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MULTI-STEP INERTIAL ACCELERATED ALGORITHMS FOR SOLVING MIXED SPLIT FEASIBILITY PROBLEMS

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Abstract. The split feasibility problem has significant applications in various fields such as medical image reconstruction, signal processing, radiation therapy, and signal recovery. With the widespread adoption of multimodal data and large-scale complex systems, efficiently solving mixed split feasibility problems which involve multiple operator constraints and linear equality conditions is now under the spotlight of research in optimization theory. In this paper, we propose two new accelerated iterative algorithms for MSFP: one accelerates the iterative process through multi-step inertial terms and proves weak convergence in Hilbert spaces; the other one further incorporates the viscosity approximation method to achieve strong convergence. We prove that our algorithm converges strongly under suitable conditions. Finally, numerical results illustrate the performances of our algorithms.

Keywords. Multi-step inertial algorithm; Split feasibility problem; Viscosity approximation method.
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1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. The description of the split feasibility problem (SFP) is to

$$\text{find } \bar{u}^* \in C \textit{ such that } A\bar{u}^* \in Q,$$

where $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a linear and bounded operator, C and Q are nonempty, convex, and closed sets in $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

In view of the wide applications of the SFP, such as medical image reconstruction [24], signal processing [6] and intensity modulated radiation therapy [7], this problem, which was first introduced by Censor and Elfving [6] in finite dimensional Hilbert spaces, has been extensively studied numerically; see, e.g., [8, 9, 19, 20, 26, 27] and the references therein. To solve the SFP, Censor and Elfving proposed an iterative algorithm based on projections, which involves the computation of the inverse of a matrix [6]. However, this approach may be inefficient in

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practice. Byrne [4, 5] introduced the celebrated CQ algorithm, which overcomes this drawback by introducing an appropriate step-size selection. Indeed, it can avoid the computation of matrix inverses and improve algorithmic efficiency. The split feasibility problem with two operators (SFPT) is an extension of the celebrated SFP, involving two bounded linear operators A and B. The problem is defined as finding $\bar{u}^* \in C$ such that

$$A\bar{u}^* \in Q \text{ and } B\bar{u}^* \in \mathscr{M},$$

where C, Q, and $\mathscr{M}$ are nonempty, convex, and closed subsets of Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$, respectively.

Recently, Jailoka et al. [12] proposed a self-adaptive CQ type algorithm for the SFPT. They proved that the sequence $\{u_k\}$ generated by their algorithm strongly converges to a solution of the SFPT. Recently, inertial methods have been widely investigated in recent years to accelerate the convergence of various algorithms. Liang proposed a multi-step inertial operator splitting method in [13]. Similar with the q-step method in [16], x_{n+1} in Liang′s method involves at most $q+1$ previous iterations $\{x_n, x_{n-1}, \cdots, x_{n-q}\}$. Let $Q = \{0, 1, \cdots, q-1\}, q \in \mathbb{N}_+$. The multi-step inertial form is as following:

$$y_n = x_n + \sum_{i \in Q} \delta_{i,n}(x_{n-i} - x_{n-i-1}).$$

The numerical example showed the superiority of Liang's method. Therefore, the ideas of alternated inertial and multi-step inertial are widely used, as in Dong, He, and Rassias [10], and Duan and Zhang [11]. Wang, Liu, and Yang [25] proposed an alternated multi-step inertial iterative algorithm for the SFPT in Hilbert spaces and strong convergence result is obtained under some mild conditions, which showed the superiority in many aspects.

The split feasibility problem with multiple output sets (SFP-MOS), which was introduced and studied by Reich, Tuyen, and Mai in [22] , is a generalization of the SFP and SFPT, involving multiple sets and multiple Hilbert spaces. Given nonempty, convex, and closed subsets $C \subset \mathscr{H}$ and $Q_i \subset \mathscr{H}_i$ $(i = 1, 2, \cdots, N)$, and linear and bounded operators B_i, the goal is to find x^* such that

$$x^* \in C \text{ and } B_i x^* \in Q_i, \ \forall\, i,$$

which extends the SFPT by considering multiple operators B_i and multiple target sets Q_i, making it applicable to scenarios where multiple constraints across different spaces need to be satisfied. The split equality problem (SEP) is another extension of the SFP, which involves finding (x, y) such that

$$x \in C, \ y \in Q \text{ and } Ax = By,$$

where C and Q are nonempty, convex, and closed subsets of Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively and $A : \mathscr{H}_1 \to \mathscr{H}_3$, $B : \mathscr{H}_2 \to \mathscr{H}_3$ are linear and bounded operators. This problem, which was first introduced by Moudafi et al. [14, 15], is applicable to fields such as game theory, decomposition methods for PDE, decision sciences, and inertial Nash equilibria [1, 2].

The mixed split feasibility problem (MSFP), which was introduced and studied by Reich et al. [21], is a further generalization of the SFP, combining the SFP-MOS and the SEP. It aims to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{N} \{x \in C : A_i x \in Q_i\} \quad \text{and} \quad \sum_{i=1}^{N} B_i(A_i x^*) = y,$$

where C and Q_i are nonempty, convex, and closed subsets of Hilbert spaces $\mathscr{H}$ and $\mathscr{H}_i$, respectively, $A_i : \mathscr{H} \to \mathscr{H}_i$ and $B_i : \mathscr{H}_i \to \mathscr{K}$ are linear and bounded operators, and y is a given element in $\mathscr{K}$. Recent studies focused on various iterative algorithms for solving the MSFP, including the methods based on unconstrained optimization approaches and self-adaptive stepsizes. The SFP-MOS can be viewed as a special case of the MSFP if B_i are zero operators for all i and $y = 0$. In this scenario, the MSFP reduces to finding a point $x^* \in C$ such that $A_i x^* \in Q_i$ for all i, which is precisely the SFP-MOS. The SEP can be derived from the MSFP under specific conditions. Specifically, when $N = 2$, $\mathscr{H} = \mathscr{H}_1 \times \mathscr{H}_2$, $C = Q_1 \times Q_2$ and the operators A_i are defined such that $A_1(x^{(1)}, x^{(2)}) = x^{(1)}$ and $A_2(x^{(1)}, x^{(2)}) = x^{(2)}$, with $B_1 = -B_2$ and $y = 0$, the MSFP reduces to the SEP. This reduction shows that the SEP is a particular instance of the MSFP, highlighting the versatility and generality of the MSFP framework.

In this paper, we also consider the MSFP. The following assumptions are imposed

(A.1) $\mathscr{H}_i (i = 1, 2, \cdots, N), \mathscr{H}$ and $\mathscr{K}$ are real Hilbert spaces; C and $Q_i (i = 1, 2, \cdots, N)$ are nonempty, convex, and closed subsets of $\mathscr{H}$ and $\mathscr{H}_i$, respectively.

(A.2) $A_i : \mathscr{H} \to \mathscr{H}_i$ and $B_i : \mathscr{H}_i \to \mathscr{K}$ $(i = 1, 2, \cdots, N)$ are linear and bounded operators.

(A.3) y is a given element in $\mathscr{K}$.

(A.4) $\Omega = \cap_{i=1}^N \{x \in C : A_i x \in Q_i\} \cap \{x \in C : \sum_{i=1}^N B_i(A_i x) = y\} \neq \emptyset$.

For each $x \in \mathscr{H}$, we define the function $F : \mathscr{H} \to \mathbb{R}$ by

$$\begin{aligned} F(x) := & \frac{\|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(x)\|_{\mathscr{H}}^2}{2} + \frac{\sum_{i=1}^N \|(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i x)\|_{\mathscr{H}_i}^2}{2} \\ & + \frac{\|\sum_{i=1}^N B_i(A_i x) - y\|_{\mathscr{K}}^2}{2}. \end{aligned}$$

It is not difficult to see that F is a differentiable convex function and that the MSFP is equivalent to the unconstrained optimization problem: $\min_{x \in \mathscr{H}} F(x)$. Consequently, x^* is a solution to the MSFP if and only if $\nabla F(x^*) = 0$, where

$$\nabla F(x) := (I^{\mathscr{H}} - P_C^{\mathscr{H}})(x) + \sum_{i=1}^N A_i^* (I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i x) + \sum_{i=1}^N A_i^* B_i^* \left(\sum_{i=1}^N B_i(A_i x) - y \right).$$

Moreover, $\nabla F(x^*) = 0$ means $x^* = x^* - \gamma \nabla F(x^*)$, where γ is a positive real number. This implies that x^* is a fixed point of the operator I-$\gamma \nabla F$.

In [21], Reich et al. proposed an iterative method (Algorithm 1) for finding a solution of the MSFP. Their algorithm reads

Algorithm 1

Step 1. Let $\rho_n \in [a, b] \subset (0, 2)$ for all $n \in \mathbb{N}$. Choose $x_0 \in \mathscr{H}$ arbitrarily, set $n := 0$.

Step 2. Given x_n, compute $x_{n+1} = x_n - \gamma_n \nabla F(x_n)$, where

$$\gamma_n = \begin{cases} \rho_n \dfrac{D_n}{E_n}, & E_n \neq 0, \\ 0, & E_n = 0, \end{cases}$$

where

$$D_n := \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(x_n)\|_{\mathscr{H}}^2 + \sum_{i=1}^N \|(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i x_n)\|_{\mathscr{H}_i}^2 + \|\sum_{i=1}^N B_i(A_i x_n) - y\|_{\mathscr{K}}^2$$

and

$$E_n := \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(x_n) + \sum_{i=1}^{N} A_i^*(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i x_n) + \sum_{i=1}^{N} A_i^* B_i^* \left(\sum_{i=1}^{N} B_i(A_i x_n) - y \right) \|_{\mathscr{H}}^2.$$

Step 3. *Set* $n \leftarrow n+1$ and go to Step 2.

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2. Preliminaries

In this section, we collect some definitions and lemmas which are used in the next section. We denote the strong convergence and the weak convergence of sequence $\{x_n\}$ to x by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Consider a nonempty, convex, and closed subset C of a real Hilbert space $\mathscr{H}$. For each $x \in \mathscr{H}$, there exists a unique point $P_C^{\mathscr{H}}(x) \in C$ satisfying

$$\|x - P_C^{\mathscr{H}}(x)\| = \inf_{w \in C} \|x - w\|. \tag{2.1}$$

The mapping $P_C^{\mathscr{H}} : \mathscr{H} \to C$ defined by (2.1) is termed the metric projection of $\mathscr{H}$ onto C. Recall that a mapping $U : C \to C$ is said to be nonexpansive if $\|U(x) - U(y)\| \leq \|x - y\|$ for all $x, y \in C$. We denote the set of fixed points of an operator $U : C \to C$ by $\mathrm{Fix}(U)$, that is, $\mathrm{Fix}(U) = \{x \in C : U(x) = x\}$. It is well established that the metric projection $P_C^{\mathscr{H}}$ is a nonexpansive mapping and satisfies $\mathrm{Fix}(P_C^{\mathscr{H}}) = C$.

Lemma 2.1. *(see [3]) Let $P_C^{\mathscr{H}}$ be the metric projection of a real Hilbert space $\mathscr{H}$ onto a nonempty, convex, and closed subset C of $\mathscr{H}$. Then the following statements hold true:*
(i) $\langle x - P_C^{\mathscr{H}}(x), y - P_C^{\mathscr{H}}(x) \rangle_{\mathscr{H}} \leq 0$ for all $x \in \mathscr{H}$ and $y \in C$;
(ii) $\langle x - y, (I^{\mathscr{H}} - P_C^{\mathscr{H}})(x) - (I^{\mathscr{H}} - P_C^{\mathscr{H}})(y) \rangle_{\mathscr{H}} \geq \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(x) - (I^{\mathscr{H}} - P_C^{\mathscr{H}})(y)\|_{\mathscr{H}}^2$ for all $x, y \in \mathscr{H}$.
It follows that $I^{\mathscr{H}} - P_C^{\mathscr{H}}$ is a firmly nonexpansive mapping.

Lemma 2.2. *(see [17]) Let $\mathscr{H}$ be a real Hilbert space ,and let $\{x_n\}$ be a sequence in $\mathscr{H}$ such that $x_n \rightharpoonup z$ as $n \to \infty$. Then $\liminf_{n\to\infty} \|x_n - z\|_{\mathscr{H}} < \liminf_{n\to\infty} \|x_n - x\|_{\mathscr{H}}$ for all $x \in \mathscr{H}$ and $x \neq z$.*

Lemma 2.3. *(see [3]) Let C be a nonempty, convex, and closed subset of a Hilbert space $\mathscr{H}$. Let $T : C \to \mathscr{H}$ be a nonexpansive mapping. Then the mapping $I^{\mathscr{H}} - T$ is demiclosed, that is, whenever $\{x_n\}$ is a sequence in C for which $x_n \rightharpoonup x \in C$ and $x_n - T(x_n) \to y \in \mathscr{H}$, it follows that $x - T(x) = y$.*

Lemma 2.4. *Let $\mathscr{H}$ be a real Hilbert space. Then the following statements hold.*
(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \ \forall x, y \in \mathscr{H}$;
(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle x+y, y \rangle, \ \forall x, y \in \mathscr{H}$;
(iii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$, *for all $\alpha \in \mathbb{R}$ and $x, y \in \mathscr{H}$.*

Lemma 2.5. *(see [23]) Let $\mathfrak{a}_{n+1} \leq (1-\mathfrak{b}_n)\mathfrak{a}_n + \mathfrak{b}_n \mathfrak{c}_n$, where $\{\mathfrak{b}_n\}$ is a real sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \mathfrak{b}_n = \infty$ and $\{\mathfrak{c}_n\}$ and $\{\mathfrak{a}_n\}$ are real positive sequences. If $\limsup_{k\to\infty} \mathfrak{c}_{n_k} \leq 0$ for every subsequence $\{\mathfrak{a}_{n_k}\}$ of $\{\mathfrak{a}_n\}$ satisfying $\liminf_{k\to\infty}(\mathfrak{a}_{n_k+1} - \mathfrak{a}_{n_k}) \geq 0$, then $\lim_{n\to\infty} \mathfrak{a}_n = 0$.*

Lemma 2.6. *(see [18]) Let $a_{n+1} \leqslant (1+\lambda_n)a_n + \mu_n$, where $\{a_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$ are nonnegative real sequence such that $\sum_{n=0}^{\infty} \lambda_n < +\infty$ and $\sum_{n=0}^{\infty} \mu_n < +\infty$. Then $\lim_{n\to\infty} a_n$ exists.*

3. MAIN RESULTS

In this section, for solving the MSFP, we propose two new accelerated iterative algorithms: Algorithm 2 accelerates the iterative process through multi-step inertial terms and proves weak convergence in Hilbert spaces; Algorithm 3 further incorporates the viscosity approximation method to achieve strong convergence.

Algorithm 2

Step 1. Let $\rho_n \in [a,b] \subset (0,2)$ for all $n \in \mathbb{N}$. Choose $x_0,\ x_{-1}, \cdots,\ x_{-q} \in \mathscr{H}$ arbitrarily, non-negative real number ε_n with $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, $\beta \geq 3$ and set $n := 0$.
Step 2. Given $x_n,\ x_{n-1}, \cdots,\ x_{n-q}$, compute

$$\begin{cases} y_n = x_n + \sum\limits_{i\in Q} \Delta_{i,n}(x_{n-i} - x_{n-i-1}), \\ x_{n+1} = y_n - \gamma_n \nabla F(y_n), \end{cases} \tag{3.1}$$

where $Q := \{0,1,\cdots,q-1\}$, $q \in N^+$, and $\Delta_{i,n}$ satisfies $0 \leq |\Delta_{i,n}| \leq \overline{\Delta}_n$, where $\overline{\Delta}_n$ is defined by

$$\overline{\Delta}_n = \begin{cases} \min\{\dfrac{n-1}{n+\beta-1}, \dfrac{\varepsilon_n}{\sum_{i\in Q}\|x_{n-i}-x_{n-i-1}\|}\}, & \sum\limits_{i\in Q}\|x_{n-i}-x_{n-i-1}\| \neq 0, \\ \dfrac{n-1}{n+\beta-1}, & \sum\limits_{i\in Q}\|x_{n-i}-x_{n-i-1}\| = 0, \end{cases}$$

$$\gamma_n = \begin{cases} \rho_n \dfrac{D_n}{E_n}, & E_n \neq 0, \\ 0, & E_n = 0, \end{cases}$$

where

$$D_n := \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(y_n)\|_{\mathscr{H}}^2 + \sum_{i=1}^{N} \|(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i y_n)\|_{\mathscr{H}_i}^2 + \|\sum_{i=1}^{N} B_i(A_i y_n) - y\|_{\mathscr{K}}^2$$

and

$$E_n := \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(y_n) + \sum_{i=1}^{N} A_i^*(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i y_n) + \sum_{i=1}^{N} A_i^* B_i^* \left(\sum_{i=1}^{N} B_i(A_i y_n) - y\right)\|_{\mathscr{H}}^2.$$

Step 3. *Set* $n \leftarrow n+1$ and go to Step 2.

The weak convergence of the sequence generated by Algorithm 2 is established in the following theorem.

Theorem 3.1 If $\Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to a solution of the MSFP.

Proof. The proof is split into several steps. Fix a pint p in Ω.
Claim 1. prove that $\{x_n\}$ is bounded.

Indeed, using (3.1), we see that

$$\begin{aligned}\|y_n-p\|_{\mathscr{H}} &\le \|x_n-p\|_{\mathscr{H}}+\sum_{i\in Q}\overline{\Delta}_n\|x_{n-i}-x_{n-i-1}\|_{\mathscr{H}}\\ &\le \|x_n-p\|_{\mathscr{H}}+\varepsilon_n.\end{aligned}$$

We now observe that

$$\begin{aligned}&\langle\nabla F(y_n),y_n-p\rangle_{\mathscr{H}}\\ &=\langle(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n),y_n-p\rangle_{\mathscr{H}}\\ &+\sum_{i=1}^{N}\langle A_i^*(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n),y_n-p\rangle_{\mathscr{H}_i}\\ &+\sum_{i=1}^{N}\left\langle A_i^*B_i^*\left(\sum_{i=1}^{N}B_i(A_iy_n)-y\right),y_n-p\right\rangle_{\mathscr{K}}\\ &=\langle(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n),y_n-p\rangle_{\mathscr{H}}+\sum_{i=1}^{N}\langle(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n),A_iy_n-A_ip\rangle_{\mathscr{H}_i}\\ &+\left\langle\sum_{i=1}^{N}B_i(A_iy_n)-y,\sum_{i=1}^{N}B_i(A_iy_n)-\sum_{i=1}^{N}B_i(A_ip)\right\rangle_{\mathscr{K}}\\ &=\langle(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n)-(I^{\mathscr{H}}-P_C^{\mathscr{H}})(p),y_n-p\rangle_{\mathscr{H}}\\ &+\sum_{i=1}^{N}\langle(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n)-(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_ip),A_iy_n-A_ip\rangle_{\mathscr{H}_i}+\|\sum_{i=1}^{N}B_i(A_iy_n)-y\|_{\mathscr{K}}^2.\end{aligned}$$

We find that

$$\begin{aligned}\langle\nabla F(y_n),y_n-p\rangle_{\mathscr{H}} &\ge \|(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n)-(I^{\mathscr{H}}-P_C^{\mathscr{H}})(p)\|_{\mathscr{H}}^2\\ &+\sum_{i=1}^{N}\|(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n)-(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_ip)\|_{\mathscr{H}_i}^2\\ &+\|\sum_{i=1}^{N}B_i(A_iy_n)-y\|_{\mathscr{K}}^2\\ &=\|(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n)\|_{\mathscr{H}}^2+\sum_{i=1}^{N}\|(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n)\|_{\mathscr{H}_i}^2\\ &+\|\sum_{i=1}^{N}B_i(A_iy_n)-y\|_{\mathscr{K}}^2\\ &=D_n.\end{aligned}\tag{3.2}$$

Note that

$$\|\nabla F(y_n)\|_{\mathscr{H}}^2=E_n.\tag{3.3}$$

Further, we have

$$\begin{aligned}\|x_{n+1}-p\|_{\mathscr{H}}^2 &= \|y_n-p\|_{\mathscr{H}}^2-2\gamma_n\langle\nabla F(y_n),y_n-p\rangle_{\mathscr{H}}+\gamma_n^2\|\nabla F(y_n)\|_{\mathscr{H}}^2\\ &\le \|y_n-p\|_{\mathscr{H}}^2-2\gamma_nD_n+\gamma_n^2E_n.\end{aligned}$$

Thus

$$\|x_{n+1}-p\|^2_{\mathscr{H}} \le \begin{cases} \|y_n-p\|^2_{\mathscr{H}}, & if\ \gamma_n=0,\\ \|y_n-p\|^2_{\mathscr{H}}-\rho_n(2-\rho_n)\dfrac{D_n^2}{E_n}, & if\ \gamma_n=\rho_n\dfrac{D_n}{E_n}.\end{cases}$$

Since $\rho_n\in[a,b]\subset(0,2)$, in both cases, we obtain

$$\|x_{n+1}-p\|_{\mathscr{H}}\le\|y_n-p\|_{\mathscr{H}}\le\|x_n-p\|_{\mathscr{H}}+\varepsilon_n. \tag{3.4}$$

Using mathematical induction, we find that $\|x_{n+1}-p\|_{\mathscr{H}}\le\|x_0-p\|_{\mathscr{H}}+\sum_{k=0}^n\varepsilon_k$, which yields that $\{x_n\}$ is bounded.

Claim 2. For each $i=1,2,\cdots,N$, show that

$$\|(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n))\|^2_{\mathscr{H}}\to0, \tag{3.5}$$

$$\|(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n)\|^2_{\mathscr{H}_i}\to0, \tag{3.6}$$

$$\|\sum_{i=1}^N B_i(A_iy_n)-y\|^2_{\mathscr{K}}\to0. \tag{3.7}$$

In order to complete the proof of this claim, we consider the following two cases.
Case 1. $\gamma_n=0$. From $E_n=0$ and (3.3), we have $\nabla F(y_n)=0$. Using (3.2), it follows that $D_n=0$.
Case 2. $\gamma_n\neq0$. Thus $\gamma_n=\rho_n\frac{D_n}{E_n}$. In this case, we have

$$\begin{aligned}\rho_n(2-\rho_n)\frac{D_n^2}{E_n}&\le\|y_n-p\|^2_{\mathscr{H}}-\|x_{n+1}-p\|^2_{\mathscr{H}}\\&\le(\|x_n-p\|_{\mathscr{H}}+\varepsilon_n)^2-\|x_{n+1}-p\|^2_{\mathscr{H}}\\&\le\|x_n-p\|^2_{\mathscr{H}}+2(\|x_n-p\|_{\mathscr{H}}+\varepsilon_n)\varepsilon_n-\|x_{n+1}-p\|^2_{\mathscr{H}}\\&\le\|x_n-p\|^2_{\mathscr{H}}-\|x_{n+1}-p\|^2_{\mathscr{H}}+M_1\varepsilon_n,\end{aligned} \tag{3.8}$$

where

$$M_1=2\sup_{n\ge0}\{\|x_n-p\|_{\mathscr{H}}+\varepsilon_n\}<\infty.$$

Using the condition $\rho_n\in[a,b]\subset(0,2)$ and the above inequality, we have $\frac{D_n^2}{E_n}\to0$. Applying the inequality $(a+b)^2\le2(a^2+b^2)$, for all $a,b\in\mathbb{R}$, we deduce

$$\begin{aligned}E_n&=\|(I^{\mathscr{H}}-P_C^{\mathscr{H}})(y_n)+\sum_{i=1}^NA_i^*(I^{\mathscr{H}_i}-P_{Q_i}^{\mathscr{H}_i})(A_iy_n)+\sum_{i=1}^NA_i^*B_i^*\left(\sum_{i=1}^NB_i(A_iy_n)-y\right)\|^2_{\mathscr{H}}\\&\le2\max\left\{1,\max_{1\le i\le N}\{\|A_i\|^2\},N\max_{1\le i\le N}\{\|A_iB_i\|^2\}\right\}D_n.\end{aligned}$$

Thus, it follows from that $D_n\to0$. Therefore, in both cases, we find that $D_n\to0$. Using the definition of D_n, we derive the limits (3.5), (3.6), (3.7), as claimed.

Claim 3. Prove that $\{x_n\}$ converges weakly to $x^*\in\Omega$.

Since $\{x_n\}$ bounded, one sees that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $x^*\in\mathscr{H}$, such that $x_{n_k}\rightharpoonup x^*$. From (3.4), it follows that $\|y_{n_k}-x_{n_k}\|\le\varepsilon_{n_k}\to0$ as $k\to\infty$. Consequently, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which $y_{n_k}\rightharpoonup x^*\in C$. Then, in light of Lemma 2.3 and (3.5), we deduce that $x^*\in C$.

On the other hand, since A_i and B_i are linear and bounded, we have $A_i y_{n_k} \rightharpoonup A_i(x^*)$ and $B_i(A_i y_{n_k}) \rightharpoonup B_i(A_i x^*)$, for all $i = 1, 2, \cdots, N$. Thus, using (3.6) and Lemma 2.3, we see that $A_i y_{n_k} \in Q_i$ for each $i = 1, 2, \cdots, N$, so $A_i x^* \in Q_i$. From (3.7), it follows that $\sum_{i=1}^{N} B_i(A_i x^*) = y$. Therefore, we conclude that $x^* \in \Omega$.

Finally, we eatablish that $x_n \rightharpoonup x^*$. Suppose, for the sake of contradiction, that there exists another subsequence $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{m_k} \rightharpoonup \bar{x}^*$ with $\bar{x}^* \neq x^*$. Using an argument which is similar to the one used above, we again find that $\bar{x}_* \in \Omega$. It follows from Lemma 2.2 and the existence of the finite limit of $\{\|x_n - x^*\|_{\mathscr{H}}\}$ that

$$\begin{aligned}\liminf_{k\to\infty} \|x_{n_k} - x^*\|_{\mathscr{H}} &< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}^*\|_{\mathscr{H}} = \liminf_{k\to\infty} \|x_{m_k} - \bar{x}^*\|_{\mathscr{H}} \\ &< \liminf_{k\to\infty} \|x_{m_k} - x^*\|_{\mathscr{H}} = \liminf_{k\to\infty} \|x_{n_k} - x^*\|_{\mathscr{H}}.\end{aligned}$$

However, this is a contradiction. Hence it follows that $x_{m_k} \rightharpoonup x^*$. Therefore, we conclude that $x_n \rightharpoonup x^*$, as claimed. This completes the proof. □

To derive a strong convergence theorem, we now combine Algorithm 2 with the viscosity approximation method. Our second algorithm is formulated as follows:

Algorithm 3

Step 1. Let $\rho_n \in [a, b] \subset (0, 2)$ for all $n \in \mathbb{N}$. Choose $x_0, x_{-1}, \cdots, x_{-q} \in \mathscr{H}$ arbitrarily, $\beta \geq 3$ and set $n := 0$.

Step 2. Given $x_n, x_{n-1}, \cdots, x_{n-q}$, compute

$$\begin{cases} y_n = x_n + \sum_{i\in Q} \Delta_{i,n}(x_{n-i} - x_{n-i-1}), \\ z_n = y_n - \gamma_n \nabla F(y_n), \\ x_{n+1} = \alpha_n h(y_n) + (1 - \alpha_n) z_n, \end{cases} \tag{3.9}$$

where $h : \mathscr{H} \to \mathscr{H}$ is a strict contraction with constant $\delta \in [0, 1)$, $Q := \{0, 1, \cdots, q-1\}$ with $q \in N^+$, and sequence $\{\alpha_n\} \subset (0, 1)$ satisfies

$$\begin{aligned} &(\mathrm{i})\alpha_n \in [a, b] \subset (0, 1), \lim_{n\to\infty} \alpha_n = 0 \textit{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \\ &(\mathrm{ii})\varepsilon_n = o(\alpha_n), i.e., \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0. \end{aligned} \tag{3.10}$$

And $\Delta_{i,n}$ satisfies $0 \leq |\Delta_{i,n}| \leq \overline{\Delta}_n$ with $\overline{\Delta}_n$ defined by

$$\overline{\Delta}_n = \begin{cases} \min\{\frac{n-1}{n+\beta-1}, \frac{\varepsilon_n}{\sum_{i\in Q} \|x_{n-i} - x_{n-i-1}\|}\}, & \sum_{i\in Q} \|x_{n-i} - x_{n-i-1}\| \neq 0, \\ \frac{n-1}{n+\beta-1}, & \sum_{i\in Q} \|x_{n-i} - x_{n-i-1}\| = 0. \end{cases}$$

$$\gamma_n = \begin{cases} \rho_n \frac{D_n}{E_n}, & E_n \neq 0, \\ 0, & E_n = 0, \end{cases} \tag{3.11}$$

where

$$D_n := \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(y_n)\|_{\mathscr{H}}^2 + \sum_{i=1}^{N} \|(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i y_n)\|_{\mathscr{H}_i}^2 + \|\sum_{i=1}^{N} B_i(A_i y_n) - y\|_{\mathscr{K}}^2$$

and

$$E_n := \|(I^{\mathscr{H}} - P_C^{\mathscr{H}})(y_n) + \sum_{i=1}^{N} A_i^*(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(A_i y_n) + \sum_{i=1}^{N} A_i^* B_i^* \left(\sum_{i=1}^{N} B_i(A_i y_n) - y\right)\|_{\mathscr{H}}^2.$$

Step 3. Set $n \leftarrow n+1$ and go to Step 2.

Theorem 3.2 If $\Omega \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to the unique solution to the equation $x^* = P_\Omega(h(x^*))$.

Proof. The proof is divided into several steps. We take any $p \in \Omega$.

Claim 1. The sequence $\{x_n\}$ is bounded.

Indeed, it follows from (3.9) and $0 < \alpha_n < 1$ that

$$\begin{aligned}
\|x_{n+1} - p\|_{\mathscr{H}} &= \|\alpha_n h(y_n) + (1-\alpha_n) z_n - p\|_{\mathscr{H}} \\
&= \|\alpha_n (h(y_n) - p) + (1-\alpha_n)(z_n - p)\|_{\mathscr{H}} \\
&\leq \alpha_n \|h(y_n) - p\|_{\mathscr{H}} + (1-\alpha_n)\|z_n - p\|_{\mathscr{H}} \\
&\leq \alpha_n [\|h(y_n) - h(p)\|_{\mathscr{H}} + \|h(p) - p\|_{\mathscr{H}}] + (1-\alpha_n)\|z_n - p\|_{\mathscr{H}} \\
&\leq \alpha_n [\delta \|y_n - p\|_{\mathscr{H}} + \|h(p) - p\|_{\mathscr{H}}] + (1-\alpha_n)\|z_n - p\|_{\mathscr{H}}.
\end{aligned} \tag{3.12}$$

Employing an argument similar to the one used in the proof of Claim 1 of Theorem 3.1, we obtain

$$\|z_n - p\|_{\mathscr{H}} \leq \|y_n - p\|_{\mathscr{H}} \leq \|x_n - p\|_{\mathscr{H}} + \varepsilon_n. \tag{3.13}$$

Using (3.4), (3.12), and (3.13), we infer that

$$\begin{aligned}
\|x_{n+1} - p\|_{\mathscr{H}} &\leq (1 - \alpha_n(1-\delta))\|y_n - p\|_{\mathscr{H}} + \alpha_n \|h(p) - p\|_{\mathscr{H}} \\
&\leq (1 - \alpha_n(1-\delta))(\|x_n - p\|_{\mathscr{H}} + \varepsilon_n) + \alpha_n(1-\delta)\frac{\|h(p) - p\|_{\mathscr{H}}}{1-\delta} \\
&\leq (1 - \alpha_n(1-\delta))\|x_n - p\|_{\mathscr{H}} + \alpha_n(1-\delta)\frac{\|h(p) - p\|_{\mathscr{H}} + \varepsilon_n/\alpha_n}{1-\delta} \\
&\leq \max\{\|x_n - p\|_{\mathscr{H}}, \frac{\|h(p) - p\|_{\mathscr{H}} + \varepsilon_n/\alpha_n}{1-\delta}\}.
\end{aligned}$$

Consequently, applying mathematical induction establishes that

$$\|x_{n+1} - p\|_{\mathscr{H}} \leq \max\{\|x_0 - p\|_{\mathscr{H}}, M_2\},$$

where $M_2 = \sup_{n\geq 0}\{\frac{\|h(p)-p\|_{\mathscr{H}} + \varepsilon_n/\alpha_n}{1-\delta}\}$. Therefore, $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{h(y_n)\}$.

Claim 2. Prove

$$\|x_{n+1} - p\|_{\mathscr{H}}^2 \leq \|x_n - p\|_{\mathscr{H}}^2 + \Theta_n \tag{3.14}$$

where

$$\Theta_n = \begin{cases} \alpha_n M_3 + M_1 \varepsilon_n, & \gamma_n = 0, \\ \alpha_n M_3 + M_1 \varepsilon_n - \rho_n(2-\rho_n)\frac{D_n^2}{E_n}, & \gamma_n = \rho_n \frac{D_n}{E_n}, \end{cases}$$

where $M_3 = \sup_{n\geq 0}\{\|h(y_n)-p\|^2_{\mathscr{H}}\} < \infty$. Indeed, using (3.9), we have

$$\begin{aligned}\|x_{n+1}-p\|^2_{\mathscr{H}} &\leq \alpha_n\|h(y_n)-p\|^2_{\mathscr{H}} + (1-\alpha_n)\|z_n-p\|^2_{\mathscr{H}} \\ &\leq \alpha_n M_3 + \|z_n-p\|^2_{\mathscr{H}}.\end{aligned} \tag{3.15}$$

Employing an argument similar to the one used in the proof of Claim 1 of the Theorem 3.1, and using (3.9) and (3.11), we find that

$$\|z_n-p\|^2_{\mathscr{H}} \leq \begin{cases}\|y_n-p\|^2_{\mathscr{H}} & if\ \gamma_n = 0, \\ \|y_n-p\|^2_{\mathscr{H}} - \rho_n(2-\rho_n)\frac{D_n^2}{E_n} & if\ \gamma_n = \rho_n\frac{D_n}{E_n}\end{cases} \tag{3.16}$$

and

$$\begin{aligned}\|y_n-p\|^2_{\mathscr{H}} &\leq (\|x_n-p\|_{\mathscr{H}}+\varepsilon_n)^2 \\ &\leq \|x_n-p\|^2_{\mathscr{H}} + 2(\|x_n-p\|_{\mathscr{H}}+\varepsilon_n)\varepsilon_n \\ &\leq \|x_n-p\|^2_{\mathscr{H}} + M_1\varepsilon_n,\end{aligned} \tag{3.17}$$

Now, using (3.15), (3.16), and (3.17), we obtain inequality (3.14), as claimed.

Claim 3. Prove the following inequality:

$$\mathfrak{a}_{n+1} \leq (1-\mathfrak{b}_n)\mathfrak{a}_n + \mathfrak{b}_n\mathfrak{c}_n, \quad \forall n \geq 1, \tag{3.18}$$

where $\mathfrak{a}_n := \|x_n-p\|^2_{\mathscr{H}}$, $\mathfrak{b}_n := \alpha_n(1-\delta)$, and

$$\mathfrak{c}_n := \frac{M_1\varepsilon_n/\alpha_n + 2\langle h(p)-p, x_{n+1}-p\rangle_{\mathscr{H}}}{(1-\delta)}.$$

Indeed, using (3.9), (3.16), (3.17), and Lemma 2.4, we see that

$$\begin{aligned}&\|x_{n+1}-p\|^2_{\mathscr{H}} \\ &= \|\alpha_n(h(y_n)-h(p)) + \alpha_n(h(p)-p)\|^2_{\mathscr{H}} + (1-\alpha_n)(z_n-p)\|^2_{\mathscr{H}} \\ &\leq \|\alpha_n(h(y_n)-h(p)) + (1-\alpha_n)(z_n-p)\|^2_{\mathscr{H}} + 2\alpha_n\langle h(p)-p, x_{n+1}-p\rangle_{\mathscr{H}} \\ &\leq \alpha_n\|h(y_n)-h(p)\|^2_{\mathscr{H}} + (1-\alpha_n)\|z_n-p\|^2_{\mathscr{H}} + 2\alpha_n\langle h(p)-p, x_{n+1}-p\rangle_{\mathscr{H}} \\ &\leq \alpha_n\delta\|y_n-p\|^2_{\mathscr{H}} + (1-\alpha_n)\|y_n-p\|^2_{\mathscr{H}} + 2\alpha_n\langle h(p)-p, x_{n+1}-p\rangle_{\mathscr{H}} \\ &= (1-\alpha_n(1-\delta))\|y_n-p\|^2_{\mathscr{H}} + 2\alpha_n\langle h(p)-p, x_{n+1}-p\rangle_{\mathscr{H}} \\ &\leq (1-\alpha_n(1-\delta))\|x_n-p\|^2_{\mathscr{H}} + \alpha_n(1-\delta)\frac{M_1\varepsilon_n/\alpha_n + 2\langle h(p)-p, x_{n+1}-p\rangle_{\mathscr{H}}}{(1-\delta)}.\end{aligned}$$

It is not hard to see that the above inequality can be rewritten in the form (3.18), as claimed.

Claim 4. Prove that $\{x_n\}$ converges strongly to x^*, which is a unique solution to $x = P_\Omega(h(x))$.

We use Claim 3 to replacing p by x^* and prove that $\mathfrak{a}_n \to 0$ by using Lemma 2.5. To begin this, we assume that $\{||x_{n_m}-x^*||^2_{\mathscr{H}}\}$ is an arbitrary subsequence of $\{\|x_n-x^*||^2_{\mathscr{H}}\}$ such that

$$\liminf_{m\to\infty}(\|x_{n_m+1}-x^*\|^2_{\mathscr{H}} - \|x_{n_m}-x^*\|^2_{\mathscr{H}}) \geq 0.$$

Next, we consider the following two cases.

Case 1. $\gamma_{n_m} = 0$.

It follows from $E_{n_m} = 0$ and (3.3) that $\nabla F(y_{n_m}) = 0$. Using (3.4), we find that $D_{n_m} = 0$. Besides, using the definition of z_n, we know that $\|z_{n_m} - y_{n_m}\|_{\mathscr{H}} = 0$.

Case 2. $\gamma_{n_m} = \rho_{n_m}\frac{D_{n_m}}{E_{n_m}}$.

It follows from Claim 2, (3.8), (3.10) and $\rho_{n_m} \in [a,b] \subset (0,2)$ that $\frac{D_{n_m}^2}{E_{n_m}} \to 0$, which implies that $D_{n_m} \to 0$. In addition, from (3.3) and (3.9), we also have

$$\|z_{n_m} - y_{n_m}\|_{\mathscr{H}}^2 = \gamma_{n_m}^2 \|\nabla F(y_{n_m})\|_{\mathscr{H}}^2 = \gamma_{n_m}^2 E_{n_m} = \rho_{n_m}^2 \frac{D_{n_m}^2}{E_{n_m}} \le b^2 \frac{D_{n_m}^2}{E_{n_m}} \to 0.$$

Therefore, in both cases, we have $\|z_{n_m} - y_{n_m}\|_{\mathscr{H}} \to 0$, and it follows from (3.13) that $\|y_n - x_n\| \le \varepsilon_n$. Thus there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $\|y_{n_m} - x_{n_m}\| \to 0$ and $\|z_{n_m} - x_{n_m}\| \to 0$. Using the definition of D_n, we infer that $\|(I^{\mathscr{H}} - P_C^{\mathscr{H}}(y_{n_m})\|_{\mathscr{H}}^2 \to 0$, $\|(I^{\mathscr{H}_i} - P_{Q_i}^{\mathscr{H}_i})(y_{n_m})\|_{\mathscr{H}_i}^2 \to 0$, and $\|\sum_{i=1}^N B_i(A_i y_{n_m}) - y\|_{\mathscr{K}}^2 \to 0$ for all $i = 1,2,\cdots,N$. Using the boundedness of $\{y_{n_m}\}$ and $\{h(y_{n_m})\}$, we see that

$$\begin{aligned}\|x_{n_m+1} - x_{n_m}\|_{\mathscr{H}} &= \|\alpha_{n_m}(h(y_{n_m}) - x_{n_m}) + (1-\alpha_{n_m})(z_{n_m} - x_{n_m})\|_{\mathscr{H}} \\ &\le \alpha_{n_m}\|h(y_{n_m}) - x_{n_m}\|_{\mathscr{H}} + (1-\alpha_{n_m})\|z_{n_m} - x_{n_m}\|_{\mathscr{H}} \\ &\le \alpha_{n_m} M_4 + (1-\alpha_{n_m})\|z_{n_m} - x_{n_m}\|_{\mathscr{H}},\end{aligned} \tag{3.19}$$

where $M_4 = \sup_m \|h(y_{n_m}) - x_{n_m}\|_{\mathscr{H}}$. Thus $\|x_{n_m+1} - x_{n_m}\|_{\mathscr{H}} \to 0$. In light of Claim 3, to apply Lemma 2.5, it is sufficient to prove $\limsup_{m\to\infty} c_{n_m} \le 0$, which is equivalent to proving that

$$\limsup_{m\to\infty} \frac{M_2 \varepsilon_{n_m}}{\alpha_{n_m}} + \langle h(x^*) - x^*, x_{n_m+1} - p\rangle_{\mathscr{H}} \le 0.$$

To achieve this, we first observe that

$$\begin{aligned}&\langle h(x^*) - x^*, x_{n_m+1} - x^*\rangle_{\mathscr{H}} \\ &= \langle h(x^*) - x^*, x_{n_m+1} - x_{n_m}\rangle_{\mathscr{H}} + \langle h(x^*) - x^*, x_{n_m} - x^*\rangle_{\mathscr{H}} \\ &\le \|h(x^*) - x^*\|_{\mathscr{H}} \|x_{n_m+1} - x_{n_m}\|_{\mathscr{H}} + \langle h(x^*) - x^*, x_{n_m} - x^*\rangle_{\mathscr{H}}.\end{aligned} \tag{3.20}$$

Since $\{x_{n_m}\}$ is a bounded sequence (Claim 1), one sees that there exists a subsequence $\{x_{n_{m_j}}\}$ of $\{x_{n_m}\}$ which converges weakly to some $z \in \mathscr{H}$ such that

$$\limsup_{m\to\infty} \langle h(x^*) - x^*, x_{n_m} - x^*\rangle_{\mathscr{H}} = \limsup_{j\to\infty} \langle h(x^*) - x^*, x_{n_{m_j}} - x^*\rangle_{\mathscr{H}} = \langle h(x^*) - x^*, z - x^*\rangle_{\mathscr{H}}.$$

Using an argument similar to one used in the proof of Claim 3 in Theorem 3.1, we find that $z \in \Omega$. Additionally, from the definition of x^* and Lemma 2.1, we obtain

$$\limsup_{m\to\infty} \langle h(x^*) - x^*, x_{n_m} - x^*\rangle_{\mathscr{H}} = \langle h(x^*) - x^*, z - x^*\rangle_{\mathscr{H}} \le 0.$$

Using (3.10) and (3.20), we find that $\limsup_{m\to\infty} c_{n_m} \le 0$. Now it is not difficult to see that all the hypotheses of Lemma 2.5 are satisfied. Hence, we obtain $\|x_n - x^*\|_{\mathscr{H}} \to 0$, specifically, $x_n \to x^*$. This completes the proof. □

4. Numerical Experiments

In this section, we verify the effectiveness and performance of the proposed algorithm through a series of numerical experiments. In the experiments, we conducted a detailed comparison of the number of iterations, errors and running times of three algorithms (Algorithm 1, Algorithm 2 and Algorithm 3) under different parameter settings. All experiments were carried out in the same computational environment to ensure the fairness and comparability of the results.

We consider the MSFP under the following settings: for $i = 1,2,3$, $\mathscr{H} = \mathbb{R}^m, \mathscr{H}_i = \mathbb{R}^{k_i}$, and $\mathscr{K} = \mathbb{R}^l$ are finite-dimensional Euclidean spaces. The sets C and Q_i are defined as:

$$C = \{x \in \mathbb{R}^m : \langle \zeta_0, x\rangle \leq \beta_0\}$$

and

$$Q_i = \{x \in \mathbb{R}^{m_i} : \langle \zeta_i, x\rangle \leq \beta_i\},$$

where $\beta_i = i+1$, $i = 0,1,\cdots,3$, and the coordinates of ζ_0 and ζ_i are randomly generated in the intervals $[0,10]$, $[5,15]$, $[10,20]$, and $[15,25]$. The bounded linear operators $A_i : \mathbb{R}^m \to \mathbb{R}^{k_i}$ and $B_i : \mathbb{R}^{k_i} \to \mathbb{R}^l$ are matrix randomly generated in the closed interval $[-20,20]$ for each $i = 1,2,3$. Setting $y = 0$ in this experiment ensures $\Omega \neq \emptyset$ because $0 \in \Omega$. We implement Algorithm 1 , Algorithm 2, and Algorithm 3 with $m = 100, k_1 = 200, k_2 = 300, k_3 = 400, l = 500$, and the coordinates of the initial point x_0, $x_{-1}, \cdots$, x_{-q} are randomly generated in the closed interval $[-5,5]$. In this case, we see $x^* = 0$. We choose the control parameter $\alpha_n = n^{-1}$ and use the stopping condition: err $= \|x_n - x^*\|$, which is required to fall below a specified tolerance TOL.

The parameters for Algorithm 2 and Algorithm 3 are similar to those of Algorithm 1, but with the addition of extra parameters related to multi-step inertia and inertia coefficient calculations in the algorithm's running parameters. Specifically, we set $\beta = 3$, $\alpha_n = \frac{1}{n+1}$, $\varepsilon_n = \frac{\alpha_n}{n^{0.01}}$, and $h(x) = 0.05x$ in Algorithm 3. In the experiments, we selected different values for the parameter ρ_n , namely 0.3, 0.5, 1.3 and 1.5, to observe their impact on the performance of the algorithms. Additionally, for each value of ρ_n , we set the tolerance error (TOL) to 10^{-5}, 10^{-7} and 10^{-9} to evaluate the algorithms' performance under different precision requirements. The specific experimental results are shown in the following table:

From the experimental results above, it can be seen that all three algorithms can effectively converge to the expected solution under different parameters ρ_n and tolerance error TOL.

Specifically, The iteration count (n): As the tolerance error TOL decreases, the iteration count for all algorithms increases. This indicates that more iterations are required to achieve higher precision. Under different ρ_n values, Algorithm 3 generally has fewer iterations, especially when TOL is small, suggesting that it has an advantage in convergence speed.

Error (*err*): The error of all algorithms gradually decreases with iterations and eventually reaches or falls below the set tolerance error TOL. Under the same parameter settings, the error of Algorithm 2 and Algorithm 3 is slightly lower than that of Algorithm 1, indicating that they may have better accuracy.

Running Time (Time): In terms of running time, Algorithm 3 generally shows faster computational speed in most cases, especially when TOL is small. This may be due to certain optimizations in its algorithm structure or implementation, enabling it to handle computational tasks more efficiently.

To more intuitively demonstrate the performance of the algorithms, we have also plotted the relationship between the iteration count and error under different ρ_n values. These figures further validate the analysis of the numerical results presented above.

Under varying parameters ρ_n and tolerance error TOL, all three algorithms effectively converge. As TOL decreases, the number of iterations increases, yet Algorithm 3 demonstrates a faster convergence rate. In terms of accuracy, Algorithms 2 and 3 outperform Algorithm 1. Regarding computation time, Algorithm 3 is faster in most cases, especially when TOL is small.

TABLE 1. Numerical results and comparisons among algorithms.

ρ_n	TOL		Algorithm 1	Algorithm 2	Algorithm 3
	1×10^{-5}	n	55	43	**36**
		err	9.36×10^{-6}	7.63×10^{-6}	$\mathbf{6.90\times10^{-6}}$
		Time	0.06	**0.04**	**0.04**
0.3	1×10^{-7}	n	72	55	**51**
		err	9.41×10^{-8}	9.95×10^{-8}	$\mathbf{8.72\times10^{-8}}$
		Time	0.06	0.05	**0.04**
	1×10^{-9}	n	95	71	**66**
		err	8.21×10^{-10}	$\mathbf{8.08\times10^{-10}}$	8.83×10^{-10}
		Time	0.05	0.05	**0.04**
	1×10^{-5}	n	46	38	**32**
		err	9.14×10^{-6}	8.39×10^{-6}	$\mathbf{7.47\times10^{-6}}$
		Time	0.03	**0.02**	0.03
0.5	1×10^{-7}	n	61	48	**45**
		err	$\mathbf{6.82\times10^{-8}}$	9.16×10^{-8}	8.90×10^{-8}
		Time	0.06	**0.02**	0.03
	1×10^{-9}	n	79	67	**63**
		err	8.07×10^{-10}	7.43×10^{-10}	$\mathbf{6.20\times10^{-10}}$
		Time	0.04	0.06	**0.03**
	1×10^{-5}	n	61	51	**40**
		err	8.71×10^{-6}	$\mathbf{7.96\times10^{-6}}$	9.74×10^{-6}
		Time	0.04	**0.03**	0.04
1.3	1×10^{-7}	n	82	60	**53**
		err	8.94×10^{-8}	8.93×10^{-8}	$\mathbf{6.86\times10^{-8}}$
		Time	0.07	**0.03**	**0.03**
	1×10^{-9}	n	108	76	**69**
		err	9.38×10^{-10}	$\mathbf{7.72\times10^{-10}}$	8.60×10^{-10}
		Time	0.09	0.06	**0.03**
	1×10^{-5}	n	62	53	**45**
		err	8.12×10^{-6}	9.00×10^{-6}	$\mathbf{7.67\times10^{-6}}$
		Time	0.04	**0.02**	**0.02**
1.5	1×10^{-7}	n	84	68	**59**
		err	9.15×10^{-8}	7.63×10^{-8}	$\mathbf{7.42\times10^{-8}}$
		Time	0.05	0.05	**0.03**
	1×10^{-9}	n	112	89	**73**
		err	8.21×10^{-10}	8.49×10^{-10}	$\mathbf{7.58\times10^{-10}}$
		Time	0.07	**0.05**	**0.05**

5. CONCLUDING REMARKS

We presented two new inertial accelerated algorithms. Our algorithms are improvements and extensions of Algorithm 1 studied by Reich et al. [21]. We adopt multi-step inertial techniques and the viscosity approximation method, and further we prove the weak and strong convergence

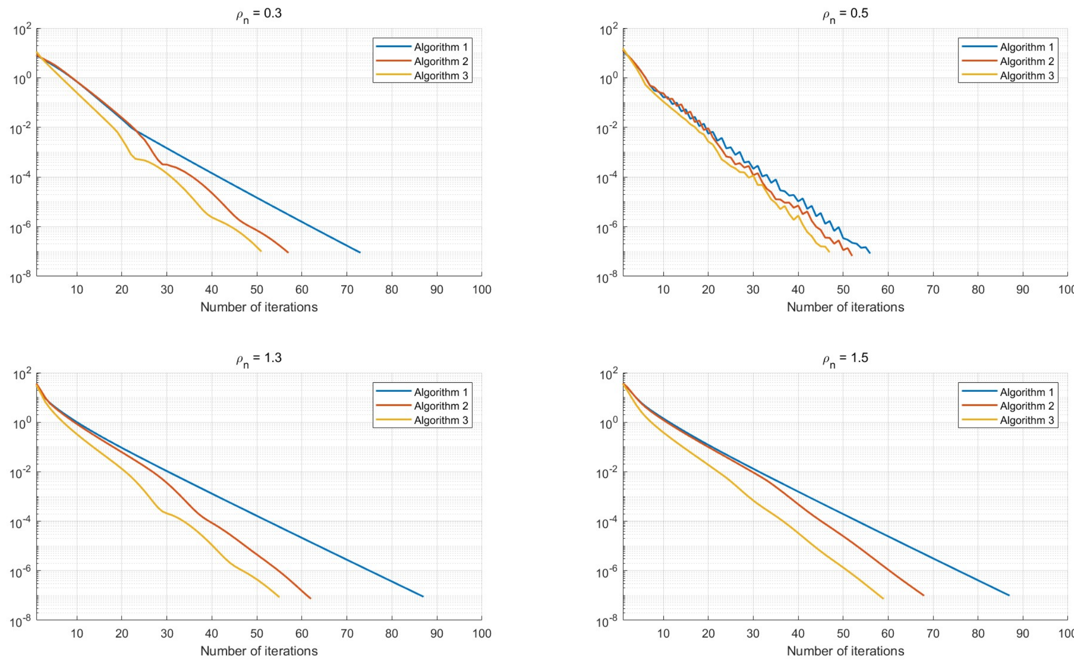

FIGURE 1. The behavior of err with TOL = 10^{-7}

of the introduced algorithms. Finally, we demonstrate the applicability and efficiency of our methods through numerical experiments.

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