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SUPERCONVERGENCE OF FEM COMBINED WITH THE $L2-1_\sigma$ SCHEME FOR TIME FRACTIONAL SCHRÖDINGER EQUATIONS

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Abstract. This paper investigates a numerical solution of the time fractional Schrödinger equation (TFSE). We first decompose the original problem into a coupled system and construct its fully discrete approximation scheme based on the $L2-1_\sigma$ formula and the finite element method (FEM). We then analyze the stability of the fully discrete scheme. We also rigorously derive its convergence and superconvergence with further regularity assumptions. Finally, we provide several numerical experiments to confirm our theoretical statements.

Keywords. Finite element method; Superconvergence; Stability; Time fractional schrödinger equations.
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1. INTRODUCTION

In this paper, we introduce the $L2-1_\sigma$ Galerkin FEM for numerically solving the following two-dimensional linear TFSE:

$$\mathrm{i}\mathscr{D}_t^\alpha y(t,\mathbf{x})+\Delta y(t,\mathbf{x})-\beta(\mathbf{x})y(t,\mathbf{x})=f(t,\mathbf{x}),\quad t\in J,\mathbf{x}\in\Omega, \tag{1.1}$$

$$y(t,\mathbf{x})=0,\quad t\in J,\mathbf{x}\in\partial\Omega, \tag{1.2}$$

$$y(0,\mathbf{x})=g(\mathbf{x}),\quad \mathbf{x}\in\Omega, \tag{1.3}$$

where $\mathrm{i}=\sqrt{-1}$, $\alpha\in(0,1)$, $J=(0,T]$ with $T>0$, $\Omega\subset\mathbb{R}^2$ is a convex polygonal with bounded by $\partial\Omega$, the potential function $\beta(\mathbf{x})\in L^\infty(\Omega)$ is real-valued, $\beta(\mathbf{x})>0$, $f(t,\mathbf{x})$ and $g(\mathbf{x})$ are given continuous functions, and $\mathscr{D}_t^\alpha$ describes the Caputo derivative

$$\mathscr{D}_t^\alpha y(t,\cdot)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{1}{(t-s)^\alpha}\frac{\partial y(s,\cdot)}{\partial s}ds,$$

where $\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt$.

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TFSEs were used to describe a variety of phenomena and processes with memory and heritability in various fields. For instance, in quantum physics, Naber initially introduced TFSEs to study a free particle under non-Markovian evolution in [1], and Tofighi explored the motion of such particles in [2], and Iomin discovered that TFSEs were a specific example of the quantum comb model in [3]. However, exact solutions to such problems are almost impossible to obtain and one has to consider their numerical solutions.

Over the past two decades, numerous effective numerical methods for the discretization of time fractional derivatives were proposed. Among the most common methods are the $L1$ scheme [4], the $L2$ scheme [5, 6], the $L2-1_\sigma$ scheme [7], fast algorithm [8, 9, 10], and so on. As stated in [11], the convergence order of $L1$ scheme is $\mathscr{O}(\tau)$ since the solution may have initial singularity. Stynes et al. [12] proposed the $L1$ scheme on graded meshes to improve the accuracy of the $L1$ scheme. Liao et al. [13] presented the non-uniform $L1$ scheme and obtained the discrete fractional Grönwall-type inequality and sharp error estimates of the $L1$ scheme. Recently, the $L2-1_\sigma$ schme was employed extensively in the solution of a variety of time fractional partial differential equations (TFPDEs) with Caputo derivative [14, 15, 16, 17, 18, 19, 20].

The numerical solution and analysis of nonlinear TFSEs received increased attention in recent years. Bhrawy et al. [21] investigated a shifted Jacobi spectral collocation method for solving one-dimensional or multi-dimensional TFSEs. Liu et al. [22] proposed a numerical method for solving one-dimensional nonlinear TFSEs. In [23, 24], the authors presented two $L1$-Galerkin FEM schemes for nonlinear TFSEs. Li and Zhao [25] developed a fast FEM for the nonlinear TFSE with a wave operator. A fully discrete FEM [26] is available for linear TFSEs. Recently, some superconvergence analysis of FEM for different TFPDEs was obtained in [27, 28, 29, 30]. The purpose of this paper is to investigate the stability, convergence, and superconvergence of the Galerkin FEM combined with the $L2-1_\sigma$ formula to approximate linear TFSEs.

The paper is structured as follows. Section 2 describes in detail the construction of the fully discrete finite element approximation scheme for problem (1.1)-(1.3). Section 3 analyzes the stability of the fully discrete scheme. Section 4 contains the convergence results. Section 5 derives the superconvergence properties. Several numerical examples are presented in Section 6 to validate our theoretical findings. The last section, Section 7, ends this paper.

2. $L2-1_\sigma$ FEM APPROXIMATION OF TFSEs

In this section, we present a $L2-1_\sigma$ FEM approximation scheme to solve the TFSE (1.1)-(1.3). Let $L^2(\Omega)$ denote the standard Lebesgue space of real-valued functions defined on Ω with inner product

$$(\varphi,\omega)=\int_\Omega \varphi(\mathbf{x})\omega(\mathbf{x})d\mathbf{x},\quad \forall\varphi,\omega\in L^2(\Omega).$$

Let $W^{m,p}(\Omega)$ be the usual Sobolev space with the semi-norm $|\omega|_{m,p}^p=\sum_{|\alpha|=m}\|\mathscr{D}^\alpha\omega\|_{L^p(\Omega)}^p$ and norm $\|\omega\|_{m,p}^p=\sum_{|\alpha|\le m}\|\mathscr{D}^\alpha\omega\|_{L^p(\Omega)}^p$. For $p=2$, we set $H^m(\Omega)=W^{m,2}(\Omega)$ with norm $\|\cdot\|_m=\|\cdot\|_{m,2}$, $H_0^m(\Omega)=\{\omega\in H^m(\Omega),\omega|_{\partial\Omega}=0\}$, $W=H_0^1(\Omega)$ and $\|\cdot\|=\|\cdot\|_{0,2}$. For $q\in[1,\infty)$, we denote by $L^q(J;W^{m,p}(\Omega))$ the Banach space of all L^q measurable functions from J into $W^{m,p}(\Omega)$ with norm

$$\|\omega\|_{L^q(J;W^{m,p}(\Omega))}^q=\int_0^T\|\omega\|_{m,p}^q dt$$

and the standard modification for $q=\infty$.

For any complex-valued function $\psi(\mathbf{x})$, we denote its real part by $\psi_1(\mathbf{x})$ and imaginary part by $\psi_2(\mathbf{x})$, namely $\psi(\mathbf{x})=\psi_1(\mathbf{x})+\mathrm{i}\psi_2(\mathbf{x})$ and

$$\|\psi\|_{m,p}=\left(\|\psi_1\|_{m,p}^p+\|\psi_2\|_{m,p}^p\right)^{1/p},$$
$$\|\psi\|_{m,\infty}=\|\psi_1\|_{m,\infty}+\|\psi_2\|_{m,\infty}.$$

Then (1.1)-(1.3) can be decomposed into the following system of equations:

$$\mathscr{D}_t^\alpha y_2-\Delta y_1+\beta y_1=-f_1,\quad t\in J,\mathbf{x}\in\Omega, \tag{2.1}$$
$$\mathscr{D}_t^\alpha y_1+\Delta y_2-\beta y_2=f_2,\quad t\in J,\mathbf{x}\in\Omega, \tag{2.2}$$
$$y_j=0, j=1,2,\quad t\in J,\mathbf{x}\in\partial\Omega, \tag{2.3}$$
$$y_j(0,\mathbf{x})=g_j(\mathbf{x}), j=1,2,\quad \mathbf{x}\in\Omega. \tag{2.4}$$

The equivalent weak form of (2.1)-(2.4) is defined as follows: Find $y_1,y_2\in W$, such that

$$(\mathscr{D}_t^\alpha y_2,\omega)+(\nabla y_1,\nabla\omega)+(\beta y_1,\omega)=-(f_1,\omega),\quad \forall t\in J,\omega\in W, \tag{2.5}$$
$$(\mathscr{D}_t^\alpha y_1,\upsilon)-(\nabla y_2,\nabla\upsilon)-(\beta y_2,\upsilon)=(f_2,\upsilon),\quad \forall t\in J,\upsilon\in W, \tag{2.6}$$
$$y_j(0,\mathbf{x})=g_j(\mathbf{x}), j=1,2,\quad \mathbf{x}\in\Omega. \tag{2.7}$$

Let $\mathscr{T}_h$ be a regular triangulation partition of Ω with $h=\max_{E\in\mathscr{T}_h}\{h_E\}$, where h_E is the diameter of E. Furthermore, we set

$$W_h=\{\omega_h\in C(\bar{\Omega}):\omega_h|_E\in\mathbb{P}_1,\forall E\in\mathscr{T}_h,\omega_h|_{\partial\Omega}=0\},$$

where $\mathbb{P}_1$ is the space of polynomials whose degree at most 1.

Hence a semi-discrete FEM approximation scheme of (2.1)-(2.4) reads as: Find $y_{1h},y_{2h}\in W_h$ such that

$$(\mathscr{D}_t^\alpha y_{2h},\omega_h)+(\nabla y_{1h},\nabla\omega_h)+(\beta y_{1h},\omega_h)=-(f_1,\omega_h),\quad \forall t\in J,\omega_h\in W_h, \tag{2.8}$$
$$(\mathscr{D}_t^\alpha y_{1h},\upsilon_h)-(\nabla y_{2h},\nabla\upsilon_h)-(\beta y_{2h},\upsilon_h)=(f_2,\upsilon_h),\quad \forall t\in J,\upsilon_h\in W_h, \tag{2.9}$$
$$y_{jh}(0,\mathbf{x})=P_hg_j(\mathbf{x}), j=1,2,\quad \mathbf{x}\in\Omega, \tag{2.10}$$

where the L^2 projector [31] $P_h:L^2(\Omega)\to W_h$ fulfills:

$$(P_hv-v,\omega_h)=0,\quad \forall\omega_h\in W_h, v\in L^2(\Omega), \tag{2.11}$$
$$\|\nabla P_hv\|\le C\|\nabla v\|,\quad \forall v\in W, \tag{2.12}$$
$$\|P_hv-v\|_s\le Ch^{2-s}\|v\|_2, s=0,1,\quad \forall v\in H^2(\Omega)\cap W. \tag{2.13}$$

For ease of error analysis later, we introduce the Ritz projector [32] $R_h:W\to W_h$ with

$$\left(\nabla\left(R_hv-v\right),\nabla\omega_h\right)=0,\quad \forall\omega_h\in W_h, v\in W, \tag{2.14}$$
$$\|v-R_hv\|+h\|v-R_hv\|_1\le Ch^{k+1}\|v\|_{k+1},\quad \forall v\in H^{k+1}(\Omega)\cap W, \tag{2.15}$$

the discrete Laplacian operator [15, 27] $\Delta_h:W_h\to W_h$ with

$$(\Delta_hv,\omega_h)=-(\nabla v,\nabla\omega_h),\quad \forall\omega_h\in W_h, \tag{2.16}$$
$$\Delta_hR_hv=P_h\Delta v,\quad \forall v\in H^2(\Omega), \tag{2.17}$$

and the Lagrange interpolation operator [32] $I_h:H^2(\Omega)\to W_h$ with $I_hv(a_i)=v(a_i)$ for each $i=1,2,3$, where $a_i\,(i=1,2,3)$ are the vertices of E, for each $E\in\mathscr{T}_h$.

Since the solution of problem (2.1)-(2.4) may have weak singularities at $t=0$, we will use the $L2-1_\sigma$ formula based on graded meshes for temporal discretization. For a positive integer N, the graded time mesh [12] is $\{t_n = T(n/N)^r\}_{n=0}^N$, where $r\geq 1$. Set $\tau_{n+1}=t_{n+1}-t_n$, $t_{n+\sigma}=\sigma t_{n+1}+(1-\sigma)t_n$ with $0\leq\sigma\leq 1$ for $n=0,1,\ldots,N-1$. Furthermore, we set $v(t_n,\cdot)=v^n$, $\bar{v}^{n+\sigma}=\sigma v^{n+1}+(1-\sigma)v^n$ and $\tilde{v}^{n+\sigma}=\bar{v}^{n+\sigma}-v^{n+\sigma}$. As in [7], $\Pi_{1,n}v(t,\cdot)$ and $\Pi_{2,n}v(t,\cdot)$ are the linear interpolation of $v(t,\cdot)$ on $[t_{n-1},t_n]$ and the quadratic interpolation of $v(t,\cdot)$ on $[t_{n-1},t_{n+1}]$, respectively.

The graded $L2-1_\sigma$ formula of Caputo derivative is as follows [7]: For $n=0,1,\ldots,N-1$,

$$\begin{aligned}\mathscr{D}_N^\alpha v^{n+\sigma} &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{(t_{n+\sigma}-s)^\alpha}\frac{\partial \Pi_{2,k}v(s,\cdot)}{\partial s}ds\\ &\quad+\frac{1}{\Gamma(1-\alpha)}\int_{t_n}^{t_{n+\sigma}}\frac{1}{(t_{n+\sigma}-s)^\alpha}\frac{\partial \Pi_{1,k}v(s,\cdot)}{\partial s}ds\\ &=\sum_{k=0}^{n}A_{n-k}^{n+1}\left(v^{k+1}-v^k\right),\end{aligned}$$

where $A_{n+1}^{n+1}=0$, $A_0^1=\tau_1^{-1}a_{0,0}$ and, for $n\geq 1$,

$$A_{n-k}^{n+1}=\begin{cases}(a_{n,0}-b_{n,0})/\tau_{k+1}, & \text{if } k=0,\\ (a_{n,k}+b_{n,k-1}-b_{n,k})/\tau_{k+1}, & \text{if } 1\leq k\leq n-1,\\ (a_{n,n}+b_{n,n-1})\tau_{k+1}, & \text{if } k=n\end{cases}$$

with

$$\begin{aligned} a_{n,k}&=\frac{1}{\Gamma(1-\alpha)}\int_{t_k}^{t_{k+1}}\frac{1}{(t_{n+\sigma}-s)^\alpha}ds, && \text{if } n\geq 1 \text{ and } 0\leq k\leq n-1,\\ a_{n,n}&=\frac{1}{\Gamma(1-\alpha)}\int_{t_n}^{t_{n+\sigma}}\frac{1}{(t_{n+\sigma}-s)^\alpha}ds, && \text{if } n\geq 0,\\ b_{n,k}&=\frac{2}{(t_{k+2}-t_k)\Gamma(1-\alpha)}\int_{t_k}^{t_{k+1}}\frac{s-t_{k+\frac12}}{(t_{n+\sigma}-s)^\alpha}ds, && \text{if } n\geq 1 \text{ and } 0\leq k\leq n-1.\end{aligned}$$

Then, imitating [20, 28], the $L2-1_\sigma$ FEM approximation scheme of (2.1)-(2.4) is given by: For $n=1,2,\ldots,N$, find $y_{1h}^n, y_{2h}^n\in W_h$, such that

$$\left(\mathscr{D}_N^\alpha y_{2h}^{n+\sigma},\omega_h\right)+\left(\nabla\bar{y}_{1h}^{n+\sigma},\nabla\omega_h\right)+\left(\beta\bar{y}_{1h}^{n+\sigma},\omega_h\right)=-\left(f_1^{n+\sigma},\omega_h\right),\quad \forall\omega_h\in W_h, \tag{2.18}$$

$$\left(\mathscr{D}_N^\alpha y_{1h}^{n+\sigma},\upsilon_h\right)-\left(\nabla\bar{y}_{2h}^{n+\sigma},\nabla\upsilon_h\right)-\left(\beta\bar{y}_{2h}^{n+\sigma},\upsilon_h\right)=\left(f_2^{n+\sigma},\upsilon_h\right),\quad \forall\upsilon_h\in W_h, \tag{2.19}$$

$$y_{jh}^0=P_hg_j,\, j=1,2. \tag{2.20}$$

In view of [30], we can construct a complementary discrete convolution kernel Q_{n-k}^{n+1} which satisfies

$$\sum_{k=j}^{n}Q_{n-k}^{n+1}A_{k-j}^{k+1}\equiv 1,\quad 0\leq k\leq n\leq N-1.$$

Rearranging this identity, one has

$$Q_{n-k}^{n+1} = \begin{cases} \dfrac{1}{A_0^{k+1}} \displaystyle\sum_{j=k+1}^{n} \left(A_{j-k-1}^{j+1} - A_{j-k}^{j+1} \right) Q_{n-j}^{n+1}, & 0 \le k \le n-1, \\ \dfrac{1}{A_0^{n+1}}, & k = n, \end{cases}$$

and

$$\max_{0 \le n \le N-1} \sum_{k=0}^{n} Q_{n-k}^{n+1} \le C t_{n+1}^{\alpha}. \tag{2.21}$$

We define the truncation error

$$\left| R_v^{n+\sigma} \right| = \left| \mathscr{D}_N^{\alpha} v^{n+\sigma} - \mathscr{D}_t^{\alpha} v^{n+\sigma} \right|$$

and

$$\begin{aligned} \psi_v^{\sigma} &= \tau_1^{\alpha} \sup_{t \in (0,t_1)} \left(t^{1-\alpha} |\mathscr{D}_N^{\alpha} v^0 - v'(t)| \right), \\ \psi_v^{n+\sigma} &= \tau_{n+1}^{3-\alpha} t_{n+\sigma}^{\alpha} \sup_{t \in (t_n, t_{n+1})} |v'''(t)|, \quad \text{for } n = 1, 2, \ldots, N-1, \\ \psi_v^{n,1} &= \tau_1^{\alpha} \sup_{t \in (0,t_1)} \left(t^{1-\alpha} |(\Pi_{2,1} v(t))' - v'(t)| \right), \quad \text{for } n = 1, 2, \ldots, N-1, \\ \psi_v^{n,k} &= \tau_{n+1}^{-\alpha} \tau_k^2 (\tau_k + \tau_{k+1}) t_k^{\alpha} \sup_{t \in (t_{k-1}, t_{k+1})} |v'''(t)|, \quad \text{for } 2 \le k \le n \le N-1. \end{aligned}$$

The following lemmas are used later in the analysis of the stability and errors.

Lemma 2.1. *([7, Corollary 1]) Let $0 \le \sigma \le 1$. Then, for any functions $v(t,\cdot)$ defined on the graded mesh $\{t_k\}_{k=0}^N$ and $n = 0, 1, \ldots, N-1$,*

$$\left(\mathscr{D}_N^{\alpha} v^{n+\sigma}, \bar{v}^{n+\sigma} \right) \ge \frac{1}{2} \sum_{k=0}^{n} A_{n-k}^{n+1} \left(\|v^{k+1}\|^2 - \|v^k\|^2 \right).$$

Lemma 2.2. *([33, Lemma 1]) Let $\sigma = 1 - \alpha/2$. Then, for any function $v(t,\cdot) \in C^3(0,T]$ and $n = 0, 1, \ldots, N-1$,*

$$\left| R_v^{n+\sigma} \right| \le C t_{n+\sigma}^{-\alpha} \Big(\psi_v^{n+\sigma} + \max_{1 \le k \le n} \{ \psi_v^{n,k} \} \Big).$$

Lemma 2.3. *([33, Lemma 7]) Let $v(t,\cdot) \in C[0,T] \cap C^3(0,T]$. Assume that $|v^{(l)}(t,\cdot)| \le C \left(1 + t^{\alpha - l} \right)$ for $l = 0, 1, 2, 3$ and $t \in (0,T]$. Then, for $p = \min\{r\alpha, 3-\alpha\}$,*

$$\begin{aligned} \psi_v^{n+\sigma} &\le C N^{-p}, \quad \textit{for } n = 0, 1, \ldots, N-1, \\ \psi_v^{n,k} &\le C N^{-p}, \quad \textit{for } k = 1, 2, \ldots, n, \textit{ when } n \ge 1. \end{aligned}$$

Lemma 2.4. *([34, Lemma 3.4],[30, Lemma 2.3]) Let $|v''(t,\cdot)| \le C \left(1 + t^{\alpha-2} \right)$ for $0 < t \le T$ and $v(t,\cdot) \in L^{\infty}(J; H_0^1(\Omega) \cap H^2(\Omega))$. Then the global consistency error satisfies*

$$\sum_{j=0}^{n-1} Q_{n-j}^{n} |\tilde{v}^{j+\sigma}| \le C \Big(\tau_1^{2\alpha} + t_n^{\alpha} \max_{1 \le k \le n-1} t_k^{\alpha-2} \tau_{k+1}^2 \Big), \quad \textit{for } 1 \le n \le N.$$

Lemma 2.5. *([30, Lemma 2.1]) Let $\frac{1}{2} < \sigma \le 1$. Let $\{\xi^{n+1}\}_{n=0}^{N-1}$, $\{\eta^{n+1}\}_{n=0}^{N-1}$, and $\{\lambda_l\}_{l=0}^{N-1}$ be the nonnegative sequences. If there exists a constant Λ such that $\sum_{l=0}^{N-1}\lambda_l \le \Lambda$, and the maximum time step size satisfies $\tau_N \le 1/\sqrt[\alpha]{2\pi_A\Gamma(2-\alpha)\Lambda}$, then, for any nonnegative grid function $\{v^{n+1}\}_{n=0}^{N-1}$ and $n \ge 0$ satisfy*

$$\sum_{k=0}^{n} A_{n-k}^{n+1}\Big(\left(v^{k+1}\right)^2-\left(v^{k}\right)^2\Big) \le \sum_{k=0}^{n}\lambda_{n-k}\left(\bar{v}^{k,\sigma}\right)^2+\xi^{n+1}\bar{v}^{n,\sigma}+\left(\eta^{n+1}\right)^2.$$

It holds that, for $n=0,1,\dots,N-1$,

$$v^{n+1} \le 2E_\alpha(2\pi_A\Lambda t_{n+1}^{\alpha})\Big(v^0+\max_{0\le k\le n-1}\sum_{j=0}^{k}Q_{k-j}^{k+1}\xi^{j+1}+\sqrt{\pi_A\Gamma(1-\alpha)}\max_{0\le k\le n}t_{k+1}^{\alpha/2}\{\eta^{k+1}\}\Big).$$

3. Stability Analysis

In this section, we discuss the stability of the approximation scheme (2.18)-(2.20).

Theorem 3.1. *($L^2(\Omega)$-Stability) Let $\left(y_{1h}^n, y_{2h}^n\right), n=1,2,\dots,N$ be the solution of (2.18)-(2.20). Assume that all the conditions in Lemma 2.5 are valid. Then,*

$$\left(\|y_{1h}^n\|^2+\|y_{2h}^n\|^2\right)^{\frac{1}{2}} \le C\Big(\left(\|y_{1h}^0\|^2+\|y_{2h}^0\|^2\right)^{\frac{1}{2}}+\max_{1\le k\le n-1}\left\{(\|f_1^{k+\sigma}\|^2+\|f_2^{k+\sigma}\|^2)^{\frac{1}{2}}\right\}\Big). \tag{3.1}$$

Proof. For $n=1,2,\dots,N-1$, taking $\omega_h=\bar{y}_{2h}^{n+\sigma}$ in (2.18) and $\upsilon_h=\bar{y}_{1h}^{n+\sigma}$ in (2.19), we obtain

$$\left(\mathscr{D}_N^\alpha y_{2h}^{n+\sigma}, \bar{y}_{2h}^{n+\sigma}\right)+\left(\nabla\bar{y}_{1h}^{n+\sigma},\nabla\bar{y}_{2h}^{n+\sigma}\right)+\left(\beta\bar{y}_{1h}^{n+\sigma},\bar{y}_{2h}^{n+\sigma}\right)=-\left(f_1^{n+\sigma},\bar{y}_{2h}^{n+\sigma}\right)$$

and

$$\left(\mathscr{D}_N^\alpha y_{1h}^{n+\sigma}, \bar{y}_{1h}^{n+\sigma}\right)-\left(\nabla\bar{y}_{2h}^{n+\sigma},\nabla\bar{y}_{1h}^{n+\sigma}\right)-\left(\beta\bar{y}_{2h}^{n+\sigma},\bar{y}_{1h}^{n+\sigma}\right)=\left(f_2^{n+\sigma},\bar{y}_{1h}^{n+\sigma}\right).$$

Adding the two equations above, we find by Lemma 2.1, the Hölder inequality, and the Cauchy-Schwarz inequality that

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|y_{1h}^{k+1}\|^2-\|y_{1h}^{k}\|^2\right)+\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|y_{2h}^{k+1}\|^2-\|y_{2h}^{k}\|^2\right)\\
\le& C\left(\|f_1^{n+\sigma}\|\|\bar{y}_{2h}^{n+\sigma}\|+\|f_2^{n+\sigma}\|\|\bar{y}_{1h}^{n+\sigma}\|\right)\\
\le& C\left(\|f_1^{n+\sigma}\|^2+\|f_2^{n+\sigma}\|^2\right)^{\frac{1}{2}}\left(\|\bar{y}_{1h}^{n+\sigma}\|^2+\|\bar{y}_{2h}^{n+\sigma}\|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.2}$$

Then (3.1) follows from (3.2) and Lemma 2.5. □

Theorem 3.2. *($H^1(\Omega)$-Stability) Let $\left(y_{1h}^n, y_{2h}^n\right), n=1,2,\dots,N$ be the solution of (2.18)-(2.20). Suppose all the conditions in Lemma 2.5 hold and $f_1, f_2 \in L^\infty(J;H^1(\Omega))$. Then,*

$$\left(\|y_{1h}^n\|_1^2+\|y_{2h}^n\|_1^2\right)^{\frac{1}{2}} \le C\Big(\left(\|y_{1h}^0\|_1^2+\|y_{2h}^0\|_1^2\right)^{\frac{1}{2}}+\max_{1\le k\le n-1}\left\{(\|f_1^{k+\sigma}\|_1^2+\|f_2^{k+\sigma}\|_1^2)^{\frac{1}{2}}\right\}\Big). \tag{3.3}$$

Proof. By choosing $\omega_h=-\Delta_h\bar{y}_{2h}^{n+\sigma}$ in (2.18) and $\upsilon_h=-\Delta_h\bar{y}_{1h}^{n+\sigma}$ in (2.19), and utilizing the definition of P_h, we have

$$\begin{aligned}
&\left(\mathscr{D}_N^\alpha y_{2h}^{n+\sigma},\Delta_h\bar{y}_{2h}^{n+\sigma}\right)-\left(\Delta_h\bar{y}_{1h}^{n+\sigma},\Delta_h\bar{y}_{2h}^{n+\sigma}\right)+\left(\beta\bar{y}_{1h}^{n+\sigma},\Delta_h\bar{y}_{2h}^{n+\sigma}\right)=-\left(P_hf_1^{n+\sigma},\Delta_h\bar{y}_{2h}^{n}\right),\\
&\left(\mathscr{D}_N^\alpha y_{1h}^{n+\sigma},\Delta_h\bar{y}_{1h}^{n+\sigma}\right)+\left(\Delta_h\bar{y}_{2h}^{n+\sigma},\Delta_h\bar{y}_{1h}^{n+\sigma}\right)-\left(\beta\bar{y}_{2h}^{n+\sigma},\Delta_h\bar{y}_{1h}^{n+\sigma}\right)=\left(P_hf_2^{n+\sigma},\Delta_h\bar{y}_{1h}^{n+\sigma}\right).
\end{aligned}$$

Recalling the definition of Δ_h, we see that

$$\left(\mathscr{D}_N^\alpha \nabla y_{2h}^{n+\sigma}, \nabla \bar{y}_{2h}^{n+\sigma}\right) - \left(\Delta_h \bar{y}_{1h}^{n+\sigma}, \Delta_h \bar{y}_{2h}^{n+\sigma}\right) = \left(\nabla(\beta \bar{y}_{1h}^{n+\sigma}) + \nabla P_h f_1^{n+\sigma}, \nabla \bar{y}_{2h}^{n+\sigma}\right), \tag{3.4}$$

$$\left(\mathscr{D}_N^\alpha \nabla y_{1h}^{n+\sigma}, \nabla \bar{y}_{1h}^{n+\sigma}\right) + \left(\Delta_h \bar{y}_{2h}^{n+\sigma}, \Delta_h \bar{y}_{1h}^{n+\sigma}\right) = -\left(\nabla(\beta \bar{y}_{2h}^{n+\sigma}) + \nabla P_h f_2^{n+\sigma}, \nabla \bar{y}_{1h}^{n+\sigma}\right). \tag{3.5}$$

Adding (3.4) and (3.5) and using Lemma 2.1, (2.19), the Hölder inequality, the Cauchy-Schwarz inequality, and the Poincaré inequality, we arrive at

$$\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{n} A_{n-k}^{n+1}\left(\|\nabla y_{1h}^{k+1}\|^2 - \|\nabla y_{1h}^{k}\|^2\right) + \frac{1}{2}\sum_{k=0}^{n} A_{n-k}^{n+1}\left(\|\nabla y_{2h}^{k+1}\|^2 - \|\nabla y_{2h}^{k}\|^2\right) \\
\leq & C\left(\|\nabla f_1^{n+\sigma}\|^2 + \|\nabla f_2^{n+\sigma}\|^2\right)^{\frac{1}{2}}\left(\|\nabla \bar{y}_{1h}^{n+\sigma}\|^2 + \|\nabla \bar{y}_{2h}^{n+\sigma}\|^2\right)^{\frac{1}{2}} \\
\leq & C\left(\|\nabla f_1^{n+\sigma}\|^2 + \|\nabla f_2^{n+\sigma}\|^2\right)^{\frac{1}{2}}\left(\|\bar{y}_{1h}^{n+\sigma}\|_1^2 + \|\bar{y}_{2h}^{n+\sigma}\|_1^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.6}$$

Then (3.3) follows from Lemma 2.5 and (3.6). □

4. Convergence Analysis

In this section, we analyze the convergence of the $L2-1_\sigma$ FEM approximation scheme (2.18)-(2.20) for problem (1.1)-(1.3).

For $j=1,2$, we set

$$\zeta_{y_j} = P_h y_j - y_{jh}, \xi_{y_j} = P_h y_j - y_j$$

and

$$\rho_{y_j} = R_h y_j - y_{jh}, \rho_{y_j} = R_h y_j - y_j.$$

Theorem 4.1. *($L^2(\Omega)$-Convergence) Let (y_1,y_2) and $\left(y_{1h}^n, y_{2h}^n\right), n=1,2,\ldots,N$ be the solutions of (2.5)-(2.7) and (2.18)-(2.20) respectively. Assume that all the conditions in Lemmas 2.2-2.5 are satisfied, $y_1,y_2 \in L^\infty(J;H_0^1(\Omega)) \cap L^\infty(J;H^2(\Omega))$, $y_{1t},y_{2t} \in L^\infty(J;H^2(\Omega))$, and*

$$\left|\frac{\partial^l y_1}{\partial t^l}\right| + \left|\frac{\partial^l y_2}{\partial t^l}\right| \leq \left(1 + t^{\alpha - l}\right), \quad l = 0,1,2.$$

Then, for $n=0,1,\ldots,N-1$,

$$\left(\left\|y_1^{n+1} - y_{1h}^{n+1}\right\|^2 + \left\|y_2^{n+1} - y_{2h}^{n+1}\right\|^2\right)^{\frac{1}{2}} \leq C\left(h^2 + N^{-\min\{r\alpha,2\}}\right). \tag{4.1}$$

Proof. Subtracting (2.18)-(2.19) from (2.5)-(2.6) and using the definition of P_h and R_h, we obtain the following error equations:

$$\begin{aligned}
&\left(\mathscr{D}_N^\alpha \zeta_{y_2}^{n+\sigma}, \omega_h\right) + \left(\nabla \bar{\rho}_{y_1}^{n+\sigma}, \nabla \omega_h\right) + \left(\beta \bar{\zeta}_{y_1}^{n+\sigma}, \omega_h\right) \\
=& \left(R_{y_2}^{n+\sigma}, \omega_h\right) + \left(\nabla \tilde{y}_1^{n+\sigma}, \nabla \omega_h\right) + \left(\beta \tilde{y}_1^{n+\sigma}, \omega_h\right), \quad \forall \omega_h \in W_h,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\left(\mathscr{D}_N^\alpha \zeta_{y_1}^{n+\sigma}, \upsilon_h\right) - \left(\nabla \bar{\rho}_{y_2}^{n+\sigma}, \nabla \upsilon_h\right) - \left(\beta \bar{\zeta}_{y_2}^{n+\sigma}, \upsilon_h\right) \\
=& \left(R_{y_1}^{n+\sigma}, \upsilon_h\right) - \left(\nabla \tilde{y}_2^{n+\sigma}, \nabla \upsilon_h\right) - \left(\beta \tilde{y}_2^{n+\sigma}, \upsilon_h\right), \quad \forall \upsilon_h \in W_h.
\end{aligned} \tag{4.3}$$

Choosing $\omega_h=\bar{\zeta}_{y_2}^{n+\sigma}$ in (4.2) and $\upsilon_h=\bar{\zeta}_{y_1}^{n+\sigma}$ in (4.3), adding the two equations and using (2.16)-(2.17), we have

$$\begin{aligned}&\left(\mathscr{D}_N^\alpha \zeta_{y_1}^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)+\left(\mathscr{D}_N^\alpha \zeta_{y_2}^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)\\ =&\left(R_{y_2}^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)+\left(\nabla\tilde{y}_1^{n+\sigma},\nabla\bar{\zeta}_{y_2}^{n+\sigma}\right)+\left(\beta\tilde{y}_1^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)\\ &+\left(R_{y_1}^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)-\left(\nabla\tilde{y}_2^{n+\sigma},\nabla\bar{\zeta}_{y_1}^{n+\sigma}\right)-\left(\beta\tilde{y}_2^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right).\end{aligned}\tag{4.4}$$

It follows from Lemma 2.1 that

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|\zeta_{y_1}^{k+1}\|^2-\|\zeta_{y_1}^{k}\|^2\right)+\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|\zeta_{y_2}^{k+1}\|^2-\|\zeta_{y_2}^{k}\|^2\right)\\ \le&\left(\mathscr{D}_N^\alpha \zeta_{y_1}^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)+\left(\mathscr{D}_N^\alpha \zeta_{y_2}^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right).\end{aligned}\tag{4.5}$$

According to Hölder inequality, Lemma 2.2, and Lemma 2.3, we obtain

$$\left(R_{y_1}^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)\le C\|R_{y_1}^{n+\sigma}\|\|\bar{\zeta}_{y_1}^{n+\sigma}\|\le Ct_{n+\sigma}^{-\alpha}N^{-p}\|\bar{\zeta}_{y_1}^{n+\sigma}\|\tag{4.6}$$

and

$$\left(R_{y_2}^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)\le C\|R_{y_2}^{n+\sigma}\|\|\bar{\zeta}_{y_2}^{n+\sigma}\|\le Ct_{n+\sigma}^{-\alpha}N^{-p}\|\bar{\zeta}_{y_2}^{n+\sigma}\|.\tag{4.7}$$

Utilizing Green formula and Hölder inequality, we have

$$\begin{aligned}\left(\nabla\tilde{y}_1^{n+\sigma},\nabla\bar{\zeta}_{y_2}^{n+\sigma}\right)+\left(\beta\tilde{y}_1^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)&=\left(\beta\tilde{y}_1^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)-\left(\Delta\tilde{y}_1^{n+\sigma},\bar{\zeta}_{y_2}^{n+\sigma}\right)\\ &\le C\left(\|\tilde{y}_1^{n+\sigma}\|+\|\Delta\tilde{y}_1^{n+\sigma}\|\right)\|\bar{\zeta}_{y_2}^{n+\sigma}\|,\end{aligned}\tag{4.8}$$

$$\begin{aligned}\left(\nabla\tilde{y}_2^{n+\sigma},\nabla\bar{\zeta}_{y_1}^{n+\sigma}\right)+\left(\beta\tilde{y}_2^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)&=\left(\beta\tilde{y}_2^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)-\left(\Delta\tilde{y}_2^{n+\sigma},\bar{\zeta}_{y_1}^{n+\sigma}\right)\\ &\le C\left(\|\tilde{y}_2^{n+\sigma}\|+\|\Delta\tilde{y}_2^{n+\sigma}\|\right)\|\bar{\zeta}_{y_1}^{n+\sigma}\|.\end{aligned}\tag{4.9}$$

Collecting (4.4)-(4.9) and employing Cauchy inequality, we arrive at

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|\zeta_{y_1}^{k+1}\|^2-\|\zeta_{y_1}^{k}\|^2\right)+\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|\zeta_{y_2}^{k+1}\|^2-\|\zeta_{y_2}^{k}\|^2\right)\\ \le&C\Big(2t_{n+\sigma}^{-\alpha}N^{-p}+\sum_{j=1}^{2}\left(\|\tilde{y}_j^{n+\sigma}\|+\|\Delta\tilde{y}_j^{n+\sigma}\|\right)\Big)\left(\|\bar{\zeta}_{y_1}^{n+\sigma}\|+\|\bar{\zeta}_{y_2}^{n+\sigma}\|\right).\end{aligned}\tag{4.10}$$

It follows from (4.10), Lemma 2.4, and Lemma 2.5 that

$$\begin{aligned}&\left(\|\zeta_{y_1}^{n+1}\|^2+\|\zeta_{y_2}^{n+1}\|^2\right)^{\frac{1}{2}}\\ \le&C\Big\{\left(\|\zeta_{y_1}^{0}\|^2+\|\zeta_{y_2}^{0}\|^2\right)^{\frac{1}{2}}+\max_{0\le k\le n-1}\sum_{l=0}^{k}Q_{k-l}^{k+1}\left(2t_{n+\sigma}^{-\alpha}N^{-p}+\sum_{j=1}^{2}(\|\tilde{y}_j^{l+\sigma}\|+\|\Delta\tilde{y}_j^{l+\sigma}\|)\right)\Big\}\\ \le&C\Big\{\left(\|\zeta_{y_1}^{0}\|^2+\|\zeta_{y_2}^{0}\|^2\right)^{\frac{1}{2}}+N^{-\min\{r\alpha,2\}}\Big\}.\end{aligned}\tag{4.11}$$

Then (4.1) follows from (2.13), (4.11), and triangle inequality. □

By taking $\omega_h=-\Delta_h\bar{\zeta}_{y_2}^{n+\sigma}$ in (4.2) and $\upsilon_h=-\Delta_h\bar{\zeta}_{y_1}^{n+\sigma}$ in (4.3) and imitating part of the proof of Theorem 4.1, we can derive the following H^1-norm convergence result.

Theorem 4.2. *($H^1(\Omega)$-Convergence) Let (y_1,y_2) and $\left(y_{1h}^n,y_{2h}^n\right),n=1,2,\ldots,N$ be the solutions of (2.5)-(2.7) and (2.18)-(2.20) respectively. Suppose that all the conditions in Theorem 4.1 hold. Then, for $n=0,1,\ldots,N-1$,*

$$\left(\|y_1^{n+1}-y_{1h}^{n+1}\|_1^2+\|y_2^{n+1}-y_{2h}^{n+1}\|_1^2\right)^{\frac{1}{2}}\leq C\left(h+N^{-\min\{r\alpha,2\}}\right).$$

5. Superclose and Superconvergence Analysis

In this section, we derive the sperclose and global superconvergence of the numerical solution of (2.18)-(2.20).

Theorem 5.1. *($H^1(\Omega)$-Supercolse) Let (y_1,y_2) and $\left(y_{1h}^n,y_{2h}^n\right),n=1,2,\ldots,N$ be the solutions of (2.5)-(2.7) and (2.18)-(2.20) respectively. Assume that all the conditions in Theorem 4.2 are valid, $y_1,y_2\in L^\infty(J;H^3(\Omega))$. Then, for $n=0,1,\ldots,N-1$,*

$$\left(\|R_hy_1^{n+1}-y_{1h}^{n+1}\|_1^2+\|R_hy_2^{n+1}-y_{2h}^{n+1}\|_1^2\right)^{\frac{1}{2}}\leq C\left(h^2+N^{-\min\{r\alpha,2\}}\right). \tag{5.1}$$

Proof. From (2.5)-(2.6) and (2.18)-(2.19), we obtain the following error equations: For any $\omega_h\in W_h$ and any $\upsilon_h\in W_h$,

$$\begin{aligned}&\left(\mathscr{D}_N^\alpha\rho_{y_2}^{n+\sigma},\omega_h\right)+\left(\nabla\bar{\rho}_{y_1}^{n+\sigma},\nabla\omega_h\right)+\left(\beta\bar{\rho}_{y_1}^{n+\sigma},\omega_h\right)\\&=\left(\mathscr{D}_N^\alpha\rho_{y_2}^{n+\sigma},\omega_h\right)+\left(\beta\bar{\rho}_{y_1}^{n+\sigma},\omega_h\right)+\left(R_{y_2}^{n+\sigma},\omega_h\right)+\left(\nabla\tilde{y}_1^{n+\sigma},\nabla\omega_h\right)+\left(\beta\tilde{y}_1^{n+\sigma},\omega_h\right),\end{aligned}\tag{5.2}$$

$$\begin{aligned}&\left(\mathscr{D}_N^\alpha\rho_{y_1}^{n+\sigma},\upsilon_h\right)-\left(\nabla\bar{\rho}_{y_2}^{n+\sigma},\nabla\upsilon_h\right)-\left(\beta\bar{\rho}_{y_2}^{n+\sigma},\upsilon_h\right)\\&=\left(\mathscr{D}_N^\alpha\rho_{y_1}^{n+\sigma},\upsilon_h\right)-\left(\beta\bar{\rho}_{y_2}^{n+\sigma},\upsilon_h\right)+\left(R_{y_1}^{n+\sigma},\upsilon_h\right)-\left(\nabla\tilde{y}_2^{n+\sigma},\nabla\upsilon_h\right)-\left(\beta\tilde{y}_2^{n+\sigma},\upsilon_h\right).\end{aligned}\tag{5.3}$$

Choosing $\omega_h=-\Delta_h\bar{\rho}_{y_2}^{n+\sigma}$ in (5.2) and $\upsilon_h=-\Delta_h\bar{\rho}_{y_1}^{n+\sigma}$ in (5.3), we arrive at

$$\begin{aligned}&\left(\mathscr{D}_N^\alpha\nabla\rho_{y_2}^{n+\sigma},\nabla\bar{\rho}_{y_2}^{n+\sigma}\right)+\left(\mathscr{D}_N^\alpha\nabla\rho_{y_1}^{n+\sigma},\nabla\bar{\rho}_{y_1}^{n+\sigma}\right)\\&=\left(\mathscr{D}_N^\alpha\nabla\rho_{y_2}^{n+\sigma},\nabla\bar{\rho}_{y_2}^{n+\sigma}\right)+\left(\nabla R_{y_2}^{n+\sigma},\nabla\bar{\rho}_{y_2}^{n+\sigma}\right)-\left(\nabla(\Delta_h\tilde{y}_1^{n+\sigma}),\nabla\bar{\rho}_{y_2}^{n+\sigma}\right)\\&\quad+\left(\nabla(\beta\tilde{y}_1^{n+\sigma}),\nabla\bar{\rho}_{y_2}^{n+\sigma}\right)+\left(\mathscr{D}_N^\alpha\nabla\rho_{y_1}^{n+\sigma},\nabla\bar{\rho}_{y_1}^{n+\sigma}\right)+\left(\nabla R_{y_1}^{n+\sigma},\nabla\bar{\rho}_{y_1}^{n+\sigma}\right)\\&\quad+\left(\nabla(\Delta_h\tilde{y}_2^{n+\sigma}),\nabla\bar{\rho}_{y_1}^{n+\sigma}\right)-\left(\nabla(\beta\tilde{y}_2^{n+\sigma}),\nabla\bar{\rho}_{y_1}^{n+\sigma}\right).\end{aligned}\tag{5.4}$$

According to (5.4), Hölder inequality, and Lemma 2.1, we derive

$$\begin{aligned}&\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|\rho_{y_1}^{k+1}\|^2-\|\rho_{y_1}^{k}\|^2\right)+\frac{1}{2}\sum_{k=0}^{n}A_{n-k}^{n+1}\left(\|\rho_{y_2}^{k+1}\|^2-\|\rho_{y_2}^{k}\|^2\right)\\&\leq\left(\mathscr{D}_N^\alpha\|\nabla\rho_{y_2}^{n+\sigma}\|+\|\nabla R_{y_2}^{n+\sigma}\|+\|\nabla(\Delta_h\tilde{y}_1^{n+\sigma})\|+\|\nabla(\beta\tilde{y}_1^{n+\sigma})\|\right)\|\nabla\bar{\rho}_{y_2}^{n+\sigma}\|\\&\quad+\left(\mathscr{D}_N^\alpha\|\nabla\rho_{y_2}^{n+\sigma}\|+\|\nabla R_{y_2}^{n+\sigma}\|+\|\nabla(\Delta_h\tilde{y}_1^{n+\sigma})\|+\|\nabla(\beta\tilde{y}_1^{n+\sigma})\|\right)\|\nabla\bar{\rho}_{y_1}^{n+\sigma}\|.\end{aligned}\tag{5.5}$$

Combining (5.5), (2.15), and lemmas 2.2-2.5, we obtain (5.1) immediately. □

Let $\mathscr{T}_H$ be a regular triangulation partition of Ω with $H=2h$ and

$$W_H=\{\omega_H\in C(\bar{\Omega}):\omega_H|_{E'}\in\mathbb{P}_2,\forall E'\in\mathscr{T}_H,\omega_H|_{\partial\Omega}=0\},$$

where $\mathbb{P}_2$ is the space of polynomials up to order 2. Furthermore, we introduce a higher order interpolation postprocessing operator [31] $I_{2h}^2 : C(\bar{\Omega}) \to W_H$. It fulfills the following properties:

$$I_{2h}^2 I_h \omega = I_{2h}^2 \omega, \quad \forall \omega \in H^2(\Omega), \tag{5.6}$$

$$\left\|\omega - I_{2h}^2 \omega\right\|_1 \le Ch^2 \|\omega\|_3, \quad \forall\, \omega \in H^3(\Omega), \tag{5.7}$$

$$\left\|I_{2h}^2 \omega\right\|_1 \le C\|\omega\|_1, \quad \forall\, \omega \in W_h. \tag{5.8}$$

Theorem 5.2. *($H^1(\Omega)$-Superconvergence) Let (y_1, y_2) and $\left(y_{1h}^n, y_{2h}^n\right), n = 1, 2, \ldots, N$ be the solutions of (2.5)-(2.7) and (2.18)-(2.20) respectively. Suppose that all the conditions in Theorem 5.1 hold. Then, for $n = 0, 1, \ldots, N-1$,*

$$\left(\left\|y_1^{n+1} - I_{2h}^2 y_{1h}^{n+1}\right\|_1^2 + \left\|y_2^{n+1} - I_{2h}^2 y_{2h}^{n+1}\right\|_1^2\right)^{\frac{1}{2}} \le C\left(h^2 + N^{-\min\{r\alpha, 2\}}\right). \tag{5.9}$$

Proof. According to [35, Theorem 2.1.1], we have

$$\left\|I_h y_{1h}^{n+1} - R_h y_1^{n+1}\right\|_1 \le Ch^2 \|y_1^{n+1}\|_3, \tag{5.10}$$

$$\left\|I_h y_{2h}^{n+1} - R_h y_2^{n+1}\right\|_1 \le Ch^2 \|y_2^{n+1}\|_3. \tag{5.11}$$

From the properties (5.6)-(5.8) of I_{2h}, (5.10)-(5.11) and Theorem 5.1, we obtain

$$\begin{aligned}
&\left\|y_1^{n+1} - I_{2h}^2 y_1^{n+1}\right\|_1 \\
=&\|y_1^{n+1} - I_{2h}^2 y_1^{n+1} + I_{2h}^2 (I_h y_1^{n+1} - R_h y_1^{n+1}) - I_{2h}^2 (R_h y_1^{n+1} - y_{1h}^{n+1})\|_0 \\
\le&\|y_1^{n+1} - I_{2h}^2 y_1^{n+1}\|_1 + \|I_{2h}^2 (I_h y_1^{n+1} - R_h y_1^{n+1})\|_1 + \|I_{2h}^2 (R_h y_1^{n+1} - y_{1h}^{n+1})\|_1 \\
\le&C\left(h^2 + N^{-\min\{r\alpha, 2\}}\right).
\end{aligned} \tag{5.12}$$

Similar to (5.12), we have

$$\left\|y_2^{n+1} - I_{2h}^2 y_2^{n+1}\right\|_1 \le C\left(h^2 + N^{-\min\{r\alpha, 2\}}\right). \tag{5.13}$$

Then (5.9) follows from (5.12) and (5.13) immediately. $\square$

6. Numerical Experiments

In this section, we perform some numerical experiments to demonstrate our theoretical findings. All numerical examples are solved by the scheme described by (2.18)-(2.20), where the program codes are based on AFEPack [36]). Let $J = (0, 1]$, $\Omega = (0, 1) \times (0, 1)$ and $\beta(\mathbf{x}) = 1$. We numerically solve the following TFSE:

$$\begin{cases}
\mathrm{i}\mathscr{D}_t^\alpha y(t, \mathbf{x}) + \Delta y(t, \mathbf{x}) - \beta(\mathbf{x}) y(t, \mathbf{x}) = f(t, \mathbf{x}), \quad t \in J, \mathbf{x} \in \Omega, \\
y(t, \mathbf{x}) = 0, \quad t \in J, \mathbf{x} \in \partial\Omega, \\
y(0, \mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega.
\end{cases}$$

For simplicity, we define $|||\psi||| = \max\limits_{0 \le n \le N} \{\|\psi^n\|\}$, $|||\psi|||_1 = \max_{0 \le n \le N} \{\|\psi^n\|_1\}$, $|||e_{y_j}||| = |||y_j - y_{jh}|||$, $|||e_{y_j}|||_1 = |||y_j - y_{jh}|||_1$, and $|||e_{I_{2h}^2 y_j}|||_1 = |||y_j - I_{2h}^2 y_{jh}|||_1$ for $j = 1, 2$. The convergence rate are computed by $Rate = \frac{\ln(e_{k+1}) - \ln(e_k)}{\ln(s_{k+1}) - \ln(s_k)}$, where $e_{k+1}\,(e_k)$ is the error with the spatial or temporal mesh step $s_{k+1}\,(s_k)$.

Example 1. The initial condition and the right function $f(t,x)$ are suitably chosen such that the exact solution

$$y(t,x) = t^2\left(\sin(2\pi x_1)\sin(2\pi x_2) + \mathrm{i}x_1(1-x_1)x_2(1-x_2)\right).$$

If the spatial step $h = \frac{1}{100}$ is fixed, the errors $|||e_{y_j}|||$, $|||e_{y_j}|||_1$, $|||e_{I_{2h}^2 y_j}|||_1$ with $j = 1,2$ and the temporal convergence orders for $\alpha = 0.4, 0.5$ and 0.8 are shown in Table 1 and Table 2. The numerical results in Table 1 and Table 2 show that the temporal convergence rate of $|||e_{y_j}|||$, $|||e_{y_j}|||_1$ and $|||e_{I_{2h}^2 y_j}|||_1$ is $\mathscr{O}\left(N^{-\min\{r\alpha,2\}}\right)$ with $r = \frac{2}{\alpha}$. Fixed $N = 1000$ and $r = \frac{2}{\alpha}$, the results in Table 3 and Table 4 reflect $|||e_{y_j}||| = \mathscr{O}\left(h^2\right)$, $|||e_{y_j}|||_1 = \mathscr{O}\left(h^2\right)$ and $|||e_{I_{2h}^2 y_j}|||_1 = \mathscr{O}\left(h^2\right)$ for $j = 1,2$. The convergence rate results are consistent with our theoretical results.

TABLE 1. Temporal errors and convergence rates of y_1 with $h = \frac{1}{100}$, Example 1.

(α, r)	N	$\|\|\|e_{y_1}\|\|\|$	Rate	$\|\|\|e_{y_1}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_1}\|\|\|_1$	Rate
(0.4, 5)	20	3.4502e-2	–	6.1548e-2	–	6.2374e-2	–
	40	8.8067e-3	1.97	1.5820e-2	1.96	1.6143e-2	1.95
	80	2.2636e-3	1.96	4.0102e-3	1.98	4.1205e-3	1.97
	160	5.7380e-4	1.98	1.0095e-3	1.99	1.0445e-3	1.98
(0.5, 4)	20	3.1648e-2	–	5.8942e-2	–	6.0385e-2	–
	40	8.0224e-3	1.98	1.5045e-2	1.97	1.5521e-2	1.96
	80	2.0336e-3	1.98	3.8138e-3	1.98	3.9618e-3	1.97
	160	5.1194-4	1.99	9.6676e-4	1.98	1.0043e-4	1.98
(0.8, 2.5)	20	3.5435e-2	–	6.0983e-2	–	6.2547e-2	–
	40	9.1078e-3	1.96	1.5674e-2	1.96	1.6188e-2	1.95
	80	2.3087e-3	1.98	4.0008e-3	1.97	4.1035e-3	1.98
	160	5.8119e-4	1.99	1.0072e-3	1.99	1.0402e-3	1.98

TABLE 2. Temporal errors and convergence rates of y_2 with $h = \frac{1}{100}$, Example 1.

(α, r)	N	$\|\|\|e_{y_2}\|\|\|$	Rate	$\|\|\|e_{y_2}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_2}\|\|\|_1$	Rate
(0.4, 5)	20	3.5163e-2	–	6.2174e-2	–	6.3156e-2	–
	40	8.9135e-3	1.98	1.5760e-2	1.98	1.6121e-2	1.97
	80	2.2752e-3	1.97	4.0228e-3	1.97	4.0583e-3	1.99
	160	5.7276e-4	1.99	1.0127e-3	1.99	1.0287e-3	1.98
(0.5, 4)	20	3.2105e-2	–	6.0857e-2	–	6.1045e-2	–
	40	8.1949e-3	1.97	1.5427e-2	1.98	1.5474e-2	1.98
	80	2.0773e-3	1.98	3.9378e-3	1.97	3.9225e-3	1.98
	160	5.2294e-4	1.99	9.9130e-4	1.99	9.8745e-4	1.99
(0.8, 2.5)	20	3.5674e-2	–	6.2765e-2	–	6.3304e-2	–
	40	9.1059e-3	1.97	1.5910e-2	1.98	1.6047e-2	1.98
	80	2.2923e-3	1.99	4.0611e-3	1.97	4.0397e-3	1.99
	160	5.8107e-3	1.98	1.0223e-3	1.99	1.0169e-3	1.99

TABLE 3. Spatial errors and convergence rates of y_1 with $N = 1000$, Example 1.

(α, r)	h	$\|\|\|e_{y_1}\|\|\|$	Rate	$\|\|\|e_{y_1}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_1}\|\|\|_1$	Rate
(0.4, 5)	1/20	1.8204e-2	–	3.1076e-2	–	3.1684e-2	–
	1/40	4.6789e-3	1.96	1.5864e-2	0.97	8.0874e-3	1.97
	1/80	1.1861e-3	1.98	8.0427e-3	0.98	2.0787e-3	1.96
	1/160	3.0066e-4	1.98	4.0493e-3	0.99	5.2693e-4	1.98
(0.5, 4)	1/20	1.6147e-2	–	3.1647e-2	–	3.2518e-2	–
	1/40	4.0931e-3	1.98	1.6044e-2	0.98	8.2430e-3	1.98
	1/80	1.0376e-3	1.98	8.1334e-3	0.98	2.0895e-3	1.98
	1/160	2.6120e-4	1.99	4.0950e-3	0.99	5.2601e-4	1.99
(0.8, 2.5)	1/20	1.8052e-2	–	3.2565e-2	–	3.2958e-2	–
	1/40	4.6399e-3	1.96	1.6740e-2	0.96	8.4126e-3	1.97
	1/80	1.1762e-3	1.98	8.5459e-3	0.97	2.1325e-3	1.98
	1/160	2.9815e-4	1.98	4.3326e-3	0.98	5.4057e-3	1.98

TABLE 4. Spatial errors and convergence rates of y_2 with $N = 1000$, Example 1.

(α, r)	h	$\|\|\|e_{y_2}\|\|\|$	Rate	$\|\|\|e_{y_2}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_2}\|\|\|_1$	Rate
(0.4, 5)	1/20	1.7891e-2	–	3.2568e-2	–	3.2895e-2	–
	1/40	4.6305e-3	1.95	1.6626e-2	0.97	8.3965e-3	1.97
	1/80	1.1819e-3	1.97	8.5467e-3	0.96	2.1284e-3	1.98
	1/160	2.9960e-4	1.98	4.3330e-3	0.98	5.3580e-4	1.99
(0.5, 4)	1/20	1.6058e-2	–	3.2085e-2	–	3.2653e-2	–
	1/40	4.0989e-3	1.97	1.6154e-2	0.99	8.2772e-3	1.98
	1/80	1.0390e-3	1.98	8.1898e-3	0.98	2.0982e-3	1.98
	1/160	2.6156e-4	1.99	4.1234e-3	0.99	5.2820e-4	1.99
(0.8, 2.5)	1/20	1.7825e-2	–	3.3548e-2	–	3.3987e-2	–
	1/40	4.5815e-3	1.96	1.7126e-2	0.97	8.7356e-3	1.96
	1/80	1.1694e-3	1.97	8.6825e-3	0.98	2.2298e-3	1.97
	1/160	2.9643e-4	1.98	4.4019e-3	0.98	5.6523e-4	1.98

Example 2. The initial condition and the right function $f(t,x)$ are suitably chosen such that the exact solution $y(t,x) = \left(t^{\alpha}+t^{3}\right)\left(x_1(1-x_1)x_2(1-x_2)+\mathrm{i}\sin(2\pi x_1)\sin(2\pi x_2)\right)$.

In Table 5 and Table 6, we also show the numerical results with the fixed spatial step $h=\frac{1}{100}$ and $r=\frac{2}{\alpha}$ for $\alpha=0.4, 0.5$ and 0.8 using the fully discrete finite element approximation scheme (2.18)-(2.20). For fixed $N=1000$ and $r=\frac{2}{\alpha}$, the numerical results are shown in Table 7 and Table 8. The numerical results are in agreement with our theoretical analysis.

TABLE 5. Temporal errors and convergence rates of y_1 with $h=\frac{1}{100}$, Example 2.

(α, r)	N	$\|\|\|e_{y_1}\|\|\|$	Rate	$\|\|\|e_{y_1}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_1}\|\|\|_1$	Rate
(0.4, 5)	20	4.5026e-2	–	8.1074e-2	–	8.1625e-2	–
	40	1.1493e-2	1.97	2.0838e-2	1.96	2.1126e-2	1.95
	80	2.9134e-3	1.98	5.3190e-3	1.97	5.4300e-3	1.96
	160	7.3342e-4	1.99	1.3390e-3	1.99	1.3764e-3	1.98
(0.5, 4)	20	4.25864e-2	–	8.0258e-2	–	8.0835e-2	–
	40	1.0795e-2	1.98	2.0486e-2	1.97	2.0777e-2	1.96
	80	2.7555e-3	1.97	5.1930e-3	1.98	5.3034e-3	1.97
	160	6.9367e-4	1.99	1.3073e-3	1.99	1.3444e-3	1.98
(0.8, 2.5)	20	4.6456e-2	–	8.2135e-2	–	8.2457e-2	–
	40	1.1858e-2	1.97	2.111e-2	1.96	2.1341e-2	1.95
	80	2.9851e-3	1.99	5.3884e-3	1.97	5.4474e-3	1.97
	160	7.5669e-4	1.99	1.3659e-3	1.98	1.3809e-3	1.98

TABLE 6. Temporal errors and convergence rates of y_2 with $h=\frac{1}{100}$, Example 2.

(α, r)	N	$\|\|\|e_{y_2}\|\|\|$	Rate	$\|\|\|e_{y_2}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_2}\|\|\|_1$	Rate
(0.4, 5)	20	4.6037e-2	–	8.1542e-2	–	8.2016e-2	–
	40	1.1493e-2	1.97	2.0814e-2	1.97	2.1080e-2	1.96
	80	2.9134e-3	1.98	5.3128e-3	1.97	5.3436e-3	1.98
	160	7.3342e-4	1.99	1.3374e-3	1.99	1.3452e-3	1.99
(0.5, 4)	20	4.1835e-2	–	8.0974e-2	–	8.1504e-2	–
	40	1.0679e-2	1.97	2.0669e-2	1.97	2.0660e-2	1.98
	80	2.7070e-3	1.98	5.2394e-3	1.98	5.2735e-3	1.97
	160	6.8620e-4	1.98	1.3190e-3	1.99	1.3275e-3	1.99
(0.8, 2.5)	20	4.5348e-2	–	8.2985e-2	–	8.3051e-2	–
	40	1.1495e-2	1.98	2.1182e-2	1.97	2.1053e-2	1.98
	80	2.9341e-3	1.97	5.3694e-3	1.98	5.3367e-3	1.98
	160	7.4376e-4	1.98	1.3517e-3	1.99	1.3435e-3	1.99

TABLE 7. Spatial errors and convergence rates of y_1 with $N = 1000$, Example 2.

(α, r)	h	$\|\|\|e_{y_1}\|\|\|$	Rate	$\|\|\|e_{y_1}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_1}\|\|\|_1$	Rate
(0.4, 5)	1/20	2.2896e-2	–	4.2106e-2	–	4.6371e-2	–
	1/40	5.8849e-3	1.96	2.1495e-2	0.97	1.1919e-2	1.96
	1/80	1.5021e-3	1.97	1.0898e-2	0.98	3.0213e-3	1.98
	1/160	3.7814e-4	1.99	5.4869e-3	0.99	7.6587e-4	1.98
(0.5, 4)	1/20	2.1985e-2	–	4.1836e-2	–	4.6493e-2	–
	1/40	5.6117e-3	1.97	2.1210e-2	0.98	1.1867e-2	1.97
	1/80	1.4324e-3	1.97	1.0753e-2	0.98	3.0082e-3	1.98
	1/160	3.6310e-4	1.98	5.4139e-3	0.99	7.5728e-4	1.99
(0.8, 2.5)	1/20	2.3067e-2	–	4.2205e-2	–	4.7035e-2	–
	1/40	5.9289e-3	1.96	2.1696e-2	0.96	1.2089e-2	1.96
	1/80	1.5134e-3	1.97	1.1076e-2	0.97	3.0858e-3	1.97
	1/160	3.8363e-4	1.98	5.5765e-3	0.99	7.8222e-4	1.98

TABLE 8. Spatial errors and convergence rates of y_1 with $N = 1000$, Example 2.

(α, r)	h	$\|\|\|e_{y_2}\|\|\|$	Rate	$\|\|\|e_{y_2}\|\|\|_1$	Rate	$\|\|\|e_{I_{2h}^2 y_2}\|\|\|_1$	Rate
(0.4, 5)	1/20	2.4724e-2	–	4.3156e-2	–	4.3617e-2	–
	1/40	6.3109e-3	1.97	2.2185e-2	0.96	1.1133e-2	1.97
	1/80	1.5997e-3	1.98	1.1404e-2	0.98	2.8221e-3	1.98
	1/160	4.0551e-4	1.98	5.7816e-3	0.98	7.1043e-4	1.99
(0.5, 4)	1/20	2.4096e-2	–	4.2387e-2	–	4.3069e-2	–
	1/40	6.1081e-3	1.98	2.1639e-2	0.97	1.0918e-2	1.98
	1/80	1.5591e-3	1.97	1.1047e-2	0.98	2.7676e-3	1.98
	1/160	3.9249e-4	1.99	5.5619e-3	0.99	6.9671e-4	1.99
(0.8, 2.5)	1/20	2.3108e-2	–	4.3365e-2	–	4.4026e-2	–
	1/40	5.9394e-3	1.96	2.2292e-2	0.96	1.1316e-2	1.96
	1/80	1.5160e-3	1.97	1.1380e-2	0.97	2.8884e-3	1.97
	1/160	3.8429e-4	1.98	5.7694e-3	0.98	7.2712e-4	1.99

7. CONCLUSION

In this paper, we developed a fully discrete approximation scheme for the linear TFSE by using the $L2-1_\sigma$ formula for temporal discretization and linear finite elements for spatial discretization. The stability, convergence, and superconvergence results of the fully discrete scheme were derived. Two experiments were provided to demonstrate the theoretical results. In the future, we will consider numerical methods for nonlinear fractional Schrödinger equations.

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