J. Nonlinear Funct. Anal. 2026 (2026) 1 https://doi.org/10.23952/jnfa.2026.1

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

EXACT NULL CONTROLLABILITY OF RIEMANN-LIOUVILLE FRACTIONAL DIFFERENTIAL EQUATIONS VIA A RESOLVENT METHOD

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Abstract. In this paper, the control systems governed by Riemann-Liouville fractional differential equations is introduced in Hilbert spaces. First, $C_{1-\alpha}$-mild solution to Riemann-Liouville fractional control system is introduced by means of fractional resolvents. Then by fractional calculus and fixed point theorem, some sufficient conditions are derived to ensure the exact null controllability of Riemann-Liouville fractional differential equations. Finally, an example is presented to illustrate our abstract results.
Keywords. Exact null controllability; Fixed point theorems; Fractional differential equations; Riemann-Liouville derivative.

1. Introduction

The study of fractional differential equations (FDEs) has received much attention in recent years due to their extensive applications in numerous fields such as physics and engineering. Fractional differential equations offers a powerful tool to model complex systems with memory and hereditary properties; see, e.g., [10, 22, 24, 27] for more details. Among the numerous aspects of FDEs, controllability stands out as a crucial topic, particularly in the context of control theory and its applications; see, e.g., [15, 26, 29]. Kalman [17] initiated a systematic investigation on controllability in 1963 when the control theory for time-invariant and time-varying control systems was developed in state-space form. For some contributions on existence and controllability results of differential systems, one can refer to previous studies [1, 14, 20] and the reference therein. Exact null controllability enables to steer the system from initial state to the origin by using a suitable control input within a finite time frame; see, e.g., [3, 7, 12, 21]. The concept is crucial in practical scenarios, such as stabilizing unstable systems or achieving specific targets in control processes. Some studies focused on transforming the controllability

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Received August 26, 2025; Accepted December 15, 2025.

problem into a fixed-point problem by using suitable functional spaces and fixed-point theorems. In [6], Dauer and Balasubramaniam discussed the sufficient conditions for the exact null controllability of integro-differential systems with infinite delay. Sathiyaraj, Feckan, and Wang [25] derived some necessary and sufficient conditions for null controllability of linear stochastic delay systems by using perturbed controllability matrix and rank correlation method. These approaches allow for the application of abstract results from functional analysis to the existence of control functions that achieve exact null controllability. For fractional differential equations, exact null controllability introduces additional challenges due to the nonlocal nature of fractional derivatives, which makes the analysis and design of controllers more complex. Authors of [8, 13] studied the exact null controllability results of some kinds of fractional differential systems with Caputo derivative. In particular, several approaches such as semigroup theory were proposed to tackle this problem, via some techniques from functional analysis, operator theory, and control theory. Unlike classical differential equations, fractional derivatives introduce a memory effect that spans over the entire history of the system. Then One of the key challenges in studying exact null controllability of FDEs lies in the representation and analysis of solutions to the corresponding control system.

To the best of our knowledge, the study on exact null controllability of Riemann-Liouville fractional control systems based on fractional resolvents is still an untreated issue in the literature. In order to fill this gap, we, in this paper, focus on the exact null controllability of the following semilinear differential equations with Riemann-Liouville fractional derivative:

$$\begin{cases} D^{\alpha}x(t) = Ax(t) + f(t,x(t)) + Bu(t),\ t \in J' := (0,b], \\ \lim_{t\to 0^+} \Gamma(\alpha)t^{1-\alpha}x(t) = x_0, \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha < 1$, $A : D(A) \subseteq X \to X$ is a linear operator and generates an $\alpha-$order fractional resolvent $\{P_\alpha(t),\ t > 0\}$ in a Hilbert space X, B is a linear bounded operator from a Hilbert space U into X, $f : J \times X \to X$, $J := [0,b]$, and the control function $u(\cdot)$ is given in $L^2(J,U)$.

By means of fractional resolvents for Caputo fractional differential equations, authors of [5, 28] investigated the nonlocal problems for Caputo fractional differential equations. Using α-order resolvents, Li and Peng [18] and Fan [11] discussed the solutions to fractional homogeneous and inhomogeneous linear differential system, respectively. It is worth noting tha,t within the framework of Riemann-Liouville fractional resolvents, the properties of fractional differential equations have not been studied extensively. We list the main contributions and address the challenges of this paper as follows. Due to the differences in their definitions, resolvent operators for Caputo derivative and Riemann-Liouville derivative have distinct operator properties, even though neither of them has the classical semigroup property. As resolvent operator $P_\alpha(t)$ for Riemann-Liouville derivative is unbounded when t is near zero, which is different from Caputo derivative, we employ some constructive approaches to tackle this problem. Moreover, it is known that, for a compact operator semigroup $T(t)$, it is continuous for $t > 0$ in the uniform operator topology. Does this property hold for fractional resolvents? Here we present some convergence results for the case of fractional resolvents to ensure the uniform continuity for fractional resolvents; see Lemma 2.5.

The paper is organized as follows. In Section 2, we recall some concepts and facts about fractional differential equations and fractional resolvents. We also obtain the definitions of mild solutions and null controllability of system (1.1) via Riemann-Liouville fractional resolvents. In Section 3, by transforming the controllability problem (1.1) into a fixed point problem, we establish the existence and null controllability results for control system (1.1). In Section 4, an example is presented to illustrate the application of our results.

2. Preliminaries

We denote by $C(J,X)$ the space of X-valued continuous functions on J with the norm $\|x\| = \sup\{\|x(t)\|, t\in J\}$, and $L^p(J,X)$ the space of X-valued Bochner integrable functions with the norm $\|f\|_{L^p} = (\int_0^b \|f(t)\|^p\,\mathrm{d}t)^{\frac{1}{p}}$, where $1\le p<\infty$ and $B(X)$ the space of all bounded linear operators from X to itself. As the solution $x(\cdot)$ to problem (1.1) is unbounded at $t=0$, we introduce the space $C_{1-\alpha}(J,X) := \{x(\cdot) : \cdot^{1-\alpha}x(\cdot)\in C(J,X), 0<\alpha<1\}$, with the norm $\|x\|_{C_{1-\alpha}} = \sup\{t^{1-\alpha}\|x(t)\|_X : t\in J\}$, where $t^{1-\alpha}x(t)|_{t=0} = \lim_{t\to 0^+} t^{1-\alpha}x(t)$. Obviously $C_{1-\alpha}(J,X)$ is a Banach space.

Now we recall some definitions and results on fractional derivative and fractional differential equations.

Definition 2.1 ([22]). For any $f\in L^1(J,X)$, the Riemann-Liouville fractional integral of order $\alpha\in\mathbb{R}^+$ with the lower limit zero for f is defined by $I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,\ t>0$, provided the right-hand side is pointwise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 ([22]). For any $f\in L^1(J,X)$, the Riemann-Liouville fractional derivative of order $\alpha\in\mathbb{R}^+$ with the lower limit zero for f is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(I^{n-\alpha} f(t)\right),\ t>0,$$

where $\alpha\in(n-1,n]$, $n\in\mathbb{N}$. If $0<\alpha<1$, then $D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} f(s)\,\mathrm{d}s,\ t>0$.

We use the symbol $*$ to represent the convolution $(f*g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}s$. For the sake of convenience, let $g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$ and $g_\alpha(t)=0$ for $t\le 0$. Then, for $0<\alpha<1$,

$$I_t^\alpha f(t) = (g_\alpha * f)(t),\ D^\alpha f(t) = \frac{\mathrm{d}}{\mathrm{d}t}(g_{1-\alpha}*f)(t).$$

Definition 2.3 ([18]). Let $0<\alpha<1$. A family $\{P_\alpha(t), t>0\}\subseteq B(X)$ is said to be a resolvent of $\alpha-$order if it satisfies the following assumptions:

(1) $P_\alpha(\cdot)x\in C(\mathbb{R}^+,X)$ and $\lim_{t\to 0^+}\Gamma(\alpha)t^{1-\alpha}P_\alpha(t)x = x, x\in X$;

(2) $P_\alpha(t)P_\alpha(s) = P_\alpha(s)P_\alpha(t),\ s,t>0$;

(3) $P_\alpha(t)I_s^\alpha P_\alpha(s) - I_t^\alpha P_\alpha(t)P_\alpha(s) = g_\alpha(t)I_s^\alpha P_\alpha(s) - g_\alpha(s)I_t^\alpha P_\alpha(t),\ s,t>0$.

The linear operator A defined by

$$Ax = \Gamma(2\alpha)\lim_{t\to 0^+}\frac{t^{1-\alpha}P_\alpha(t)x - \frac{1}{\Gamma(\alpha)}x}{t^\alpha},\ x\in D(A)$$

is called the infinitesimal generator of fractional resolvent $P_\alpha(t)$, where

$$D(A) = \left\{ x\in X : \lim_{t\to 0+}\frac{t^{1-\alpha}P_\alpha(t)x - \frac{1}{\Gamma(\alpha)}x}{t^\alpha} \text{ exists} \right\}.$$

Notice that $P_\alpha(t)$ is unbounded when t is near zero point, but $t^{1-\alpha}P_\alpha(t)$ is bounded on $J=[0,b]$. We denote by $M = \sup_{t\in J}\left\|t^{1-\alpha}P_\alpha(t)\right\|$.

Lemma 2.4 ([18]). *Let $\{P_\alpha(t), t>0\}$ be a resolvent of $\alpha-$order and A be its infinitesimal generator. Then*

(1) $P_\alpha(t)x \in D(A)$ and $AP_\alpha(t)x = P_\alpha(t)Ax$ for all $x \in D(A), t>0$.

(2) For all $x \in X, t>0$, $P_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + AI_t^\alpha P_\alpha(t)x$.

(3) For all $x \in D(A), t>0$, $P_\alpha(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)}x + I_t^\alpha P_\alpha(t)Ax$.

(4) A is closed and densely defined.

It is known that, for a compact operator semigroup $T(t)$, it is continuous for $t>0$ in the uniform operator topology. But it is not true for a fractional resolvent $\{P_\alpha(t), t>0\}$, so we need the following convergence results for the fractional case.

Lemma 2.5. *Let $\{t^{1-\alpha}P_\alpha(t), t>0\}$ be equicontinuous and compact. Then, for every $t>0$,*

(1) $\lim_{h\to 0^+}\left\|(t+h)^{1-\alpha}P_\alpha(t+h) - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)\right\| = 0$;

(2) $\lim_{h\to 0^+}\left\|t^{1-\alpha}P_\alpha(t) - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot (t-h)^{1-\alpha}P_\alpha(t-h)\right\| = 0$.

Proof. From the compactness of $t^{1-\alpha}P_\alpha(t)$ for $t>0$, we have the set $R_t := \{t^{1-\alpha}P_\alpha(t)x : \|x\| \le 1\}$ is relatively compact in X for every $t>0$. Then we can find a finite family $\{t^{1-\alpha}P_\alpha(t)x_i : \|x_i\| \le 1\}_{i=1}^m \subset R_t$ satisfying, for every x, $\|x\| \le 1$, and there exists x_i, $i=1,\cdots,m$, such that

$$\|t^{1-\alpha}P_\alpha(t)x - t^{1-\alpha}P_\alpha(t)x_i\| < \frac{\varepsilon}{3(1+\Gamma(\alpha)M)}. \tag{2.1}$$

According to Definition 2.3 (1), there exists $l_1 > 0$ such that

$$\|t^{1-\alpha}P_\alpha(t)x_i - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x_i\| < \frac{\varepsilon}{3}, \tag{2.2}$$

for every $0<h\le l_1$ and $1\le i\le m$. As $t^{1-\alpha}P_\alpha(t)$ is equicontinuous for $t>0$, we can find $l_2>0$ such that

$$\|(t+h)^{1-\alpha}P_\alpha(t+h)x - t^{1-\alpha}P_\alpha(t)x\| < \frac{\varepsilon}{3}, \tag{2.3}$$

for every $0<h\le l_2$ and $\|x\|\le 1$. Now, for $0<h\le \min\{l_1,l_2\}$ and $\|x\|\le 1$, it follows from (2.1)-(2.3) that

$$\begin{aligned}
& \left\|(t+h)^{1-\alpha}P_\alpha(t+h)x - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x\right\| \\
\le\ & \left\|(t+h)^{1-\alpha}P_\alpha(t+h)x - t^{1-\alpha}P_\alpha(t)x\right\| + \left\|t^{1-\alpha}P_\alpha(t)x - t^{1-\alpha}P_\alpha(t)x_i\right\| \\
& + \left\|t^{1-\alpha}P_\alpha(t)x_i - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x_i\right\| \\
& + \left\|\Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x_i - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x\right\| \\
<\ & \frac{\varepsilon}{3} + (1+\Gamma(\alpha)M)\frac{\varepsilon}{3(1+\Gamma(\alpha)M)} + \frac{\varepsilon}{3} \le \varepsilon,
\end{aligned}$$

which implies that, for every $t>0$,

$$\lim_{h\to 0}\left\|(t+h)^{1-\alpha}P_\alpha(t+h) - \Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)\right\| = 0.$$

(2) Let $t>0$ and $0<h<\min\{t,b\}$. Then, for $\|x\|\le 1$, we have

$$\begin{aligned}
& \left\|t^{1-\alpha}P_\alpha(t)x-\Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot(t-h)^{1-\alpha}P_\alpha(t-h)x\right\| \\
= & \left\|t^{1-\alpha}P_\alpha(t)x-(t+h)^{1-\alpha}P_\alpha(t+h)x\right\| \\
& +\left\|(t+h)^{1-\alpha}P_\alpha(t+h)x-\Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x\right\| \\
& +\|\Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot[t^{1-\alpha}P_\alpha(t)x-(t-h)^{1-\alpha}P_\alpha(t-h)x]\| \\
\le & \left\|t^{1-\alpha}P_\alpha(t)x-(t+h)^{1-\alpha}P_\alpha(t+h)x\right\| \\
& +\left\|(t+h)^{1-\alpha}P_\alpha(t+h)x-\Gamma(\alpha)h^{1-\alpha}P_\alpha(h)\cdot t^{1-\alpha}P_\alpha(t)x\right\| \\
& +\Gamma(\alpha)M\|t^{1-\alpha}P_\alpha(t)x-(t-h)^{1-\alpha}P_\alpha(t-h)x\|,
\end{aligned}$$

which implies the desired result by the conclusion of Lemma 2.5 (1) and the equicontinuity of $\{t^{1-\alpha}P_\alpha(t), t>0\}$. □

The exact null controllability for differential equations is connected with the solution to differential systems. Now we give the definition of mild solutions to differential control system (1.1).

Definition 2.6. A function $x\in C_{1-\alpha}(J,X)$ is said to be a mild solution to problem (1.1) if it satisfies $x(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x_0+AI_t^\alpha x(t)+I_t^\alpha[f(t,x(t))+Bu(t)], t\in J'$, for some control function $u\in L^2(J,U)$.

Lemma 2.7. *Let $\alpha\in(0,1)$ and $h\in L^2(J,X)$. Then $x\in C_{1-\alpha}(J,X)$ is a solution to integral equation*

$$x(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x_0+AI_t^\alpha x(t)+I_t^\alpha h(t), t\in J', \tag{2.4}$$

if and only if x satisfies:

$$x(t)=P_\alpha(t)x_0+\int_0^t P_\alpha(t-s)h(s)\mathrm{d}s, t\in J'. \tag{2.5}$$

Proof. In view of Lemma 2.4 (2) $P_\alpha(t)x=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x+AI_t^\alpha P_\alpha(t)x$ and $g_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, we have, for $t>0$, $g_\alpha(t)=P_\alpha(t)-(Ag_\alpha*P_\alpha)(t)$. Let $x(\cdot)$ be a solution to equation (2.4). Then

$$g_\alpha*x=(P_\alpha-Ag_\alpha*P_\alpha)*x=P_\alpha*(x-Ag_\alpha*x)=g_\alpha*\Big(P_\alpha x_0+P_\alpha*h(\cdot)\Big),$$

which implies $x(t)=P_\alpha(t)x_0+\int_0^t P_\alpha(t-s)h(s)\mathrm{d}s$. On the other hand, suppose that $x(\cdot)$ satisfies (2.5). Considering the structure of $AI_t^\alpha x(t)$, by Definition 2.3 (3), we have

$$\begin{aligned}
\left(s^{1-\alpha}P_\alpha(s)-\frac{1}{\Gamma(\alpha)}\right)I_t^\alpha x(t) &= s^{1-\alpha}[P_\alpha(s)I_t^\alpha P_\alpha(t)x_0-g_\alpha(s)I_t^\alpha P_\alpha(t)x_0] \\
&\quad +s^{1-\alpha}[P_\alpha(s)\cdot(I_t^\alpha P_\alpha)*h(t)-g_\alpha(s)\cdot(I_t^\alpha P_\alpha)*h(t)] \\
&= s^{1-\alpha}[P_\alpha(t)I_s^\alpha P_\alpha(s)x_0-g_\alpha(t)I_s^\alpha P_\alpha(s)x_0] \\
&\quad +s^{1-\alpha}[I_s^\alpha P_\alpha(s)P_\alpha(t)-I_s^\alpha P_\alpha(s)g_\alpha(t)]*h(t) \\
&= s^{1-\alpha}I_s^\alpha P_\alpha(s)\left[x(t)-g_\alpha(t)x_0-I_t^\alpha h(t)\right].
\end{aligned}$$

It follows that

$$\begin{aligned} AI_t^\alpha x(t) &= \lim_{s\to 0^+}\Gamma(2\alpha)\frac{\left(s^{1-\alpha}P_\alpha(s)-\frac{1}{\Gamma(\alpha)}\right)I_t^\alpha x(t)}{s^\alpha} \\ &= \lim_{s\to 0^+}\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha P_\alpha(s)\left[x(t)-g_\alpha(t)x_0-I_t^\alpha h(t)\right]. \end{aligned} \tag{2.6}$$

For $x\in X$, we have

$$\begin{aligned} &\left\|\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha P_\alpha(s)x-x\right\| \\ = &\left\|\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1 s^{1-\alpha}\Gamma(\alpha)(1-\tau)^{\alpha-1}P_\alpha(s\tau)x\mathrm{d}\tau-x\right\| \\ = &\left\|\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1(1-\tau)^{\alpha-1}\tau^{\alpha-1}\Gamma(\alpha)(s\tau)^{1-\alpha}P_\alpha(s\tau)x\mathrm{d}\tau-\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1(1-\tau)^{\alpha-1}\tau^{\alpha-1}x\mathrm{d}\tau\right\| \\ \le &\frac{\Gamma(2\alpha)}{[\Gamma(\alpha)]^2}\int_0^1(1-\tau)^{\alpha-1}\tau^{\alpha-1}\mathrm{d}\tau\cdot\sup_{\tau\in[0,1]}\left\|\Gamma(\alpha)(s\tau)^{1-\alpha}P_\alpha(s\tau)x-x\right\| \\ \le &\sup_{\tau\in[0,1]}\left\|\Gamma(\alpha)(s\tau)^{1-\alpha}P_\alpha(s\tau)x-x\right\|. \end{aligned}$$

From Definition 2.3 (1), we have

$$\left\|\Gamma(2\alpha)s^{1-2\alpha}I_s^\alpha P_\alpha(s)x-x\right\|\to 0,\ \text{as } s\to 0^+. \tag{2.7}$$

It follows from (2.6) and (2.7) that $x(t)=g_\alpha(t)x_0+AI_t^\alpha x(t)+I_t^\alpha h(t)$, which shows that x is a solution to equation (2.4). □

According to Lemma 2.7, we can give the definition of mild solution via fractional resolvents.

Definition 2.8. A function $x\in C_{1-\alpha}(J,X)$ is a mild solution to problem (1.1) if it satisfies the following fractional integral equation $x(t)=P_\alpha(t)x_0+\int_0^t P_\alpha(t-s)[f(s,x(s))+Bu(s)]\mathrm{d}s, t\in J'$.

Consider the fractional linear control system associated with system (1.1),

$$\begin{cases} D^\alpha y(t)=Ay(t)+f(t)+Bu(t),\ t\in J':=(0,b], \\ \lim_{t\to 0^+}\Gamma(\alpha)t^{1-\alpha}y(t)=y_0, \end{cases} \tag{2.8}$$

where $f\in L^2(J,X)$.

We define two relevant operators $L_0^b: L^2(J,U)\to X$ and $N_0^b: X\times L^2(J,U)\to X$ by

$$L_0^b u=\int_0^b P_\alpha(b-s)Bu(s)\,\mathrm{d}s,\ u\in L^2(J,U),$$

$$N_0^b(y_0,f)=P_\alpha(b)y_0+\int_0^b P_\alpha(b-s)f(s)\,\mathrm{d}s,\ (y_0,f)\in X\times L^2(J,X).$$

Definition 2.9. Fractional linear differential system (2.8) is called exactly null controllable on J if $\operatorname{Im} L_0^b\supset\operatorname{Im} N_0^b$.

Remark 2.10. From [9], it is known that linear system (2.8) is exactly null controllable on $[0,b]$ if and only if there exists a number $k>0$ such that $\|(L_0^b)^*z\|\ge k\|(N_0^b)^*z\|$ for all $z\in X$.

Similar to the proof of Lemma 3 in [7], where the null controllability of semilinear integrodifferential systems is discussed, we have the following result.

Lemma 2.11. *Suppose that linear system (2.8) is exactly null controllable on* $[0,b]$*. Then the linear operator* $H := (L_0)^{-1}(N_0^b) : X \times L^2(J,X) \to L^2(J,U)$ *is bounded and the control*

$$u(t) = -H(y_0, f)(t) = -(L_0)^{-1}\left(P_\alpha(b)y_0 + \int_0^b P_\alpha(b-s)f(s)\,\mathrm{d}s\right)(t)$$

transfers system (2.8) from y_0 *to* 0*, where* L_0 *is the restriction of* L_0^b *to* $[\mathrm{Ker}\ L_0^b]^\perp$.

Lemma 2.12 ([30])**.** *Let* $f \in L^p(J,X)$ *with* $1 \le p < \infty$*. Then* $\lim_{h\to 0}\int_0^b \|f(t+h) - f(t)\|^p \mathrm{d}t = 0$*, where* $f(t) = 0$ *for* $t \neq J$*.*

3. Results on Exact Null Controllability

In this section, we discuss the sufficient conditions for exact null controllability of control system (1.1). For this purpose, we need some hypotheses for the control differential system (1.1). Let $M_B = \|B\|$, $M_H = \|H\|$, r be a finite positive constant, and set $W_r = \{x \in C_{1-\alpha}(J,X) : \|x\|_{1-\alpha} \le r\}$.

(H1) $\{t^{1-\alpha}P_\alpha(t),\ t > 0\}$ is equicontinuous and compact.

(H2) The function $f(t,\cdot) : X \to X$ is continuous for a.e. $t \in [0,b]$ and $f(\cdot,x) : [0,b] \to X$ is measurable for all $x \in X$. Moreover, for any $r > 0$, there exists a function $\rho_r \in L^2(J,\mathbb{R}^+)$ such that $\|f(t,x)\| \le \rho_r(t)$ for a.e. $t \in [0,b]$ and $x \in X$ satisfying $\|x\| \le r$.

(H3) Linear system (2.8) is exactly null controllable on $[0,b]$.

Due to Definition 2.8, we define the solution operator $Q : C_{1-\alpha}(J,X) \to C_{1-\alpha}(J,X)$ by

$$(Qx)(t) = P_\alpha(t)x_0 + \int_0^t P_\alpha(t-s)[f(s,x(s)) + Bu(s)]\mathrm{d}s, \tag{3.1}$$

where

$$u(t) = -H(x_0, f) = -(L_0)^{-1}\big(P_\alpha(b)x_0 + \int_0^b P_\alpha(b-s)f(s,x(s))\,\mathrm{d}s\big)(t), \tag{3.2}$$

with $(Q_1x)(t) = P_\alpha(t)x_0$ and $(Q_2x)(t) = \int_0^t P_\alpha(t-s)[f(s,x(s)) + Bu(s)]\,\mathrm{d}s$. Under the control $u(t)$ defined by (3.2), it is easy to see that

$$\begin{aligned} x(b) &= P_\alpha(b)x_0 + \int_0^b P_\alpha(b-s)f(s,x(s))\,\mathrm{d}s \\ &\quad - \int_0^b P_\alpha(b-s)B(L_0)^{-1}\Big(P_\alpha(b)x_0 + \int_0^b P_\alpha(b-s)f(s,x(s))\,\mathrm{d}s\Big)(s)\,\mathrm{d}s = 0, \end{aligned}$$

which implies that system (1.1) is exactly null controllable.

Now, we discuss the compactness property of Cauchy operator to system (1.1).

Lemma 3.1. *Suppose that conditions* (H1) – (H3) *are satisfied. Then the mapping* $Q_2 : W_r \to C_{1-\alpha}(J,X)$ *defined by* $(Q_2x)(t) = \int_0^t P_\alpha(t-s)[f(s,x(s)) - BH(x_0,f)(s)]\,\mathrm{d}s$ *is compact.*

Proof. By the relationship between $(C_{1-\alpha}(J,X), \|\cdot\|_{C_{1-\alpha}})$ and $(C(J,X), \|\cdot\|_C)$, we only need to prove the set $G = \{y \in C(J,X) : y(t) = t^{1-\alpha}(Q_2x)(t),\ x \in W_r,\ t \in J\}$ is precompact in $C(J,X)$.

Firstly, we show that, for each $t \in [0,b]$, $G(t) = \{t^{1-\alpha}(Q_2x)(t) :\ x \in W_r\}$ is relatively compact in X. If $t = 0$, it is easy to see $G(0) = 0$ is relatively compact in X. For $t \in (0,b]$ and $\varepsilon \in (0,t)$, we define a set $G^\varepsilon(t) = \{y^\varepsilon(t) :\ x \in W_r\} \subseteq X$, where

$$y^\varepsilon(t) = \varepsilon^{1-\alpha}P_\alpha(\varepsilon)\cdot\Gamma(\alpha)t^{1-\alpha}\int_0^{t-\varepsilon} P_\alpha(t-s-\varepsilon)[f(s,x(s)) - BH(x_0,f)(s)]\,\mathrm{d}s.$$

By Lemma 2.11, we know

$$\|H(x_0,f)\|_{L^2} \leq M_H\left(\|x_0\|+\|f\|_{L^2}\right). \tag{3.3}$$

In view of (3.3), for $x \in W_r$, $t \in (0,b]$, we have

$$\begin{aligned}
&\|t^{1-\alpha}\int_0^{t-\varepsilon} P_\alpha(t-s-\varepsilon)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\| \\
\leq\ & b^{1-\alpha}\int_0^{t-\varepsilon} \|(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon)\cdot(t-s-\varepsilon)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\|\,\mathrm{d}s \\
\leq\ & Mb^{1-\alpha}\int_0^{t-\varepsilon} \|(t-s-\varepsilon)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\|\,\mathrm{d}s \\
\leq\ & Mb^{1-\alpha}\Big[\int_0^{t-\varepsilon}(t-s-\varepsilon)^{\alpha-1}\|f(s,x(s))\|\,\mathrm{d}s+\int_0^{t-\varepsilon}(t-s-\varepsilon)^{\alpha-1}M_B\|H(x_0,f)(s)\|\,\mathrm{d}s\Big] \\
\leq\ & Mb^{1-\alpha}\left[\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}\|f\|_{L^2}+M_B\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}M_H\left(\|x_0\|+\|f\|_{L^2}\right)\right] \\
\leq\ & Mb^{1-\alpha}\left[\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}\|\rho_r\|_{L^2}+M_B\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}M_H\left(\|x_0\|+\|\rho_r\|_{L^2}\right)\right]<\infty.
\end{aligned} \tag{3.4}$$

Then $G^\varepsilon(t)$ is relatively compact in X for each $t\in(0,b]$ as the operator $\varepsilon^{1-\alpha}P_\alpha(\varepsilon)$ is compact for $\varepsilon>0$. Let $t\in(0,b]$ and $\delta\in(\varepsilon,t)$. We have

$$\begin{aligned}
&\|y(t)-y^\varepsilon(t)\| \\
\leq\ & t^{1-\alpha}\Big[\Big\|\int_0^{t-\varepsilon}(t-s)^{1-\alpha}P_\alpha(t-s)\cdot(t-s)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \\
& -\varepsilon^{1-\alpha}P_\alpha(\varepsilon)\Gamma(\alpha)\int_0^{t-\varepsilon}(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon)\cdot(t-s)^{\alpha-1} \\
& [f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\|+\Big\|\varepsilon^{1-\alpha}P_\alpha(\varepsilon)\Gamma(\alpha)\int_0^{t-\varepsilon}(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon) \\
& ((t-s)^{\alpha-1}-(t-s-\varepsilon)^{\alpha-1})[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\| \\
& +\Big\|\int_{t-\varepsilon}^{t}(t-s)^{1-\alpha}P_\alpha(t-s)\cdot(t-s)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\|\Big] \\
\leq\ & b^{1-\alpha}\int_0^{t-\delta}\|(t-s)^{1-\alpha}P_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}P_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon)\| \\
& (t-s)^{\alpha-1}\big\|f(s,x(s))-BH(x_0,f)(s)\big\|\,\mathrm{d}s \\
& +b^{1-\alpha}\int_{t-\delta}^{t-\varepsilon}\|(t-s)^{1-\alpha}P_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}P_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon)\| \\
& (t-s)^{\alpha-1}\big\|f(s,x(s))-BH(x_0,f)(s)\big\|\,\mathrm{d}s
\end{aligned}$$

$$+b^{1-\alpha}M^2\Gamma(\alpha)\left(\int_0^{t-\varepsilon}\left[(t-s)^{\alpha-1}-(t-s-\varepsilon)^{\alpha-1}\right]^2\mathrm{d}s\right)^{\frac{1}{2}}$$
$$\left(\|\rho_r\|_{L^2}(1+M_BM_H)+M_BM_H\|x_0\|\right)$$
$$+b^{1-\alpha}M\int_{t-\varepsilon}^{t}\|(t-s)^{\alpha-1}\left[f(s,x(s))-BH(x_0,f)(s)\right]\|\,\mathrm{d}s:=J_1+J_2+J_3+J_4,$$

where

$$\begin{aligned}
J_1 &= b^{1-\alpha}\int_0^{t-\delta}\|(t-s)^{1-\alpha}P_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}P_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon)\| \\
&\quad (t-s)^{\alpha-1}\left\|f(s,x(s))-BH(x_0,f)(s)\right\|\mathrm{d}s,\\
J_2 &= b^{1-\alpha}\int_{t-\delta}^{t-\varepsilon}\|(t-s)^{1-\alpha}P_\alpha(t-s)-\Gamma(\alpha)\varepsilon^{1-\alpha}P_\alpha(\varepsilon)(t-s-\varepsilon)^{1-\alpha}P_\alpha(t-s-\varepsilon)\| \\
&\quad (t-s)^{\alpha-1}\left\|f(s,x(s))-BH(x_0,f)(s)\right\|\mathrm{d}s,\\
J_3 &= b^{1-\alpha}M^2\Gamma(\alpha)\left(\int_0^{t-\varepsilon}\left[(t-s)^{\alpha-1}-(t-s-\varepsilon)^{\alpha-1}\right]^2\mathrm{d}s\right)^{\frac{1}{2}} \\
&\quad \left(\|\rho_r\|_{L^2}(1+M_BM_H)+M_BM_H\|x_0\|\right),
\end{aligned}$$

and

$$J_4=b^{1-\alpha}M\int_{t-\varepsilon}^{t}\left\|(t-s)^{\alpha-1}\left[f(s,x(s))-BH(x_0,f)(s)\right]\right\|\mathrm{d}s.$$

From Lemma 2.5, we know that $J_1\to 0$, as $\varepsilon\to 0^+$. By the arbitrariness of ε, δ and absolute continuity of integral, $J_2\to 0$, $J_4\to 0$ are obtained as $\varepsilon,\delta\to 0^+$. From Lemma 2.12, we have $J_3\to 0$, as $\varepsilon\to 0^+$. Now for $t\in J'$, we have $\lim_{\varepsilon\to 0^+}\|y(t)-y^{\varepsilon}(t)\|=0$, which implies that $G(t)=\{y(t):y\in G\}$ is precompact in X as there is a family of precompact sets arbitrarily close to it.

Next, we show the equicontinuity of G on J. For $t\in J'$, $x\in W_r$, we have

$$\begin{aligned}
&\left\|\int_0^t P_\alpha(t-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\right\| \\
\leq\ & \int_0^t\|(t-s)^{1-\alpha}P_\alpha(t-s)\cdot(t-s)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\|\,\mathrm{d}s \\
\leq\ & M\int_0^t(t-s)^{\alpha-1}\|f(s,x(s))\|\,\mathrm{d}s+M\int_0^t(t-s)^{\alpha-1}\|BH(x_0,f)(s)\|\,\mathrm{d}s \\
\leq\ & M\|\rho_r\|_{L^2}\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}+MM_B\left(\int_0^t\|H(x_0,f)(s)\|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}} \\
\leq\ & M\|\rho_r\|_{L^2}\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}}+MM_BM_H(\|x_0\|+\|\rho_r\|_{L^2})\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac{1}{2}} \\
:=\ & F_r<\infty. \qquad (3.5)
\end{aligned}$$

Let $0<\varepsilon<t_1<t_2\le b$, by (3.5), we have

$$
\begin{aligned}
& \Big\| \int_0^{t_2} P_\alpha(t_2-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \\
& -\int_0^{t_1} P_\alpha(t_1-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \Big\| \\
\le\ & \Big\| \int_0^{t_1-\varepsilon} [(t_2-s)^{1-\alpha}P_\alpha(t_2-s)-(t_1-s)^{1-\alpha}P_\alpha(t_1-s)] \\
& (t_2-s)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \Big\| \\
& +\Big\| \int_{t_1-\varepsilon}^{t_1} [(t_2-s)^{1-\alpha}P_\alpha(t_2-s)-(t_1-s)^{1-\alpha}P_\alpha(t_1-s)] \\
& (t_2-s)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \Big\| \\
& +\Big\| \int_0^{t_1} (t_1-s)^{1-\alpha}P_\alpha(t_1-s)\cdot[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}] \\
& [f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \Big\| \\
& +\left\| \int_{t_1}^{t_2} (t_2-s)^{1-\alpha}P_\alpha(t_2-s)\cdot(t_2-s)^{\alpha-1}[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \right\| \\
\le\ & \sup_{s\in[0,t_1-\varepsilon]} \|(t_2-s)^{1-\alpha}P_\alpha(t_2-s)-(t_1-s)^{1-\alpha}P_\alpha(t_1-s)\|\cdot\frac{F_r}{M} \\
& +2M\|\rho_r\|_{L^2}\left(\frac{1}{2\alpha-1}\right)^{\frac12}\left[(t_2-t_1+\varepsilon)^{2\alpha-1}-(t_2-t_1)^{2\alpha-1}\right]^{\frac12} \\
& +2MM_BM_H(\|x_0\|+\|\rho_r\|_{L^2})\left(\frac{1}{2\alpha-1}\right)^{\frac12}\left[(t_2-t_1+\varepsilon)^{2\alpha-1}-(t_2-t_1)^{2\alpha-1}\right]^{\frac12} \\
& +M\Big[\|\rho_r\|_{L^2}+M_BM_H(\|x_0\|+\|\rho_r\|_{L^2})\Big]\left(\int_0^{t_1}\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]^2\mathrm{d}s\right)^{\frac12} \\
& +M\big[\|\rho_r\|_{L^2}+M_BM_H(\|x_0\|+\|\rho_r\|_{L^2})\big]\left(\frac{1}{2\alpha-1}\right)^{\frac12}(t_2-t_1)^{\alpha-\frac12}.
\end{aligned}
$$

Due to the equicontinuity of $\{t^{1-\alpha}P_\alpha(t),t>0\}$, Lemma 2.12, and the arbitrariness of ε, we have

$$
\begin{aligned}
& \Big\| \int_0^{t_2} P_\alpha(t_2-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \\
& -\int_0^{t_1} P_\alpha(t_1-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \Big\| \to 0
\end{aligned}
\tag{3.6}
$$

as $t_1\to t_2$ independent of $x\in W_r$.

Now, for $y \in G$, $0 \leq t_1 < t_2 \leq b$, we have

$$\begin{aligned} \|y(t_2)-y(t_1)\| &= \Big\|t_2^{1-\alpha}\int_0^{t_2} P_\alpha(t_2-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \\ &\quad -t_1^{1-\alpha}\int_0^{t_1} P_\alpha(t_1-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\| \\ &\leq \Big\|(t_2^{1-\alpha}-t_1^{1-\alpha})\int_0^{t_2} P_\alpha(t_2-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\| \\ &\quad +t_1^{1-\alpha}\Big\|\int_0^{t_2} P_\alpha(t_2-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \\ &\quad -\int_0^{t_1} P_\alpha(t_1-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\| \\ &\leq (t_2^{1-\alpha}-t_1^{1-\alpha})F_r+b^{1-\alpha}\Big\|\int_0^{t_2} P_\alpha(t_2-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s \\ &\quad -\int_0^{t_1} P_\alpha(t_1-s)[f(s,x(s))-BH(x_0,f)(s)]\Big\|. \end{aligned}$$

Then it follows from (3.5) and (3.6) that $\lim_{t_1\to t_2}\|y(t_2)-y(t_1)\|=0$, which leads to the equicontinuity of G on J. By Ascoli-Arzela Theorem, the set G is precompact in $C(J,X)$. Then $Q_2 : W_r \to C_{1-\alpha}(J,X)$ is a compact mapping. The proof is completed. □

Now, we are able to state and prove our main result.

Theorem 3.2. *Assume that the hypotheses* $(\mathrm{H1})-(\mathrm{H3})$ *are satisfied. Then system* (1.1) *is exactly null controllable on* $[0,b]$.

Proof. We consider the operator $Q : C_{1-\alpha}(J,X) \to C_{1-\alpha}(J,X)$ defined by (3.1). It is easy to see that if we can get the fixed point of Q, then differential system (1.1) is exactly null controllable on $[0,b]$. The proof is divided into three steps.

Firstly, there exists a number $r > 0$ such that Q maps $W_r \subset C_{1-\alpha}(J,X)$ into itself, where

$$r \geq M\|x_0\|+b^{1-\alpha}\left[M\|\rho_r\|_{L^2}\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac12}+MM_BM_H(\|x_0\|+\|\rho_r\|_{L^2})\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac12}\right].$$

In fact, for $x \in W_r$, $t \in J$, we have by (3.5) that

$$\begin{aligned} &\|t^{1-\alpha}Qx(t)\| \\ \leq\ & \|t^{1-\alpha}P_\alpha(t)x_0\|+b^{1-\alpha}\Big\|\int_0^t P_\alpha(t-s)[f(s,x(s))-BH(x_0,f)(s)]\,\mathrm{d}s\Big\| \\ \leq\ & M\|x_0\|+b^{1-\alpha}\left[M\|\rho_r\|_{L^2}\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac12}+MM_BM_H(\|x_0\|+\|\rho_r\|_{L^2})\left(\frac{b^{2\alpha-1}}{2\alpha-1}\right)^{\frac12}\right] \\ \leq\ & r. \end{aligned}$$

Secondly, Q is continuous on $W_r \subset C_{1-\alpha}(J,X)$. For this purpose, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in W_r with $\lim\limits_{n\to\infty} x_n = x$. From hypotheses (H2), (H3), and Lemma 2.11, for $t \in J$, we have

$$(t-s)^{\alpha-1}[f(s,x_n(s))-f(s,x(s))] \to 0, \ \text{ a.e. } s \in [0,t],$$

and

$$\begin{aligned} & (t-s)^{\alpha-1}\left\|BH\big(x_0,f(s,x_n(s))\big)(s)-BH\big(x_0,f(s,x(s))\big)(s)\right\| \\ \leq\quad & (t-s)^{\alpha-1}M_B\left\|H\big(x_0,f(s,x_n(s))\big)(s)-H\big(x_0,f(s,x(s))\big)(s)\right\| \\ \to 0\quad & \text{a.e. } s\in[0,t], \end{aligned}$$

as $n\to\infty$. From the Lebesgue dominated convergence theorem, we see that

$$\begin{aligned} & t^{1-\alpha}\|(Qx_n)(t)-(Qx)(t)\| \\ \leq\quad & b^{1-\alpha}\int_0^t\|(t-s)^{1-\alpha}P_\alpha(t-s)\|\cdot(t-s)^{\alpha-1} \\ & \Big[\|f(s,x_n(s))-f(s,x(s))\|+\|BH(x_0,f(s,x_n(s)))(s)-BH(x_0,f(s,x(s)))(s)\|\Big]\mathrm{d}s \\ \leq\quad & Mb^{1-\alpha}\Big[\int_0^t(t-s)^{\alpha-1}\|f(s,x_n(s))-f(s,x(s))\|\mathrm{d}s \\ & +\int_0^t(t-s)^{\alpha-1}\|BH(x_0,f(s,x_n(s)))(s)-BH(x_0,f(s,x(s)))(s)\|\mathrm{d}s\Big] \\ \to 0\quad & ,\ n\to\infty. \end{aligned}$$

Thus

$$\|Qx_n-Qx\|_{C_{1-\alpha}}=\sup_{t\in J}t^{1-\alpha}\|(Qx_n)(t)-(Qx)(t)\|\to 0,$$

as $n\to\infty$, which implies the continuity of Q on W_r.

At last, it remains to prove that Q is a compact operator on W_r. From Lemma 3.1, it is known that $Q_2: W_r\to C_{1-\alpha}(J,X)$ is compact. For the compactness of $Q_1=P_\alpha(\cdot)x_0$, it is sufficient to check the set

$$R=\{u\in C(J,X): u(t)=t^{1-\alpha}P_\alpha(t)x_0, x_0\in X, t\in J\},$$

is precompact in $C(J,X)$. Obviously, $R(0)=\{\frac{x_0}{\Gamma(\alpha)}\}$, $R(t)=\{t^{1-\alpha}P_\alpha(t)x_0\}$, $t>0$ is precompact in X. Suppose that $0\leq t_1<t_2\leq b$. If $t_1=0$, in view of Definition 2.3(a), we have

$$\|u(t_2)-u(0)\|=\|t_2^{1-\alpha}P_\alpha(t_2)x_0-\frac{x_0}{\Gamma(\alpha)}\|\to 0,$$

as $t_2\to 0$. If $t_1>0$, by hypothesis (H1), and the fact that $\{t^{1-\alpha}P_\alpha(t), t>0\}$ is equicontinuous, we see that

$$\|u(t_2)-u(t_1)\|\leq\|t_2^{1-\alpha}P_\alpha(t_2)x_0-t_1^{1-\alpha}P_\alpha(t_1)x_0\|\to 0.$$

From the Ascoli-Arzela theorem, we get that R is precompact in $C(J,X)$. So $Q=Q_1+Q_2$ is a compact operator on W_r.

Hence, by applying Schauder's fixed point theorem, we see that Q has at least a fixed point in W_r, which infers that system (1.1) is exactly null controllable on $[0,b]$. This completes the proof. □

In the following, we discuss the nonlocal differential equations with Riemann-Liouville fractional derivative:

$$\begin{cases} D^\alpha x(t)=Ax(t)+f(t,x(t))+Bu(t),\ t\in J':=(0,b], \\ \lim_{t\to 0^+}\Gamma(\alpha)t^{1-\alpha}x(t)=g(x), \end{cases}\tag{3.7}$$

where the nonlocal item g satisfies the hypothesis, (H4) $g : C_{1-\alpha}(J,X) \to X$ is a continuous and compact mapping. There exists a positive constant N such that $\|g(x)\| \le N$ for all $x \in C_{1-\alpha}(J,X)$.

As the nonlocal differential equations with initial condition $x(0) = g(x)$ are found to have better effects in some applications than the classical ones($x(0) = x_0$), various types of differential equations with nonlocal initial conditions have been studied in the literatures; see, e.g., [2, 4, 16, 19]. When the method of operator semigroup is applied to discuss the nonlocal problems in Banach spaces, some methods, including approximate solutions and measure of noncompactness, were used to discuss this problem; see, e.g., [5, 16]. By using approximation method, Fu and Zhang [12] studied the exact null controllability of nonlocal functional differential systems.

Theorem 3.3. *Assume that the hypotheses* (H1) − (H4) *are satisfied. Then the nonlocal differential system* (3.7) *is exactly null controllable on* $[0,b]$.

Proof. We define the operator $K : C_{1-\alpha}(J,X) \to C_{1-\alpha}(J,X)$ by

$$(Kx)(t) = P_\alpha(t)g(x) + \int_0^t P_\alpha(t-s)[f(s,x(s)) + Bu(s)]\mathrm{d}s.$$

Similar to the proof of Theorem 3.2, it is easy to see that G is continuous, and there exists $r > 0$ such that G maps $W_r \subset C_{1-\alpha}(J,X)$ into itself. We prove K is compact on W_r. By Lemma 3.1, it is known that

$$(Q_2x)(\cdot) = \int_0^{\cdot} P_\alpha(\cdot - s)[f(s,x(s)) + Bu(s)]\mathrm{d}s$$

is a compact mapping on W_r. It remains to prove the compactness of $(K - Q_2)x(\cdot) = P_\alpha(\cdot)g(x)$. It is equivalent to the compactness of

$$R' = \{u \in C(J,X) : u(t) = t^{1-\alpha}P_\alpha(t)g(x), x \in W_r, t \in J\}$$

in $C(J,X)$. In fact, $R'(0) = \{\frac{g(x)}{\Gamma(\alpha)} : x \in W_r\}$ is precompact in X as g is a compact mapping. For $t > 0$, due to the compactness of $t^{1-\alpha}P_\alpha(t)$, we have that

$$R'(t) = \{t^{1-\alpha}P_\alpha(t)g(x) : x \in W_r\} \subset X$$

is precompact in X. For $0 < t_1 < t_2 \le b$, the equicontinuity of $t^{1-\alpha}P_\alpha(t)$ implies that

$$\|u(t_2) - u(t_1)\| = \|t_2^{1-\alpha}P_\alpha(t_2)g(x) - t_1^{1-\alpha}P_\alpha(t_1)g(x)\| \to 0,$$

as $t_1 \to t_2$ independent of x. On the other hand, as g is compact and $g(W_r) = \{g(x) : x \in W_r\}$ is precompact in X, for any $\varepsilon > 0$ we can find a finite family $\{y_i\}_{i=1}^m \subset X$ such that, for any $x \in W_r$, there exists y_i such that

$$\|g(x) - y_i\| < \frac{\varepsilon}{3(M + \frac{1}{\Gamma(\alpha)})}. \tag{3.8}$$

From Definition 2.3, it follows that $\lim_{t\to 0^+} \Gamma(\alpha)t^{1-\alpha}P_\alpha(t)x = x$, for any $x \in X$. Then, for any $\varepsilon > 0$, there exists a constant $\delta > 0$, independent of y_i, $i = 1,2,\cdots,m$, such that, for any t with $|t| < \delta$,

$$\|t^{1-\alpha}P_\alpha(t)y_i - \frac{1}{\Gamma(\alpha)}y_i\| < \frac{\varepsilon}{3}. \tag{3.9}$$

Now by (3.8) and (3.9), it follows that, for any $\varepsilon > 0$, there exists a constant $\delta > 0$, independent of $x \in W_r$, such that, for any $0 < |t| < \delta$,

$$\begin{aligned}\|u(t)-u(0)\| &= \|t^{1-\alpha}P_\alpha(t)g(x) - \frac{1}{\Gamma(\alpha)}g(x)\| \\ &\leq \|t^{1-\alpha}P_\alpha(t)g(x) - t^{1-\alpha}P_\alpha(t)y_i\| + \|t^{1-\alpha}P_\alpha(t)y_i - \frac{1}{\Gamma(\alpha)}y_i\| \\ &\quad + \|\frac{1}{\Gamma(\alpha)}y_i - \frac{1}{\Gamma(\alpha)}g(x)\| \\ &\leq M\|g(x)-y_i\| + \|t^{1-\alpha}P_\alpha(t)y_i - \frac{1}{\Gamma(\alpha)}y_i\| + \frac{1}{\Gamma(\alpha)}\|y_i - g(x)\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3},\end{aligned}$$

which leads to the equicontinuity of R' at $t = 0$. By Ascoli-Arzela theorem, we see that R' is relatively compact in $C(J,X)$. Thus K is a compact operator on W_r. From Schauder's fixed point theorem, we can conclude that K has a fixed point on W_r, which infers that system (1.1) is exactly null controllable on $[0,b]$. This completes the proof. □

4. Application

We consider the following partial differential control system to illustrate our abstract results:

$$\begin{cases} D^\alpha x(t,\theta) = \frac{\partial^2}{\partial\theta^2}x(t,\theta) + f(t,x(t,\theta)) + u(t,\theta),\ 0 < t \leq b,\ 0 \leq \theta \leq 1, \\ x(t,0) = x(t,1) = 0, \\ \lim_{t\to 0^+}\Gamma(\alpha)t^{1-\alpha}x(t,\theta) = x_0(\theta), \end{cases} \tag{4.1}$$

where $1/2 < \alpha < 1$, and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous.

Let $X = L^2([0,1])$ and the operator $A : D(A) \subseteq X \to X$ be defined by $Az = z''$, with

$$D(A) = \{z \in X : z, z' \text{ are absolutely continuous, } z'' \in X, z(0) = z(1) = 0\}.$$

From Pazy [23], it is known that A generates an analytic semigroup $S(t)$, $t \geq 0$. The operator A has the eigenvalues $\lambda_n = -n^2\pi^2$, $n \in \mathbb{N}$, and the corresponding eigenvectors $e_n(\theta) = \sqrt{2}\sin(n\pi\theta)$ for $n \geq 1$, $e_0 = 1$, form an orthogonal basis for $L^2([0,1])$. Then $S(t)$ is given by

$$S(t)z = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < z, e_n > e_n.$$

If $u_0(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x$, then $S(t)u_0(x) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} c_n \sin n\pi x$. On the other hand, from [18], A is known as the infinitesimal generator of an $\alpha-$order fractional resolvent $P_\alpha(t)$ and

$$P_\alpha(t)u_0(x) = \sum_{n=1}^{\infty} t^{\alpha-1}E_{\alpha,\alpha}\left(-n^2\pi^2 t^\alpha\right) c_n \sin n\pi x.$$

Using Laplace transformation and probability density functions, similar to the method in [27], we have

$$t^{1-\alpha}P_\alpha(t)u_0(x) = \alpha\int_0^\infty \theta\xi_\alpha(\theta)S(t^\alpha\theta)u_0(x)\mathrm{d}\theta, \tag{4.2}$$

for any $u_0 \in X$, where

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha\left(\theta^{-\frac{1}{\alpha}}\right),$$

$$\varpi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty).$$

It follows from (4.2) that

$$t^{1-\alpha}P_\alpha(t)=\alpha\int_0^\infty \theta\xi_\alpha(\theta)S(t^\alpha\theta)\,\mathrm{d}\theta.$$

Using [27, Lemma 2.9], we see that the family of operators $\{t^{1-\alpha}P_\alpha(t):t>0\}$ is equicontinuous, compact and it satisfies

$$\|t^{1-\alpha}P_\alpha(t)\|\le\frac{\alpha M'}{\Gamma(1+\alpha)}:=M,$$

where $M'=\sup\{\|S(t)\|:0\le t\le 1\}$. Then hypothesis of (H1) is satisfied. For the differential system (4.1), let $u\in L^2([0,1],X)$, $B=I$. Then $B^*=I$. Firstly, we discuss the exact null controllability of the corresponding linear system with additive term $f\in L^2([0,b],X)$. That is

$$\begin{cases} D^\alpha x(t,\theta)=\frac{\partial^2}{\partial\theta^2}x(t,\theta)+f(t,\theta)+u(t,\theta),\ 0<t\le b,\ 0\le\theta\le 1,\\ x(t,0)=x(t,1)=0,\\ \lim_{t\to0^+}\Gamma(\alpha)t^{1-\alpha}x(t,\theta)=x_0(\theta).\end{cases}\tag{4.3}$$

By Remark 2.10, the exact null controllability of system (4.3) is equal to the existence of a number $k>0$ such that

$$\int_0^b\|B^*P_\alpha^*(b-s)z\|^2\,\mathrm{d}s\ge k\Big(\|P_\alpha^*(b)z\|^2+\int_0^b\|P_\alpha^*(b-s)z\|^2\,\mathrm{d}s\Big),$$

which is equivalent to

$$\int_0^b\|P_\alpha(b-s)z\|^2\,\mathrm{d}s\ge k\Big(\|P_\alpha(b)z\|^2+\int_0^b\|P_\alpha(b-s)z\|^2\,\mathrm{d}s\Big).$$

For the the linear control system (4.3) with $f=0$, it was shown in [9] that it is exactly null controllable if

$$\int_0^b\|T(b-s)z\|^2\,\mathrm{d}s\ge b\|T(b)z\|^2.$$

It follows that

$$\frac{1}{1+b}\int_0^b\|T(b-s)z\|^2\,\mathrm{d}s\ge\frac{b}{1+b}\|T(b)z\|^2,$$

which infers

$$\int_0^b\|T(b-s)z\|^2\,\mathrm{d}s\ge\frac{b}{1+b}\Big(\|T(b)z\|^2+\int_0^b\|T(b-s)z\|^2\,\mathrm{d}s\Big).$$

Thus linear system (4.2) is exactly null controllable on $[0,b]$ with $k=\frac{b}{1+b}$.

In addition, we may assume that the function $f:[0,b]\times X\to X$ is continuous and there exists $\rho\in L^2([0,b],\mathbb{R}^+)$ such that $\|f(t,z)\|\le\rho(t)(\|z\|^{\frac{1}{3}}+1)$ for $(t,z)\in[0,b]\times X$. Let function

$$g(x(t,\theta))=\sum_{j=1}^{q}c_j\sqrt[3]{x(t_j,\theta)}.$$

Here hypotheses (H2)and (H4) are satisfied. Then by Theorem 3.3, differential system (4.1) is exactly null controllable on $[0,b]$.

5. Conclusions

In this paper, we investigate the exact null controllability of Riemann-Liouville fractional differential equations based on fractional resolvents and fixed point theorems. To see the results, the solution to Riemann-Liouville fractional differential equations is defined under the frame of the Banach space $C_{1-\alpha}(J,X)$. Moreover, some convergence results are presented to ensure the uniform continuity for fractional resolvents when $\{t^{1-\alpha}P_\alpha(t), t>0\}$ is equicontinuous and compact. Finally the controllability of fractional differential equations is transformed into a fixed point problem for discussion.

Acknowledgements
The paper was supported by National Natural Science Foundation of China (No. 11601178) and Philosophy and Social Science Foundation of Jiangsu Province Universities (No. 2024SJYB1408).

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