + sp(\kappa, \tau) + sp(A\tau, B\kappa)}p(\kappa, B\kappa)\}, \end{aligned}$$

Obviously, $p(\kappa, \tau)$ satisfies condition $(p_{\mathfrak{M}}^1)$ and $p(\kappa, \tau) > 0$ for all $\kappa \neq \tau$. For $\alpha(B\kappa, A\tau) \geq s^5$, one can see that $B\kappa, A\tau \in [0, \frac{1}{12}]$, which implies that $\kappa, \tau \in [0, 1]$. It is easy to prove that (A, B) is composite α -admissible. For $\kappa, \tau \in [0, 2]$, letting $\delta(\varepsilon) = 8\varepsilon$, we have

$$18p(B\kappa, A\tau) \leq \max\{\kappa^2, \tau^2\} \leq p(\kappa, \tau) \leq p(\kappa, \tau) < \varepsilon + 8\varepsilon,$$

which implies $p(B\kappa, A\tau) < \frac{\varepsilon}{2}$. It follows that Definition 3.3 is met about Theorem 3.13. Let $\xi \in \Xi$ be given by $\xi(t, s) = \frac{4s}{5} - t$. Now we consider the following cases:

For $\kappa, \tau \in [0, 1]$, $\kappa \neq \tau$, we have

$$\begin{aligned} p(B\kappa, A\tau) &= \max\{\frac{1}{144}\kappa^2, \frac{1}{196}\tau^2\} + (\frac{1}{14}\kappa - \frac{1}{12}\tau)^2, \\ p(\kappa, \tau) &= \max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2. \end{aligned}$$

Thus

$$\begin{aligned}
& \frac{4}{5} \psi(p(\kappa, \tau)) - \varphi((\alpha(B\kappa, A\tau) + s^{-q})p(B\kappa, A\tau)) \\
& \geq \frac{4}{5} \psi(p(\kappa, \tau)) - \varphi((\alpha(B\kappa, A\tau) + s^{-q})p(B\kappa, A\tau)) \\
& \geq \frac{3}{5} (\max\{\kappa^2, \tau^2\} + (\kappa - \tau)^2) - \left(\frac{4}{9} + 2^{-q-3}\frac{1}{9}\right) (\max\{\kappa^2, \tau^2\}) \geq 0.
\end{aligned}$$

For other cases, one can obtain $\alpha(B\kappa, A\tau) = 0$, which implies

$$\varphi((\alpha(B\kappa, A\tau) + s^{-q})p(B\kappa, A\tau)) \leq 2^{-q-1}\frac{1}{9} \max\{\kappa^2, \tau^2\}.$$

Consequently,

$$\begin{aligned}
& \frac{4}{5} \psi(p(\kappa, \tau)) - \varphi((\alpha(B\kappa, A\tau) + s^{-q})p(B\kappa, A\tau)) \\
& \geq \frac{3}{5} \max\{\kappa^2, \tau^2\} - 2^{-q-1}\frac{1}{9} \max\{\kappa^2, \tau^2\} \geq 0.
\end{aligned}$$

In summary, all the conditions of Theorem 3.13 are fulfilled. Here, 0 is the unique common fixed point of A, B .

Theorem 3.15. *Let $(\mathbb{G}, p)$ be a complete partial b -metric space with parameter $s \geq 1$. Let $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ and $A, B : \mathbb{G} \rightarrow \mathbb{G}$ be given mappings. Suppose $\varphi \in \Phi$, $\psi \in \Psi$, $\xi \in \Xi$ and $q > 1$. If the following conditions are satisfied:*

(i) A, B are the generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type II, where the inequality becomes to*

$$\xi(\varphi((\alpha(B\kappa, A\tau))p(B\kappa, A\tau), \psi(p(\kappa, \tau))) \geq 0,$$

- (ii) (A, B) is α -admissible mapping pairs,*
 - (iii) there is $\kappa_0 \in \mathbb{G}$ with $\alpha(A\kappa_0, B\kappa_0) \geq s^5$,*
 - (iv) properties (a') and (b') are satisfied,*
 - (v) $p(A\kappa, B\kappa) \leq p(\kappa, A\kappa)$ for all $\kappa \in \mathbb{G}$,*
- then A, B have a unique common fixed point.*

Proof. Since the proof of Theorem 3.15 is similar to that of Theorem 3.13, we also omit it. $\square$

4. APPLICATIONS

In this section, we present the existence of solutions to an integral equation by means of one of our results.

Consider the following integral equation

$$\kappa(l) = \int_0^L K(l, w, \kappa(w))dw, \quad (4.1)$$

where $l \in [0, L]$, $L > 0$ and $K : [0, L] \times [0, L] \times \mathbb{R} \rightarrow \mathbb{R}$. The aim of this section is to give an existence theorem for a solution to the above integral equation by use of Theorem 3.8.

Let $\mathbb{G} = C[0, L]$ and define $p : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{R}^+$ by

$$p(\kappa, \tau) = \sup_{l \in [0, L]} |\kappa(l) - \tau(l)|^p + (\max\{\sup_{l \in [0, L]} |\kappa(l)|, \sup_{l \in [0, L]} |\tau(l)|\})^p. (p \geq 2)$$

It is obvious that $(\mathbb{G}, \mathfrak{p})$ is a p_b -complete partial b -metric space with $s = 2^{p-1}$.

Consider the mapping $A, B : \mathbb{G} \rightarrow \mathbb{G}$ defined by

$$A\kappa(l) = \int_0^L K_1(l, w, \kappa(w))dw, \text{ and } B\kappa(l) = \int_0^L K_2(l, w, \kappa(w))dw.$$

Let $\xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given function.

Theorem 4.1. *Let A, B be self mappings on a partial b -metric space $(\mathbb{G}, \mathfrak{p})$. Suppose the following hypotheses hold:*

- (i) $K_1, K_2 : [0, L] \times [0, L] \times \mathbb{R} \rightarrow \mathbb{R}$,
- (ii) $A\mathbb{G} \subseteq B\mathbb{G}$ and $B\mathbb{G}$ is p_b -complete,
- (iii) $AB\kappa = BA\kappa$, whenever $A\kappa = B\kappa$ for some $\kappa \in C[0, L]$,
- (iv) there exists $\kappa_0 \in \mathbb{G}$ such that $\xi(A\kappa_0(l), B\kappa_0(l)) \geq 0$ for all $l \in [0, L]$,
- (v) for all $l \in [0, L]$ and $\kappa, \tau \in \mathbb{G}$, $\xi(B\kappa(l), B\tau(l)) \geq 0$ implies $\xi(A\kappa(l), A\tau(l)) \geq 0$,
- (vi) properties (a), (b) and (c) are satisfied,
- (vii) there exists a continuous function $h : [0, L] \times [0, L] \rightarrow \mathbb{R}^+$ such that $\int_0^L h(l, w)dw \leq \delta$ ($\frac{1}{4} < \delta \leq 1$),
- (viii) for each $u, v \in \mathbb{R}$, $0 < k < 1$ and each $l, s \in [0, L]$,

$$|K_1(l, w, \kappa(w)) - K_1(l, v, \tau(w))| \leq \sqrt[p]{\frac{k}{s^3}} h(l, w) |B\kappa - B\tau|$$

and

$$\max\{|K_1(l, w, \kappa(w))|, |K_1(l, w, \tau(w))|\} \leq \sqrt[p]{\frac{k}{4s^2}} \max\{|K_2(l, w, \kappa(w))|, |K_2(l, w, \tau(w))|\}.$$

Then integral equation (4.1) has a unique solution $u \in C[0, L]$.

Proof. Define $p(\kappa, \tau) = M(\kappa, \tau)$, $\varphi(t) = t$, $\psi(t) = \zeta t$, and $\xi(t, s) = as - t$, where $(0 < \zeta, a < 1)$ and $\zeta a = k\delta$. Meanwhile, define $\alpha : \mathbb{G} \times \mathbb{G} \rightarrow [0, +\infty)$ by

$$\alpha(\kappa, \tau) = \begin{cases} s^2, & \text{if } \xi(\kappa(l), \tau(l)) \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

It is easy to prove that A is (B, α) -admissible and $A\mathbb{G} \subseteq B\mathbb{G}$ and $B\mathbb{G}$ is p_b -complete. For $\kappa, \tau \in \mathbb{G}$, by virtue of assumptions (i) – (viii), one see that there exists $\delta(\varepsilon) = \varepsilon$ such that

$$\begin{aligned} 2s\mathfrak{p}(A\kappa, A\tau) &= 2s \left(\sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \kappa(w))dw - \int_0^L K_1(l, w, \tau(w))dw \right|^p \right. \\ &\quad \left. + \left(\max\left\{ \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \kappa(w))dw \right|, \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \tau(w))dw \right| \right\} \right)^p \right) \\ &\leq 2s \sup_{l \in [0, L]} \left(\int_0^L |K_1(l, w, \kappa(w)) - K_1(l, w, \tau(w))|dw \right)^p \\ &\quad + \left(\max\left\{ \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \kappa(w))dw \right|, \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \tau(w))dw \right| \right\} \right)^p \\ &\leq \sup_{l \in [0, L]} |B\kappa(l) - B\tau(l)|^p + \left(\max\left\{ \sup_{l \in [0, L]} |B\kappa(l)|, \sup_{l \in [0, L]} |B\tau(l)| \right\} \right)^p \\ &\leq \mathfrak{p}(B\kappa, B\tau) \leq p(\kappa, \tau) < \varepsilon + \varepsilon, \end{aligned}$$

which implies $d(A\kappa, A\tau) < \frac{\varepsilon}{s}$. Then the condition (i) of Definition 3.7 is satisfied, and we also have

$$\begin{aligned}
& s^2 \mathfrak{p}(A\kappa, A\tau) \\
&= s^2 \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \kappa(w)) dw - \int_0^L K_1(l, w, \tau(w)) dw \right|^p \\
&\quad + s^2 (\max\{ \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \kappa(w)) ds \right|, \sup_{l \in [0, L]} \left| \int_0^L K_1(l, w, \tau(w)) ds \right| \})^p \\
&\leq s^2 \sup_{l \in [0, L]} \left(\int_0^L |K_1(l, w, \kappa(w)) - K_1(l, w, \tau(w))| dw \right)^p \\
&\quad + s^2 \frac{\kappa}{4s^2} (\max\{ \sup_{l \in [0, L]} \left| \int_0^L K_2(l, w, \kappa(w)) dw \right|, \sup_{l \in [0, L]} \left| \int_0^L K_2(l, w, \tau(w)) dw \right| \})^p \\
&\leq s^2 \sup_{l \in [0, L]} \left(\int_0^L \sqrt[p]{\frac{\kappa}{s^2}} h(l, w) dw \right)^p \sup_{l \in [0, L]} |B\kappa(l) - B\tau(l)|^p \\
&\quad + \frac{\kappa}{4} (\max\{ \sup_{l \in [0, L]} \left| \int_0^L K_2(l, w, \kappa(w)) dw \right|, \sup_{l \in [0, L]} \left| \int_0^L K_2(l, w, \tau(w)) dw \right| \})^p \\
&\leq s^2 \sup_{l \in [0, L]} \left(\int_0^L \sqrt[p]{\frac{\kappa}{s^2}} h(l, w) dw \right)^p \sup_{l \in [0, L]} |B\kappa(l) - B\tau(l)|^p \\
&\quad + \frac{\kappa}{4} (\max\{ \sup_{l \in [0, L]} |B\kappa(l)|, \sup_{l \in [0, L]} |B\tau(l)| \})^p.
\end{aligned}$$

By hypothesis (vii), there exists $\frac{1}{4} < \delta \leq 1$ such that $\int_0^L h(l, w) dw \leq \delta$. Thus

$$\begin{aligned}
s^2 \mathfrak{p}(A\kappa, A\tau) &\leq k\delta \left(\sup_{l \in [0, L]} |B\kappa(l) - B\tau(l)|^p + (\max\{ \sup_{l \in [0, L]} |B\kappa(l)|, \sup_{l \in [0, L]} |B\tau(l)| \})^p \right) \\
&= k\delta \mathfrak{p}(B\kappa, B\tau) \leq k\delta p(\kappa, \tau) = \varsigma ap(\kappa, \tau) = \varsigma aM(\kappa, \tau),
\end{aligned}$$

so $\xi(\varphi(\alpha(B\kappa, B\tau)\mathfrak{p}(A\kappa, A\tau)), \psi(M(\kappa, \tau))) \geq 0$. Therefore, all the conditions of Theorem 3.8 hold. As a result, A and B have unique point $u \in C[0, L]$, which is a solution to integral equation (4.1). $\square$

5. CONCLUSIONS

In this paper, we introduced the notions of generalized (α, φ, ψ) -Meir-Keeler hybrid contractive mappings of type I and II via simulation functions, and gave the sufficient conditions for the existence and uniqueness of common fixed points for such mappings in the framework of complete partial b -metric spaces. Moreover, we provided examples that elaborated the usability of our results. As an application, we presented the existence of solutions to an integral equation.

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