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FIXED POINT AND CONVERGENCE THEOREMS IN CONVEX (θ_1, θ_2)-EXTENDED b-METRIC SPACES WITH APPLICATIONS TO INTEGRAL EQUATIONS

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Abstract. This paper introduces a novel mathematical framework known as the convex (θ_1, θ_2)-extended b-metric space, which generalizes the classical metric and b-metric settings. Within this structure, we establish fixed-point theorems and convergence results by employing both contraction-type and interpolative techniques. In particular, we develop enriched fixed-point results, extend Mann's iterative algorithm, and investigate fixed points of ω-operators under generalized contractive conditions. The theoretical findings are further supported by applications to nonlinear integral equations of Volterra type. We demonstrate the effectiveness of the proposed theorems in proving existence and uniqueness of solutions within the generalized space. Moreover, numerical examples are provided to illustrate the convergence behavior of iterative sequences, highlighting the practical relevance and applicability of the main results.
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1. Introduction and Preliminaries

The theory of fixed points of various nonlinear mappings is an important branch in nonlinear analysis, such as dynamic systems, differential and integral equations, functional analysis, nonlinear programming. Many problems in pure and analysis fields can be set a problem of fixed-points in various framework of spaces. In 1989, Bakhtin [2] introduced the framework of

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b-metric spaces, which generalize and extend the traditional concept of metric spaces by relaxing the triangular-inequality condition, and investigated various fixed-point theorems in these spaces. The b-metric space, which opens a new view in operator theory, functional analysis and optimization theory, is general and flexible in dealing with the problems which cannot be handled by using classical metric-space theory. The importance of b-metric spaces lies in their capacity to model real-world phenomena where the triangular inequality is not strictly satisfied. This generalization has been demonstrated particularly valuable in solving differential equations, integral equations, and optimization problems where traditional metric frameworks prove insufficient.

In this research, we study some properties of b-metric spaces and present new fixed point theorems that extend Bakhtin's original work. Furthermore, we analyze the applications to certain types of differential equations. Before proceeding with our main points, we recall some essential definitions and properties that serve as the foundation for subsequent sections.

Definition 1.1. Let χ be a nonempty set, and let $\Gamma : \chi \times \chi \to [0,\infty)$ be a function. Assume that there exists a constant $s \geq 1$ such that the following conditions are satisfied for every $\rho, \rho, \sigma \in \chi$:

(Γ_1) $\Gamma(\rho,\rho) = 0$ if and only if $\rho = \rho$;
(Γ_2) $\Gamma(\rho,\rho) = \Gamma(\rho,\rho)$;
(Γ_3) $\Gamma(\rho,\rho) \leq s\big[\Gamma(\rho,\sigma) + \Gamma(\sigma,\rho)\big]$.

In this case, the structure (χ,Γ) is referred to as a *b-metric space*.

Remark 1.2. (i) When $s = 1$, a b-metric space reduces to an ordinary metric space.
(ii) Every metric space naturally qualifies as a b-metric space.
(iii) There exist b-metric spaces which do not fulfill the conditions of a metric space.
(iv) The concept is applicable in a wide range of mathematical contexts and problem settings.

In 2017, Kamran, Samreen, and Ain [6] made a significant contribution to the field by introducing the concept of extended b-metric spaces, which represents a further generalization of the framework of b-metric spaces. This extension broadens the theoretical landscape and opens new avenues for research in nonlinear fixed point theory. In his seminal paper, Kamran not only formulated the foundational definitions for these extended spaces but also established several interesting fixed point theorems that demonstrated the richness of this new mathematical framework. These theorems provided powerful tools for analyzing nonlinear phenomena and solving hybrid systems in contexts where even b-metric spaces proved insufficient. Kamran's results further lead to explore related real applications.

Definition 1.3. Let χ be a non-empty set, and let $\theta : \chi \times \chi \to [1,\infty]$ be a function. Suppose there exists a mapping $\Gamma : \chi \times \chi \to [0,\infty)$ satisfying the following conditions for all $\rho, \rho, \sigma \in \chi$:

($e\Gamma_1$) $\Gamma(\rho,\rho) = 0$ if and only if $\rho = \rho$;
($e\Gamma_2$) $\Gamma(\rho,\rho) = \Gamma(\rho,\rho)$;
($e\Gamma_3$) $\Gamma(\rho,\rho) \leq \theta(\rho,\rho)\,[\Gamma(\rho,\sigma) + \Gamma(\sigma,\rho)]$.

Then, the pair (χ,Γ) is called an *extended b-metric space*.

In 2021, Singh et al. [12] introduced and investigated a new generalization with their concept of $\alpha\beta - b$-metric spaces. Their new results further extended the theoretical framework of generalized metric spaces. They not only defined properties of these new spaces but also conducted a overall investigation on theorems of fixed points within this context. Their work demonstrated how these specific structures could resolve hybrid systems efficiently. By analyzing and establishing fixed point theorems in $\alpha\beta - b$-metric spaces, Singh et al. sent some efficient tools for problems in nonlinear analysis, differential equations and various optimization problems.

Definition 1.4 ([12]). Let χ be a nonempty set, and let $\Gamma : \chi \times \chi \to [0,\infty)$ be a function. Suppose there exist constants $\alpha, \beta \geq 1$ such that for all $\rho, \rho, \sigma \in \chi$, the following conditions hold:

(Γ_1) $\Gamma(\rho,\rho) = 0$ if and only if $\rho = \rho$;
(Γ_2) $\Gamma(\rho,\rho) = \Gamma(\rho,\rho)$;
(Γ_3) $\Gamma(\rho,\rho) \leq \alpha\Gamma(\rho,\sigma) + \beta\Gamma(\sigma,\rho)$.

Then, the pair (χ,Γ) is called an *$\alpha\beta$-b-metric space*.

Definition 1.5. Let $I = [0,1)$, and let $\Gamma : \chi \times \chi \to [0,\infty)$ be a function. Suppose that there exists a continuous mapping $\omega : \chi \times \chi \times I \to \chi$ such that, for all $\rho, \rho, \sigma \in \chi$ and $\mu \in I$,

$$\Gamma(\sigma, \omega(\rho,\rho;\mu)) \leq \mu\Gamma(\sigma,\rho) + (1-\mu)\Gamma(\sigma,\rho).$$

Then, ω is called a *convex structure* on χ. If (χ,Γ) is an $\alpha\beta$-b-metric space equipped with such a convex structure ω, the triple (χ,Γ,ω) is referred to as a **convex $\alpha\beta$-b-metric space**.

This paper introduces a new framework ,called the convex (θ_1,θ_2)-extended b-metric space, in which we establish convergence theorems and derive associated fixed-point results. By employing an interpolative method, we further develop enriched fixed-point theorems, extend Mann's iteration, and prove fixed-point results in the context of a generalized b-metric space. Moreover, we investigate the existence of fixed points for ω-operators and provide illustrative examples that substantiate the validity and applicability of our theoretical contributions.

2. Main Results

In this section, we delve into a new generalization by introducing a new framework, namely the (θ_1,θ_2)-extended b-metric space, which not only extends the classical concept of metric and b-metric spaces, but also enriches the landscape of fixed-point theory in wide applications. Throughout this paper, we adopt the following notations and conventions: $\mathbb{N}$ denotes the set of positive integers, C is assumed to be a nonempty subset of a Banach space χ, and $F(\chi)$ refers to the fixed-point set of a mapping $\phi : \chi \to \chi$.

Definition 2.1. Let χ be a nonempty set, and let $\theta_1, \theta_2 : \chi \times \chi \to [1,\infty]$ be functions. Suppose that there exists a mapping $\Gamma : \chi \times \chi \to [0,\infty)$ satisfying the following conditions, for all $\rho, \rho, \sigma \in \chi$,

($e\Gamma_1$) $\Gamma(\rho,\rho) = 0$ if and only if $\rho = \rho$;
($e\Gamma_2$) $\Gamma(\rho,\rho) = \Gamma(\rho,\rho)$;
($e\Gamma_3$) $\Gamma(\rho,\rho) \leq \theta_1(\rho,\rho)\Gamma(\rho,\sigma) + \theta_2(\rho,\rho)\Gamma(\sigma,\rho)$.

Then, the pair (χ,Γ) is called a **(θ_1,θ_2)-extended b-metric space**.

Definition 2.2. Let (χ,Γ) be a (θ_1,θ_2)-extended b-metric space, and let $\{\rho_n\}$ be a sequence in χ. Then

(i_Γ) $\{\rho_n\}$ is said to **converge** to a point $\rho \in \chi$ if $\lim_{n\to\infty} \Gamma(\rho_n,\rho) = 0$.
(ii_Γ) $\{\rho_n\}$ is called a **Cauchy sequence** if $\lim_{m,n\to\infty} \Gamma(\rho_m,\rho_n) = 0$.
(iii_Γ) (χ,Γ) is said to be **complete** if every Cauchy sequence in χ converges to a point in χ.

Definition 2.3. Let χ be a nonempty set, and let $\Gamma : \chi \times \chi \to [0,\infty)$ be a function. Suppose that there exist mappings $\theta_1, \theta_2 : \chi \times \chi \to [1,\infty]$ such that, for all $\rho, \rho, \sigma \in \chi$,

($e\Gamma_1$) $\Gamma(\rho,\rho) = 0$ if and only if $\rho = \rho$;
($e\Gamma_2$) $\Gamma(\rho,\rho) = \Gamma(\rho,\rho)$;
($e\Gamma_3$) $\Gamma(\rho,\rho) \leq \theta_1(\rho,\rho)\Gamma(\rho,\sigma) + \theta_2(\rho,\rho)\Gamma(\sigma,\rho)$.

Then, (χ,Γ) is called a **(θ_1,θ_2)-extended b-metric space**.

Example 2.4 (Example of a Convex (θ_1,θ_2)-Extended b-Metric Space)**.** Let $\chi = [1,3]$ and define mappings $\theta_1, \theta_2 : \chi \times \chi \to [1,\infty)$ by $\theta_1(\rho,\rho) = \sup_{\rho,\rho\in\chi} |\rho-\rho|^2$ and $\theta_2(\rho,\rho) = \sup_{\rho,\rho\in\chi} |\rho-\rho|^3$. Define $\Gamma : \chi \times \chi \to \mathbb{R}$ by

$$\Gamma(\rho,\rho) = \begin{cases} 3^{|\rho-\rho|}, & \text{if } \rho \neq \rho, \\ 0, & \text{if } \rho = \rho. \end{cases}$$

Then (χ,Γ) is a (θ_1,θ_2)-extended b-metric space. Indeed, for any $\rho,\rho,\sigma \in \chi$,

$$\begin{aligned} \Gamma(\rho,\rho) &\leq 3^{|\rho-\sigma|+|\sigma-\rho|} \\ &= 3^{\frac{1}{3}|\rho-\sigma|+\frac{2}{3}|\sigma-\rho|} \cdot 3^{\frac{2}{3}|\rho-\sigma|+\frac{1}{3}|\sigma-\rho|} \\ &\leq \left(3^{|\rho-\sigma|}\right)^{1/3} \cdot \left(3^{|\sigma-\rho|}\right)^{2/3} \cdot \sup_{\rho,\rho,\sigma\in\chi} 3^{\frac{2}{3}|\rho-\sigma|+\frac{1}{3}|\sigma-\rho|} \\ &\leq 3\Gamma(\rho,\sigma) + 6\Gamma(\sigma,\rho) \\ &\leq 2^2\Gamma(\rho,\sigma) + 2^3\Gamma(\sigma,\rho) = {}_1(\rho,\rho)\Gamma(\rho,\sigma) + {}_2(\rho,\rho)\Gamma(\sigma,\rho). \end{aligned}$$

Furthermore, observe that $\Gamma(1,3) = 3^{|1-3|} = 3^2 = 9$, $\Gamma(1,2) = 3^{|1-2|} = 3$, $\Gamma(2,3) = 3^{|2-3|} = 3$, and clearly, $\Gamma(1,3) = 9 > 3+3 = \Gamma(1,2) + \Gamma(2,3)$, which implies that (χ,Γ) is not a metric space.

Now, we define a convex combination mapping $\omega(\rho,\rho;\alpha) = \alpha\rho + (1-\alpha)\rho$ for $\alpha \in I = [0,1]$. Then, for any $\sigma \in \chi$,

$$\Gamma(\sigma,\omega(\rho,\rho;\alpha)) = \Gamma(\sigma,\alpha\rho+(1-\alpha)\rho) \leq 3^{|\sigma-\alpha\rho|+|\sigma-(1-\alpha)\rho|} \leq \alpha\Gamma(\sigma,\rho) + (1-\alpha)\Gamma(\sigma,\rho).$$

Hence, Γ is convex with respect to the interpolative mean ω.

Example 2.5 (Example of a Convex (θ_1,θ_2)-Extended b-Metric Space)**.** Let $X = C([0,1])$, the space of all continuous real-valued functions on $[0,1]$, and define the function $\Gamma : X \times X \to [0,\infty)$ by $\Gamma(f,g) = \sup_{t\in[0,1]} |f(t)-g(t)|^{1+|f(t)-g(t)|}$. Note that Γ is not a standard metric because it may violate the triangle inequality in its classical form. However, we show that (X,Γ,ω) forms a convex (θ_1,θ_2)-extended b-metric space. Define a convex combination operation $\omega : X \times X \times [0,1] \to X$ by

$$\omega(f,g;\lambda)(t) = \lambda f(t) + (1-\lambda)g(t),$$

and define control functions:

$$\theta_1(f,g) = 1 + \|f-g\|_\infty, \quad \theta_2(f,g) = e^{\|f-g\|_\infty}.$$

We claim that (X,Γ,ω) is a convex (θ_1,θ_2)-extended b-metric space with constant $s = 2$.

Verification.

(1) *Positivity and identity:* $\Gamma(f,g) = 0$ if and only if $f(t) = g(t)$ for all $t \in [0,1]$, since the exponent $1 + |f(t)-g(t)|$ vanishes iff the base does.

(2) *Symmetry:* $\Gamma(f,g) = \Gamma(g,f)$ by absolute value.
(3) *Generalized triangle inequality:* for all $f,g,h \in X$,

$$\Gamma(f,h) \le s\left(\theta_1(f,g)\Gamma(f,g) + \theta_2(g,h)\Gamma(g,h)\right),$$

which holds due to the superadditive behavior of the exponent in Γ, and the choice of θ_1, θ_2 controlling growth.
(4) *Convexity:* For any $\lambda \in [0,1]$, $\Gamma(f, \omega(f,g;\lambda)) = \sup_{t\in[0,1]} |\lambda(f(t)-g(t))|^{1+|\lambda(f(t)-g(t))|} \le \Gamma(f,g)$, which implies that the space is convex with respect to ω.

Therefore, (X,Γ,ω) satisfies the conditions of Definition 2.5 and thus forms a convex (θ_1,θ_2)-extended b-metric space.

Lemma 2.6. *Let $\{\rho_n\}$ be a sequence in a (θ_1,θ_2)-extended Γ-metric space (χ,Γ), and suppose that $\lim_{m,n\to\infty} \theta_i(\rho_m,\rho_n) < \infty$ for $i = 1,2$. If the sequence $\{\rho_n\}$ converges, then it is a Cauchy sequence.*

Proof. Assume that the sequence $\{\rho_n\}$ converges to some point $\rho \in \chi$. Then, for any $m,n \in \mathbb{N}$, we have

$$\Gamma(\rho_m,\rho_n) \le \theta_1(\rho_m,\rho_n)\Gamma(\rho_m,\rho) + \theta_2(\rho_m,\rho_n)\Gamma(\rho,\rho_n). \tag{2.1}$$

Since $\rho_n \to \rho$ as $n \to \infty$, it follows that both $\Gamma(\rho_m,\rho) \to 0$ and $\Gamma(\rho,\rho_n) \to 0$. Moreover, the functions θ_1 and θ_2 are bounded in the limit by assumption.

Hence, the right-hand side of (2.1) tends to zero as $m,n \to \infty$, which implies that $\Gamma(\rho_m,\rho_n) \to 0$. Therefore, $\{\rho_n\}$ is a Cauchy sequence. □

Theorem 2.7. *Let (χ,Γ,ω) be a complete convex (θ_1,θ_2)-extended b-metric space, and let $\phi : \chi \to \chi$ be a contraction mapping. That is, there exists $\rho \in [0,1)$ such that, for all $\rho,\rho \in \chi$ $\Gamma(\phi\rho,\phi\rho) \le \rho\,\Gamma(\rho,\rho)$. Let $\rho_0 \in \chi$ be such that $\Gamma(\rho_0,\phi\rho_0) < \infty$, and define the sequence $\{\rho_n\}$ iteratively by $\rho_n = \omega(\rho_{n-1},\phi\rho_{n-1};\mu_{n-1})$, where, for each $n \in \mathbb{N}$,*

- $\theta_1(\rho_n,\phi\rho_n)\mu_{n-1} + \rho\,\theta_2(\rho_n,\phi\rho_n)(1-\mu_{n-1}) < 1$,
- $\theta_i(\rho_n,\rho_m) \le \theta_i(\rho_{n-1},\rho_m)$ *for all $m \in \mathbb{N}$ and $i = 1,2$,*
- $\sum_{k=0}^{\infty} \theta(\rho_n,\rho_m)^{k+1}\Gamma(\rho_{n+k},\rho_{n+k+1}) < \infty$, *where $\theta(\rho_n,\rho_m) = \max\{\theta_1(\rho_n,\rho_m), \theta_2(\rho_n,\rho_m)\}$.*

Then, ϕ has a unique fixed point in χ.

Proof. Consider any $\rho_0 \in \chi$, and generate the sequence $\{\rho_n\}$ vua (2.3). If there exist some $n_0 \in \mathbb{N}$ where $\rho_{n_0} = \rho_{n_0+1}$, then evidently ρ_{n_0} constitutes a fixed point of ϕ. Alternatively, we assume that $\rho_n \ne \rho_{n+1}$ for every $n \ge 0$. Consequently, for all $n \ge 0$, we have $\rho_n, \rho_{n+1} \notin F(\phi)$. Employing the convexity characteristic and contractivity of ϕ, we derive

$$\begin{aligned}
\Gamma(\rho_n,\rho_{n+1}) &= \Gamma(\rho_n, \omega(\rho_n,\phi\rho_n;\mu_{n-1})) \\
&\le (1-\mu_{n-1})\Gamma(\rho_n,\phi\rho_n) \\
&\le \Gamma(\rho_n,\phi\rho_n) \\
&\le \theta_1(\rho_n,\phi\rho_n)\Gamma(\rho_n,\phi\rho_{n-1}) + \theta_2(\rho_n,\phi\rho_n)\Gamma(\phi\rho_{n-1},\phi\rho_n) \\
&\le \theta_1(\rho_n,\phi\rho_n)\mu_{n-1}\Gamma(\rho_{n-1},\phi\rho_{n-1}) + \rho\,\theta_2(\rho_n,\phi\rho_n)\Gamma(\rho_{n-1},\rho_n) \\
&\le [\theta_1(\rho_n,\phi\rho_n)\mu_{n-1} + \rho\,\theta_2(\rho_n,\phi\rho_n)(1-\mu_{n-1})]\,\Gamma(\rho_{n-1},\phi\rho_{n-1}) \\
&\le \Gamma(\rho_{n-1},\phi\rho_{n-1}).
\end{aligned}$$

This calculation establishes that $\{\Gamma(\rho_n, \phi\rho_n)\}$ is monotonically decreasing and has a lower bound of zero. Consequently, it converges to some value $r \geq 0$. If we suppose $r > 0$, taking limits in the inequality above yields a contradiction. Therefore, $r = 0$, which implies

$$\lim_{n\to\infty} \Gamma(\rho_n, \rho_{n+1}) = \lim_{n\to\infty} \Gamma(\rho_n, \phi\rho_n) = 0.$$

We now demonstrate that $\{\rho_n\}$ forms a Cauchy sequence. For any $m > n$, by repeatedly applying the triangle-type inequality:

$$\Gamma(\rho_n, \rho_m) \leq \sum_{k=0}^{m-n-1} \theta(\rho_n, \rho_m)^{k+1} \Gamma(\rho_{n+k}, \rho_{n+k+1}) \leq \sum_{k=0}^{\infty} \theta(\rho_n, \rho_m)^{k+1} \Gamma(\rho_{n+k}, \rho_{n+k+1}) < \infty.$$

Thus, $\{\rho_n\}$ is indeed a Cauchy sequence. Given that (χ, Γ) is complete, there must exist some $\rho \in \chi$ such that $\rho_n \to \rho$.

Next, we establish that ρ represents a fixed point of ϕ by utilizing the limit and the properties of θ_1, θ_2, along with the contractive mapping:

$$\begin{aligned}\Gamma(\rho, \phi\rho) &\leq \theta_1(\rho, \phi\rho)\Gamma(\rho, \rho_n) + \theta_2(\rho, \phi\rho)\Gamma(\rho_n, \phi\rho) \\ &\leq \theta_1(\rho, \phi\rho)\Gamma(\rho, \rho_n) + \theta_2(\rho, \phi\rho)\left[\theta_1(\rho_n, \phi\rho)\Gamma(\rho_n, \phi\rho_n) + \theta_2(\rho_n, \phi\rho)\rho\,\Gamma(\rho_n, \rho)\right].\end{aligned}$$

As n approaches the infinity, each term on the right side tends to zero. Hence, $\Gamma(\rho, \phi\rho) = 0$, which necessarily means $\rho = \phi\rho$.

To verify uniqueness, let us suppose there exists another fixed point $\rho \in \chi$. Then

$$\Gamma(\rho, \rho) = \Gamma(\phi\rho, \phi\rho) \leq \rho\,\Gamma(\rho, \rho),$$

which necessarily implies $\Gamma(\rho, \rho) = 0$ because $\rho < 1$. Consequently, $\rho = \rho$, confirming the fixed point is indeed unique. □

Theorem 2.8. *Let (χ, Γ, ω) represent a complete convex (θ_1, θ_2)-extended b-metric space, and let $\phi : \chi \to \chi$ be a Kannan-type mapping; specifically, there exists $\rho \in \left[0, \frac{1}{2}\right)$ such that, for all $\rho, \rho \in \chi$,*

$$\Gamma(\phi\rho, \phi\rho) \leq \rho\,\Gamma(\rho, \phi\rho) + \rho\,\Gamma(\rho, \phi\rho). \tag{2.2}$$

Let $\rho_0 \in \chi$ satisfy $\Gamma(\rho_0, \phi\rho_0) < \infty$, and construct the sequence $\{\rho_n\}$ via

$$\rho_n = \omega(\rho_{n-1}, \phi\rho_{n-1}; \mu_{n-1}), \tag{2.3}$$

where, for each $n \in \mathbb{N}$,

(C1) $\theta_1(\rho_n, \phi\rho_n)\mu_{n-1} + \dfrac{\rho}{1-\rho}\theta_2(\rho_n, \phi\rho_n)(1-\mu_{n-1}) < 1$,

(C2) $\theta_i(\rho_n, \rho_m) \leq \theta_i(\rho_{n-1}, \rho_m)$ *for* $i = 1, 2$,

(C3) $\sum_{k=0}^{\infty} \theta(\rho_n, \rho_m)^{k+1}\Gamma(\rho_{n+k}, \rho_{n+k+1}) < \infty$.

Then ϕ possesses exactly one fixed point in χ.

Proof. Select any $\rho_0 \in \chi$, and formulate the sequence $\{\rho_n\}$ via (2.3). If, for some $n_0 \in \mathbb{N}$, $\rho_{n_0} = \rho_{n_0+1}$, then, $\rho_{n_0} = \omega(\rho_{n_0}, \phi\rho_{n_0}; \mu_{n_0}) = \phi\rho_{n_0}$, identifying a fixed point. Let us now suppose that $\rho_n \neq \rho_{n+1}$ for all $n \geq 0$. In this case, $\rho_n, \rho_{n+1} \notin F(\phi)$, where $F(\phi)$ represents the collection of fixed points. From the Kannan condition (2.2), we can write

$$\Gamma(\rho_1, \rho_2) \leq \rho\Gamma(\rho_0, \phi\rho_0) + \rho\Gamma(\rho_1, \phi\rho_1) \leq \rho\Gamma(\rho_0, \rho_1) + \rho\Gamma(\rho_1, \rho_2),$$

which leads to $\Gamma(\rho_1,\rho_2) \leq \frac{\rho}{1-\rho}\Gamma(\rho_0,\rho_1)$. In a similar fashion,

$$\Gamma(\rho_2,\rho_3) \leq \frac{\rho}{1-\rho}\Gamma(\rho_1,\rho_2) \leq \left(\frac{\rho}{1-\rho}\right)^2 \Gamma(\rho_0,\rho_1).$$

Using mathematical induction, we arrive at

$$\Gamma(\rho_n,\rho_{n+1}) \leq \left(\frac{\rho}{1-\rho}\right)^n \Gamma(\rho_0,\rho_1) = \beta^n\Gamma(\rho_0,\rho_1),$$

where $\beta = \frac{\rho}{1-\rho} < 1$. Now, we examine

$$\Gamma(\rho_n,\rho_{n+1}) = \Gamma(\rho_n,\omega(\rho_n,\phi\rho_n;\mu_{n-1})) \leq (1-\mu_{n-1})\Gamma(\rho_n,\phi\rho_n) \leq \Gamma(\rho_n,\phi\rho_n).$$

Applying the extended b-metric inequality,

$$\Gamma(\rho_n,\phi\rho_n) \leq \theta_1\Gamma(\rho_n,\phi\rho_n)\Gamma(\rho_n,\phi\rho_{n-1}) + \theta_2\Gamma(\rho_n,\phi\rho_n)\Gamma(\phi\rho_{n-1},\phi\rho_n).$$

Notice that $\Gamma(\rho_n,\phi\rho_{n-1}) \leq \mu_{n-1}\Gamma(\rho_{n-1},\phi\rho_{n-1})$ and

$$\Gamma(\phi\rho_{n-1},\phi\rho_n) \leq \rho\Gamma(\rho_{n-1},\phi\rho_{n-1}) + \rho\Gamma(\rho_n,\phi\rho_n).$$

From these results, we obtain

$$\Gamma(\rho_n,\rho_{n+1}) \leq [\theta_1\mu_{n-1} + \beta\theta_2(1-\mu_{n-1})]\,\Gamma(\rho_{n-1},\phi\rho_{n-1}). \tag{2.4}$$

According to condition (C1), $\{\Gamma(\rho_n,\phi\rho_n)\}$ exhibits strict monotonic decrease and is bounded below by 0. Thus, there must exist some $r \geq 0$ satisfying $\lim_{n\to\infty}\Gamma(\rho_n,\phi\rho_n) = \lim_{n\to\infty}\Gamma(\rho_n,\rho_{n+1}) = r > 0$. Inequality (2.4) can imply $r < r$, an obvious contradiction. Hence, $r = 0$. We proceed to show that $\{\rho_n\}$ constitutes a Cauchy sequence. Consider $m > n$. By successively applying the extended triangle inequality, we derive

$$\Gamma(\rho_n,\rho_m) \leq \sum_{k=0}^{m-n-1} \theta(\rho_n,\rho_m)^{k+1}\Gamma(\rho_{n+k},\rho_{n+k+1}),$$

where $\theta(\rho_n,\rho_m) = \max\{\theta_1(\rho_n,\rho_m),\theta_2(\rho_n,\rho_m)\}$. By assumption (C3), this series converges, confirming that $\{\rho_n\}$ is indeed Cauchy. Given the completeness of χ, there must exist some $\rho \in \chi$ where $\rho_n \to \rho$. To verify that ρ represents a fixed point, we observe

$$\begin{aligned}\Gamma(\rho,\phi\rho) &\leq \theta_1(\rho,\phi\rho)\Gamma(\rho,\rho_n) + \theta_2(\rho,\phi\rho)\Gamma(\rho_n,\phi\rho)\\ &\leq \theta_1(\rho,\phi\rho)\Gamma(\rho,\rho_n) + \theta_2(\rho,\phi\rho)\big[\theta_1\Gamma(\rho_n,\phi\rho_n) + \theta_2\Gamma(\phi\rho_n,\phi\rho)\big].\end{aligned}$$

Since all terms approach 0, we conclude that $\Gamma(\rho,\phi\rho) = 0$, thus $\rho = \phi\rho$. To the uniqueness, we assume $\rho \in \chi$ represents another fixed point. It follows that

$$\Gamma(\rho,\rho) = \Gamma(\phi\rho,\phi\rho) \leq \rho\Gamma(\rho,\phi\rho) + \rho\Gamma(\rho,\phi\rho) = 0.$$

Therefore, $\rho = \rho$, which demonstrates that the fixed point is indeed unique. $\square$

The notion of the MCRR contraction was first established by Debnath and La Sen [4], and the MCRR made important contributions to the advancement of fixed-point theory within generalized metric spaces.

Definition 2.9 ([4])**.** Consider a b-metric space (χ,Γ). A mapping $\phi:\chi\to\chi$ is termed an *interpolative CRR-type contraction* (**I-CRR-C**) if there exist parameters $k\in[0,1)$, and $a,b\in(0,1)$ where $a+b<1$, such that $\Gamma(\phi\rho,\phi\rho)\leq k\Gamma(\rho,\rho)^a\Gamma(\rho,\phi\rho)^b\Gamma(\rho,\phi\rho)^{1-a-b}$ for every $\rho,\rho\in\chi\setminus F(\phi)$, where $F(\phi)$ signifies the collection of fixed points of ϕ.

Theorem 2.10 ([4])**.** *Let (χ,Γ) represent a complete b-metric space with continuous b-metric δ. If $\phi:\chi\to\chi$ constitutes an interpolative MCRR-type contraction, then ϕ possesses a fixed point in χ.*

Our methodology expands upon this work and draws further inspiration from developments found in [1, 8, 9], which have notably shaped our theoretical framework.

Definition 2.11. Let (χ,Γ) denote a (θ_1,θ_2)-extended b-metric space. A mapping $\phi:\chi\to\chi$ is categorized as an *interpolative Ćirić-Reich-Rus-θ-type contraction* (**I-CRR-θ-C**) if there exist constants

$$k\in\left[0,\frac{1}{\sup\{\theta_1(\rho,\rho),\,\theta_2(\rho,\rho),\,2\}}\right),\ a,b\in(0,1),\ \text{with } a+b<1,$$

such that

$$\Gamma(\phi\rho,\phi\rho)\leq k\Gamma(\rho,\rho)^a\Gamma(\rho,\phi\rho)^b\Gamma(\rho,\phi\rho)^{1-a-b} \tag{2.5}$$

for every $\rho,\rho\in\chi\setminus F(\phi)$.

Definition 2.12. Let (χ,Γ) represent a (θ_1,θ_2)-extended b-metric space. For any $\rho\in X\setminus F(\phi)$, there exist $\rho_n\in X$ such that $\rho_n\to\rho$. (X,b) has *condition P*.

Theorem 2.13. *Suppose that (χ,Γ) is a complete (θ_1,θ_2)-extended b-metric space. Assume that $\phi:\chi\to\chi$ represents an interpolative Ćirić-Reich-Rus-θ-type contraction (I-CRR-θ-C) mapping. Select an initial element $\rho_0\in\chi$, where $\Gamma(\rho_0,\phi\rho_0)<\infty$, and construct a sequence $\{\rho_n\}$ iteratively by for all $n\in\mathbb{N}$, $\rho_n=\phi(\rho_{n-1})$ where the following conditions and restrictions are satisfied for each $n\in\mathbb{N}$:*

- $\theta_i(\rho_n,\rho_m)\leq\theta_i(\rho_{n-1},\rho_m)$ *for* $i=1,2$,
- $\sum_{k=0}^{\infty}\theta(\rho_n,\rho_m)^{k+1}\Gamma(\rho_{n+k},\rho_{n+k+1})<\infty$.

Then ϕ contains a fixed point in χ. In addition, if (χ,Γ) fulfills condition P, then ϕ possesses exactly one fixed point in χ.

Proof. The first step is to establish the existence. Take $\rho_0\in X$ and generate the iterative sequence $\{\rho_n\}$ by, for all $n\in\mathbb{N}$, $\rho_n=\phi^n(\rho_0)$. If there exists $n_0\in\mathbb{N}$ such that $\rho_{n_0}=\rho_{n_{0+1}}$, then ρ_{n_0} evidently constitutes a fixed point of ϕ.

The second step is to obtain the fixed point. Let us suppose that, for all $n\geq 0$, $\rho_n\neq\rho_{n+1}$. Consequently, for all $n\geq 0$, $\rho_n,\rho_{n+1}\in X\setminus F(\phi)$.

$$\begin{aligned}\Gamma(\rho_n,\rho_{n+1})=\Gamma(\phi\rho_{n-1},\phi\rho_n)&\leq k[\Gamma(\rho_{n-1},\rho_n)]^b[\Gamma(\rho_{n-1},\phi\rho_{n-1})]^q\Gamma(\rho_n,\phi\rho_n)^{1-a-b}\\&\leq k\Gamma(\rho_{n-1},\rho_n)^b\Gamma(\rho_{n-1},\rho_n)^a\Gamma(\rho_n,\rho_{n+1})^{1-a-b}.\end{aligned}$$

From this, we derive $\Gamma(\rho_n,\rho_{n+1})^{a+b}\leq\Gamma(\rho_{n-1},\rho_n)^{a+b}$ which implies $\Gamma(\rho_n,\rho_{n+1})\leq\Gamma(\rho_{n-1},\rho_n)$ for all $n\geq 0$. From the two inequalities, we obtain $\Gamma(\rho_n,\rho_{n+1})\leq k\Gamma(\rho_{n-1},\rho_n)\leq k\Gamma(\rho_{n-1},\rho_n)$

for all $n \geq 0$. Thus, for any $n \geq 0$,

$$\Gamma(\rho_n, \rho_{n+1}) \leq k^n \Gamma(\rho_0, \rho_1). \tag{2.6}$$

Observe that $\{\Gamma(\rho_n, \rho_{n+1})\}$ forms a decreasing sequence of non-negative real values. Hence, there must exist $r \geq 0$ such that $\lim_{n\to\infty} \Gamma(\rho_n, \rho_{n+1}) = r$. If we assume $r > 0$ then by letting $n \to \infty$ in (2.6) we find $r \leq 0$, an obvious contradiction. We now demonstrate that $\{\rho_n\}$ constitutes a Cauchy sequence

$$\begin{aligned}
\Gamma(\rho_n, \rho_m) &\leq \theta_1 \Gamma(\rho_n, \rho_m)\Gamma(\rho_n, \rho_{n+1}) + \theta_2 \Gamma(\rho_n, \rho_m)\Gamma(\rho_{n+1}, \rho_m) \\
&\leq \theta_1(\rho_n, \rho_m)\Gamma(\rho_n, \rho_{n+1}) + \theta_2(\rho_n, \rho_m)[\theta_1(\rho_{n+1}, \rho_m)\Gamma(\rho_{n+1}, \rho_{n+2}) \\
&\quad + \theta_2(\rho_{n+1}, \rho_m)\Gamma(\rho_{n+2}, \rho_m)] \\
&\leq \theta_1(\rho_n, \rho_m)\Gamma(\rho_n, \rho_{n+1}) + \theta_2(\rho_n, \rho_m)\theta_1(\rho_{n+1}, \rho_m)\Gamma(\rho_{n+1}, \rho_{n+2}) + \dots \\
&\quad + \Gamma(\rho_{m-1}, \rho_m) \prod_{i=n}^{m-2} \theta_2(\rho_i, \rho_m) \\
&\leq \sum_{k=0}^{m-n-1} \theta(\rho_n, \rho_m)^{k+1} \Gamma(\rho_{n+k}, \rho_{n+k+1}) \\
&\leq \sum_{k=0}^{\infty} \theta(\rho_n, \rho_m)^{k+1} \Gamma(\rho_{n+k}, \rho_{n+k+1}) \leq \infty.
\end{aligned}$$

where $\theta(\rho_n, \rho_m) = \max\{\theta_1(\rho_n, \rho_m), \theta_2(\rho_n, \rho_m)\}$, which confirms that $\{x_n\}$ is indeed a Cauchy sequence. Through the completeness property of χ, there exists $\rho \in \chi$ such that $\rho_{k_n} \to \rho$, where $\{\rho_{k_n}\} \subset \{\rho_n\}$. We now proceed to confirm that ρ represents a fixed point of ϕ

$$\begin{aligned}
\Gamma(\rho, \phi\rho) &\leq \theta_1(\rho, \phi\rho)\Gamma(\rho, \rho_{k_n}) + \theta_2(\rho, \phi\rho)\Gamma(\rho_{k_n}, \phi\rho) \\
&\leq \theta_1(\rho, \phi\rho)\Gamma(\rho, \rho_{k_n}) + \theta_2(\rho, \phi\rho)[\theta_1(\rho_{k_n}, \phi\rho)\Gamma(\rho_{k_n}, \phi\rho_{k_n}) + \theta_2(\rho_{k_n}, \phi\rho)\Gamma(\phi\rho_{k_n}, \phi\rho)] \\
&\leq \theta_1(\rho, \phi\rho)\Gamma(\rho, \rho_{k_n}) + \theta_2(\rho, \phi\rho)[\theta_1(\rho_{k_n}, \phi\rho)\Gamma(\rho_{k_n}, \phi\rho_{k_n}) \\
&\quad + \theta_2(\rho_{k_n}, \phi\rho)(k\Gamma(\rho_{k_n}, \rho)^a \Gamma(\rho_{k_n}, \phi\rho)^b \Gamma(\rho, \phi\rho)^{1-a-b}).
\end{aligned} \tag{2.7}$$

As $n \to \infty$ in the equation above, we determine that $\Gamma(\rho, \phi\rho) = 0$. Thus $\rho = \phi\rho$. If we consider ρ as another fixed point of ϕ, where $\rho_n \to \rho$, then

$$\begin{aligned}
\Gamma(\rho_{k_{n+1}}, \rho_{n+1}) \leq \Gamma(\phi\rho_{k_n}, \phi\rho_n) &\leq k\Gamma(\rho_{k_n}, \rho_n)^b \Gamma(\rho_{k_n}, \phi\rho_{k_n})^a \Gamma(\rho_n, \phi\rho_n)^{1-a-b} \\
&\leq k\Gamma(\rho_{k_n}, \rho_n)^b \Gamma(\rho_{k_n}, \rho_{k_{n+1}})^a \Gamma(\rho_n, \rho_{n+1})^{1-a-b} \\
&\to 0.
\end{aligned} \tag{2.8}$$

Thus $0 \leq \Gamma(\rho, \rho) \leq 0$. Consequently, the fixed point is indeed unique. □

We may utilize fixed point methods, as formulated in both classical and contemporary works [3, 5, 7, 10, 11] to establish the existence and uniqueness of solutions to the integral equation.

3. Application to Integral Equations

In this section, we employ Theorem 2.7 and 2.13 to demonstrate the uniqueness-existence of a solution to a nonlinear Volterra-type integral equation. The fixed point result presented in Theorem 2.7 is structured in the framework of a convex (θ_1, θ_2)-extended b-metric space, offering greater adaptability compared to traditional Banach spaces.

3.1. **The Integral Equation.** We examine the nonlinear integral equation:

$$\mu(t) = \int_0^t K(t,s,\mu(s))\,ds, \quad t \in [0,1], \tag{3.1}$$

where $K : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ exhibits continuity. We establish $\chi = C([0,1])$, the Banach space comprising continuous real-valued functions on $[0,1]$, and furnish it with the generalized distance function $\Gamma : \chi \times \chi \to [0,\infty)$ specified by: $\Gamma(\mu,\nu) = \sup_{t\in[0,1]} |\mu(t) - \nu(t)|$. We investigate this equation under two distinct fixed point frameworks. We introduce an operator $\phi : \chi \to \chi$ on the space $\chi = C([0,1])$, the collection of continuous real-valued functions on $[0,1]$, defined as: $\phi(\mu)(t) = \int_0^t K(t,s,\mu(s))\,ds$. Let $\omega(\mu,\nu;\lambda) = \lambda\mu + (1-\lambda)\nu$, with $\lambda \in [0,1]$. Then (χ,Γ,ω) represents a convex (θ_1,θ_2)-extended b-metric space if we define:

$$\theta_1(\mu,\nu) = \theta_2(\mu,\nu) = 1 \quad \text{for all } \mu,\nu \in \chi.$$

Assumptions on the Kernel. Assume the kernel $K(t,s,x)$ satisfies these conditions:

- K has the continuity on $[0,1] \times [0,1] \times \mathbb{R}$,
- there exists a constant $L \in [0,1)$ such that $|K(t,s,x) - K(t,s,y)| \le L|x-y|$ for all t and s in $[0,1]$, and x and y in $\mathbb{R}$.

3.2. **Application of Theorem 2.7.** We show that the operator ϕ satisfies the contractive condition in Theorem 2.7, $\Gamma(\phi\mu,\phi\nu) \le \rho\Gamma(\mu,\nu)$ for all μ and ν in χ, with $\rho = L < 1$. Indeed, for all $t \in [0,1]$, we have

$$\begin{aligned}
|\phi(\mu)(t) - \phi(\nu)(t)| &= \left| \int_0^t [K(t,s,\mu(s)) - K(t,s,\nu(s))]\,ds \right| \\
&\le \int_0^t L|\mu(s) - \nu(s)|\,ds \le L\int_0^t \Gamma(\mu,\nu)\,ds \\
&= Lt\Gamma(\mu,\nu) \le L\Gamma(\mu,\nu).
\end{aligned}$$

Taking the supremum over $t \in [0,1]$, we obtain $\Gamma(\phi\mu,\phi\nu) \le L\Gamma(\mu,\nu)$, so ϕ is a contraction mapping. Moreover, let us define the sequence $\mu_0 \in \chi$, $\mu_n = \omega(\mu_{n-1},\phi(\mu_{n-1});\lambda_{n-1})$, where $\lambda_{n-1} \in [0,1]$ is chosen such that

$$\theta_1(\mu_n,\phi(\mu_n))\lambda_{n-1} + \rho\theta_2(\mu_n,\phi(\mu_n))(1-\lambda_{n-1}) < 1.$$

This sequence satisfies the requirements of Theorem 2.7 (namely the interpolative iterative process), and since χ is complete and Γ satisfies the properties of a (θ_1,θ_2)-extended b-metric, the theorem guarantees

- $\{\mu_n\}$ converges to a fixed point $\mu^* \in \chi$,
- $\phi(\mu^*) = \mu^*$, i.e., μ^* is a solution to the integral equation,
- the solution is unique.

By applying Theorem 2.7, within the framework of convex (θ_1,θ_2)-extended b-metric spaces, we have shown that the nonlinear Volterra-type integral equation $\mu(t) = \int_0^t K(t,s,\mu(s))\,ds$ admits a unique continuous solution under mild assumptions on the kernel K, such as continuity and a Lipschitz-type condition.

3.3. **Application of Theorem 2.13 (I-CRR-θ-type contraction).** Now, we consider a more general case where K satisfies an interpolative inequality, and ϕ is not necessarily a contraction but satisfies:

$$\Gamma(\phi\mu,\phi\nu) \le k\Gamma(\mu,\nu)^a\Gamma(\mu,\phi\mu)^b\Gamma(\nu,\phi\nu)^{1-a-b},$$

for all $\mu,\nu \in \chi \setminus F(\phi)$, where $k \in [0,1)$, $a,b \in (0,1)$, and $a+b<1$. Under such conditions, we can no longer apply Banach's theorem directly. Suppose further that the mapping ϕ is defined as before and satisfies the above condition. Let us define a sequence: $\mu_0 \in \chi$, $\mu_n = \phi(\mu_{n-1})$, $n \in \mathbb{N}$. Then, under the assumptions of Theorem 2.13, including monotonicity of θ_i and summability of the form

$$\sum_{k=0}^{\infty} \theta(\mu_n,\mu_m)^{k+1}\Gamma(\mu_{n+k},\mu_{n+k+1}) < \infty,$$

this sequence converges to a fixed point $\mu^* \in \chi$, and μ^* is the unique solution of equation (3.1).

We have demonstrated that $\mu(t) = \int_0^t K(t,s,\mu(s))\,ds$ admits a unique continuous solution under both classical contraction assumptions (via Theorem 2.7) and more general interpolative contractive mappings (via Theorem 2.13). Our convex framework provides a natural and robust generalization of Banach's theory, enabling the treatment of a wider class of nonlinear integral equations. These results underscore the strength of abstract fixed point theory in solving functional equations in generalized metric spaces.

3.4. **Numerical examples.** We present a concrete numerical example to illustrate the iterative convergence to a fixed point, as guaranteed by Theorem 2.7 and Theorem 2.13.

Example for Theorem 2.7 (Contraction Mapping). Let us consider the integral equation:

$$\mu(t) = \int_0^t (t-s)\mu(s)\,ds, \quad t \in [0,1].$$

Define $\phi(\mu)(t) = \int_0^t (t-s)\mu(s)\,ds$. We consider the initial function $\mu_0(t) = 1$ and define the sequence: $\mu_n(t) = \phi(\mu_{n-1})(t)$. One can compute iteratively:

$$\mu_1(t) = \int_0^t (t-s)(1)\,ds = \int_0^t (t-s)\,ds = \frac{t^2}{2},\ \mu_2(t) = \int_0^t (t-s)\frac{s^2}{2}\,ds = \frac{t^4}{8},$$
$$\mu_3(t) = \int_0^t (t-s)\frac{s^4}{8}\,ds = \frac{t^6}{48},\ \mu_4(t) = \int_0^t (t-s)\frac{s^6}{48}\,ds = \frac{t^8}{384}.$$

Thus the sequence converges pointwise to the zero function $\mu^*(t) = 0$, which is the unique fixed point of ϕ. We summarize values at $t = 0.5$ below: $n=0$, $\mu_n = 1$; $n=1$, $\mu_n = 0.125$; $n=2$, $\mu_n = 0.0078$; $n=3$, $\mu_n = 0.000651$; $n=4$, $\mu_n = 0.000065$. This rapid convergence shows how the contraction mapping ensures exponential decay to the fixed point.

Example for Theorem 2.13 (I-CRR-θ-Type Contraction). Now consider the operator:

$$\phi(\mu)(t) = \int_0^t \frac{\mu(s)^2}{1+\mu(s)^2}\,ds,$$

which is not a contraction in the classical sense but satisfies an interpolative condition:

$$\Gamma(\phi\mu,\phi\nu) \le k\Gamma(\mu,\nu)^a\Gamma(\mu,\phi\mu)^b\Gamma(\nu,\phi\nu)^{1-a-b},$$

for some constants $k = 0.5$, $a = 0.4$, $b = 0.3$. Let $\mu_0(t) = t$, and we compute the first few iterates numerically at $t = 0.5$; $n = 0$, $\mu_0 = 0.5$; $n = 1$, $\mu_1 = \int_0^{0.5} \frac{s^2}{1+s^2}\, ds \approx 0.112$; $n = 1$, $\mu_2 = \int_0^{0.5} \frac{(\mu_1(s))^2}{1+(\mu_1(s))^2}\, ds \approx 0.012$; $n = 3$, $\mu_3 \approx 0.0007$, and $n = 4$, $\mu_n < 10^{-4}$.

In both examples, we observe that $\{\mu_n\}$ converges rapidly to a fixed point. The first example satisfies the classical contraction conditions (Theorem 2.7), while the second satisfies the more general I-CRR-θ-type contraction (Theorem 2.13). These concrete iterations affirm the theoretical results and demonstrate convergence behavior.

4. Conclusion

This article introduces a new mathematical framework, the convex (θ_1, θ_2)-extended b-metric space. In this broader framework, we proved convergence theorems and established corresponding fixed-point results. With the aid of an interpolation, we investigated classical fixed-point theorems and Mann's iteration in the generalized b-metric context. Moreover, we considered fixed points of ω-operators with concrete examples.

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