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INFINITELY MANY SOLUTIONS FOR A CLASS OF ELLIPTIC BOUNDARY VALUE PROBLEMS WITH (p,q)-KIRCHHOFF TYPE

ZONGXI LI[1], WANTING QI[1], XINGYONG ZHANG[1,2,*]

[1]Faculty of Science, Kunming University of Science and Technology, Kunming 650500, China
[2]Research Center for Mathematics and Interdisciplinary Sciences, Kunming University of Science and Technology, Kunming 650500, China

Abstract. In this paper, we investigate the existence of infinitely many solutions for the following elliptic boundary value problem with (p,q)-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u_1|^p dx\right)\right]^{p-1}\Delta_p u_1 + \left[M_3\left(\int_\Omega a_1(x)|u_1|^p dx\right)\right]^{p-1} a_1(x)|u_1|^{p-2}u_1 = G_{u_1}(x,u_1,u_2) & \text{in } \Omega,\\ -\left[M_2\left(\int_\Omega |\nabla u_2|^q dx\right)\right]^{q-1}\Delta_q u_2 + \left[M_4\left(\int_\Omega a_2(x)|u_2|^q dx\right)\right]^{q-1} a_2(x)|u_2|^{q-2}u_2 = G_{u_2}(x,u_1,u_2) & \text{in } \Omega,\\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

By using a critical point theorem due to Ding in [Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Anal. 25 (1995) 1095-1113], we obtain that the system has infinitely many solutions when the nonlinear terms satisfy the asymptotically-(p,q) conditions.

Keywords. Elliptic boundary value problems; (p,q)-Kirchhoff type; Infinitely many solutions; Variational methods.

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*Corresponding author.
E-mail address: zhangxingyong1@163.com (X. Zhang).
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1. INTRODUCTION

In this paper, we investigate the following nonlocal elliptic boundary value problem with (p,q)-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u_1|^p dx\right)\right]^{p-1}\Delta_p u_1 + \left[M_3\left(\int_\Omega a_1(x)|u_1|^p dx\right)\right]^{p-1} a_1(x)|u_1|^{p-2}u_1 = G_{u_1}(x,u_1,u_2) \\ \text{in } \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla u_2|^q dx\right)\right]^{q-1}\Delta_q u_2 + \left[M_4\left(\int_\Omega a_2(x)|u_2|^q dx\right)\right]^{q-1} a_2(x)|u_2|^{q-2}u_2 = G_{u_2}(x,u_1,u_2) \\ \text{in } \Omega, \\ u_1 = u_2 = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p \le \frac{pN}{N-p}$, $1 < q \le \frac{qN}{N-q}$, N is a integer with $N > \max\{p,q\}$, $\Delta_p u_1 = \operatorname{div}(|\nabla u_1|^{p-2}\nabla u_1)$, $\Delta_q u_2 = \operatorname{div}(|\nabla u_2|^{q-2}\nabla u_2)$, Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $M_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1,2,3,4$ are continuous functions, and $a_i \in C(\bar{\Omega}, \mathbb{R}^+)$, $i = 1,2$.

In 2011, Cheng, Wu and Liu [13] investigated the following nonlocal elliptic system of (p,q)-Kirchhoff type with a parameter λ:

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u = \lambda F_u(x,u,v) \ \text{ in } \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^q dx\right)\right]^{q-1}\Delta_q v = \lambda F_v(x,u,v) \ \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \ge 1)$ is a bounded smooth domain, $\lambda \in (0,+\infty)$, $p > N$, $q > N$, $M_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1,2$ are continuous functions with bounded conditions. Under some reasonable conditions, by using a critical point theorem due to Bonanno in [4], they obtained that system (1.2) has at least two weak solutions. By using an equivalent formulation [5, Theorem 2.3] of a three critical points theorem due to Ricceri in [22], they obtained that system (1.2) has at least three weak solutions.

Subsequently, Chen et al. [12] and Massar and Talbi [20] both investigated the following the following nonlocal elliptic system of (p,q)-Kirchhoff type with two parameters λ and μ:

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1}\Delta_p u = \lambda F_u(x,u,v) + \mu G_u(x,u,v) \ \text{ in } \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^q dx\right)\right]^{q-1}\Delta_q v = \lambda F_v(x,u,v) + \mu G_v(x,u,v) \ \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Under different conditions for F and G, by using a critical point theorem due to Ricceri in [23], they obtained that system (1.3) has at least three weak solutions and they generalized the corresponding result in [13]. Then, the results of [12] were extended to a Dirichlet boundary problem involving the $(p_1,...,p_n)$-Kirchhoff type systems in [15].

Moreover, in [9], Chung investigated the following system with a parameter λ:

$$\begin{cases} -M_1\left(\int_\Omega |\nabla u|^p dx\right)\Delta_p u = \lambda a(x) f(u,v) \ \text{ in } \Omega, \\ -M_2\left(\int_\Omega |\nabla v|^q dx\right)\Delta_q v = \lambda b(x) g(u,v) \ \text{ in } \Omega, \\ u = v = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $a,b \in C(\bar{\Omega})$. By using sub and supersolutions method, under some reasonable conditions for f and g, the author obtained that system (1.4) has a positive solutions when $\lambda > \lambda^*$ for some $\lambda^* > 0$. For problem (1.4), there are many similar results. We refer to [24, 25] for pertinent results. As pointed out in [13], (1.2) is related to the Kirchhoff equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ, P_0, h, E, and L are parameters with special meanings. Kirchhoff problems were studied extensively and numerous interesting results were obtained by using variational method recently; see, e.g., [2, 3, 6, 10, 11, 19, 26].

However, existing studies on (p,q)-Kirchhoff systems have the following gaps: First, the results in [9, 12, 13] only proved *finite solutions* (one, two, or three solutions) via Bonanno / Ricceri theorems or sub and supersolutions method (there are no results on *infinitely many solutions*). Second, previous works rely on the conditions: either external parameters (e.g., λ, μ in [12]) or *sub-(p,q)* growth of nonlinear terms (nonlinearity grows slower than $|z_1|^p + |z_2|^q$) (e.g., [12] and [13]) which excludes *asymptotically-(p,q) case* (the nonlinearity grows almost uniformly like $|z_1|^p + |z_2|^q$). Third, tools like sub-supersolution methods or Bonanno/Ricceri theorems in [4, 22] are unsuitable for infinite many solutions, while Ding's theorem [14] for infinitely may critical points has not been applied to (p,q)-Kirchhoff systems.

In this paper, motivated by [13, 12, 20], we investigate the existence of infinitely many solutions for system (1.1). Via a critical point theorem due to Ding in [14], we establish two results when the nonlinear terms satisfy the asymptotically-(p,q) conditions. To be precise, we obtain the following results:

Theorem 1.1. *Assume that the following conditions hold:*
($\mathscr{M}$) there exist positive constants $\theta \in (0, \min\{p,q\})$, m_*, and m^* such that

$$\frac{m_*}{m^*} \geq \max\left\{ \left(\frac{\theta \max\limits_{x\in\bar{\Omega}} a_1(x)}{p \min\limits_{x\in\bar{\Omega}} a_1(x)} \right)^{\frac{1}{p-1}}, \left(\frac{\theta \max\limits_{x\in\bar{\Omega}} a_2(x)}{q \min\limits_{x\in\bar{\Omega}} a_2(x)} \right)^{\frac{1}{q-1}} \right\}$$

and

$$m_* \leq M_i(s) \leq m^*, \forall s \geq 0, i = 1,2,3,4;$$

($\mathscr{A}$) $a_i \in C(\bar{\Omega}, \mathbb{R}^+)$ and $\inf\limits_{\Omega} a_i(x) > 0$, $i = 1,2$;
(G0) $G(x,0,0) \equiv 0$ and $G(x,z_1,z_2) = G(x,-z_1,-z_2)$;
(G1) $G(x,z_1,z_2) \in C^1(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_1, c_2 > 0$, $s_1 \in \left[p, \frac{(p-1)N}{N-p}\right)$ and $s_2 \in \left[q, \frac{(q-1)N}{N-q}\right)$ such that

$$|G_{z_1}(x,z_1,z_2)| \leq c_1(1+|z_1|^{s_1}), \quad \textit{for all } x \in \Omega \times \mathbb{R} \times \mathbb{R},$$
$$|G_{z_2}(x,z_1,z_2)| \leq c_2(1+|z_2|^{s_2}), \quad \textit{for all } x \in \Omega \times \mathbb{R} \times \mathbb{R};$$

(G2)

$$\lim_{|z_1|+|z_2|\to\infty} \frac{G(x,z_1,z_2)}{|z_1|^p + |z_2|^q} < \min\left\{ \frac{m_*^{p-1} \inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1} \inf\limits_{x\in\Omega} a_2(x)}{q} \right\} \quad \textit{uniformly for } x \in \Omega;$$

(G3) there exist $\gamma_1 \in [1,p)$, $\gamma_2 \in [1,q)$ and $C^* > 0$ such that

$$G(x,z_1,z_2) \ge C^*(|z_1|^{\gamma_1} + |z_2|^{\gamma_2}), \quad \text{for all } (x,z_1,z_2) \in \Omega \times \mathbb{R} \times \mathbb{R};$$

(G4) there exists a function $h \in L^1(\Omega,\mathbb{R})$ such that

$$\theta G(x,z_1,z_2) - G_{z_1}(x,z_1,z_2)z_1 - G_{z_2}(x,z_1,z_2)z_2 \ge h(t) \quad \text{for a.e. } x \in \Omega,$$

and

$$\lim_{|z_1|+|z_2|\to\infty} [\theta G(x,z_1,z_2) - G_{z_1}(x,z_1,z_2)z_1 - G_{z_2}(x,z_1,z_2)z_2] = +\infty, \ \text{for a.e. } x \in \Omega.$$

Then system (1.1) has infinitely many nontrivial solutions.

Theorem 1.2. *Assume that $(\mathscr{M})$, $(\mathscr{A})$, (G0), (G1), (G3), (G4) and the following condition hold: (G2)$'$*

$$\lim_{|z_1|+|z_2|\to\infty} \frac{G(x,z_1,z_2)}{|z_1|^p + |z_2|^q} < \min\left\{ \frac{\min\{m_*, m_*^{p-1} \inf_{x\in\Omega} a_1(x)\}}{p\tau_{p,p}}, \frac{\min\{m_*, m_*^{q-1} \inf_{x\in\Omega} a_2(x)\}}{q\tau_{q,q}} \right\}$$

uniformly for $x \in \Omega$, where $\tau_{p,p}$ and $\tau_{q,q}$ are the embedding constants in $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and $W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$, respectively. Then system (1.1) has infinitely many nontrivial solutions.

2. Preliminaries

Let $r > 1$. On $W_0^{1,r} := W_0^{1,r}(\Omega)$, we define the norm

$$\|u\|_{1,r} = \left(\int_\Omega |\nabla u|^r dx + \int_\Omega |u|^r dx \right)^{1/r}.$$

Then $(W_0^{1,r}(\Omega), \|\cdot\|_{1,r})$ is a reflexive and separable Banach space. It is well known that there exist $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,r}(\Omega)$ such that $\overline{\text{span}}\{v_n; n \in \mathbb{N}\} = W_0^{1,r}(\Omega)$. Let $X_{r,j} = \mathbb{R}v_j$. Then $W_0^{1,r}(\Omega) = \oplus_{j\ge1} X_{r,j}$. Define

$$E_{r,k}^{(1)} = \oplus_{j=1}^k X_{r,j}, \quad E_{r,k}^{(2)} = \overline{\oplus_{j\ge k} X_{r,j}}.$$

Then $W_0^{1,r}(\Omega) = E_{r,k}^{(1)} \oplus E_{r,k}^{(2)}$ (see [18]). Let $\mathscr{W} = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Then

$$\mathscr{W} = \left(E_{p,1}^{(1)} \oplus E_{p,1}^{(2)}\right) \times \left(E_{q,1}^{(1)} \oplus E_{q,1}^{(2)}\right) = \left(E_{p,1}^{(1)} \times E_{q,1}^{(1)}\right) \oplus \left(E_{p,1}^{(2)} \times E_{q,1}^{(2)}\right).$$

On $\mathscr{W}$, we define the norm

$$\|(u_1,u_2)\| = \|u_1\|_{1,p} + \|u_2\|_{1,q},$$

where $u_1 \in W_0^{1,p}(\Omega)$ and $u_2 \in W_0^{1,q}(\Omega)$. Then $(\mathscr{W}, \|\cdot\|)$ is also a reflexive Banach space.

Lemma 2.1. [1] *$W_0^{1,p}$ ($W_0^{1,q}$) is compactly embedded in $L^r := L^r(\Omega)$ for $p \le r < \frac{pN}{N-p}$ ($q \le r < \frac{qN}{N-q}$) and continuously embedded in $L^{\frac{pN}{N-p}} := L^{\frac{pN}{N-p}}(\Omega)$ $\left(L^{\frac{qN}{N-q}} := L^{\frac{qN}{N-q}}(\Omega)\right)$, and hence for every $p \le r \le \frac{pN}{N-p}$ $\left(q \le r \le \frac{qN}{N-q}\right)$, there exists $\tau_{r,p} > 0$ ($\tau_{r,q} > 0$) such that*

$$|u|_r \le \tau_{r,p}\|u\|_{1,p} \quad (|u|_r \le \tau_{r,q}\|u\|_{1,q}), \quad \forall u \in W_0^{1,p} (u \in W_0^{1,q}),$$

where $|\cdot|_r$ denotes the usual norm in L^r for all $p \le r \le \frac{pN}{N-p}$ $\left(q \le r \le \frac{qN}{N-q}\right)$.

Define

$$\hat{M}_1(t)=\int_0^t [M_1(s)]^{p-1}ds, \quad \hat{M}_2(t)=\int_0^t [M_2(s)]^{q-1}ds,$$
$$\hat{M}_3(t)=\int_0^t [M_3(s)]^{p-1}ds, \quad \hat{M}_4(t)=\int_0^t [M_4(s)]^{q-1}ds, \quad \forall t\geq 0.$$

On $\mathscr{W}$, we define the functional $I:\mathscr{W}\to\mathbb{R}$ by

$$\begin{aligned}
I(u) &= I(u_1,u_2)\\
&= \varphi(u)+\psi(u)\\
&= \varphi(u_1,u_2)+\psi(u_1,u_2)\\
&= -\frac{1}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1|^p dx\right)-\frac{1}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2|^q dx\right)\\
&\quad -\frac{1}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1|^p dx\right)-\frac{1}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2|^q dx\right)+\int_\Omega G(x,u_1,u_2)dx,
\end{aligned}$$

for all $u=(u_1,u_2)\in\mathscr{W}$, where

$$\begin{aligned}
\varphi(u)=\varphi(u_1,u_2) == \frac{1}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1|^p dx\right)-\frac{1}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2|^q dx\right)\\
-\frac{1}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1|^p dx\right)-\frac{1}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2|^q dx\right),
\end{aligned}$$

and

$$\psi(u)=\psi(u_1,u_2)=\int_\Omega G(x,u_1,u_2)dx, \quad \forall u=(u_1,u_2)\in\mathscr{W}.$$

Lemma 2.2. *Suppose that the following condition holds:*
(G1)′ $G(x,z_1,z_2)\in C^1(\bar{\Omega}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ *and there exist constants* $c_1,c_2>0$, $s_1\in\left[p,\frac{(p-1)N+p}{N-p}\right)$ *and* $s_2\in\left[q,\frac{(q-1)N+q}{N-q}\right)$ *such that*

$$\begin{aligned}
|G_{z_1}(x,z_1,z_2)| &\leq c_1(1+|z_1|^{s_1}), \quad \text{for all } x\in\Omega\times\mathbb{R}\times\mathbb{R},\\
|G_{z_2}(x,z_1,z_2)| &\leq c_2(1+|z_2|^{s_2}), \quad \text{for all } x\in\Omega\times\mathbb{R}\times\mathbb{R}.
\end{aligned}$$

Then $\psi\in C^1(\mathscr{W},\mathbb{R})$ *and* $\psi':\mathscr{W}\to\mathscr{W}^*$ *is compact, and hence* $\varphi\in C^1(\mathscr{W},\mathbb{R})$. *Moreover,*

$$\begin{aligned}
\langle\varphi'(u_1,u_2),(v_1,v_2)\rangle &= -\left[M_1\left(\int_\Omega |\nabla u_1|^p dx\right)\right]^{p-1}\int_\Omega |\nabla u_1|^{p-2}(\nabla u_1,\nabla v_1)dx\\
&\quad -\left[M_2\left(\int_\Omega |\nabla u_2|^q dx\right)\right]^{q-1}\int_\Omega |\nabla u_2|^{q-2}(\nabla u_2,\nabla v_2)dx\\
&\quad -\left[M_3\left(\int_\Omega a_1(x)|u_1|^p dx\right)\right]^{p-1}\int_\Omega a_1(x)|u_1|^{p-2}(u_1,v_1)dx\\
&\quad -\left[M_4\left(\int_\Omega a_2(x)|u_2|^q dx\right)\right]^{q-1}\int_\Omega a_2(x)|u_2|^{q-2}(u_2,v_2)dx,
\end{aligned}$$

$$\langle\psi'(u_1,u_2),(v_1,v_2)\rangle = \int_\Omega G_{z_1}(x,u_1,u_2)v_1dx+\int_\Omega G_{z_2}(x,u_1,u_2)v_2dx$$

and

$$\langle I'(u_1,u_2),(v_1,v_2)\rangle = \langle \varphi'(u_1,u_2),(v_1,v_2)\rangle + \langle \psi'(u_1,u_2),(v_1,v_2)\rangle \tag{2.1}$$

for all $(u_1,u_2),(v_1,v_2) \in \mathscr{W}$ *and critical points of* I *are solutions of system (1.1).*

Proof. The proof is essentially identical to that of Proposition B.10 in [21]. Hence we omit the details. □

Remark 2.3. Obviously, (G1) implies that (G1)′. Hence, Lemma 2.2 also holds under (G1).

Lemma 2.4. [14] *Let* E *be an infinite dimensional Banach space and let* $f \in C^1(E,\mathbb{R})$ *be even, satisfy (PS), and* $f(0) = 0$. *If* $E = E_1 \oplus E_2$, *where* E_1 *is finite dimensional, and* f *satisfies*

(f_1) f *is bounded from above on* E_2,

(f_2) for each finite dimensional subspace $\tilde{E} \subset E$, *there are positive constants* $\rho = \rho(\tilde{E})$ *and* $\sigma = \sigma(\tilde{E})$ *such that* $f \geq 0$ *on* $B_\rho \cap \tilde{E}$ *and* $f|_{\partial B_\rho \cap \tilde{E}} \geq \sigma$ *where* $B_\rho = \{x \in E; \|x\| \leq \rho\}$,

then f *possesses infinitely many nontrivial critical points.*

Remark 2.5. As demonstrated in [7], a deformation lemma can be proved with replacing the usual (PS)-condition with the (C)-condition introduced by Cerami in [8], and it turns out that Lemma 2.2 are true under the (C)-condition. We say that φ satisfies the (C)-condition, i.e., for every sequence $\{u_n\} \subset E$, $\{u_n\}$ has a convergent subsequence if $\varphi(u_n)$ is bounded and $(1+\|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.

Lemma 2.6. [17, Lemma 2.3] *Let* $\Omega \subseteq \mathbb{R}^N$ *be a bounded domain with* C^1 *boundary,* $1 < r < \infty$, *and let* X *be a closed subspace of the Sobolev space* $W^{1,r}(\Omega)$ *satisfying* $W_0^{1,r}(\Omega) \subseteq X \subseteq W^{1,p}(\Omega)$ *(where* $W_0^{1,r}(\Omega)$ *denotes functions in* $W^{1,r}(\Omega)$ *that vanish on* $\partial\Omega$ *in the trace sense). Let* $\beta \in L^\infty(\partial\Omega)$ *be a nonnegative function (i.e.,* $\beta \geq 0$ *almost everywhere on* $\partial\Omega$*), and define the* C^1*-functional* $G : X \to \mathbb{R}$ *by:*

$$G(u) = \int_\Omega (|\nabla u|^r + |u|^r)\,dx + \int_{\partial\Omega} \beta |u|^r ds,$$

where ∇u *is the weak gradient of* u, dx *denotes the Lebesgue measure on* Ω, *and* ds *denotes the surface measure on* $\partial\Omega$. *Let* $G' : X \to X^*$ *(with* X^* *the topological dual space of* X*) be the Fréchet derivative of* G, *and define the operator* $B : X \to X^*$ *as* $B = \frac{1}{p}G'$. *The duality pairing of* B *with any* $v \in X$ *is given by:*

$$\langle Bu, v\rangle = \int_\Omega \left(|\nabla u|^{r-2}\nabla u \cdot \nabla v + |u|^{r-2}uv\right) dx + \int_{\partial\Omega} \beta |u|^{r-2} uv\, ds.$$

Then, for any $u, v \in X$, *the following assertions hold:*

(1) *Inequality property:*

$$\langle Bu - Bv, u - v\rangle \geq \left(\|u\|^{r-1} - \|v\|^{r-1}\right)(\|u\| - \|v\|),$$

where $\|u\| = (\int_\Omega (|\nabla u|^r + |u|^r)dx)^{1/r}$ *is the standard norm on* $W^{1,r}(\Omega)$, *which is consistent with the norm definition in* [17] *(Lemma 2.3);*

(2) *Equality condition:* $\langle Bu - Bv, u - v\rangle = 0$ *if and only if* $u = v$ *almost everywhere in* Ω.

3. Main Results

Lemma 3.1. *Assume that (G2) holds. Then I is bounded from above on $\mathscr{W}$.*

Proof. It follows from (G2) that there exist positive constants

$$\varepsilon < \min\left\{ \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q} \right\}$$

and $r_0 > 0$ such that

$$G(x,z_1,z_2) \le \varepsilon(|z_1|^p + |z_2|^q), \quad \text{for all } |z_1|+|z_2| > r_0 \text{ and all } x\in\Omega. \tag{3.1}$$

Then (3.1) and the continuity of G imply that there exists a positive constant C_0 such that

$$G(x,z_1,z_2) \le \varepsilon(|z_1|^p + |z_2|^q) + C_0, \quad \text{for all } (x,z_1,z_2)\in\Omega\times\mathbb{R}\times\mathbb{R}. \tag{3.2}$$

Hence, by $(\mathscr{M})$, $(\mathscr{A})$, (3.2), and Lemma 2.1, for all $u\in\mathscr{W}$, we have

$$\begin{aligned}
I(u) &= \varphi(u_1,u_2)+\psi(u_1,u_2)\\
&= -\frac{1}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1|^p dx\right) - \frac{1}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2|^q dx\right)\\
&\quad -\frac{1}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1|^p dx\right) - \frac{1}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2|^q dx\right) + \int_\Omega G(x,u_1,u_2)dx\\
&\le -\frac{m_*^{p-1}}{p}\int_\Omega |\nabla u_1|^p dx - \frac{m_*^{q-1}}{q}\int_\Omega |\nabla u_2|^q dx\\
&\quad -\frac{m_*^{p-1}}{p}\int_\Omega a_1(x)|u_1|^p dx - \frac{m_*^{q-1}}{q}\int_\Omega a_2(x)|u_2|^q dx\\
&\quad +\varepsilon\int_\Omega |u_1|^p dx + \varepsilon\int_\Omega |u_2|^q dx + C_0\Omega\\
&\le -\frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}\int_\Omega |u_1|^p dx - \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q}\int\limits_\Omega |u_2|^q dx\\
&\quad +\varepsilon\int_\Omega |u_1|^p dx + \varepsilon\int_\Omega |u_2|^q dx + C_0\Omega.
\end{aligned} \tag{3.3}$$

Note that

$$\varepsilon < \min\left\{ \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q} \right\}.$$

Hence, (3.3) implies the conclusion holds. □

Lemma 3.2. *Assume that (G2)$'$ holds. Then I is bounded from above on $\mathscr{W}$.*

Proof. It follows from (G2)$'$ that there exist positive constants

$$\varepsilon' < \min\left\{ \frac{\min\{m_*, m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)\}}{p\tau_{p,p}}, \frac{\min\{m_*, m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)\}}{q\tau_{q,q}} \right\}$$

and $r_0' > 0$ such that

$$G(x,z_1,z_2) \le \varepsilon'(|z_1|^p + |z_2|^q), \quad \text{for all } |z_1| + |z_2| > r_0' \text{ and all } x \in \Omega. \tag{3.4}$$

Then (3.4) and the continuity of G imply that there exists a positve constant C_0' such that

$$G(x,z_1,z_2) \le \varepsilon'(|z_1|^p + |z_2|^q) + C_0', \quad \text{for all } (x,z_1,z_2) \in \Omega \times \mathbb{R} \times \mathbb{R}. \tag{3.5}$$

Hence, by $(\mathscr{M})$, $(\mathscr{A})$, (3.5), and Lemma 2.1, for all $u \in \mathscr{W}$, we have

$$\begin{aligned}
I(u) = &-\frac{1}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1|^p dx\right) - \frac{1}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2|^q dx\right) \\
&- \frac{1}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1|^p dx\right) - \frac{1}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2|^q dx\right) + \int_\Omega G(x,u_1,u_2)dx \\
\le &-\frac{m_*^{p-1}}{p}\int_\Omega |\nabla u_1|^p dx - \frac{m_*^{q-1}}{q}\int_\Omega |\nabla u_2|^q dx \\
&- \frac{m_*^{p-1}}{p}\int_\Omega a_1(x)|u_1|^p dx - \frac{m_*^{q-1}}{q}\int_\Omega a_2(x)|u_2|^q dx \\
&+ \varepsilon'\int_\Omega |u_1|^p dx + \varepsilon'\int_\Omega |u_2|^q dx + C_0'\Omega \\
\le &-\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}\right\}\|u_1\|_{1,p}^p - \min\left\{\frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\}\|u_2\|_{1,q}^q \\
&+ \varepsilon'\tau_{p,p}^p\|u_1\|_{1,p}^p + \varepsilon'\tau_{q,q}^q\|u_2\|_{1,q}^q + C_0'\Omega.
\end{aligned} \tag{3.6}$$

Note that

$$\varepsilon' < \min\left\{\frac{\min\{m_*, m_*^{p-1}\inf_{x\in\Omega} a_1(x)\}}{p\tau_{p,p}}, \frac{\min\{m_*, m_*^{q-1}\inf_{x\in\Omega} a_2(x)\}}{q\tau_{q,q}}\right\}.$$

Hence, (3.6) implies the conclusion holds. □

Lemma 3.3. *Assume that (G3) holds. Then, for each finite dimensional subspace $\tilde{\mathscr{W}} \subset \mathscr{W}$, there exist positive constants $\rho = \rho(\tilde{\mathscr{W}})$ and $\sigma = \sigma(\tilde{\mathscr{W}})$ such that $I \ge 0$ on $B_\rho \cap \tilde{\mathscr{W}}$ and $I|_{\partial B_\rho \cap \tilde{\mathscr{W}}} \ge \sigma$.*

Proof. Since $\tilde{\mathscr{W}}$ is finite dimensional, all norms on $\tilde{\mathscr{W}}$ are equivalent. Hence there exist $d_i = d_i(\tilde{\mathscr{W}}) > 0, i = 1,2,3,4$ such that, for all $u = (u_1,u_2) \in \tilde{\mathscr{W}}$,

$$d_1\|u_1\|_{1,p} \le \left(\int_\Omega |u_1|^{\gamma_1} dx\right)^{1/\gamma_1} \le d_2\|u_1\|_{1,p}$$

and

$$d_3\|u_2\|_{1,q} \le \left(\int_\Omega |u_2|^{\gamma_2} dx\right)^{1/\gamma_2} \le d_4\|u_2\|_{1,q}.$$

It follows from (G3) that

$$\begin{aligned}
I(u) &= -\frac{1}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1|^p dx\right) - \frac{1}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2|^q dx\right)\\
&\quad - \frac{1}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1|^p dx\right) - \frac{1}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2|^q dx\right) + \int_\Omega G(x,u_1,u_2)dx\\
&\geq -\frac{(m^*)^{p-1}}{p}\int_\Omega |\nabla u_1|^p dx - \frac{(m^*)^{q-1}}{q}\int_\Omega |\nabla u_2|^q dx\\
&\quad - \frac{(m^*)^{p-1}}{p}\int_\Omega a_1(x)|u_1|^p dx - \frac{(m^*)^{q-1}}{q}\int_\Omega a_2(x)|u_2|^q dx\\
&\quad + C^*\int_\Omega |u_1|^{\gamma_1} dx + C^*\int_\Omega |u_2|^{\gamma_2} dx\\
&\geq -\frac{(m^*)^{p-1}}{p}\left(1+\max_{x\in\bar{\Omega}} a_1(x)\right)\left(\int_\Omega |\nabla u_1|^p dx + \int_\Omega |u_1|^p dx\right)\\
&\quad - \frac{(m^*)^{q-1}}{q}\left(1+\max_{x\in\bar{\Omega}} a_2(x)\right)\left(\int_\Omega |\nabla u_2|^q dx + \int_\Omega |u_2|^q dx\right)\\
&\quad + C^* d_1^{\gamma_1}\|u_1\|_{1,p}^{\gamma_1} + C^* d_3^{\gamma_2}\|u_2\|_{1,q}^{\gamma_2}\\
&= -\frac{(m^*)^{p-1}}{p}\left(1+\max_{x\in\bar{\Omega}} a_1(x)\right)\|u_1\|_{1,p}^p - \frac{(m^*)^{q-1}}{q}\left(1+\max_{x\in\bar{\Omega}} a_2(x)\right)\|u_2\|_{1,q}^q\\
&\quad + C^* d_1^{\gamma_1}\|u_1\|_{1,p}^{\gamma_1} + C^* d_3^{\gamma_1}\|u_2\|_{1,q}^{\gamma_2}\\
&= \left[C^* d_1^{\gamma_1} - \frac{(m^*)^{p-1}}{p}\left(1+\max_{x\in\bar{\Omega}} a_1(x)\right)\|u_1\|_{1,p}^{p-\gamma_1}\right]\|u_1\|_{1,p}^{\gamma_1}\\
&\quad + \left[C^* d_3^{\gamma_2} - \frac{(m^*)^{q-1}}{q}\left(1+\max_{x\in\bar{\Omega}} a_2(x)\right)\|u_2\|_{1,q}^{q-\gamma_2}\right]\|u_2\|_{1,q}^{\gamma_2}.
\end{aligned}$$

Let

$$\rho = \frac{1}{2}\min\left\{1, \left(\frac{pC^* d_1^{\gamma_1}}{(m^*)^{p-1}\left(1+\max_{x\in\bar{\Omega}} a_1(x)\right)}\right)^{1/(p-\gamma_1)}, \left(\frac{qC^* d_3^{\gamma_2}}{(m^*)^{q-1}\left(1+\max_{x\in\bar{\Omega}} a_2(x)\right)}\right)^{1/(q-\gamma_2)}\right\}.$$

Then $\rho \in (0,1)$ and $\|u_1\|_{1,p} < 1$ and $\|u_2\|_{1,q} < 1$ for all $u \in B_\rho \cap \tilde{\mathscr{W}}$. Thus

$$\begin{aligned}
I(u) \geq \frac{C^* d_1^{\gamma_1}}{2}\|u_1\|_{1,p}^{\gamma_1} + \frac{C^* d_3^{\gamma_2}}{2}\|u_2\|_{1,q}^{\gamma_2} &\geq C_1(\|u_1\|_{1,p}^{\gamma_1} + \|u_2\|_{1,q}^{\gamma_2})\\
&\geq C_1\left(\|u_1\|_{1,p}^{\max\{\gamma_1,\gamma_2\}} + \|u_2\|_{1,q}^{\max\{\gamma_1,\gamma_2\}}\right)\\
&\geq 2^{-\max\{\gamma_1,\gamma_2\}} C_1(\|u_1\|_{1,p} + \|u_2\|_{1,q})^{\max\{\gamma_1,\gamma_2\}}\\
&= 2^{-\max\{\gamma_1,\gamma_2\}} C_1\|u\|^{\max\{\gamma_1,\gamma_2\}}\\
&\geq 0, \quad \forall\, u \in B_\rho \cap \tilde{\mathscr{W}}, \qquad (3.7)
\end{aligned}$$

where $C_1 = \min\{\frac{C^* d_1^{\gamma_1}}{2}, \frac{C^* d_3^{\gamma_2}}{2}\}$. Let $\sigma = 2^{-\max\{\gamma_1,\gamma_2\}} C_1 \rho^{\max\{\gamma_1,\gamma_2\}}$. Therefore, from (3.7), it is easy to see that $I|_{\partial B_\rho \cap \tilde{\mathscr{W}}} \geq \sigma > 0$. This completes the proof. □

Lemma 3.4. *Assume that* $(G2)$ *and (G4) hold. Then* I *satisfies the (C)-condition.*

Proof. For every sequence $\{u^{[n]}\}_{n\in\mathbb{N}} = \{(u_1^{[n]}, u_2^{[n]})\}_{n\in\mathbb{N}} \subset \mathscr{W}$, assume that $\{I(u^{[n]})\}$ is bounded and $\left(1+\|u^{[n]}\|\right)\|I'(u^{[n]})\| \to 0$, as $n\to\infty$. Then there exists a constant $C_2 > 0$ such that

$$|I(u^{[n]})| = |I(u_1^{[n]}, u_2^{[n]})| \leq C_2, \quad \left(1+\|u^{[n]}\|\right)\|I'(u^{[n]})\| \leq C_2, \ \text{ for all } n \in \mathbb{N}. \tag{3.8}$$

Then, by $(\mathscr{M})$,

$$\begin{aligned}
&(\theta+1)C_2 \\
&\geq \theta I(u^{[n]}) - \langle I'(u^{[n]}), u^{[n]}\rangle \\
&= -\frac{\theta}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1^{[n]}|^p dx\right) - \frac{\theta}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2^{[n]}|^q dx\right) - \frac{\theta}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1^{[n]}|^p dx\right) \\
&\quad - \frac{\theta}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2^{[n]}|^q dx\right) + \theta\int_\Omega G(x,u_1^{[n]},u_2^{[n]})dx \\
&\quad + \left[M_1\left(\int_\Omega |\nabla u_1^{[n]}|^p dx\right)\right]^{p-1}\int_\Omega |\nabla u_1^{[n]}|^p dx + \left[M_2\left(\int_\Omega |\nabla u_2^{[n]}|^q dx\right)\right]^{q-1}\int_\Omega |\nabla u_2^{[n]}|^q dx \\
&\quad + \left[M_3\left(\int_\Omega a_1(x)|u_1^{[n]}|^p dx\right)\right]^{p-1}\int_\Omega a_1(x)|u_1^{[n]}|^p dx \\
&\quad + \left[M_4\left(\int_\Omega a_2(x)|u_2^{[n]}|^q dx\right)\right]^{q-1}\int_\Omega a_2(x)|u_2^{[n]}|^q dx \\
&\quad - \int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]}dx - \int_\Omega G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}dx \\
&\geq -\frac{\theta(m^*)^{p-1}}{p}\int_\Omega |\nabla u_1^{[n]}|^p dx - \frac{\theta(m^*)^{q-1}}{q}\int_\Omega |\nabla u_2^{[n]}|^q dx - \frac{\theta(m^*)^{p-1}\max\limits_{x\in\bar\Omega} a_1(x)}{p}\int_\Omega |u_1^{[n]}|^p dx \\
&\quad - \frac{\theta(m^*)^{q-1}\max\limits_{x\in\bar\Omega} a_2(x)}{q}\int_\Omega |u_2^{[n]}|^q dx + \theta\int_\Omega G(x,u_1^{[n]},u_2^{[n]})dx + m_*^{p-1}\int_\Omega |\nabla u_1^{[n]}|^p dx \\
&\quad + m_*^{q-1}\int_\Omega |\nabla u_2^{[n]}|^q dx + m_*^{p-1}\min_{x\in\bar\Omega} a_1(x)\int_\Omega |u_1^{[n]}|^p dx \\
&\quad + m_*^{q-1}\min_{x\in\bar\Omega} a_2(x)\int_\Omega |u_2^{[n]}|^q dx - \int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]}dx - \int_\Omega G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}dx \\
&\geq \theta\int_\Omega G(x,u_1^{[n]},u_2^{[n]})dx - \int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]}dx - \int_\Omega G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}dx.
\end{aligned} \tag{3.9}$$

Next, we prove that $\{u^{[n]}\} = \{(u_1^{[n]}, u_2^{[n]})\}$ is bounded. Assume that

$$\|u^{[n]}\| = \|u_1^{[n]}\|_{1,p} + \|u_2^{[n]}\|_{1,q} \to \infty, \ \text{ as } n \to \infty. \tag{3.10}$$

Let

$$z^{[n]} = (z_1^{[n]}, z_2^{[n]}) = \left(\frac{u_1^{[n]}}{\|u_1^{[n]}\|_{1,p}}, \frac{u_2^{[n]}}{\|u_2^{[n]}\|_{1,q}} \right).$$

Then $\|z_1^{[n]}\|_{1,p} = 1$, $\|z_2^{[n]}\|_{1,q} = 1$ and hence $\|z^{[n]}\| = 2$, where

$$z_1^{[n]} = \frac{u_1^{[n]}}{\|u_1^{[n]}\|_{1,p}} \text{ and } z_2^{[n]} = \frac{u_2^{[n]}}{\|u_2^{[n]}\|_{1,q}}.$$

So there exist subsequences, still denoted by $\{z_1^{[n]}\}$ and $\{z_2^{[n]}\}$, such that $z_1^{[n]} \rightharpoonup z_1$ and $z_2^{[n]} \rightharpoonup z_2$ on $W_0^{1,p}$ and $W_0^{1,q}$, respectively. Then by Lemma 2.1, we have

$$z_1^{[n]} \to z_1 \text{ in } L^r \text{ for } p \le r < \frac{pN}{N-p} \text{ and a.e. } x \in \Omega \tag{3.11}$$

and

$$z_2^{[n]} \to z_2 \text{ in } L^r \text{ for } q \le r < \frac{qN}{N-q} \text{ and a.e. } x \in \Omega. \tag{3.12}$$

It follows by (3.2) that

$$\begin{aligned}
&I(u^{[n]}) \\
&= -\frac{1}{p}\hat{M}_1\left(\int_\Omega |\nabla u_1^{[n]}|^p dx\right) - \frac{1}{q}\hat{M}_2\left(\int_\Omega |\nabla u_2^{[n]}|^q dx\right) \\
&- \frac{1}{p}\hat{M}_3\left(\int_\Omega a_1(x)|u_1^{[n]}|^p dx\right) - \frac{1}{q}\hat{M}_4\left(\int_\Omega a_2(x)|u_2^{[n]}|^q dx\right) + \int_\Omega G(x,u_1^{[n]},u_2^{[n]})dx \\
&\le -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}\right\}\|u_1^{[n]}\|_{1,p}^p - \min\left\{\frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\}\|u_2^{[n]}\|_{1,q}^q \\
&+ \varepsilon\int_\Omega |u_1^{[n]}|^p dx + \varepsilon\int_\Omega |u_2^{[n]}|^q dx + C_0 \text{meas}\Omega \\
&\le -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\}\left(\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^q\right) \\
&+ \varepsilon\int_\Omega |u_1^{[n]}|^p dx + \varepsilon\int_\Omega |u_2^{[n]}|^q dx + C_0 \text{meas}\Omega.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{I(u^{[n]})}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^q} &\le -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\} \\
&\quad + \frac{\varepsilon\int_\Omega |u_1^{[n]}|^p dx}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^q} + \frac{\varepsilon\int_\Omega |u_2^{[n]}|^q dx}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^q} \\
&\quad + \frac{C_0 \text{meas}\Omega}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^q}
\end{aligned}$$

$$
\begin{aligned}
&\le -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q}\right\} \\
&\quad + \frac{\varepsilon\int_\Omega |u_1^{[n]}|^p dx}{\|u_1^{[n]}\|_{1,p}^p} + \frac{\varepsilon\int_\Omega |u_2^{[n]}|^q dx}{\|u_2^{[n]}\|_{1,q}^q} + \frac{C_0 \text{meas}\Omega}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^p} \\
&= -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q}\right\} \\
&\quad + \varepsilon\int_\Omega |z_1^{[n]}|^p dx + \varepsilon\int_\Omega |z_2^{[n]}|^q dx + \frac{C_0 \text{meas}\Omega}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^p} \\
&= -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q}\right\} \\
&\quad + \varepsilon\int_\Omega |z_1^{[n]} - z_1 + z_1|^p dx + \varepsilon\int_\Omega |z_2^{[n]} - z_2 + z_2|^q dx + \frac{C_0 \text{meas}\Omega}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^p} \\
&\le -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q}\right\} \\
&\quad + 2^{p-1}\varepsilon\int_\Omega |z_1^{[n]} - z_1|^p dx + 2^{p-1}\varepsilon\int_\Omega |z_1|^p dx + 2^{q-1}\varepsilon\int_\Omega |z_2^{[n]} - z_2|^q dx \\
&\quad + 2^{q-1}\varepsilon\int_\Omega |z_2|^q dx + \frac{C_0 \text{meas}\Omega}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^p}.
\end{aligned} \tag{3.13}
$$

In view of (3.8), (3.11), (3.12), and (3.13), we have

$$
\begin{aligned}
&\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf\limits_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf\limits_{x\in\Omega} a_2(x)}{q}\right\} \\
&\le (2^{p-1} + 2^{q-1})\varepsilon\left(\int_\Omega |z_1|^p dx + \int_\Omega |z_2|^p dx\right).
\end{aligned}
$$

Then it follows that $\int_\Omega |z_1|^p dx + \int_\Omega |z_2|^q dx > 0$, so $z = (z_1, z_2) \neq 0$.

Next, we claim that the following three conclusions hold:

(i) $\|u_1^{[n]}\|_{1,p} \to \infty$ if $z_1 \neq 0$ and $z_2 \equiv 0$;

(ii) $\|u_2^{[n]}\|_{1,q} \to \infty$ if $z_2 \neq 0$ and $z_1 \equiv 0$;

(iii) $\|u_1^{[n]}\|_{1,p} \to \infty$ or $\|u_2^{[n]}\|_{1,q} \to \infty$ if $z_1 \neq 0$ and $z_2 \neq 0$.

Indeed, for conclusion (i), if $\|u_1^{[n]}\|_{1,p}$ is bounded, then $\int_\Omega |u_1^{[n]}|^p dx$ is also bounded. Hence, by

(3.10), we have

$$\begin{aligned} 0 &\leq -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\} + \frac{\varepsilon\int_\Omega |u_2^{[n]}|^q dx}{\|u_1^{[n]}\|_{1,p}^p + \|u_2^{[n]}\|_{1,q}^q} \\ &\leq -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\} + \frac{\varepsilon\int_\Omega |u_2^{[n]}|^q dx}{\|u_2^{[n]}\|_{1,q}^q}, \end{aligned}$$

which implies that $\int_\Omega |z_2|^q dx > 0$. This contradicts $z_2 \equiv 0$. Similar to the argument of conclusion (i), it is easy to obtain conclusion (ii). For conclusion (iii), if $\|u_1^{[n]}\|_{1,p}$ and $\|u_2^{[n]}\|_{1,p}$ are bounded, then $\int_\Omega |u_1^{[n]}|^p dx$ and $\int_\Omega |u_2^{[n]}|^p dx$ are also bounded. Hence, (3.10) yields

$$0 \leq -\min\left\{\frac{m_*^{p-1}}{p}, \frac{m_*^{p-1}\inf_{x\in\Omega} a_1(x)}{p}, \frac{m_*^{q-1}}{q}, \frac{m_*^{q-1}\inf_{x\in\Omega} a_2(x)}{q}\right\},$$

which is a contradiction.

Let

$$S = \{x \in \Omega : \lim_{|z_1|+|z_2|\to\infty} [\theta G(x,z_1,z_2) - G_{z_1}(x,z_1,z_2)z_1 - G_{z_2}(x,z_1,z_2)z_2] = +\infty\}$$

and

$$S_1 = \{x \in \Omega : z(x) = (z_1(x), z_2(x)) \neq 0\}.$$

Then by (G4), we have meas $S > 0$. Then conclusion (i), (ii), and (iii) imply that

$$\lim_{n\to\infty}(|u_1^{[n]}(x)| + |u_2^{[n]}(x)|) = \lim_{n\to\infty}(|z_1^{[n]}(x)|\cdot\|u_1^{[n]}\|_{1,p} + |z_2^{[n]}(x)|\cdot\|u_2^{[n]}\|_{1,q}) = +\infty \tag{3.14}$$

for $x \in S_1$. Let

$$J_n(x) = \theta G(x, u_1^{[n]}(x), u_2^{[n]}(x)) - G_{z_1}(x, u_1^{[n]}(x), u_2^{[n]}(x))u_1^{[n]}(x) - G_{z_2}(x, u_1^{[n]}(x), u_2^{[n]}(x))u_2^{[n]}(x).$$

Then (3.14) and (G4) imply that

$$\lim_{n\to\infty} J_n(x) = +\infty \ \text{ for } x \in S_1. \tag{3.15}$$

It follows from (3.15) and [27, Lemma 1] that there exists a subset S_2 of S_1 with meas $S_2 > 0$ such that

$$\lim_{n\to\infty} J_n(x) = +\infty \ \text{ uniformly for } x \in S_2. \tag{3.16}$$

By (G4), we have

$$\begin{aligned} &\theta\int_\Omega G(x,u_1^{[n]},u_2^{[n]})dx - \int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]}dx - \int_\Omega G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}dx \\ = &\int_{S_2}[\theta G(x,u_1^{[n]},u_2^{[n]}) - G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]} - G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}]dx \\ &+ \int_{\Omega/S_2}[\theta G(x,u_1^{[n]},u_2^{[n]}) - G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]} - G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}]dx \\ \geq &\int_{S_2}[\theta G(x,u_1^{[n]},u_2^{[n]}) - G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]} - G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}]dx + \int_{\Omega/S_2} h(t)dt. \end{aligned}$$

From Fatou's lemma and (3.16), we have

$$\int_\Omega [\theta G(x,u_1^{[n]},u_2^{[n]}) - G_{z_1}(x,u_1^{[n]},u_2^{[n]})u_1^{[n]} - G_{z_2}(x,u_1^{[n]},u_2^{[n]})u_2^{[n]}]dx \to +\infty$$

which contradicts (3.9). Hence $\{u^{[n]}\} = \{(u_1^{[n]},u_2^{[n]})\}$ is bounded, so $\{u_1^{[n]}\}$ and $\{u_2^{[n]}\}$ are also bounded in $W_0^{1,p}$ and $W_0^{1,q}$, respectively. We may assume

$$u_1^{[n]} \rightharpoonup u_1 \text{ in } W_0^{1,p} \text{ and } u_2^{[n]} \rightharpoonup u_2 \text{ in } W_0^{1,q}. \tag{3.17}$$

Lemma 2.1 yields

$$u_1^{[n]} \to u_1 \text{ in } L^p(\Omega) \quad \text{and } u_2^{[n]} \to u_2 \text{ in } L^q(\Omega). \tag{3.18}$$

By (2.1), we have

$$\begin{aligned}
& \langle I'(u_1^{[n]},u_2^{[n]}),(u_1^{[n]}-u_1,u_2^{[n]}-u_2)\rangle \\
= & -\left[M_1\left(\int_\Omega |\nabla u_1^{[n]}|^p dx\right)\right]^{p-1}\int_\Omega |\nabla u_1^{[n]}|^{p-2}(\nabla u_1^{[n]},\nabla u_1^{[n]}-\nabla u_1)dx \\
& -\left[M_2\left(\int_\Omega |\nabla u_2^{[n]}|^q dx\right)\right]^{q-1}\int_\Omega |\nabla u_2^{[n]}|^{q-2}(\nabla u_2^{[n]},\nabla u_2^{[n]}-\nabla u_2)dx \\
& -\left[M_3\left(\int_\Omega a_1(x)|u_1^{[n]}|^p dx\right)\right]^{p-1}\int_\Omega a_1(x)|u_1^{[n]}|^{p-2}(u_1^{[n]},u_1^{[n]}-u_1)dx \\
& -\left[M_4\left(\int_\Omega a_2(x)|u_2^{[n]}|^q dx\right)\right]^{q-1}\int_\Omega a_2(x)|u_2^{[n]}|^{q-2}(u_2^{[n]},u_2^{[n]}-u_2)dx \\
& +\int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})(u_1^{[n]}-u_1)dx + \int_\Omega G_{z_2}(x,u_1^{[n]},u_2^{[n]})(u_2^{[n]}-u_2)dx.
\end{aligned} \tag{3.19}$$

The boundedness of $\{u_1^{[n]}\}$ and $\{u_2^{[n]}\}$ and $I'(u_1^{[n]},u_2^{[n]}) \to 0$ as $n \to \infty$ imply that

$$|\langle I'(u_1^{[n]},u_2^{[n]}),(u_1^{[n]}-u_1,u_2^{[n]}-u_2)\rangle| \le \|I'(u_1^{[n]},u_2^{[n]})\|\|(u_1^{[n]}-u_1,u_2^{[n]}-u_2)\| \to 0 \tag{3.20}$$

as $n \to \infty$. Moreover, by (G1) and Hölder inequality, we have

$$\begin{aligned}
& \left|\int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})(u_1^{[n]}-u_1)dx\right| \\
\le & \int_\Omega c_1(1+|u_1^{[n]}|^{s_1})|u_1^{[n]}-u_1|dx \\
\le & \left(\int_\Omega c_1^{p'}(1+|u_1^{[n]}|^{s_1})^{p'}ds\right)^{1/p'}\left(\int_\Omega |u_1^{[n]}-u_1|^p dx\right)^{1/p} \\
\le & 2^{\frac{p'-1}{p'}}c_1\left(\int_\Omega (1+|u_1^{[n]}|^{p's_1})dx\right)^{1/p'}\left(\int_\Omega |u_1^{[n]}-u_1|^p dx\right)^{1/p}.
\end{aligned} \tag{3.21}$$

Since $p's_1 \le \frac{pN}{N-p}$, where $p' > 1$ satisfies $\frac{1}{p'}+\frac{1}{p} = 1$, then Lemma 2.1 and the boundedness of $\{u_1^{[n]}\}$ imply that $\int_\Omega |u_1^{[n]}|^{p's_1}dx$ is bounded. Thus by (3.18) and (3.21), we obtain that

$$\int_\Omega G_{z_1}(x,u_1^{[n]},u_2^{[n]})(u_1^{[n]}-u_1)dx \to 0, \quad \text{as } n \to \infty. \tag{3.22}$$

Similarly, noting that $q's_2 \le \frac{qN}{N-q}$, where $q' > 1$ satisfies $\frac{1}{q'} + \frac{1}{q} = 1$, we obtain that

$$\int_\Omega G_{z_2}(x, u_1^{[n]}, u_2^{[n]})(u_2^{[n]} - u_2)dx \to 0, \quad \text{as } n \to \infty. \tag{3.23}$$

Note that $a_1 \in C(\bar{\Omega}, \mathbb{R})$. Then by $(\mathscr{M})$ and Hölder inequality, we have

$$\begin{aligned}
&\left| \left[M_3 \left(\int_\Omega a_1(x)|u_1^{[n]}|^p dx \right) \right]^{p-1} \int_\Omega a_1(x)|u_1^{[n]}|^{p-2}(u_1^{[n]}, u_1^{[n]} - u_1)dx \right| \\
\le\ & (m^*)^{p-1} \max_{x\in\bar{\Omega}} a_1(x) \int_\Omega |u_1^{[n]}|^{p-1}|u_1^{[n]} - u_1|dx \\
\le\ & (m^*)^{p-1} \max_{x\in\bar{\Omega}} a_1(x) \left(\int_\Omega |u_1^{[n]}|^p dx \right)^{1/p'} \left(\int_\Omega |u_1^{[n]} - u_1|^p dx \right)^{1/p}.
\end{aligned}$$

Then Lemma 2.1 and (3.18) imply that

$$\left| \left[M_3 \left(\int_\Omega a_1(x)|u_1^{[n]}|^p dx \right) \right]^{p-1} \int_\Omega a_1(x)|u_1^{[n]}|^{p-2}(u_1^{[n]}, u_1^{[n]} - u_1)dx \right| \to 0, \quad \text{as } n \to \infty. \tag{3.24}$$

Similarly, we also have

$$\left| \left[M_4 \left(\int_\Omega a_2(x)|u_2^{[n]}|^q dx \right) \right]^{q-1} \int_\Omega a_2(x)|u_2^{[n]}|^{q-2}(u_2^{[n]}, u_2^{[n]} - u_2)dx \right| \to 0, \quad \text{as } n \to \infty. \tag{3.25}$$

Thus, (3.19), (3.20), (3.22), (3.23), (3.24), and (3.25) imply that

$$\begin{aligned}
&\left[M_1 \left(\int_\Omega |\nabla u_1^{[n]}|^p dx \right) \right]^{p-1} \int_\Omega |\nabla u_1^{[n]}|^{p-2}(\nabla u_1^{[n]}, \nabla u_1^{[n]} - \nabla u_1)dx \\
&+ \left[M_2 \left(\int_\Omega |\nabla u_2^{[n]}|^q dx \right) \right]^{q-1} \int_\Omega |\nabla u_2^{[n]}|^{q-2}(\nabla u_2^{[n]}, \nabla u_2^{[n]} - \nabla u_2)dx \\
&\to 0, \text{ as } n \to \infty.
\end{aligned} \tag{3.26}$$

Then by $(\mathscr{M})$, we have

$$\begin{aligned}
&\frac{1}{\max\{(m^*)^{p-1}, (m^*)^{q-1}\}} \left[M_1 \left(\int_\Omega |\nabla u_1^{[n]}|^p dx \right) \right]^{p-1} \int_\Omega |\nabla u_1^{[n]}|^{p-2}(\nabla u_1^{[n]}, \nabla u_1^{[n]} - \nabla u_1)dx \\
&+ \frac{1}{\max\{(m^*)^{p-1}, (m^*)^{q-1}\}} \left[M_2 \left(\int_\Omega |\nabla u_2^{[n]}|^q dx \right) \right]^{q-1} \int_\Omega |\nabla u_2^{[n]}|^{q-2}(\nabla u_2^{[n]}, \nabla u_2^{[n]} - \nabla u_2)dx \\
&\le \int_\Omega |\nabla u_1^{[n]}|^{p-2}(\nabla u_1^{[n]}, \nabla u_1^{[n]} - \nabla u_1)dx + \int_\Omega |\nabla u_2^{[n]}|^{q-2}(\nabla u_2^{[n]}, \nabla u_2^{[n]} - \nabla u_2)dx \\
&\le \frac{1}{\min\{(m_*)^{p-1}, (m_*)^{q-1}\}} \left[M_1 \left(\int_\Omega |\nabla u_1^{[n]}|^p dx \right) \right]^{p-1} \int_\Omega |\nabla u_1^{[n]}|^{p-2}(\nabla u_1^{[n]}, \nabla u_1^{[n]} - \nabla u_1)dx \\
&+ \frac{1}{\min\{(m_*)^{p-1}, (m_*)^{q-1}\}} \left[M_2 \left(\int_\Omega |\nabla u_2^{[n]}|^q dx \right) \right]^{q-1} \int_\Omega |\nabla u_2^{[n]}|^{q-2}(\nabla u_2^{[n]}, \nabla u_2^{[n]} - \nabla u_2)dx.
\end{aligned}$$

which together with (3.26) implies that

$$\int_\Omega |\nabla u_1^{[n]}|^{p-2}(\nabla u_1^{[n]}, \nabla u_1^{[n]} - \nabla u_1)dx + \int_\Omega |\nabla u_2^{[n]}|^{q-2}(\nabla u_2^{[n]}, \nabla u_2^{[n]} - \nabla u_2)dx \to 0 \tag{3.27}$$

as $n \to \infty$. Moreover, by Hölder inequality, we have

$$\begin{aligned}
&\left| \int_\Omega |u_1^{[n]}|^{p-2}(u_1^{[n]}, u_1^{[n]} - u_1)dx + \int_\Omega |u_2^{[n]}|^{q-2}(u_2^{[n]}, u_2^{[n]} - u_2)dx \right| \\
&\leq \int_\Omega |u_1^{[n]}|^{p-1}|u_1^{[n]} - u_1|dx + \int_\Omega |u_2^{[n]}|^{q-1}||u_2^{[n]} - u_2|dx \\
&\leq \left(\int_\Omega |u_1^{[n]}|^p dx\right)^{\frac{1}{p'}} \left(\int_\Omega |u_1^{[n]} - u_1|^p dx\right)^{\frac{1}{p}} + \left(\int_\Omega |u_2^{[n]}|^q dx\right)^{\frac{1}{q'}} \left(\int_\Omega |u_2^{[n]} - u_2|^q dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Then (3.18) implies that

$$\int_\Omega |u_1^{[n]}|^{p-2}(u_1^{[n]}, u_1^{[n]} - u_1)dx + \int_\Omega |u_2^{[n]}|^{q-2}(u_2^{[n]}, u_2^{[n]} - u_2)dx \to 0, \text{ as } n \to \infty. \tag{3.28}$$

Set

$$\begin{aligned}
J(u) &= \frac{1}{p}\left(\int_\Omega |u_1|^p dx + \int_\Omega |\nabla u_1|^p dx\right) + \frac{1}{q}\left(\int_\Omega |u_2|^q dx + \int_\Omega |\nabla u_2|^q dx\right) \\
&= J_1(u_1) + J_2(u_2),
\end{aligned}$$

where

$$\begin{aligned}
J_1(u_1) &= \frac{1}{p}\left(\int_\Omega |u_1|^p dx + \int_\Omega |\nabla u_1|^p dx\right), \quad \forall u_1 \in W_0^{1,p}, \\
J_2(u_2) &= \frac{1}{q}\left(\int_\Omega |u_2|^q dx + \int_\Omega |\nabla u_2|^q dx\right), \quad \forall u_2 \in W_0^{1,q}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\langle J'(u^{[n]}), u^{[n]} - u\rangle &= \langle J'(u_1^{[n]}, u_2^{[n]}), (u_1^{[n]} - u_1, u_2^{[n]} - u_2)\rangle \\
&= \int_\Omega (|u_1^{[n]}|^{p-2}u_1^{[n]}, u_1^{[n]} - u_1)dx + \int_\Omega (|\nabla u_1^{[n]}|^{p-2}\nabla u_1^{[n]}, \nabla u_1^{[n]} - \nabla u_1)dx \\
&\quad + \int_\Omega (|u_2^{[n]}|^{q-2}u_2^{[n]}, u_2^{[n]} - u_2)dx + \int_\Omega (|\nabla u_2^{[n]}|^{q-2}\nabla u_2^{[n]}, \nabla u_2^{[n]} - \nabla u_2)dx \\
&= \langle J_1'(u_1^{[n]}), u_1^{[n]} - u_1\rangle + \langle J_2'(u_2^{[n]}), u_2^{[n]} - u_2\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle J'(u), u^{[n]} - u\rangle &= \langle J'(u_1, u_2), (u_1^{[n]} - u_1, u_2^{[n]} - u_2)\rangle \\
&= \int_\Omega (|u_1|^{p-2}u_1, u_1^{[n]} - u_1)dx + \int_\Omega (|\nabla u_1|^{p-2}\nabla u_1, \nabla u_1^{[n]} - \nabla u_1)dx \\
&\quad + \int_\Omega (|u_2|^{q-2}u_2, u_2^{[n]} - u_2)dx + \int_\Omega (|\nabla u_2|^{q-2}\nabla u_2, \nabla u_2^{[n]} - \nabla u_2)dx \\
&= \langle J_1'(u_1), u_1^{[n]} - u_1\rangle + \langle J_2'(u_2), u_2^{[n]} - u_2\rangle.
\end{aligned}$$

In view of (3.27) and (3.28), one has that

$$\langle J'(u^{[n]}), u^{[n]} - u\rangle = \langle J_1'(u_1^{[n]}), u_1^{[n]} - u_1\rangle + \langle J_2'(u_2^{[n]}), u_2^{[n]} - u_2\rangle \to 0, \text{ as } n \to \infty \tag{3.29}$$

and (3.17) implies that

$$\langle J'(u), u^{[n]} - u\rangle = \langle J_1'(u_1), u_1^{[n]} - u_1\rangle + \langle J_2'(u_2), u_2^{[n]} - u_2\rangle \to 0, \text{ as } n \to \infty. \tag{3.30}$$

By Lemma 2.6, we know that

$$\langle J_1'(u_1^{[n]}) - J_1'(u_1), u_1^{[n]} - u_1\rangle \geq (\|u_1^{[n]}\|_{1,p}^{p-1} - \|u_1\|_{1,p}^{p-1})(\|u_1^{[n]}\|_{1,p} - \|u_1\|_{1,p}) \tag{3.31}$$

and

$$\langle J_2'(u_2^{[n]}) - J_2'(u_2), u_2^{[n]} - u_2\rangle \geq (\|u_2^{[n]}\|_{1,q}^{q-1} - \|u_2\|_{1,q}^{q-1})(\|u_2^{[n]}\|_{1,q} - \|u_2\|_{1,q}). \tag{3.32}$$

Then it follows from (3.31) and (3.32) that

$$\begin{aligned} 0 &\leq (\|u_1^{[n]}\|_{1,p}^{p-1} - \|u_1\|_{1,p}^{p-1})(\|u_1^{[n]}\|_{1,p} - \|u_1\|_{1,p}) + (\|u_2^{[n]}\|_{1,q}^{q-1} - \|u_2\|_{1,q}^{q-1})(\|u_2^{[n]}\|_{1,q} - \|u_2\|_{1,q}) \\ &\leq \langle J'(u^{[n]}) - J'(u), u^{[n]} - u\rangle, \end{aligned}$$

which, together with (3.29) and (3.30), yields $\|u_1^{[n]}\|_{1,p} \to \|u_1\|_{1,p}$ and $\|u_2^{[n]}\|_{1,q} \to \|u_2\|_{1,q}$ as $n \to \infty$. Note that $W_0^{1,p}$ and $W_0^{1,q}$ are reflexive and uniformly convex (see [1]). Then it follows from Kadec-Klee property (see [16]) that $u_1^{[n]} \to u_1$ strongly in $W_0^{1,p}$ and $u_2^{[n]} \to u_2$ strongly in $W_0^{1,q}$ and so $u^{[n]} \to u$ strongly in $\mathscr{W}$. Thus we have verified that I satisfies the (C)-condition. This completes the proof. □

Lemma 3.5. *Assume that (G2)′ and (G4) hold. Then I satisfies the (C)-condition.*

Proof. The proof is the same as that of Lemma 3.4 by replacing (3.2) with (3.5). □

Proof of Theorem 1.1. Obviously, by (G0), we have $I(0) = 0$ and I is even. Let $E = \mathscr{W}$, $E_1 = E_{p,1}^{(1)} \times E_{q,1}^{(1)}$ and $E_2 = E_{p,1}^{(2)} \times E_{q,1}^{(2)}$. Then E_1 is finite dimensional. Combining Lemma 3.1, Lemma 3.3, and Lemma 3.4, we obtain that system (1.1) has infinitely many nontrivial solutions.

Proof of Theorem 1.2. The proof is the same as that of Theorem 1.1 by replacing Lemma 3.2 with Lemma 3.1 and replacing Lemma 3.4 with Lemma 3.5.

4. Conclusion

This paper studies infinitely many nontrivial solutions for (p,q)-Kirchhoff type elliptic systems via variational methods and Ding's critical point theorem in [14]. The key contributions are summarized as follows:

(1) Unlike previous works (e.g., [12] and [13]) that only obtain two or three weak solutions, we obtain *infinitely many nontrivial solutions* for the (p,q)-Kirchhoff type elliptic system (1.1) without parameters.
(2) Unlike the works in [12] and [13], the nonlinear terms in system (1.1) satisfy the asymptotically-(p,q) condition rather than the sub-(p,q) condition because of the appearance of the following two terms in system (1.1):
$\left[M_3\left(\int_\Omega a_1(x)|u_1|^p dx\right)\right]^{p-1} a_1(x)|u_1|^{p-2}u_1$ and $\left[M_4\left(\int_\Omega a_2(x)|u_2|^q dx\right)\right]^{q-1} a_2(x)|u_2|^{q-2}u_2$.
(3) Our results can be extended to higher-dimensional $(p_1,\ldots,p_n)$-Kirchhoff systems or unbounded domains if more restrictions on $a_i(x), i = 1,2$ are added.

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