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# ON A COUPLED SYSTEM OF FRACTIONAL SYMMETRIC HAHN DIFFERENCE EQUATIONS WITH NONLOCAL FRACTIONAL SYMMETRIC HAHN INTEGRAL BOUNDARY VALUE CONDITIONS

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**Abstract.** This paper focuses on a coupled system of fractional symmetric Hahn difference equations with fractional symmetric Hahn integral boundary conditions. The existence and uniqueness of solutions are addressed through the application of Banach's fixed point theorem. Moreover, the Schauder fixed point theorem is used to establish the existence of at least one solution.

**Keywords.** Boundary value problems; Fractional symmetric Hahn integral; Fractional symmetric Hahn difference; Existence of solutions.

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## 1. INTRODUCTION

Quantum calculus, which is a generalization of classical calculus, has been under the spotlight in the last three decades. Among its variants, $q$-calculus, pioneered by Jackson [1, 2], has found wide applications in quantum mechanics, the calculus of variations, and particle physics [3]-[8]. In [9], Hahn extended quantum calculus by introducing a two-parameter $(q,\omega)$ framework, known as Hahn calculus. The Hahn difference operator, which is a unification of the forward and $q$-difference operators, is defined as follows

$$D_{q,\omega}f(t) := \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega},\ t \neq \omega_0 := \frac{\omega}{1-q}.$$

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The Hahn difference operator has been the subject of extensive research, particularly regarding its right inverse and various properties (see [10, 11]). The applications span diverse areas, including Hahn quantum variational calculus [12, 13], initial value problems [14, 15], boundary value problems [16, 17], and the theory of orthogonal polynomial families [18, 19]. In 2017, Brikshavana and Sitthiwirattham [20] introduced fractional Hahn difference operators, which have numerous applications in boundary value problems; see, e.g., [21, 22, 23, 24]. The symmetric Hahn difference operator, $\tilde{D}_{q,\omega}$, which was subsequently introduced by Artur et al. [25], is defined by

$$\tilde{D}_{q,\omega}f(t) := \frac{f(qt+\omega)-f(q^{-1}(t-\omega))}{(q-q^{-1})t+(1+q^{-1})\omega} \quad \text{for } t \neq \omega_0.$$

The fractional symmetric Hahn integral and the corresponding Riemann-Liouville and Caputo fractional symmetric Hahn difference operators, introduced by Patanarapeelert and Sitthiwirattham [26], offer a novel approach to analyzing discrete fractional systems. Despite the establishment of their fundamental properties, the application of these operators to boundary value problems is limited. Existing studies on this topic [27, 28, 29] indicate a promising avenue for research, highlighting the need for further investigation to fully realize their potential in both theoretical and practical contexts. Since the boundary value problem for systems of fractional symmetric Hahn difference equations has not been previously studied, this research is dedicated to investigating this area. We focus on the specific form of the boundary value problem for these systems, as follows:

$$\tilde{D}^{\alpha}_{q,\omega}u_1(t) = F_1\left[t,u_1(t),u_2(t)\right], \tilde{D}^{\alpha}_{q,\omega}u_2(t) = F_2\left[t,u_1(t),u_2(t)\right], \quad t \in I^T_{q,\omega}, \tag{1.1}$$

with the nonlocal three-point fractional Hahn integral boundary value conditions

$$\begin{aligned} u_1(\omega_0) &= \phi_1(u_1,u_2),\ u_1(T) = \mu_2\tilde{\mathscr{I}}^{\theta_2}_{q,\omega}u_2(\eta_2), \\ u_2(\omega_0) &= \phi_2(u_1,u_2),\ u_2(T) = \mu_1\tilde{\mathscr{I}}^{\theta_1}_{q,\omega}u_1(\eta_1), \end{aligned} \tag{1.2}$$

where $I^T_{q,\omega} := \{q^kT+\omega[k]_q :\ k\in\mathbb{N}_0\}\cup\{\omega_0\}$ ; $\alpha\in(1,2],\ \omega>0,\ q\in(0,1)$. For $i=1,2$, $\theta_i \in (0,1],\ \eta_i \in I^T_{q,\omega}-\{\omega_0,T\}$, $F_i \in C\left(I^T_{q,\omega}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\right)$ are given functions, and $\phi_i : C\left(I^T_{q,\omega},\mathbb{R}\right)\times C\left(I^T_{q,\omega},\mathbb{R}\right)\to\mathbb{R}$ are given functionals.

This paper is structured as follows. In Section 2, we provide the necessary background on fractional symmetric Hahn calculus, including relevant definitions, lemmas, and properties of the fractional symmetric Hahn integral. In addition, we recast nonlinear problem (1.1) as a fixed-point problem through linearization. The fixed-point operator is then defined, and the existence is established by using classical fixed-point theorems. In Section 3, we apply both the Banach and Schauder fixed point theorems to analyze the existence and uniqueness of solutions to problem (1.1). Finally, an example is presented to demonstrate the applicability of the obtained results.

## 2. Preliminaries

This section introduces the fundamental principles of fractional symmetric Hahn difference calculus, based on [25]-[31].

For $0<q<1$, $\omega>0$, $\omega_0=\frac{\omega}{1-q}$, and $[k]_q=\frac{1-q^k}{1-q}$, we present some notations as follows

$$\widetilde{[k]}_q := \begin{cases} \dfrac{1-q^{2k}}{1-q^2}=[k]_{q^2}, & k\in\mathbb{N}\\ 1, & k=0,\end{cases}$$

$$\widetilde{[k]}_q! := \begin{cases} \widetilde{[k]}_q\widetilde{[k-1]}_q\cdots\widetilde{[1]}_q=\displaystyle\prod_{i=1}^{k}\frac{1-q^{2i}}{1-q^2}, & k\in\mathbb{N}\\ 1, & k=0.\end{cases}$$

For $k\in\mathbb{N}$, the $q,\omega$-forward jump and $q,\omega$-backward jump operators are defined by

$$\sigma^k_{q,\omega}(t):=q^kt+\omega[k]_q \ \text{ and } \ \rho^k_{q,\omega}(t):=\frac{t-\omega[k]_q}{q^k}, \ \text{respectively.}$$

For $n\in\mathbb{N}_0:=\{0,1,2,...\}$, and $a,b\in\mathbb{R}$, the power functions are defined by

• The $q$-analogue of the power function

$$(a-b)_q^{\underline{0}}:=1, \qquad (a-b)_q^{\underline{n}}:=\prod_{i=0}^{n-1}(a-bq^i).$$

• The $q$-symmetric analogue of the power function

$$\widetilde{(a-b)}_q^{\underline{0}}:=1, \qquad \widetilde{(a-b)}_q^{\underline{n}}:=\prod_{i=0}^{n-1}(a-bq^{2i+1}).$$

• The $q,\omega$-symmetric analogue of the power function

$$\widetilde{(a-b)}_{q,\omega}^{\underline{0}}:=1, \qquad \widetilde{(a-b)}_{q,\omega}^{\underline{n}}:=\prod_{i=0}^{n-1}\left[a-\sigma^{2i+1}_{q,\omega}(b)\right].$$

For $\alpha\in\mathbb{R}$, we define

$$(a-b)_q^{\underline{\alpha}}=a^\alpha\prod_{i=0}^{\infty}\frac{1-\left(\frac{b}{a}\right)q^i}{1-\left(\frac{b}{a}\right)q^{\alpha+i}},\ a\neq 0,$$

$$\widetilde{(a-b)}_q^{\underline{\alpha}}=a^\alpha\prod_{i=0}^{\infty}\frac{1-\left(\frac{b}{a}\right)q^{2i+1}}{1-\left(\frac{b}{a}\right)q^{2(\alpha+i)+1}},\ a\neq 0,$$

$$\widetilde{(a-b)})_{q,\omega}^{\underline{\alpha}}=\left(\widetilde{(a-\omega_0)-(b-\omega_0)}\right)_q^{\underline{\alpha}}=(a-\omega_0)^\alpha\prod_{i=0}^{\infty}\frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2i+1}}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{2(\alpha+i)+1}},\ a\neq\omega_0.$$

If $b=0$, $a_q^{\underline{\alpha}}=\widetilde{a}_q^{\underline{\alpha}}=a^\alpha$ and $\widetilde{(a-\omega_0)}_{q,\omega}^{\underline{\alpha}}=(a-\omega_0)^\alpha$. If $a=b$, $(0)_q^{\underline{\alpha}}=\widetilde{(0)}_q^{\underline{\alpha}}=\widetilde{(\omega_0)}_{q,\omega}^{\underline{\alpha}}=0$ for $\alpha>0$. The $q$-symmetric gamma and $q$-symmetric beta functions are defined by

$$\tilde{\Gamma}_q(x) := \begin{cases} \frac{(1-q^2)_q^{\underline{x-1}}}{(1-q^2)^{x-1}}=\frac{\widetilde{(1-q)}_q^{\underline{x-1}}}{(1-q^2)^{x-1}}, & x\in\mathbb{R}\setminus\{0,-1,-2,...\}\\ \widetilde{[x-1]}_q!, & x\in\mathbb{N}\end{cases}$$

$$\tilde{B}_q(x,y) := \int_0^1\left(q^{-1}s\right)^{x-1}\widetilde{(1-s)}_q^{\underline{y-1}}\tilde{d}_qs=\frac{\tilde{\Gamma}_q(x)\tilde{\Gamma}_q(y)}{\tilde{\Gamma}_q(x+y)},$$

respectively.

**Lemma 2.1.** [26] *For* $m,n \in \mathbb{N}_0$ *and* $\alpha \in \mathbb{R}$,

$(a)$ $\widetilde{(x-\sigma^n_{q,\omega}(x))}^{\underline{\alpha}}_{q,\omega} = (x-\omega_0)^k \widetilde{(1-q^n)}^{\underline{\alpha}}_q$,

$(b)$ $\widetilde{(\sigma^m_{q,\omega}(x)) - \sigma^n_{q,\omega}(x))}^{\underline{\alpha}}_{q,\omega} = q^{m\alpha}(x-\omega_0)^{\alpha}\widetilde{(1-q^{n-m})}^{\underline{\alpha}}_q$.

We next provide the definitions of the symmetric Hahn difference and symmetric Hahn integral as follows.

**Definition 2.2.** [25] For $q \in (0,1)$, $\omega > 0$, and $f$, a function defined on $I^T_{q,\omega} \subseteq \mathbb{R}$, the symmetric Hahn difference of $f$ is defined by

$$\tilde{D}_{q,\omega} f(t) := \frac{f(\sigma_{q,\omega}(t)) - f(\rho_{q,\omega}(t))}{\sigma_{q,\omega}(t) - \rho_{q,\omega}(t)} \quad t \in I^T_{q,\omega} - \{\omega_0\},$$
$$\tilde{D}_{q,\omega} f(\omega_0) = f'(\omega_0), \text{ where } f \text{ is differentiable at } \omega_0.$$

$\tilde{D}_{q,\omega} f$ is called $q,\omega$-symmetric derivative of $f$, and $f$ is $q,\omega$-symmetric differentiable on $I^T_{q,\omega}$.

For $N \in \mathbb{N}$, we define $\tilde{D}^0_{q,\omega} f(x) = f(x)$ and $\tilde{D}^N_{q,\omega} f(x) = \tilde{D}_{q,\omega}\tilde{D}^{N-1}_{q,\omega} f(x)$.

**Definition 2.3.** [25] Let $I$ be any closed interval of $\mathbb{R}$ containing $a, b$, and $\omega_0$, and let $f : I \to \mathbb{R}$ be a given function. The symmetric Hahn integral of $f$ from $a$ to $b$ is defined by

$$\int_a^b f(t)\tilde{d}_{q,\omega}t := \int_{\omega_0}^b f(t)\tilde{d}_{q,\omega}t - \int_{\omega_0}^a f(t)\tilde{d}_{q,\omega}t,$$

where

$$\tilde{\mathscr{I}}_{q,\omega} f(t) = \int_{\omega_0}^x f(t)\tilde{d}_{q,\omega}t := (1-q^2)(x-\omega_0)\sum_{k=0}^{\infty} q^{2k} f\left(\sigma^{2k+1}_{q,\omega}(x)\right),\ x \in I.$$

$f$ is called symmetric Hahn integrable on $[a,b]$ when the series converges at $x = a$ and $x = b$. In addition, $f$ is symmetric Hahn integrable on $I$ if it is symmetric Hahn integrable on $[a,b]$ for all $a, b \in I$.

For $N \in \mathbb{N}$, one has $\tilde{\mathscr{I}}^0_{q,\omega} f(x) = f(x)$, $\tilde{\mathscr{I}}^N_{q,\omega} f(x) = \tilde{\mathscr{I}}_{q,\omega}\tilde{\mathscr{I}}^{N-1}_{q,\omega} f(x)$, $\tilde{D}_{q,\omega}\tilde{\mathscr{I}}_{q,\omega} f(x) = f(x)$, and $\tilde{\mathscr{I}}_{q,\omega}\tilde{D}_{q,\omega} f(x) = f(x) - f(\omega_0)$. The fractional symmetric Hahn integral and fractional symmetric Hahn difference of Riemann-Liouville type are defined as follows.

**Definition 2.4.** [26] Let $\alpha, \omega > 0$, $0 < q < 1$, and $f$ be a function defined on $I^T_{q,\omega}$. The fractional symmetric Hahn integral is defined by

$$\begin{aligned}
\widetilde{\mathscr{I}}^{\alpha}_{q,\omega} f(t) &:= \frac{q^{\binom{\alpha}{2}}}{\widetilde{\Gamma}_q(\alpha)}\int_{\omega_0}^t \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega} f\left(\sigma^{\alpha-1}_{q,\omega}(s)\right)\tilde{d}_{q,\omega}s \\
&= \frac{(1-q^2)q^{\binom{\alpha}{2}}(t-\omega_0)}{\widetilde{\Gamma}_q(\alpha)}\sum_{k=0}^{\infty} q^{2k}\left(t-\sigma^{2k+1}_{q,\omega}(t)\right)^{\underline{\alpha-1}}_{q,\omega} f\left(\sigma^{2k+\alpha}_{q,\omega}(t)\right) \\
&= \frac{(1-q^2)q^{\binom{\alpha}{2}}(t-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha)}\sum_{k=0}^{\infty} q^{2k}\widetilde{\left(1-q^{2k+1}\right)}^{\underline{\alpha-1}}_q f\left(\sigma^{2k+\alpha}_{q,\omega}(t)\right)
\end{aligned}$$

and $\widetilde{\mathscr{I}}^0_{q,\omega} f(t) = f(t)$.

**Definition 2.5.** [26] For $\alpha,\omega>0,\ 0<q<1$ and $f$ defined on $I^T_{q,\omega}$, the fractional symmetric Hahn difference operator of Riemann-Liouville type of order $\alpha$ is defined by

$$\widetilde{D}^{\alpha}_{q,\omega}f(t) \ := \ \widetilde{D}^{N}_{q,\omega}\widetilde{\mathscr{I}}^{N-\alpha}_{q,\omega}f(t)=\frac{q^{\binom{-\alpha}{2}}}{\widetilde{\Gamma}_q(-\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}^{\,-\alpha-1}_{q,\omega}f\left(\sigma^{-\alpha-1}_{q,\omega}(s)\right)\tilde{d}_{q,\omega}s$$

and $\widetilde{D}^{0}_{q,\omega}f(t)=f(t)$, where $N-1<\alpha<N,\ N\in\mathbb{N}$.

Next, we introduce lemmas that are used in the main results.

**Lemma 2.6.** [26] *Let* $\alpha,\omega>0,\ 0<q<1,$ *and* $f:I^T_{q,\omega}\to\mathbb{R}$. *Then,*

$$\widetilde{\mathscr{I}}^{\alpha}_{q,\omega}\tilde{D}^{\alpha}_{q,\omega}f(t)=f(t)+C_1(t-\omega_0)^{\alpha-1}+C_2(t-\omega_0)^{\alpha-2}+\dots+C_N(t-\omega_0)^{\alpha-N}$$

*for some* $C_i\in\mathbb{R}, i=1,2,\dots,N$ *and* $N-1<\alpha<N$ *for* $N\in\mathbb{N}$.

**Lemma 2.7.** [32] *(Arzelá-Ascoli theorem) A set of function in* $C[a,b]$ *with the sup norm is relatively compact if and only if it is uniformly bounded and equicontinuous on* $[a,b]$.

**Lemma 2.8.** [32] *If a set is closed and relatively compact, then it is compact.*

**Lemma 2.9.** [33] *(Schauder's fixed point theorem) Let* $(D,d)$ *be a complete metric space,* $U$ *be a closed convex subset of* $D$*, and* $T:D\to D$ *be the map such that* $Tu:u\in U$ *is relatively compact in* $D$*. Then* $T$ *has at least one fixed point* $u^*\in U$*:* $Tu^*=u^*$.

**Lemma 2.10.** [27] *Let* $\alpha>0, q\in(0,1), \omega>0,$ *and* $n\in\mathbb{N}$. *Then,*

$$(i)\quad \int_{\omega_0}^{t}\tilde{d}_{q,\omega}s=t-\omega_0,$$

$$(ii)\quad \int_{\omega_0}^{t}(s-\omega_0)^n\,\tilde{d}_{q,\omega}s=\frac{q^n}{\widetilde{[n+1]}_q}(t-\omega_0)^{n+1},$$

$$(iii)\quad \int_{\omega_0}^{t}\widetilde{(t-s)}^{\,\alpha-1}_{q,\omega}\,\tilde{d}_{q,\omega}s \ = \ \frac{(t-\omega_0)^{\alpha}}{\widetilde{[\alpha]}_q},$$

$$(iv)\quad \int_{\omega_0}^{t}\widetilde{(t-s)}^{\,\alpha-1}_{q,\omega}\left(\sigma^{\alpha-1}_{q,\omega}(s)-\omega_0\right)^{\beta}\tilde{d}_{q,\omega}s=q^{\alpha\beta}\,(t-\omega_0)^{\alpha+\beta}\,\tilde{B}_q(\beta+1,\alpha).$$

We next provide some lemmas that are used further in our main results.

**Lemma 2.11.** [28] *Let* $\alpha,\beta>0,\ q\in(0,1),\ \omega>0,$ *and* $n\in\mathbb{N}$. *Then,*

$$(i)\quad \int_{\omega_0}^{t}\widetilde{(t-s)}^{\,-\beta-1}_{q,\omega}\left(\sigma^{-\beta-1}_{q,\omega}(s)-\omega_0\right)^{\alpha-n}\tilde{d}_{q,\omega}s \ = \ \frac{(t-\omega_0)^{\alpha-\beta-n}}{q^{\beta(\alpha-n)}}\tilde{B}_q(\alpha-n+1,-\beta),$$

$$(ii)\quad \int_{\omega_0}^{t}\widetilde{(t-s)}^{\,\theta-1}_{q,\omega}\left(\sigma^{\theta-1}_{q,\omega}(s)-\omega_0\right)^{\alpha-n}\tilde{d}_{q,\omega}s \ = \ \frac{(t-\omega_0)^{\alpha+\theta-n}}{q^{-\theta(\alpha-n)}}\tilde{B}_q(\alpha-n+1,\theta),$$

$$(iii)\quad \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma^{-\beta-1}_{q,\omega}(x)}\widetilde{(t-x)}^{\,-\beta-1}_{q,\omega}\left(\widetilde{\sigma^{-\beta-1}_{q,\omega}(x)-s}\right)^{\alpha-1}_{q,\omega}\tilde{d}_{q,\omega}s\,\tilde{d}_{p,\omega}x=\frac{(t-\omega_0)^{\alpha-\beta}}{[\tilde{\alpha}]_q q^{\alpha\beta}}\tilde{B}_q(\alpha+1,-\beta),$$

$$(iv)\quad \int_{\omega_0}^{t}\int_{\omega_0}^{\sigma^{\theta-1}_{q,\omega}(x)}\widetilde{(t-x)}^{\,\theta-1}_{q,\omega}\left(\widetilde{\sigma^{\theta-1}_{q,\omega}(x)-s}\right)^{\alpha-1}_{q,\omega}\tilde{d}_{q,\omega}s\,\tilde{d}_{p,\omega}x=\frac{(t-\omega_0)^{\alpha+\theta}}{[\tilde{\alpha}]_q q^{-\alpha\theta}}\tilde{B}_q(\alpha+1,\theta)$$

We initiate our study of the existence and uniqueness of solutions for nonlinear problem (1.1) by considering the corresponding linear problem and its solution, as detailed in the lemma below.

**Lemma 2.12.** *Let $\omega > 0$, $q \in (0,1)$, $\alpha \in (1,2]$, for $i = 1,2$, $\theta_i \in (0,1]$, $\mu_i \in \mathbb{R}^+$, $h_i \in C\big(I^T_{q,\omega}, \mathbb{R}\big)$ be given functions; and $\phi_i : C\left(I^T_{q,\omega}, \mathbb{R}\right) \times C\left(I^T_{q,\omega}, \mathbb{R}\right) \to \mathbb{R}$ be given functionals. Then, the problem*

$$
\begin{aligned}
&\tilde{D}^{\alpha}_{q,\omega} u_1(t) = h_1(t), \\
&\tilde{D}^{\alpha}_{q,\omega} u_2(t) = h_2(t),\ t \in I^T_{q,\omega}, \\
&u_1(\omega_0) = \phi_1(u_1,u_2),\ u_1(T) = \mu_2 \tilde{\mathscr{I}}^{\theta_2}_{q,\omega} u_2(\eta_2),\ \eta_2 \in I^T_{q,\omega} - \{\omega_0, T\}, \\
&u_2(\omega_0) = \phi_2(u_1,u_2),\ u_2(T) = \mu_1 \tilde{\mathscr{I}}^{\theta_1}_{q,\omega} u_1(\eta_1),\ \eta_1 \in I^T_{q,\omega} - \{\omega_0, T\},
\end{aligned}
\tag{2.1}
$$

*has the unique solution*

$$
\begin{aligned}
u_1(t) &= \phi_1(u_1,u_2) + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega} h_1\big(\sigma^{\alpha-1}_{q,\omega}\big)\, \tilde{d}_{q,\omega}s \\
&\quad + (t-\omega_0)\left\{ \frac{\mathscr{P}(h_1,h_2)\mu_1 q^{\binom{\theta_1}{2}}}{\Lambda(T-\omega_0)\tilde{\Gamma}_q(\theta_1)} \int_{\omega_0}^{\eta_1} \widetilde{(\eta_1-s)}^{\underline{\theta_1-1}}_{q,\omega} \big(\sigma^{\theta_1-1}_{q,\omega}(s)-\omega_0\big) \tilde{d}_{q,\omega}s \right. \\
&\quad \left. - \frac{\mathscr{Q}(h_1,h_2)\mu_2 q^{\binom{\theta_2}{2}}}{\Lambda(T-\omega_0)\tilde{\Gamma}_q(\theta_2)} \int_{\omega_0}^{\eta_2} \widetilde{(\eta_2-s)}^{\underline{\theta_2-1}}_{q,\omega} \big(\sigma^{\theta_2-1}_{q,\omega}(s)-\omega_0\big) \tilde{d}_{q,\omega}s \right\}
\end{aligned}
\tag{2.2}
$$

$$
u_2(t) = \phi_2(u_1,u_2) + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}^{\underline{\alpha-1}}_{q,\omega} h_2\big(\sigma^{\alpha-1}_{q,\omega}\big)\, \tilde{d}_{q,\omega}s + (t-\omega_0)\, \frac{\mathscr{P}(h_1,h_2)-\mathscr{Q}(h_1,h_2)}{\Lambda},
\tag{2.3}
$$

*where*

$$
\begin{aligned}
\Lambda &= \frac{\mu_2 q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)} \int_{\omega_0}^{\eta_2} \widetilde{(\eta_2-s)}^{\underline{\theta_2-1}}_{q,\omega} \big(\sigma^{\theta_2-1}_{q,\omega}(s)-\omega_0\big)\, \tilde{d}_{q,\omega}s \\
&\quad - \frac{\mu_1 q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)} \int_{\omega_0}^{\eta_1} \widetilde{(\eta_1-s)}^{\underline{\theta_1-1}}_{q,\omega} \big(\sigma^{\theta_1-1}_{q,\omega}(s)-\omega_0\big)\, \tilde{d}_{q,\omega}s,
\end{aligned}
\tag{2.4}
$$

*and*

$$
\begin{aligned}
\mathscr{P}(h_1,h_2) &= \phi_1(u_1,u_2) + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{T} \widetilde{(T-s)}^{\underline{\alpha-1}}_{q,\omega} h_1\big(\sigma^{\alpha-1}_{q,\omega}(s)\big) \tilde{d}_{q,\omega}s \\
&\quad - \frac{\mu_2 \phi_2(u_1,u_2) q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)} \int_{\omega_0}^{\eta_2} \widetilde{(\eta_2-s)}^{\underline{\theta_2-1}}_{q,\omega} \tilde{d}_{q,\omega}s - \frac{\mu_2 q^{\binom{\alpha}{2}} q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_2)} \times \\
&\quad \int_{\omega_0}^{\eta_2} \int_{\omega_0}^{\sigma^{\theta_2-1}_{q,\omega}(\xi)} \widetilde{(\eta_2-\xi)}^{\underline{\theta_2-1}}_{q,\omega} \widetilde{\big(\sigma^{\theta_2-1}_{q,\omega}(\xi)-s\big)}^{\underline{\alpha-1}}_{q,\omega} h_2\big(\sigma^{\alpha-1}_{q,\omega}(s)\big)\, \tilde{d}_{q,\omega}s\, \tilde{d}_{q,\omega}\xi,
\end{aligned}
\tag{2.5}
$$

$$\begin{aligned}\mathscr{Q}(h_1,h_2) &= \phi_2(u_1,u_2)+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}h_2\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s \\ &\quad -\frac{\mu_1\phi_1(u_1,u_2)q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\tilde{d}_{q,\omega}s-\frac{\mu_1q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_1)}\times \\ &\quad \int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(\xi)}\widetilde{(\eta_1-\xi)}_{q,\omega}^{\underline{\theta_1-1}}\widetilde{\big(\sigma_{q,\omega}^{\theta_1-1}(\xi)-s\big)}_{q,\omega}^{\underline{\alpha-1}}h_1\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi.\end{aligned} \tag{2.6}$$

*Proof.* For $i,j\in\{1,2\}$ and $i\neq j$, by using lemma 2.6 and the fractional symmetric Hahn integral of order $\alpha$ for (2.1), we have

$$u_i(t)=C_{1i}+C_{2i}(t-\omega_0)+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}h_i\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s,\ t\in I_{q,\omega}^{T}.$$

Using the boundary condition of (2.1) at $t=\omega_0$, we find $C_{1i}=\phi_i(u_1,u_2)$. Therefore,

$$\begin{aligned}u_i(t) &= \phi_i(u_1,u_2)+C_{2i}(t-\omega_0) \\ &\quad +\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}h_i\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s.\end{aligned} \tag{2.7}$$

Taking the fractional Hahn integral of order $0<\theta_i\leq 1;\ i=1,2$ for (2.7), we have

$$\begin{aligned}&\tilde{\mathscr{I}}_{q,\omega}^{\theta_i}u_i(t) \\ &=\frac{\phi_i(u_1,u_2)q^{\binom{\theta}{2}}}{\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\theta_i-1}}\tilde{d}_{q,\omega}s+\frac{C_{2i}q^{\binom{\theta}{2}}}{\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\theta_i-1}}\big(\sigma_{q,\omega}^{\theta_i-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s \\ &\quad+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\frac{q^{\binom{\theta_i}{2}}}{\tilde{\Gamma}_q(\theta_i)}\int_{\omega_0}^{t}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_i-1}(\xi)}\widetilde{(t-\xi)}_{q,\omega}^{\underline{\theta_i-1}}\widetilde{\big(\sigma_{q,\omega}^{\theta_i-1}(\xi)-s\big)}_{q,\omega}^{\underline{\alpha-1}}h_i\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi,\end{aligned}$$

for $t\in I_{q,\omega}^{T}$. From boundary condition (2.1), we have

$$\begin{aligned}&\phi_1(u_1,u_2)+C_{21}(T-\omega_0)+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}h_1\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s \\ &=\frac{\mu_2\phi_2(u_1,u_2)q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\tilde{d}_{q,\omega}s \\ &\quad+\frac{\mu_2C_{22}q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\big(\sigma_{q,\omega}^{\theta_2-1}(s)-\omega_0\big)\tilde{d}_{q,\omega}s \\ &\quad+\frac{\mu_2q^{\binom{\alpha}{2}}q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_2-1}(\xi)}\widetilde{(\eta_2-\xi)}_{q,\omega}^{\underline{\theta_2-1}}\widetilde{\big(\sigma_{q,\omega}^{\theta_2-1}(\xi)-s\big)}_{q,\omega}^{\underline{\alpha-1}}h_2(\sigma_{q,\omega}^{\alpha-1}(s))\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi,\end{aligned} \tag{2.8}$$

and

$$\phi_2(u_1,u_2)+C_{22}(T-\omega_0)+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}h_2\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s \tag{2.9}$$
$$=\frac{\mu_1\phi_1(u_1,u_2)q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\tilde{d}_{q,\omega}s$$
$$+\frac{\mu_1 C_{21}q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\big(\widetilde{\sigma_{q,\omega}^{\theta_1-1}}-\omega_0\big)\,\tilde{d}_{q,\omega}s$$
$$+\frac{\mu_1 q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(\xi)}\widetilde{(\eta_1-\xi)}_{q,\omega}^{\underline{\theta_1-1}}\big(\widetilde{\sigma_{q,\omega}^{\theta_1-1}(\xi)}-s\big)_{q,\omega}^{\underline{\alpha-1}}h_1\big(\sigma_{q,\omega}^{\alpha-1}(s)\big)\,\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi.$$

Finally, the constants $C_{11}$ and $C_{12}$ are obtained by solving the system of equations (2.8) and (2.9) as

$$C_{21}=\frac{\mathscr{P}(h_1,h_2)\mu_1 q^{\binom{\theta_1}{2}}}{\Lambda(T-\omega_0)\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\big(\sigma_{q,\omega}^{\theta_1-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s$$
$$-\frac{\mathscr{Q}(h_1,h_2)\mu_2 q^{\binom{\theta_2}{2}}}{\Lambda(T-\omega_0)\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\big(\sigma_{q,\omega}^{\theta_2-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s,$$

and

$$C_{22}=\frac{\mathscr{P}(h_1,h_2)-\mathscr{Q}(h_1,h_2)}{\Lambda},$$

where $\Lambda$, $\mathscr{P}(h_1,h_2)$ and $\mathscr{Q}(h_1,h_2)$ are defined as (2.4)-(2.6), respectively. Substituting $C_{11}$ and $C_{12}$ into (2.7), we obtain (2.2) and (2.3) immediately. □

## 3. Existence-Uniqueness Result

In this section, we aim to prove the existence result for problem (1.1)-(1.2). Here, we let $E:C\left(I_{q,\omega}^{T},\mathbb{R}\right)$ be the Banach space for all continuous functions on $I_{q,\omega}^{T}$. Clearly, the product space $\mathscr{C}=E\times E$ is the Banach space. We set the spaces $\|(u_1,u_2)\|_{\mathscr{C}}=\|u_1\|+\|u_2\|$, where $\|u_i\|=\max_{t\in I_{q,\omega}^{T}}|u_i(t)|$, $i=1,2$.

Next, we define the operator $\mathscr{A}:\mathscr{C}\to\mathscr{C}$ by

$$\big(\mathscr{A}(u_1,u_2)\big)(t)=\Big(\big(\mathscr{A}_1(u_1,u_2)\big)(t),\big(\mathscr{A}_2(u_1,u_2)\big)(t)\Big).$$

$$(\mathscr{A}_1(u_1,u_2))(t)=\phi_1(u_1,u_2)+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),u\big)\,\tilde{d}_{q,\omega}s$$
$$+\frac{(t-\omega_0)}{\Lambda}\left\{\frac{\mathscr{P}_u(F_1^*,F_2^*)\mu_1 q^{\binom{\theta_1}{2}}}{\Lambda(T-\omega_0)\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\big(\sigma_{q,\omega}^{\theta_1-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s\right.$$
$$\left.-\frac{\mathscr{Q}_u(F_1^*,F_2^*)\mu_2 q^{\binom{\theta_2}{2}}}{\Lambda(T-\omega_0)\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\big(\sigma_{q,\omega}^{\theta_2-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s\right\}$$

and

$$(\mathscr{A}_2(u_1,u_2))(t) = \phi_2(u_1,u_2) + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{t} \widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}} F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) \tilde{d}_{q,\omega}s$$
$$+(t-\omega_0)\frac{\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{Q}_u(F_1^*,F_2^*)}{\Lambda},$$

where $\Lambda$ is defined as (2.4), and the functionals $\mathscr{P}_u(F_1^*,F_2^*),\mathscr{Q}_u(F_1^*,F_2^*)$ are defined by

$$\mathscr{P}_u(F_1^*,F_2^*) = \phi_1(u_1,u_2) + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha_1)} \int_{\omega_0}^{T} \widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}} F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) \tilde{d}_{q,\omega}s$$
$$-\frac{\mu_2\phi_2(u_1,u_2)q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2} \widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}} \tilde{d}_{q,\omega}s - \frac{\mu_2 q^{\binom{\alpha}{2}}q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_2)}\times$$
$$\int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_2-1}(\xi)} \widetilde{(\eta_2-\xi)}_{q,\omega}^{\underline{\theta_2-1}} \widetilde{\left(\sigma_{q,\omega}^{\theta_2-1}(\xi)-s\right)}_{q,\omega}^{\underline{\alpha-1}} F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) \tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi,$$

$$\mathscr{Q}_u(F_1^*,F_2^*) = \phi_2(u_1,u_2) + \frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)} \int_{\omega_0}^{T} \widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}} F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) \tilde{d}_{q,\omega}s$$
$$-\frac{\mu_1\phi_1(u_1,u_2)q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1} \widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}} \tilde{d}_{q,\omega}s - \frac{\mu_1 q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_1)}\times$$
$$\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(\xi)} \widetilde{(\eta_1-\xi)}_{q,\omega}^{\underline{\theta_1-1}} \widetilde{\left(\sigma_{q,\omega}^{\theta_1-1}(\xi)-s\right)}_{q,\omega}^{\underline{\alpha-1}} F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) \tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi,$$

where

$$F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) = F_1\left(\sigma_{q,\omega}^{\alpha-1}(s),u_1(t),u_2(t)\right)$$

and

$$F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right) = F_2\left(\sigma_{q,\omega}^{\alpha-1}(s),u_1(t),u_2(t)\right).$$

We note that problem (1.1)-(1.2) has solutions if and only if $\mathscr{A}$ has fixed points.

**Theorem 3.1.** *For each $i,j\in\{1,2\}; i\neq j$, assume that $F_i\in C\left(I_{q,\omega}^T\times\mathbb{R}\times\mathbb{R},\mathbb{R}\right)$ and $\phi_i : C\left(I_{q,\omega}^T,\mathbb{R}\right)\times C\left(I_{q,\omega}^T,\mathbb{R}\right)\to\mathbb{R}$ are given functionals. Suppose that*

$(H_1)$ *There exist constants $M_1,M_2,N_1,N_2>0$ such that, for each $t\in I_{q,\omega}^T$,*

$$\left|F_i\left(t,u_i,u_j\right)-F_i\left(t,v_i,v_j\right)\right| \le M_i\left|u_i-v_i\right|+N_j\left|u_j-v_j\right|.$$

$(H_2)$ *There exist constants $K_1,K_2,L_1,L_2>0$ such that, for each $(u_1,u_2),(v_1,v_2)\in\mathscr{C}$,*

$$\begin{aligned}|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)| &\le K_1\|u_1-v_1\|+K_2\|u_2-v_2\|,\\ \text{and } |\phi_2(u_1,u_2)-\phi_2(v_1,v_2)| &\le L_1\|u_1-v_1\|+L_2\|u_2-v_2\|.\end{aligned}$$

*Then, problem (1.1)-(1.2) has a unique solution provided that*

$$\chi := \max\left\{K_1\Theta_1+L_1\Theta_2+M_1\Theta_3+N_1\Theta_4, K_2\Theta_1+L_2\Theta_2+N_2\Theta_3+M_2\Theta_4,\right.$$
$$\left.L_1\bar{\Theta}_1+K_1\bar{\Theta}_2+M_1\bar{\Theta}_3+N_1\bar{\Theta}_4, L_2\bar{\Theta}_1+K_2\bar{\Theta}_2+N_2\bar{\Theta}_3+M_2\bar{\Theta}_4\right\}<1,$$

*where*

$$\Theta_1 = 1+\frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)}+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}\mu_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)\tilde{\Gamma}_q(1+\theta_1)} \tag{3.1}$$

$$\Theta_2 = \frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}\mu_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)\tilde{\Gamma}_q(1+\theta_2)}+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)} \tag{3.2}$$

$$\begin{aligned}\Theta_3 &= \frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}+\frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1} q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{|\Lambda|\tilde{\Gamma}_q(\alpha_2+1)\tilde{\Gamma}_q(1+\alpha)}\\ &\quad+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}\mu_1 q^{\binom{\alpha}{2}} q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2) q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Theta_4 &= \frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}\mu_2 q^{\binom{\alpha}{2}+\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1) q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}\\ &\quad+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1} q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)\tilde{\Gamma}_q(1+\alpha)}\end{aligned} \tag{3.4}$$

$$\bar{\Theta}_1 = 1+\frac{(T-\omega_0)}{|\Lambda|}\left[1+\frac{\mu_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{\tilde{\Gamma}_q(1+\theta_2)}\right] \tag{3.5}$$

$$\bar{\Theta}_2 = \frac{(T-\omega_0)}{|\Lambda|}\left[1+\frac{\mu_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{\tilde{\Gamma}_q(1+\theta_1)}\right] \tag{3.6}$$

$$\bar{\Theta}_3 = \frac{(T-\omega_0)}{|\Lambda|}\left[\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}+\frac{\mu_1 q^{\binom{\alpha}{2}} q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}\right] \tag{3.7}$$

$$\bar{\Theta}_4 = \frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(\alpha+1)}+\frac{(T-\omega_0)}{|\Lambda|}\left[\frac{\mu_2 q^{\binom{\alpha}{2}+\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}\right]. \tag{3.8}$$

*Proof.* We need to prove that $\mathscr{A}$ is a contraction mapping. Letting $t\in I_{q,\omega}^{T}$ and $(u_1,u_2),(v_1,v_2)\in\mathscr{C}$, we obtain

$$\begin{aligned}&\left|\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{P}_v(F_1^*,F_2^*)\right|\\ &\leq\left|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right|+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}\left|F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),u\big)-F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),v\big)\right|\tilde{d}_{q,\omega}s\\ &+\frac{\mu_2 q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)}\left|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\right|\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\tilde{d}_{q,\omega}s+\frac{\mu_2 q^{\binom{\alpha}{2}} q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_2)}\\ &\times\int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_2-1}(\xi)}\widetilde{(\eta_2-\xi)}_{q,\omega}^{\underline{\theta_2-1}}\widetilde{\big(\sigma_{q,\omega}^{\theta_2-1}(\xi)-s\big)}_{q,\omega}^{\underline{\alpha-1}}\left|F_2^*\big(\sigma_{q,\omega}^{\alpha-1}(s),u\big)-F_2^*\big(\sigma_{q,\omega}^{\alpha-1}(s),v\big)\right|\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi\\ &\leq K_1\left\|u_1-v_1\right\|+K_2\left\|u_2-v_2\right\|+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}\Big\{M_1\left|u_1-v_1\right|+N_2\left|u_2-v_2\right|\Big\}\end{aligned}$$

$$+\frac{\mu_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(1+\theta_2)}\Big\{L_1\|u_1-v_1\|+L_2\|u_2-v_2\|\Big\}$$
$$+\frac{\mu_2 q^{\binom{\alpha}{2}+\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}\Big\{M_2|u_2-v_2|+N_1|u_1-v_1|\Big\}$$

and

$$\begin{aligned}
&\Big|\mathscr{Q}_u(F_1^*,F_2^*)-\mathscr{Q}_v(F_1^*,F_2^*)\Big|\\
&\le \big|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\big|+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}\big|F_2^*\big(\sigma_{q,\omega}^{\alpha-1}(s),u\big)-F_2^*\big(\sigma_{q,\omega}^{\alpha-1}(s),v\big)\big|\,\tilde{d}_{q,\omega}s\\
&+\frac{\mu_1 q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\big|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\big|\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\tilde{d}_{q,\omega}s+\frac{\mu_1 q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_1)}\times\\
&\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(\xi)}\widetilde{(\eta_1-\xi)}_{q,\omega}^{\underline{\theta_1-1}}\big(\widetilde{\sigma_{q,\omega}^{\theta_1-1}(\xi)-s}\big)_{q,\omega}^{\underline{\alpha-1}}\big|F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),u\big)-F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),v\big)\big|\,\tilde{d}_{q,\omega}s\,\tilde{d}_{q,\omega}\xi\\
&\le L_1\|u_1-v_1\|+L_2\|u_2-v_2\|+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}(1+\alpha)}\Big(M_1|u_2-v_2|+N_1|u_1-v_1|\Big)\\
&+\frac{\mu_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\binom{\theta_1}{2}}}{\tilde{\Gamma}(1+\theta_1)}\Big(K_1\|u_1-v_1\|+K_2\|u_2-v_2\|\Big)\\
&+\frac{\mu_1 q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{q^{-\alpha\theta_1}\tilde{\Gamma}(\alpha+1+\theta_1)}\Big(M_1|u_1-v_1|+N_2|u_2-v_2|\Big).
\end{aligned}$$

These yield

$$\begin{aligned}
&\Big|\left(\mathscr{A}_1(u_1,u_2)\right)(t)-\left(\mathscr{A}_1(v_1,v_2)\right)(t)\Big|\\
&\le\Big|\phi_1(u_1,u_2)+\phi_1(v_1-v_2)\Big|+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\big|F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),u\big)-F_1^*\big(\sigma_{q,\omega}^{\alpha-1}(s),v\big)\big|\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\\
&+\frac{1}{|\Lambda|}\left\{\frac{\mu_1 q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\Big|\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{P}_v(F_1^*,F_2^*)\Big|\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\big(\sigma_{q,\omega}^{\theta_1-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s\right.\\
&\left.+\frac{\mu_2 q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)}\Big|\mathscr{Q}_u(F_1^*,F_2^*)-\mathscr{Q}_v(F_1^*,F_2^*)\Big|\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\big(\sigma_{q,\omega}^{\theta_2-1}(s)-\omega_0\big)\,\tilde{d}_{q,\omega}s\right\}\\
&\le\Big(K_1\|u_1-v_1\|+K_2\|u_2-v_2\|\Big)\\
&\times\left\{1+\frac{\mu_1 q^{\binom{\theta_1}{2}}q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)}+\frac{\mu_2 q^{\binom{\theta_2}{2}}q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}\mu_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)\tilde{\Gamma}_q(1+\theta_1)}\right\}\\
&+\Big(L_1\|u_1-v_1\|+L_2\|u_2-v_2\|\Big)\left\{\frac{\mu_1 q^{\binom{\theta_1}{2}}q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}\mu_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)\tilde{\Gamma}_q(1+\theta_2)}\right.
\end{aligned}$$

$$+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)}\Bigg\}+\Big(M_1|u_1-v_1|+N_2|u_2-v_2|\Big)\Bigg\{\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}+$$
$$\frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1} q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)\tilde{\Gamma}_q(1+\alpha)}+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}\mu_1 q^{\binom{\alpha}{2}} q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}\Bigg\}$$
$$+\Big(M_2|u_2-v_2|+N_1|u_1-v_1|\Big)\Bigg\{\frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}\mu_2 q^{\binom{\alpha}{2}+\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}$$
$$+\frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1} q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)\tilde{\Gamma}_q(1+\alpha)}\Bigg\}$$
$$\leq \|u_1-v_1\|_{\mathscr{C}}\left[K_1\Theta_1+L_1\Theta_2+M_1\Theta_3+N_1\Theta_4\right]+\|u_2-v_2\|_{\mathscr{C}}\left[K_2\Theta_1+L_2\Theta_2+N_2\Theta_3+M_2\Theta_4\right].$$

Therefore,

$$\left\|\mathscr{A}_1(u_1,u_2)-\mathscr{A}_1(v_1,v_2)\right\| \leq \|(u_1-v_1,u_2-v_2)\|_{\mathscr{C}}\max\Big\{K_1\Theta_1+L_1\Theta_2+M_1\Theta_3+N_1\Theta_4,$$
$$+K_2\Theta_1+L_2\Theta_2+N_2\Theta_3+M_2\Theta_4\Big\}.$$

Hence

$$\|\mathscr{A}_1(u_1,u_2)-\mathscr{A}_1(v_1,v_2)\|_{\mathscr{C}} \leq \chi\|(u_1-v_1,u_2-v_2)\|_{\mathscr{C}} \tag{3.9}$$

Similarly, we obtain

$$\left\|\mathscr{A}_2(u_1,u_2)-\mathscr{A}_2(v_1,v_2)\right\|$$
$$\leq \Big|\phi_2(u_1,u_2)+\phi_2(v_1-v_2)\Big|+\frac{q^{\binom{\alpha}{2}}}{\tilde{\Gamma}_q(\alpha)}\left|F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right)-F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),v\right)\right|\int_{\omega_0}^{t}\widetilde{(t-s)}_{q,\omega}^{\alpha-1}\tilde{d}_{q,\omega}s$$
$$+\frac{(T-\omega_0)}{|\Lambda|}\Big|\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{P}_v(F_1^*,F_2^*)\Big|+\frac{(T-\omega_0)}{|\Lambda|}\Big|\mathscr{Q}_u(F_1^*,F_2^*)-\mathscr{Q}_v(F_1^*,F_2^*)\Big|$$
$$\leq \|(u_1-v_1,u_2-v_2)\|_{\mathscr{C}}\Big[L_1\bar{\Theta}_1+K_1\bar{\Theta}_2+M_1\bar{\Theta}_3+N_1\bar{\Theta}_4+L_2\bar{\Theta}_1$$
$$+K_2\bar{\Theta}_2+N_2\bar{\Theta}_3+M_2\bar{\Theta}_4\Big].$$

Hence $\|\mathscr{A}_2(u_1,u_2)-\mathscr{A}_2(v_1,v_2)\|_{\mathscr{C}} \leq \chi\|(u_1-v_1,u_2-v_2)\|_{\mathscr{C}}$, which together with (3.9) concludes that $\left\|\mathscr{A}(u_1,u_2)-\mathscr{A}(v_1,v_2)\right\|_{\mathscr{C}} < \chi\|(u_1-v_1,u_2-v_2)\|_{\mathscr{C}}$. Since $\chi<1$, then $\mathscr{A}$ is a contraction mapping. Hence, from Banach fixed point theorem, we can conclude that $\mathscr{A}$ has a fixed point. Therefore problem (1.1)-(1.2) has a unique solution. □

## 4. Existence of At Least One Solution

In this section, we further present the existence of at least one solution to (1.1)-(1.2) by using Schauder's fixed point theorem.

**Theorem 4.1.** *Suppose that $(H_1)-(H_2)$ hold. Then, problem (1.2) has at least one solution on $I_{q,\omega}^T$.*

*Proof.* We divide the proof into three steps as follows

Step 1. Verify that $\mathscr{A}$ map bounded sets into bounded sets in $B_R = \{(u_1,u_2) \in \mathscr{C} : \|(u_1,u_2)\|_{\mathscr{C}} \leq R\}$.

Let $\max_{t\in I^T_{q,\omega}} |F_i(t,u_1,u_2)| = A_i$, $\sup_{(u_1,u_2)\in\mathscr{U}} |\phi_i(u_1,u_2)| = B_i$ for $i = 1,2$ and choose a constant $L = \max\{A_1\Theta_3 + B_1\Theta_1 + A_2\Theta_4 + B_2\Theta_2,\ A_1\bar{\Theta}_3 + B_1\bar{\Theta}_2 + A_2\bar{\Theta}_4 + B_2\bar{\Theta}_1\}$, where $\Theta_i, \bar{\Theta}_i$, $i = 1,2,3,4$ are defined as (3.1)-(3.8),respectively. For each $t \in I^T_{q,\omega}$ and $(u_1,u_2) \in B_R$, we obtain

$$\begin{aligned} \left|\mathscr{P}_u(F_1^*,F_2^*)\right| \leq B_1 &+ \frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}A_1}{\tilde{\Gamma}_q(1+\alpha)} + \frac{\mu_2 B_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{\tilde{\Gamma}_q(1+\theta_2)} \\ &+ \frac{\mu_2 q^{\binom{\alpha}{2}} q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}A_2}{q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \left|\mathscr{Q}_u(F_1^*,F_2^*)\right| \leq B_2 &+ \frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}A_2}{\tilde{\Gamma}_q(1+\alpha)} + \frac{\mu_1 B_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{\tilde{\Gamma}_q(1+\theta_1)} \\ &+ \frac{\mu_1 q^{\binom{\alpha}{2}} q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}A_1}{q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}. \end{aligned} \tag{4.2}$$

From (4.1)-(4.2), we find that

$$\begin{aligned} &\left|\left(\mathscr{A}_1(u_1,u_2)\right)(t)\right| \\ \leq\, & A_1\left[\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)} + \frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1} q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)\tilde{\Gamma}_q(1+\alpha)}\right. \\ &\left. + \frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}\mu_1 q^{\binom{\alpha}{2}} q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}\right] \\ & + B_1\left[1 + \frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)} + \frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}\mu_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)\tilde{\Gamma}_q(1+\theta_1)}\right] \\ & + A_2\left[\frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}\mu_2 q^{\binom{\alpha}{2}+\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}\right. \\ &\left. + \frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1} q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)\tilde{\Gamma}_q(1+\alpha)}\right] \\ & + B_2\left[\frac{\mu_1 q^{\binom{\theta_1}{2}} q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}\mu_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)\tilde{\Gamma}_q(1+\theta_2)} + \frac{\mu_2 q^{\binom{\theta_2}{2}} q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)}\right] \\ =\, & A_1\Theta_3 + B_1\Theta_1 + A_2\Theta_4 + B_2\Theta_2. \end{aligned} \tag{4.3}$$

Hence

$$\|\mathscr{A}_1(u_1,u_2)(t)\|_{\mathscr{C}} < A_1\Theta_3 + A_1\Theta_1 + A_2\Theta_4 + B_2\Theta_2. \tag{4.4}$$

Furthermore, we have

$$
\begin{aligned}
&\left|\mathscr{A}_2(u_1,u_2)(t)\right|\\
&\leq\ B_2+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}A_2}{\tilde{\Gamma}_q(1+\alpha)}+\frac{(T-\omega_0)}{|\Lambda|}\left\{B_1+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}A_1}{\tilde{\Gamma}_q(1+\alpha)}+\frac{\mu_2B_2q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{\tilde{\Gamma}_q(1+\theta_2)}\right.\\
&\quad+\frac{\mu_2q^{\binom{\alpha}{2}}q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}A_2}{q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}+B_2+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}A_2}{\tilde{\Gamma}_q(1+\alpha)}++\frac{\mu_1B_1q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{\tilde{\Gamma}_q(1+\theta_1)}\\
&\quad\left.+\frac{\mu_1q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}A_1}{q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}\right\}\\
&=A_1\frac{(T-\omega_0)}{|\Lambda|}\left[\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha_1)}+\frac{\mu_1q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}\right]\\
&\quad+B_1\frac{(T-\omega_0)}{|\Lambda|}\left[1+\frac{\mu_1q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{\tilde{\Gamma}_q(1+\theta_1)}\right]\\
&\quad+A_2\left\{\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}+\frac{(T-\omega_0)}{|\Lambda|}\left[\frac{\mu_2q^{\binom{\alpha}{2}+\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}+\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}\right]\right\}\\
&\quad+B_2\left[1+\frac{(T-\omega_0)}{|\Lambda|}\frac{\mu_2q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{\tilde{\Gamma}_q(1+\theta_2)}+\frac{(T-\omega_0)}{|\Lambda|}\right]\\
&=\ A_1\bar{\Theta}_3+B_1\bar{\Theta}_2+A_2\bar{\Theta}_4+B_2\bar{\Theta}_1. \qquad (4.5)
\end{aligned}
$$

Thus

$$\|\mathscr{A}_2(u_1,u_2)(t)\|_{\mathscr{C}}<A_1\bar{\Theta}_3+B_1\bar{\Theta}_2+A_2\bar{\Theta}_4+B_2\bar{\Theta}_1. \qquad (4.6)$$

Therefore, from (4.4) and (4.6), we can conclude $\left\|\mathscr{A}(u_1,u_2)\right\|_{\mathscr{C}}\leq L\leq\infty$. Therefore $\mathscr{A}$ is uniformly bounded.

Step 2. Show that $\mathscr{A}$ is continuous on $B_R$.

Let $\varepsilon>0$. Then one sees that there exists $\delta=\max\{\delta_1,\delta_2,\delta_3,,\delta_4\}>0$ such that, for each $t\in I^T_{q,\omega}$ and $(u_1,u_2),(v_1,v_2)\in B_R$ with $|F_1^*(t,u)-F_1^*(t,v)|<\min\left\{\frac{\varepsilon}{4\Theta_3},\frac{\varepsilon}{4\bar{\Theta}_3}\right\}$ whenever $|u_1-v_1|+|u_2-v_2|<\delta_1$, $|F_2^*(t,u)-F_2^*(t,v)|<\min\left\{\frac{\varepsilon}{4\Theta_4},\frac{\varepsilon}{4\bar{\Theta}_4}\right\}$ whenever $|u_1-v_1|+|u_2-v_2|<\delta_2$, $|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)|<\min\left\{\frac{\varepsilon}{4\Theta_1},\frac{\varepsilon}{4\bar{\Theta}_2}\right\}$ whenever $|u_1-v_1|+|u_2-v_2|<\delta_3$,

$$|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)|<\min\left\{\frac{\varepsilon}{4\Theta_2},\frac{\varepsilon}{4\bar{\Theta}_1}\right\}$$

whenever $|u_1-v_1|+|u_2-v_2|<\delta_4$.

Consider

$$
\begin{aligned}
&\Big|\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{P}_v(F_1^*,F_2^*)\Big|\\
&\leq \|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\|+\frac{q^{\binom{\alpha}{2}}\left\|F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right)-F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),v\right)\right\|}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(t-s)}_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\\
&+\frac{\mu_2 q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\theta_2)}\left\|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\right\|\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\tilde{d}_{q,\omega}s+\frac{\mu_2 q^{\binom{\alpha}{2}}q^{\binom{\theta_2}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_2)}\times\\
&\left\|F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right)-F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),v\right)\right\|\int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_2-1}(\xi)}\widetilde{(\eta_2-\xi)}_{q,\omega}^{\underline{\theta_2-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_2-1}(\xi)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}\xi \qquad (4.7)\\
&\leq \|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\|+\|F_1^*(t,u)-F_1^*(t,v)\|\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}\\
&+\|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\|\frac{\mu_2 q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\theta_2}}{\tilde{\Gamma}_q(1+\theta_2)}\\
&+\|F_2^*(t,u)-F_2^*(t,v)\|\frac{\mu_2 q^{\binom{\alpha}{2}}q^{\binom{\theta_2}{2}}(\eta_2-\omega_0)^{\alpha+\theta_2}}{q^{-\alpha\theta_2}\tilde{\Gamma}_q(\alpha+1+\theta_2)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big|\mathscr{Q}_u(F_1^*,F_2^*)-\mathscr{Q}_v(F_1^*,F_2^*)\Big|\\
&\leq \|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\;|+\frac{q^{\binom{\alpha}{2}}\left\|F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right)-F_2^*\left(\sigma_{q,\omega}^{\alpha-1}(s),v\right)\right\|}{\tilde{\Gamma}_q(\alpha)}\int_{\omega_0}^{T}\widetilde{(T-s)}_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\\
&+\frac{\mu_1 q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\theta_1)}\left\|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right\|\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\tilde{d}_{q,\omega}s+\frac{\mu_1 q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}}{\tilde{\Gamma}_q(\alpha)\tilde{\Gamma}_q(\theta_1)}\times\\
&\left\|F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),u\right)-F_1^*\left(\sigma_{q,\omega}^{\alpha-1}(s),v\right)\right\|\int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\sigma_{q,\omega}^{\theta_1-1}(\xi)}\widetilde{(\eta_1-\xi)}_{q,\omega}^{\underline{\theta_1-1}}\left(\widetilde{\sigma_{q,\omega}^{\theta_1-1}(\xi)-s}\right)_{q,\omega}^{\underline{\alpha-1}}\tilde{d}_{q,\omega}s\tilde{d}_{q,\omega}\xi\\
&\leq \|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\|+\|F_2^*(t,u)-F_2^*(t,v)\|\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}\\
&+\|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\|\frac{\mu_1 q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\theta_1}}{\Gamma_q(1+\theta_1)}+\|F_1^*(t,u)-F_1^*(t,v)\|\frac{\mu_1 q^{\binom{\alpha}{2}}q^{\binom{\theta_1}{2}}(\eta_1-\omega_0)^{\alpha+\theta_1}}{q^{-\alpha\theta_1}\tilde{\Gamma}_q(\alpha+1+\theta_1)}. \qquad (4.8)
\end{aligned}
$$

From (4.7) – (4.8), we find that

$$
\begin{aligned}
&\Big|\left(\mathscr{A}_1(u_1,u_2)\right)(t)-\left(\mathscr{A}_1(v_1,v_2)\right)(t)\Big|\\
&\qquad\leq \|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\|+\|F_1^*(t,u)-F_1^*(t,v)\|\frac{q^{\binom{\alpha}{2}}(T-\omega_0)^{\alpha}}{\tilde{\Gamma}_q(1+\alpha)}\\
&\qquad+\|\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{P}_v(F_1^*,F_2^*)\|\frac{\mu_1 q^{\binom{\theta_1}{2}}q^{\theta_1}(\eta_1-\omega_0)^{\theta_1+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_1)}\\
&\qquad+\|\mathscr{Q}_u(F_1^*,F_2^*)-\mathscr{Q}_v(F_1^*,F_2^*)\|\frac{\mu_2 q^{\binom{\theta_2}{2}}q^{\theta_2}(\eta_2-\omega_0)^{\theta_2+1}}{|\Lambda|\tilde{\Gamma}_q(2+\theta_2)}. \qquad (4.9)
\end{aligned}
$$

Therefore,

$$\begin{aligned}\left\|\mathscr{A}_1(u_1,u_2)-\mathscr{A}_1(v_1,v_2)\right\|_{\mathscr{C}} \leq &\left\|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right\|\Theta_1+\left\|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\right\|\Theta_2+\\ &\left\|F_1^*(t,u)-F_1^*(t,v)\right\|\Theta_3+\left\|F_2^*(t,u)-F_2^*(t,v)\right\|\Theta_4\\ <&\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon. \end{aligned} \tag{4.10}$$

Similarly, we obtain

$$\begin{aligned}\left\|\mathscr{A}_2(u_1,u_2)-\mathscr{A}_2(v_1,v_2)\right\|_{\mathscr{C}} \leq &\left\|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right\|\bar{\Theta}_2+\left\|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\right\|\bar{\Theta}_1+\\ &\left\|F_1^*(t,u)-F_1^*(t,v)\right\|\bar{\Theta}_3+\left\|F_2^*(\sigma_{q,\omega}^{\alpha-1}(t),u)-F_2^*(\sigma_{q,\omega}^{\alpha-1}(t),v)\right\|\bar{\Theta}_4\\ <&\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon. \end{aligned} \tag{4.11}$$

Therefore, from (4.10) and (4.11), we can conclude that $\left\|\mathscr{A}(u_1,u_2)-\mathscr{A}(v_1,v_2)\right\|_{\mathscr{C}} < \varepsilon$. This means that $\mathscr{A}$ is continuous on $B_R$.

Step 2. Prove that $\mathscr{A}$ is equicontinuous with $B_R$.
For any $t_1,t_2 \in I_{q,\omega}^T$ with $t_1<t_2$, we have

$$\begin{aligned}&\left|(\mathscr{A}_1(u_1,u_2))(t_2)-(\mathscr{A}_1(u_1,u_2))(t_1)\right|\\ &\quad\leq \frac{q^{\binom{\alpha}{2}}\|F_1^*\|}{\tilde{\Gamma}_q(1+\alpha)}\left|(t_2-\omega_0)^{\alpha}-(t_1-\omega_0)^{\alpha}\right|\\ &\quad+\frac{|t_2-t_1|}{|\Lambda|}\left\{\frac{\mu_1 q^{\binom{\theta_1}{2}}\mathscr{P}_u(F_1^*,F_2^*)}{(T-\omega_0)\tilde{\Gamma}_q(\theta_1)}\int_{\omega_0}^{\eta_1}\widetilde{(\eta_1-s)}_{q,\omega}^{\underline{\theta_1-1}}\left(\sigma_{q,\omega}^{\theta_1-1}(s)-\omega_0\right)\tilde{d}_{q,\omega}s\right.\\ &\quad\left.-\frac{\mu_2 q^{\binom{\theta_2}{2}}\mathscr{P}_u(F_1^*,F_2^*)}{(T-\omega_0)\tilde{\Gamma}_q(\theta_2)}\int_{\omega_0}^{\eta_2}\widetilde{(\eta_2-s)}_{q,\omega}^{\underline{\theta_2-1}}\left(\sigma_{q,\omega}^{\theta_2-1}(s)-\omega_0\right)\tilde{d}_{q,\omega}s\right\}, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\left|(\mathscr{A}_2(u_1,u_2))(t_2)-(\mathscr{A}_2(u_1,u_2))(t_1)\right|\\ &\quad\leq \frac{q^{\binom{\alpha}{2}}\|F_2^*\|}{\tilde{\Gamma}_q(1+\alpha)}\left|(t_2-\omega_0)^{\alpha}-(t_1-\omega_0)^{\alpha}\right|+\frac{|t_2-t_1|}{|\Lambda|}\left(\mathscr{P}_u(F_1^*,F_2^*)-\mathscr{Q}_u(F_1^*,F_2^*)\right). \end{aligned} \tag{4.13}$$

When $|t_2-t_1|\to 0$, the right-hand side of inequalities (4.12) and (4.13) tends be zero. So, $\mathscr{A}$ is relatively compact on $B_R$.

As a result, $\mathscr{G}(B_R)$ exhibits equicontinuity. By applying the Arzelá-Ascoli theorem to the steps above, we demonstrate that $\mathscr{A}:\mathscr{C}\to\mathscr{C}$ is completely continuous. Therefore, the Schauder fixed-point theorem allows us to conclude that problem (1.1)-(1.2) possesses at least one solution. □

## 5. EXAMPLE

We illustrate the application of our findings with the following examples. Consider the system of fractional symmetric Hahn difference equations

$$\begin{aligned}
\tilde{D}^{\frac{3}{2}}_{\frac{1}{2},\frac{3}{4}}u_1(t) &= \frac{\sin^2(t+\pi)}{1000(t^2+3t)}|u_1(t)|+\frac{e^{-2t}}{1000t^3}|u_2(t)|,\\
\tilde{D}^{\frac{3}{2}}_{\frac{1}{2},\frac{3}{4}}u_2(t) &= \frac{\cos^2(t+\pi)}{1000(t^3+2t)}|u_2(t)|+\frac{e^{-t}}{1000(t^2+t)}|u_1(t)|,\ t\in\Big[\frac{3}{2},30\Big]_{\frac{1}{2},\frac{3}{4}}\\
u_1\left(\frac{3}{2}\right) &= \frac{|u_1+u_2|}{e^2} \qquad , \qquad u_1(30)=3\tilde{\mathscr{I}}^{\frac{1}{5}}_{\frac{1}{2},\frac{3}{4}}\big[u_2(15)\big],\\
u_2\left(\frac{3}{2}\right) &= \frac{|u_1+u_2|}{e^3} \qquad , \qquad u_2(30)=5\tilde{\mathscr{I}}^{\frac{1}{5}}_{\frac{1}{2},\frac{3}{4}}\big[u_2(10)\big]. \qquad (5.1)
\end{aligned}$$

Here, we set $q=\frac{1}{2},\ \omega=\frac{3}{4},\ \omega_0=\frac{\omega}{1-q}=\frac{3}{2},\ \alpha=\frac{3}{2},\ \theta_1=\frac{1}{10},\ \theta_2=\frac{1}{5},\ T=30,\ \eta_1=10,\ \eta_2=15,\ \mu_1=3,\ \mu_2=5,\ \phi_1(u_1,u_2)=\frac{|u_1+u_2|}{1000(e^2+100)},\ \phi_2(u_1,u_2)=\frac{|u_1+u_2|}{1000(e^3+200)}$ and

$$\begin{aligned}
F_1(t,u_1(t),u_2(t)) &= \frac{\sin^2(t+\pi)}{1000(t^2+3t)}|u_1(t)|+\frac{e^{-2t}}{1000t^3}|u_2(t)|,\\
F_2(t,u_1(t),u_2(t)) &= \frac{\cos^2(t+\pi)}{1000(t^3+2t)}|u_2(t)|+\frac{e^{-t}}{1000(t^2+t)}|u_1(t)|.
\end{aligned}$$

For all $t\in\big[\frac{3}{2},30\big]_{\frac{1}{2},\frac{3}{4}}$ and $u,v\in\mathbb{R}$, it is clear that

$$\begin{aligned}
|F_1(t,u_1,u_2)-F_1(t,v_1,v_2)| &\le \frac{1}{6750}|u_1-v_1|+\frac{2}{3375e^3}|u_2-v_2|,\\
|F_2(t,u_1,u_2)-F_2(t,v_1,v_2)| &\le \frac{2}{6375}|u_1-v_1|+\frac{1}{37505e^{\frac{3}{2}}}|u_2-v_2|.
\end{aligned}$$

So, $(H_1)$ holds with $M_1=1.4815\times10^{-4}$, $M_2=3.1373\times10^{-4}$ and $N_1=5.9501\times10^{-5}$, $N_2=5.9259\times10^{-4}$.

For all $u,v\in\mathscr{C}$, we have

$$\begin{aligned}
|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)| &\le \frac{1}{1000e^3}\|u_1-v_1\|+\frac{1}{500e^2}\|u_2-v_2\|,\\
|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)| &\le \frac{1}{500e^2}\|u_1-v_1\|+\frac{1}{1000e^3}\|u_2-v_2\|.
\end{aligned}$$

Thus, $(H_2)$ holds with $K_1=1.3534\times10^{-4}$, $K_2=2.7067\times10^{-4}$ and $L_1=9.9574\times10^{-5}$, $L_2=4.9787\times10^{-5}$. Also, we find that

$$\begin{gathered}
|\Lambda|=72.1339,\ \Theta_1=6.9128,\ \Theta_2=5.2021,\ \Theta_3=3763.3,\ \Theta_4=260.2858,\\
\bar{\Theta}_1=4.9798,\ \bar{\Theta}_2=1.9274,\ \bar{\Theta}_3=69.8312,\ \bar{\Theta}_4=264.7638
\end{gathered}$$

and then

$$\begin{aligned}
\chi = {} & \max\big\{K_1\Theta_1+L_1\Theta_2+M_1\Theta_3+N_1\Theta_4+K_2\Theta_1+L_2\Theta_2+N_2\Theta_3+M_2\Theta_4,\\
& L_1\bar{\Theta}_1+K_1\bar{\Theta}_2+M_1\bar{\Theta}_3+N_1\bar{\Theta}_4+L_2\bar{\Theta}_1+K_2\bar{\Theta}_2+N_2\bar{\Theta}_3+M_2\bar{\Theta}_4\big\}\\
= {} & 0.7429\ <\ 1.
\end{aligned}$$

Therefore, we can conclude from Theorem 3.1 that problem (5.1) has a unique solution. □

We present the solution graphs for problem $(1.1)-(1.2)$, with $q=\frac{1}{2}, \omega=\frac{3}{4}, \omega_0=\frac{3}{2}, \eta_1=10, \eta_2=15, \mu_1=3, \mu_2=5, \phi_1=2, \phi_2=4, F_1=F_2=1$ and $T=30$. In Figure 1, when $\alpha=1.5, \theta_1=0.1, \theta_2=0.2$, it can be observed that the solution graphs of $u_1$ and $u_2$ initially decrease. However, after $t>15$, the values of $u_1$ and $u_2$ gradually increase. In Figure 2, when $\alpha=1.1, \theta_1=0.7, \theta_2=0.1$, the solution graphs of $u_1$ and $u_2$ exhibit a decreasing function behavior.

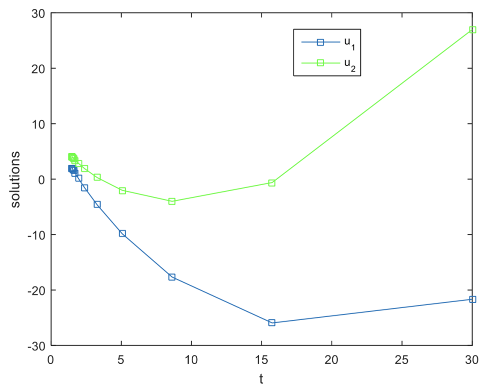


FIGURE 1. Graph of the solutions to problem $(1.1)-(1.2)$, when $\alpha=1.5, \theta_1=0.1, \theta_2=0.2$.

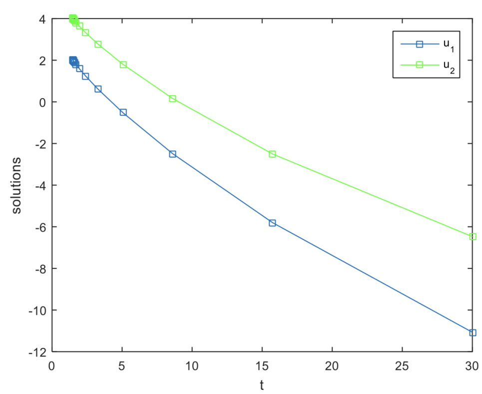


FIGURE 2. Graph of the solutions to problem $(1.1)-(1.2)$, when $\alpha=1.1, \theta_1=0.7, \theta_2=0.1$.

## 6. CONCLUSIONS

This paper analyzed the existence and uniqueness of solutions for a coupled system of fractional symmetric Hahn difference equations with nonlocal fractional symmetric Hahn integral boundary conditions. Banach's fixed point theorem serves as a primary tool to ascertain the conditions for a unique solution. Schauder fixed point theorem was then used to determine conditions for the existence of at least one solution.

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