J. Nonlinear Funct. Anal. 2026 (2026) 11 https://doi.org/10.23952/jnfa.2026.11

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

# GLOBAL WELL-POSEDNESS FOR THE 2D INCOMPRESSIBLE CAHN-HILLIARD-MHD EQUATIONS WITH MIXED PARTIAL VISCOSITY, MAGNETIC DIFFUSION, AND MOBILITY

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**Abstract.** This paper focuses on the Cauchy problem of the 2D incompressible Cahn-Hilliard-MHD equations. We construct the global well-posedness of strong solutions to the model without full viscosity, magnetic diffusion, and mobility.

**Keywords.** Cauchy problem; Incompressible Cahn-Hilliard-MHD equations; Global well-posedness.

## 1. Introduction

The incompressible Magnetohydrodynamics (MHD) equations describe the motion of an electrically conducting fluid in the presence of the magnetic field which essentially needs to consider the interaction between the fluid velocity and the magnetic field [5, 6, 13]. The classical incompressible MHD system takes the following form

$$\begin{cases} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = B\cdot\nabla B, \\ \partial_t B + u\cdot\nabla B - \eta\Delta B = B\cdot\nabla u, \\ \operatorname{div} u = \operatorname{div} B = 0, \end{cases} \tag{1.1}$$

where $u$, $B$, and $p$ describe the velocity field, the magnetic field and the pressure respectively, $\nu \geq 0$ is the kinematic viscosity and $\eta \geq 0$ is the magnetic diffusivity. Here, it should point out that system (1.1) is only a single-phase MHD model. However, there is a variety of different multi-phase flow phenomena in real-world applications, such as the Aluminum electrolysis cells, metallurgical industry, pump accelerators, MHD generators and fusion reactors [15, 27]. Thus one needs to study the interaction of electromagnetic fields with two incompressible, immiscible and electrically conducting fluids, i.e., a two-phase MHD problem. For example, in metallurgy processes, bubbles are always injected into the molten metal for stirring and homogenizing the liquid metal and the magnetic field is imposed to control the bubble motion in a

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Received October 13, 2025; Accepted February 22, 2026.

contactless method. In MHD generators and pump accelerators, some experimental and analytical studies on the flow of two immiscible fluids in a channel under an external magnetic field are carried out [28, 30]. At present, one fundamental problem for two-phase MHD problem is the interfacial dynamics between two different incompressible fluids. In numerous situations, it may not be convenient or accurate for the classical sharp interface model to describe the topological transitions of interfaces, such as self-intersection, pinch-off, reconnection and splitting during the evolution of interface [2, 16]. In the last decades, the diffuse interface (phase field) method has been widely applied to model and simulate the topological transitions of interfaces. This method assumes that the fluids are mixed and store the mixing (elastic) energy within the thin layer of finite thickness. The surface tension force on the fluids was extensively derived via the variational approach (see, e.g., [9, 23, 37]). It was known that the sharp interface model can be recovered in the limit as the interface thickness approaches zero [3, 26]. For the extensive study on the phase field approach, we refer to [14, 24] and the references therein. For the phase field $\varphi$, the free energy of two-phase fluids is

$$E(\varphi)=\int_\Omega\left(\frac{1}{2}|\nabla\varphi|^2+\frac{1}{\varepsilon^2}F(\varphi)\right)\mathrm{d}x,$$

where $F(\varphi)$ models the immiscibility of the fluid components. The two minima of $F(\varphi)$, i.e., $\varphi=\pm1$, correspond to two stable phases of the fluids. The first term (i.e., the gradient energy) and second term (i.e., the bulk energy) of $E(\varphi)$, respectively, represent the hydrophilic and hydrophobic parts of the free energy. It is known that the Allen-Cahn equation is the $L^2$-gradient flow of the free energy $E(\varphi)$ and the Cahn-Hilliard equation is the $H^{-1}$-gradient flow of $E(\varphi)$ [17]. To preserve the mass conservation, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\varphi(x,t)\mathrm{d}x=0$, one needs to consider the following Cahn-Hilliard equations

$$\begin{cases}\varphi_t &=\operatorname{div}\left(\kappa\nabla\frac{\partial E}{\partial\varphi}\right)=\kappa\Delta\mu,\\ \mu &=\frac{\partial E}{\partial\varphi}=-\Delta\varphi+\frac{1}{\varepsilon^2}f(\varphi),\end{cases}$$

where $\mu$ represents the chemical potential which is given by the variational derivative of the energy $E$ with respect to $\varphi$, $f(\varphi)=F'(\varphi)$, and $\kappa$, $\varepsilon$ denote the mobility of the mixture and width of the interfacial layer, respectively.

Then combining the physics of MHD fluids and the phase field approach, researchers [11,32] proposed a new diffuse interface model to describe the flow of two incompressible, immiscible and electrically conducting fluids with different viscosities and electric conductivities, which is called the Cahn-Hilliard-MHD ( CH-MHD for short) model. The governing equation consists of the Cahn-Hilliard equations, Navier-Stokes equations, and the Maxwell's equations, which are coupled through convection, stresses, and Lorentz forces. More precisely, the CH-MHD system reads as follows

$$\begin{cases}\partial_t u+u\cdot\nabla u-\nu\Delta u+\nabla p=B\cdot\nabla B-\lambda\operatorname{div}(\nabla\varphi\otimes\nabla\varphi),\\ \partial_t B+u\cdot\nabla B-\eta\Delta B=B\cdot\nabla u,\\ \varphi_t+u\cdot\nabla\varphi=\kappa\Delta\mu,\\ \mu=-\Delta\varphi+\dfrac{1}{\varepsilon^2}f(\varphi),\\ \operatorname{div}u=0,\operatorname{div}B=0,\end{cases}\tag{1.2}$$

where the surface tension $\mathrm{div}(\nabla\varphi\otimes\nabla\varphi)$ is often called the Korteweg force, and $\lambda$ is capillary coefficient. From the viewpoint of partial differential equations, system (1.2) is a highly nonlinear coupled system. Such a model has extensive application prospects in the fields of nuclear fusion, metallurgy, liquid metal magnetic pumps, aluminum electrolysi,s and so on [13,22,27]. We here point out that system (1.2) includes several important models as special cases. When $B=0$, it turns into the known incompressible Navier-Stokes-Cahn-Hilliard ( NS-CH for short) system, which was studied by many researchers (see, e.g., [1, 4, 18, 21]) and the references therein. When $\varphi$ is absence, system (1.2) reduces the classical incompressible MHD equations (1.1). Due to its mathematical challenges and broad physical applications, there are extensive results on the well-posedness of solutions for this system (see, e.g., [10, 33, 36]). At present, the study on the CH-MHD model can be traced back to [11, 32], which mainly focused on the numerical scheme (see, e.g., [11, 34, 35]).

However, to the best of our knowledge, there is no result regarding on the mathematical analysis of system (1.2). Recently, there are numerous results focusing on the global well-posedness of the anisotropic viscosity for the incompressible fluids, for example, MHD equations [8, 10], Boussinesq equations [7, 12] and Navier-Stokes-Cahn-Hilliard system [4]. It should be also emphasized that experimental evidence [25, 29] shows that, in certain regimes and after suitable rescaling, the vertical/horizontal mobility is negligible with respect to the horizontal/vertical mobility. Therefore, these results motivate us to study the global well-posedness of the following 2D CH-MHD model in $\mathbb{R}_+\times\mathbb{R}^2$:

$$\begin{cases} u_t+u\cdot\nabla u+\nabla p=\nu_1 u_{xx}+\nu_2 u_{yy}+B\cdot\nabla B-\lambda\varphi\nabla\mu,\\ B_t+u\cdot\nabla B-B\cdot\nabla u=\eta_1 B_{xx}+\eta_2 B_{yy},\\ \varphi_t+u\cdot\nabla\varphi=\kappa_1\mu_{xx}+\kappa_2\mu_{yy},\\ \mu=-\Delta\varphi+\dfrac{1}{\varepsilon^2}f(\varphi),\\ \mathrm{div}\,u=0,\ \mathrm{div}\,B=0, \end{cases} \tag{1.3}$$

where the term $\lambda\varphi\nabla\mu$ in $(1.3)_1$ is the continuum surface tension force in the potential form [16,23,26], which is more convenient to get energy estimate (see, e.g., [1,19,20]) and originates from the phase induced force in the stress form

$$\lambda\,\mathrm{div}(\nabla\varphi\otimes\nabla\varphi)=\lambda\Delta\varphi\nabla\varphi+\frac{\lambda}{2}\nabla|\nabla\varphi|^2=\lambda\varphi\nabla\mu+\nabla\Big(\frac{\lambda}{\varepsilon^2}F(\varphi)-\lambda\mu\varphi+\frac{\lambda}{2}|\nabla\varphi|^2\Big),$$

where $\nabla\varphi\otimes\nabla\varphi$ is the induced elastic stress due to the mixing of the different phases. Then, the pressure in $(1.3)_1$ is given by $p+\frac{\lambda}{\varepsilon^2}F(\varphi)-\lambda\mu\varphi+\frac{\lambda}{2}|\nabla\varphi|^2$ (still denote by $p$ for simplicity). Here, we are concerned with the Cauchy problem of the system (1.3) subject to the initial data

$$u_{|t=0}=u_0,\quad B_{|t=0}=B_0,\quad \varphi_{|t=0}=\varphi_0. \tag{1.4}$$

The goal of the this paper is to establish the global regularity and uniqueness of strong solution to Cauchy problem (1.3) -(1.4). We first assume the following conditions on the regular potential $F$:

(i) $F\in C^3(\mathbb{R},\mathbb{R})$ satisfies

$$\lim_{|s|\to\infty}\inf\frac{F(s)}{s^2}>0.$$

(ii) $f = F'$ satisfies $\lim_{|s|\to\infty} \inf f'(s) > 0$, and there exists a constant $s_0 > 0$ such that, for all $|s| \geqslant s_0$, $|f''(s)| \leqslant C_f |s|^p$, for some $C_f > 0$ and $p \geqslant 1$.

Now we state our main results as follows.

**Theorem 1.1.** *Consider the 2D system (1.3) with $\nu_2 = \kappa_2 = \eta_1 = 0$, $\nu_1 > 0$, $\kappa_1 > 0$, $\eta_2 > 0$. Assume that* (i)-(ii) *hold. For any initial data $(u_0, B_0, \varphi_0) \in H^1_{\rm div}(\mathbb{R}^2) \times H^1_{\rm div}(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, the Cauchy problem (1.3) -(1.4) admits a unique global strong solution $(u, \varphi, B)$ which satisfies the equations in the sense of distributions. Moreover,*

$$
\begin{aligned}
&(u, \varphi, B) \in L^\infty([0,\infty); H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \\
&\partial_t \varphi \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \mu \in L^\infty([0,\infty); L^2(\mathbb{R}^2)), \\
&\mu_x, \quad \mu_{xx} \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \omega_x, \quad h_y \in L^2([0,\infty); L^2(\mathbb{R}^2)),
\end{aligned}
$$

*where $\omega = \nabla \times u$, $h = \nabla \times B$ represent the vorticity and the current density, respectively.*

**Theorem 1.2.** *Consider the 2D system (1.3) with $\nu_1 = \kappa_1 = \eta_2 = 0$, $\nu_2 > 0$, $\kappa_2 > 0$, $\eta_1 > 0$. Assume that* (i)-(ii) *hold. For any initial data $(u_0, B_0, \varphi_0) \in H^1_{\rm div}(\mathbb{R}^2) \times H^1_{\rm div}(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, Cauchy problem (1.3) -(1.4) admits a unique global strong solution $(u, \varphi, B)$ which satisfies the equations in the sense of distributions. Moreover,*

$$
\begin{aligned}
&(u, \varphi, B) \in L^\infty([0,\infty); H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \\
&\partial_t \varphi \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \mu \in L^\infty([0,\infty); L^2(\mathbb{R}^2)), \\
&\mu_y, \quad \mu_{yy} \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \omega_y, \quad h_x \in L^2([0,\infty); L^2(\mathbb{R}^2)).
\end{aligned}
$$

**Theorem 1.3.** *Consider the 2D system (1.3) with $\nu_1 = \kappa_2 = \eta_2 = 0$, $\nu_2 > 0$, $\kappa_1 > 0$, $\eta_1 > 0$. Assume that* (i)-(ii) *hold. For any initial data $(u_0, B_0, \varphi_0) \in H^1_{\rm div}(\mathbb{R}^2) \times H^1_{\rm div}(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, Cauchy problem (1.3) -(1.4) admits a unique global strong solution $(u, \varphi, B)$ which satisfies the equations in the sense of distributions. Moreover,*

$$
\begin{aligned}
&(u, \varphi, B) \in L^\infty([0,\infty); H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \\
&\partial_t \varphi \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \mu \in L^\infty([0,\infty); L^2(\mathbb{R}^2)), \\
&\mu_x, \quad \mu_{xx} \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \omega_y, \quad h_x \in L^2([0,\infty); L^2(\mathbb{R}^2)).
\end{aligned}
$$

**Theorem 1.4.** *Consider the 2D system (1.3) with $\nu_2 = \kappa_1 = \eta_1 = 0$, $\nu_1 > 0$, $\kappa_2 > 0$, $\eta_2 > 0$. Assume that* (i)-(ii) *hold. For any initial data $(u_0, B_0, \varphi_0) \in H^1_{\rm div}(\mathbb{R}^2) \times H^1_{\rm div}(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$, the Cauchy problem (1.3) -(1.4) admits a unique global strong solution $(u, \varphi, B)$ which satisfies the equations in the sense of distributions. Moreover, we have*

$$
\begin{aligned}
&(u, \varphi, B) \in L^\infty([0,\infty); H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times H^2(\mathbb{R}^2)) \\
&\partial_t \varphi \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \mu \in L^\infty([0,\infty); L^2(\mathbb{R}^2)), \\
&\mu_y, \quad \mu_{yy} \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad \omega_x, \quad h_y \in L^2([0,\infty); L^2(\mathbb{R}^2)).
\end{aligned}
$$

## 2. PRELIMINARIES

We first need to fix some notation. Throughout this paper, the $L^p-$norm is denoted by $\|\cdot\|_p$, the $H^s-$norm by $\|\cdot\|_{H^s}$ and the norm in the Sobolev space $W^{s,p}$ by $\|\cdot\|_{W^{s,p}}$. $\langle\cdot,\cdot\rangle_2$ and $\langle\cdot,\cdot\rangle_{H^s}$ denote the usual scalar product in $L^2$ and $H^s$, respectively. Let $H^s_{\rm div}(\mathbb{R}^2) =$ the closure of $\{u \in$

$C_0^\infty(\mathbb{R}^2)$ : div $u=0\}$ with respect to $H^s(\mathbb{R}^2)$-norm for any $s \geqslant 0$ (of course, the convention is that $H^0=L^2$). In order to establish sufficiently strong *a priori* estimates for solutions of Cauchy problem (1.3) -(1.4), the following elementary inequalities play an important role.

**Lemma 2.1.** *[10] Assume that $f, g, g_y, h$, and $h_x$ belong to $L^2(\mathbb{R}^2)$. Then, the following estimate holds:*

$$\int\int |fgh| \mathrm{d}x\, \mathrm{d}y \leqslant C\|f\|_2\|g\|_2^{1/2}\|g_y\|_2^{1/2}\|h\|_2^{1/2}\|h_x\|_2^{1/2}. \tag{2.1}$$

**Lemma 2.2.** *[12] (i) Assume that $f \in H^1(\mathbb{R}^2)$. The following inequality holds:*

$$\|f\|_4 \leqslant C\|f\|_2^{1/2}\|f_x\|_2^{1/4}\|f_y\|_2^{1/4}. \tag{2.2}$$

*(ii) Assume that $f \in H^2(\mathbb{R}^2)$. The following inequalities hold:*

$$\|f\|_6 \leqslant C\|f\|_2^{5/12}\|f_x\|_2^{1/4}\|f_y\|_2^{1/4}\|f_{xx}\|_2^{1/48}\|f_{yy}\|_2^{1/48}\|f_{xy}\|_2^{1/24}, \tag{2.3}$$

*and for $f \in H^s(\mathbb{R}^2)$ with $s=\max(s_1,s_2)>0$,*

$$\|f\|_\infty \leqslant C\big(\|f\|_2+\|\partial_x^{s_1} f\|_2+\|\partial_y^{s_2} f\|_2\big), \tag{2.4}$$

*for any $s_1, s_2 > 0$ such that $1/s_1+1/s_2<2$.*

## 3. Main Results

This section is devoted to proving Theorem 1.1. For a clear presentation, we split the proof into the following three propositions. For the sake of simplicity, we now set $\varepsilon=1$.

**Proposition 3.1.** *Under the assumptions of Theorem 1.1, let $(u,\varphi,B)$ be a smooth enough solution of the Cauchy problem (1.3) -(1.4). Then*

$$\begin{aligned}\|u(t)\|_2^2+\|\varphi(t)\|_{H^1}^2+\|B(t)\|_2^2+\int_{\mathbb{R}^2} F(\varphi)\mathrm{d}x\, \mathrm{d}y+\int_0^t \big(\nu_1\|u_x(s)\|_2^2+\kappa_1\|\mu_x(s)\|_2^2+\eta_2\|B_y\|_2^2\big)\mathrm{d}s \\ \le Q_T\big(\|u_0\|_2+\|\varphi_0\|_{H^1}+\|B_0\|_2\big),\end{aligned} \tag{3.1}$$

*for all $t\in[0,T]$, where the positive monotone function $Q_T$ depends only on $\nu_1, \kappa_1, \eta_2$.*

*Proof.* Taking the inner product in $L^2(\mathbb{R}^2)$ of $(1.3)_1$ with $u$, after integrating by parts, and owing to div$\,u=0$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\mathrm{d}x=-\nu_1\int |u_x|^2\mathrm{d}x+\int B\cdot\nabla B\cdot u\mathrm{d}x+\int \mu\nabla\varphi\cdot u\mathrm{d}x. \tag{3.2}$$

Similarly,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |B|^2\mathrm{d}x+\int u\cdot\nabla B\cdot B\mathrm{d}x=-\eta_2\int |B_y|^2\mathrm{d}x. \tag{3.3}$$

Taking the inner product in $L^2(\mathbb{R}^2)$ of $(1.3)_4$ with $\partial_t\varphi$, after integrating by parts, and owing to $F'=f$, we see that

$$\int \varphi_t\cdot\mu\mathrm{d}x=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\varphi|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int F(\varphi)\mathrm{d}x. \tag{3.4}$$

Taking the inner product in $L^2(\mathbb{R}^2)$ of $(1.3)_3$ with $\mu$, after integrating by parts, we see that

$$\int \varphi_t\cdot\mu\mathrm{d}x+\int u\cdot\nabla\varphi\cdot\mu\mathrm{d}x=-\kappa_1\int |\mu_x|^2\mathrm{d}x. \tag{3.5}$$

By summing (3.2), (3.3), (3.4), and (3.5), we arrive at

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u(t)\|_2^2+\|\nabla\varphi(t)\|_2^2+\|B(t)\|_2^2\\&\qquad+2\int_{\mathbb{R}^2}F(\varphi)\mathrm{d}x\,\mathrm{d}y\big)+2\nu_1\|u_x(t)\|_2^2+2\kappa_1\|\mu_x(t)\|_2^2+2\eta_2\|B_y(t)\|_2^2=0.\end{aligned}\tag{3.6}$$

Integrate now (3.6) over $(0,t)$ and recall that $C_{F,1}|s|^2\leqslant F(s)\leqslant C_{F,2}|s|^{p+3}$, for all $|y|\geqslant y_0$, for some $C_{F,1},C_{F,2}>0$ (these inequalities are an immediate consequence of assumptions (i) and (ii)). It then follows from the fact $H^1\subset L^s(s\geqslant 2)$ that

$$\begin{aligned}&\|u(t)\|_2^2+\|\nabla\varphi(t)\|_2^2+\|B(t)\|_2^2+2\int\int_{\mathbb{R}^2}F(\varphi(t))\mathrm{d}x\,\mathrm{d}y\\&\qquad+\int_0^t(2\nu_1\|u_x(s)\|_2^2+2\kappa_1\|\mu_x(s)\|_2^2+2\eta_2\|B_y(s)\|_2^2)\mathrm{d}s\leqslant C,\end{aligned}$$

for all $t\in[0,T]$. Moreover, taking the inner product in $L^2(\mathbb{R}^2)$ of $(1.3)_3$ with $\varphi$ and integrating by parts, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|_2^2=-2\kappa_1\langle\mu_x(t),\varphi_x(t)\rangle_2\leqslant\kappa_1(\|\mu_x(t)\|_2^2+\|\varphi_x(t)\|_2^2).\tag{3.7}$$

Finally, integrating (3.7) over $(0,t)$, the desired inequality (3.1) follows easily in view of the preceding estimate. □

In order to deduce higher-order dissipative estimates, we need exploit some estimates for the chemical potential $\mu$ as follows.

**Proposition 3.2.** *Under the assumptions of Theorem 1.1, let $(u,\varphi,B)$ be a smooth enough solution of the Cauchy problem (1.3)-(1.4). Then, for all $t\in[0,T]$,*

$$\int_0^t\|\Delta\varphi_x(s)\|_2^2\mathrm{d}s\leqslant C,\tag{3.8}$$

*where the positive constant $C$ depends only on $\nu_1,\kappa_1,\eta_2,T$, and the initial data.*

*Proof.* First, multiplying $(1.3)_4$ by $\varphi_{xx}$, integrating over $\mathbb{R}^2$, and employing the assumption (ii), Hölder's and Young's inequalities, we conclude by (2.2) that

$$\begin{aligned}\|\nabla\varphi_x\|_2^2&\leqslant C\big(\|\varphi_x\mu_x\|_1+\|(f(\varphi))_x\varphi_x\|_1\big)\\&\leqslant C\big(\|\mu_x\|_2^2+\|\varphi_x\|_2^2+\|\varphi\|_{2(p+1)}^{(p+1)}\|\varphi_x\|_4^2\big)\\&\leqslant C\big(\|\mu_x\|_2^2+\|\varphi_x\|_2^2+\|\varphi\|_{2(p+1)}^{2(p+1)}+\delta\|\varphi_x\|_4^4\big)\\&\leqslant C\big(\|\mu_x\|_2^2+\|\varphi_x\|_2^2+\|\varphi\|_{2(p+1)}^{2(p+1)}\big)+C\delta\|\varphi_x\|_2^2\|\varphi_{xx}\|_2\|\varphi_{xy}\|_2.\end{aligned}$$

Taking a sufficiently small positive $\delta$ and using $H^1\subset L^s(s\geqslant 2)$ and (3.1) imply that

$$\nabla\varphi_x\in L^2([0,T];L^2(\mathbb{R}^2)).\tag{3.9}$$

In order to deduce a proper upper bound for the $L^2$-norm of $\Delta\varphi_x$, we exploit equation (1.3)$_4$ once more. Using (3.1) and (2.2), and recalling the fact that $f$ satisfies the bound (ii), we have

$$\begin{aligned}\|\Delta\varphi_x\|_2^2 &\leqslant C(\|\mu_x\|_2^2+\|(f(\varphi))_x\|_2^2)\\ &\leqslant C\|\mu_x\|_2^2+C\|\varphi\|_{4(p+1)}^{2(p+1)}\|\varphi_x\|_4^2\\ &\leqslant C\|\mu_x\|_2^2+C\big(1+\|\varphi_{xx}\|_2^2+\|\varphi_{xy}\|_2^2\big),\end{aligned}$$

which together with (3.1) and (3.9) implies that (3.8) holds.

**Proposition 3.3.** *Under the assumptions of Theorem 1.1, let $(u,\varphi,B)$ be a smooth enough solution of the Cauchy problem (1.3)-(1.4). Then, for all $t\in[0,T]$,*

$$\|\omega(t)\|_2^2+\|\varphi(t)\|_{H^2}^2+\|h(t)\|_2^2+\int_0^t(\nu_1\|\omega_x(s)\|_2^2+\kappa_1\|\mu_{xx}(s)\|_2^2+\eta_2\|h_y(s)\|_2^2)\mathrm{d}s\leqslant C, \tag{3.10}$$

*where the positive constant $C$ depends only on $\nu_1,\kappa_1,\eta_2,T$, and the initial data.*

**Proof.** Observe that $\omega,h$ and $j=\Delta\varphi$ solve the following system

$$\begin{cases}\partial_t\omega+u\cdot\nabla\omega=\nu_1\omega_{xx}-d\cdot\nabla j+B\cdot\nabla h,\\ h_t+u\cdot\nabla h=\eta_2h_{yy}+B\cdot\nabla\omega+2\partial_xB_1(\partial_xu_2+\partial_yu_1)-2\partial_xu_1(\partial_xB_2+\partial_yB_1),\\ \partial_tj+\kappa_1\Delta j_{xx}+u\cdot\nabla j+d\cdot\nabla\omega=-2\nabla u_1\cdot\nabla\varphi_x-2\nabla u_2\cdot\nabla\varphi_y+\kappa_1\Delta(f(\varphi))_{xx},\end{cases} \tag{3.11}$$

where $d=\nabla^\perp\varphi,\quad \nabla^\perp=(\partial_y,-\partial_x)$. Denote

$$\mathscr{E}(t):=\|\omega(t)\|_2^2+\|j(t)\|_2^2+\|h(t)\|_2^2,\quad \mathscr{R}(t):=C(1+\|j_x\|_2^2+\|u_x\|_2^2+\|B_y\|_2^2).$$

Taking the $L^2$-scalar product of (3.11)$_1$, (3.11)$_2$, and (3.11)$_3$ with $\omega$, $j$ and $h$ respectively, we conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t)+2\nu_1\|\omega_x\|_2^2+2\kappa_1\|\nabla j_x\|_2^2+2\eta_2\|h_y\|_2^2\\ &\quad=-2\int\int(\nabla u_2\cdot\nabla\varphi_y)j\,\mathrm{d}x\,\mathrm{d}y-2\int\int(\nabla u_1\cdot\nabla\varphi_x)j\,\mathrm{d}x\,\mathrm{d}y+2\kappa_1\int\int\Delta(f(\varphi))_{xx}j\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad+2\int[\partial_xB_1(\partial_xu_2+\partial_yu_1)-\partial_xu_1(\partial_xB_2+\partial_yB_1)]h\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\triangleq I_1+I_2+I_3+I_4.\end{aligned} \tag{3.12}$$

We bound term by term above in what follows. For $I_3$, we integrate by parts twice and divide $I_3$ further into the following two terms:

$$\begin{aligned}I_3&=2\kappa_1\int\int f'(\varphi)\nabla\varphi_x\cdot\nabla j_x\mathrm{d}x\,\mathrm{d}y+2\kappa_1\int\int f''(\varphi)\varphi_x\nabla\varphi\cdot\nabla j_x\mathrm{d}x\,\mathrm{d}y\\ &\triangleq I_{31}+I_{32}.\end{aligned}$$

Using (2.1), (2.2), the assumption (ii) on $f$, Young's inequality, $H^1 \subset L^s(s \geqslant 2)$, and (3.3), we obtain

$$\begin{aligned}|I_{31}| &\leqslant C\|\nabla j_x\|_2 \|f'(\varphi)\|_2^{1/2}\|f''(\varphi)\varphi_y\|_2^{1/2}\|\nabla\varphi_x\|_2^{1/2}\|\nabla\varphi_{xx}\|_2^{1/2}\\ &\leqslant \frac{\kappa_1}{8}\|\nabla j_x\|_2^2 + C\|f''(\varphi)\|_2\|f'(\phi)\varphi_y\|_2\|\nabla\varphi_x\|_2\|\nabla\varphi_{xx}\|_2\\ &\leqslant \frac{\kappa_1}{8}\|\nabla j_x\|_2^2 + C(1+\|\varphi\|_{2(p+1)}^{p+1})\\ &\quad\times [C\|\varphi\|_{4p}^p\|\varphi_y\|_2^{1/4}\|\varphi_{xy}\|_2^{1/4}\|\varphi_{yy}\|_2^{1/4}]\|\nabla\varphi_x\|_2\|\nabla\varphi_{xx}\|_2.\end{aligned}$$

It follows from (2.1), (2.2), (2.3), assumption (ii) for the function $f$ and Young's inequality, thanks to $H^1 \subset L^s(s \geqslant 2)$ and (3.3), we deduce that

$$\begin{aligned}|I_{32}| &\leqslant C(\|\nabla\varphi\|_4\|\varphi_x\|_2\|\nabla j_x\|_2 + \|\phi\|_{12(2p+1)}^{2p+1}\|\nabla\varphi\|_4\|\varphi_x\|_6\|\nabla j_x\|_2)\\ &\leqslant C\|\nabla\varphi\|_2^{1/2}\|\nabla\varphi_x\|_2^{1/4}\|\nabla\varphi_y\|_2^{1/4}\|\nabla j_x\|_2\\ &+C\|\varphi\|_{12(2p+1)}^{2p+1}[\|\nabla\varphi\|_2^{1/2}\|\nabla\varphi_x\|_2^{1/4}\|\nabla\varphi_y\|_2^{1/4}\|\varphi_x\|_2^{5/12}\|\varphi_{xx}\|_2^{1/4}\|\varphi_{xy}\|_2^{1/4}]\\ &\times\|\varphi_{xxx}\|_2^{1/48}\|\varphi_{xyy}\|_2^{1/48}\|\varphi_{xxy}\|_2^{1/24}\|\nabla j_x\|_2\\ &\leqslant C\|\nabla j_x\|_2\|j\|_2^{1/2} + C\|j\|_2\|\Delta\varphi_x\|_2^{1/12}.\end{aligned}$$

In order to deal with the term $I_1$, we split $I_1$ into the following two parts:

$$I_1 = -2\int\int u_{2x}\varphi_{xy} j \,\mathrm{d}x\,\mathrm{d}y - 2\int\int u_{2y}\varphi_{yy} j \,\mathrm{d}x\,\mathrm{d}y \triangleq I_{11} + I_{12}.$$

Applying (2.1) to $I_{11}$ yields

$$\begin{aligned}|I_{11}| &\leqslant C\|j\|_2\|u_{2x}\|_2^{1/2}\|u_{2xx}\|_2^{1/2}\|\varphi_{xy}\|_2^{1/2}\|\varphi_{xyy}\|_2^{1/2}\\ &\leqslant C\|j\|_2^{3/2}\|u_{2x}\|_2^{1/2}\|\omega_x\|_2^{1/2}\|\Delta\varphi_x\|_2^{1/2}\\ &\leqslant \frac{\nu_1}{8}\|\omega_x\|_2^2 + C\|j\|_2^2\|u_{2x}\|_2^{2/3}\|\Delta\varphi_x\|_2^{2/3}.\end{aligned}$$

Using (2.1) to $I_{12}$ again and the incompressible condition $\partial_x u_1 + \partial_y u_2 = 0$, we have

$$\begin{aligned}|I_{12}| &\leqslant C\|j\|_2\|u_{2y}\|_2^{1/2}\|u_{2yy}\|_2^{1/2}\|\varphi_{yy}\|_2^{1/2}\|\varphi_{xyy}\|_2^{1/2}\\ &\leqslant C\|j\|_2^{3/2}\|u_{2y}\|_2^{1/2}\|\omega_x\|_2^{1/2}\|\Delta\varphi_x\|_2^{1/2}\\ &\leqslant \frac{\nu_1}{8}\|\omega_x\|_2^2 + C\|j\|_2^2\|u_{1x}\|_2^{2/3}\|\Delta\varphi_x\|_2^{2/3}.\end{aligned}$$

In what follows, we bound the term $I_2$ on the right-hand side of (3.12). We rewrite $I_2$ into the following form

$$I_2 = -2\int\int u_{1x}\varphi_{xx} j \,\mathrm{d}x\,\mathrm{d}y - 2\int\int u_{1y}\varphi_{xy} j \,\mathrm{d}x\,\mathrm{d}y \triangleq I_{21} + I_{22}.$$

For $I_{21}$, employing Lemma 2.1, (3.3) and (3.8), we deduce that

$$\begin{aligned}|I_{21}| &\leqslant C\|j\|_2\|u_{1x}\|_2^{1/2}\|u_{1xy}\|_2^{1/2}\|\varphi_{xx}\|_2^{1/2}\|\varphi_{xxx}\|_2^{1/2}\\ &\leqslant C\|j\|_2^{3/2}\|u_{1x}\|_2^{1/2}\|\omega_x\|_2^{1/2}\|\Delta\varphi_x\|_2^{1/2}\\ &\leqslant \frac{\nu_1}{8}\|\omega_x\|_2^2 + C\|j\|_2^2(1+\|u_{1x}\|_2^2+\|\Delta\varphi_x\|_2^2).\end{aligned}$$

In order to estimate $I_{22}$, we split it into two parts and integrate by parts twice

$$\begin{aligned}I_{22} &= 2\int\int (u_{1y}j)_x\varphi_y \mathrm{d}x\,\mathrm{d}y = 2\int\int u_{1xy}j\varphi_y \mathrm{d}x\,\mathrm{d}y + 2\int\int u_{1y}j_x\varphi_y \mathrm{d}x\,\mathrm{d}y\\ &= 2\int\int u_{1xy}j\varphi_y \mathrm{d}x\,\mathrm{d}y - 2\int\int u_1 j_{xy}\varphi_y \mathrm{d}x\,\mathrm{d}y - 2\int\int u_1 j_x\varphi_{yy} \mathrm{d}x\,\mathrm{d}y\\ &\triangleq I_{221} + I_{222} + I_{223}.\end{aligned}$$

Due to the incompressible condition $\partial_x u_1 + \partial_y u_2 = 0$, thanks to (2.1) and (3.1), we have

$$\begin{aligned}|I_{221}| &\leqslant C\|u_{1xy}\|_2\|j\|_2^{1/2}\|j_x\|_2^{1/2}\|\varphi_y\|_2^{1/2}\|\varphi_{yy}\|_2^{1/2}\\ &\leqslant C\|\omega_x\|_2\|j\|_2\|\Delta\varphi_x\|_2^{1/2}\\ &\leqslant \frac{\nu_1}{8}\|\omega_x\|_2^2 + C\|j\|_2^2\|\Delta\varphi_x\|_2.\end{aligned}$$

Similarly,

$$\begin{aligned}|I_{222}| &\leqslant C\|j_{xy}\|_2\|u_1\|_2^{1/2}\|u_{1x}\|_2^{1/2}\|\varphi_y\|_2^{1/2}\|\varphi_{yy}\|_2^{1/2}\\ &\leqslant C\|\nabla j_x\|_2\|u_{1x}\|_2^{1/2}\|j\|_2^{1/2}\\ &\leqslant \frac{\kappa_1}{4}\|\nabla j_x\|_2^2 + C\|j\|_2\|u_{1x}\|_2,\end{aligned}$$

and

$$\begin{aligned}|I_{223}| &\leqslant C\|\varphi_{yy}\|_2\|u_1\|_2^{1/2}\|u_{1x}\|_2^{1/2}\|j_x\|_2^{1/2}\|j_{xy}\|_2^{1/2}\\ &\leqslant C\|j\|_2\|u_{1x}\|_2^{1/2}\|\Delta\varphi_x\|_2^{1/2}\|\nabla j_x\|_2^{1/2}\\ &\leqslant \frac{\kappa_1}{4}\|\nabla j_x\|_2^2 + C(\|j\|_2^2\|u_{1x}\|_2 + \|\Delta\varphi_x\|_2^2),\end{aligned}$$

Let us turn to the estimate of $I_4$. Denote

$$\begin{aligned}I_4 &= \int \partial_x B_1\partial_x u_2 h \mathrm{d}x\,\mathrm{d}y + \int \partial_x B_1\partial_y u_1 h \mathrm{d}x\,\mathrm{d}y\\ &\quad + \int \partial_x u_1\partial_x B_2 h \mathrm{d}x\,\mathrm{d}y + \int \partial_x u_1\partial_y B_1 h \mathrm{d}x\,\mathrm{d}y\\ &\triangleq I_{41} + I_{42} + I_{43} + I_{44}.\end{aligned}$$

For $I_{41}$, due to the incompressible condition $\partial_x B_1 + \partial_y B_2 = 0$, thanks to (2.1) and (3.1), we have

$$\begin{aligned}|I_{41}| &= \left|\int \partial_x B_1\partial_x u_2 h \mathrm{d}x\,\mathrm{d}y\right|\\ &= \left|-\int \partial_y B_2\partial_x u_2 h \mathrm{d}x\,\mathrm{d}y\right|\\ &\leqslant C\|\partial_x u_2\|_2^{1/2}\|\partial_{xx}u_2\|_2^{1/2}\|h\|_2^{1/2}\|h_y\|_2^{1/2}\|\partial_y B_2\|_2\\ &\leqslant \frac{\nu_1}{8}\|\partial_{xx}u_2\|_2^2 + \frac{\eta_2}{8}\|h_y\|_2^2 + C\|\partial_x u_2\|_2\|\partial_y B_2\|^2\|h\|_2\\ &\leqslant \frac{\nu_1}{8}\|\omega_x\|_2^2 + \frac{\eta_2}{8}\|h_y\|_2^2 + C\|\omega\|_2\|\partial_y B_2\|_2^2\|h\|_2,\end{aligned}$$

$$\begin{aligned}
|I_{42}| &= \left|\int \partial_x B_1 \partial_y u_1 h \mathrm{d}x\, \mathrm{d}y\right| \\
&= \left|\int (u_1 \partial_{xy} B_1 h + u_1 \partial_x B_1 h_y) \mathrm{d}x\, \mathrm{d}y\right| \\
&\leqslant C\|u_1\|_2^{\frac{1}{2}} \|\partial_x u_1\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_y\|_2^{\frac{1}{2}} \|\partial_{xy} B_1\|_2 \\
&\quad + C\|u_1\|_2^{\frac{1}{2}} \|\partial_x u_1\|_2^{\frac{1}{2}} \|\partial_x B_1\|_2^{\frac{1}{2}} \|\partial_{xy} B_1\|_2^{\frac{1}{2}} \|h_y\|_2 \\
&\leqslant C\|u_1\|_2^{\frac{1}{2}} \|\partial_x u_1\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_y\|_2^{\frac{3}{2}} + C\|u_1\|_2^{\frac{1}{2}} \|\partial_x u_1\|_2^{\frac{1}{2}} \|\partial_x B_1\|_2^{\frac{1}{2}} \|h_y\|_2^{\frac{3}{2}} \\
&\leqslant \frac{\eta_2}{8} \|h_y\|_2^2 + C\|u_1\|_2^2 \|\partial_x u_1\|_2^2 \|h\|_2^2 + C\|u_1\|_2^2 \|\partial_x u_1\|_2^2 \|\partial_x B_1\|_2^2 \\
&\leqslant \frac{\eta_2}{8} \|h_y\|_2^2 + C\|u_1\|_2^2 \|\partial_x u_1\|_2^2 \|h\|_2^2, \\
|I_{43}| &= \left|\int \partial_x u_1 \partial_x B_2 h \mathrm{d}x\, \mathrm{d}y\right| \\
&= \left|\int \partial_y u_2 \partial_x B_2 h \mathrm{d}x\, \mathrm{d}y\right| \\
&= \left|\int (u_2 \partial_{xy} B_2 h + u_2 \partial_x B_2 h_y) \mathrm{d}x\, \mathrm{d}y\right| \\
&\leqslant C\|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_y\|_2^{\frac{1}{2}} \|\partial_{xy} B_2\|_2 \\
&\quad + C\|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_x B_2\|_2^{\frac{1}{2}} \|\partial_{xy} B_2\|_2^{\frac{1}{2}} \|h_y\|_2 \\
&\leqslant C\|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_y\|_2^{\frac{3}{2}} + C\|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_x B_2\|_2^{\frac{1}{2}} \|h_y\|_2^{\frac{3}{2}} \\
&\leqslant \frac{\eta_2}{8} \|h_y\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|h\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|\partial_x B_2\|_2^2 \\
&\leqslant \frac{\eta_2}{8} \|h_y\|_2^2 + C\|u_2\|_2^2 \|\partial_x u_2\|_2^2 \|h\|_2^2,
\end{aligned}$$

and

$$\begin{aligned}
|I_{44}| &= \left|\int \partial_x u_1 \partial_y B_1 h \mathrm{d}x\, \mathrm{d}y\right| \\
&\leqslant C\|\partial_x u_1\|_2^{1/2} \|\partial_{xx} u_1\|_2^{1/2} \|h\|_2^{1/2} \|h_y\|_2^{1/2} \|\partial_y B_1\|_2 \\
&\leqslant \frac{\nu_1}{8} \|\partial_{xx} u_1\|_2^2 + \frac{\eta_2}{8} \|h_y\|_2^2 + C\|\partial_x u_2\|_2 \|\partial_y B_1\|^2 \|h\|_2 \\
&\leqslant \frac{\nu_1}{8} \|\omega_x\|_2^2 + \frac{\eta_2}{8} \|h_y\|_2^2 + C\|\omega\|_2 \|\partial_y B_1\|_2^2 \|h\|_2.
\end{aligned}$$

Inserting all the above estimates into the right-hand side of (3.12), for all $t \geqslant 0$, we finally conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) + \nu_1 \|\omega_x(t)\|_2^2 + \kappa_1 \|\nabla j_x\|_2^2 + \eta_2 \|h_y\|_2^2 \leqslant C\mathscr{R}(t)\mathscr{E}(t) + \mathscr{R}(t). \tag{3.13}$$

Due to (3.1), we easily deduce

$$\int_0^T \mathscr{R}(s)\mathrm{d}s \leqslant C, \quad \forall T > 0, \tag{3.14}$$

for some positive constant $C$ that depends only on the initial data $(u_0, \varphi_0)$. Combining with (3.13) and (3.14), and applying Gronwall's inequality implies that (3.10) holds for all $t \in [0,T]$. □

On account of (3.10), it is not difficult to find the following estimates.

**Corollary 3.4.** *Under the assumptions of Theorem 1.1, let $(u,\varphi,B)$ be a smooth enough solution of the Cauchy problem (1.3)-(1.4). Then, for all $t \in [0,T]$,*

$$\int_0^t \|\partial_{xx}\mu(s)\|_2^2 \mathrm{d}s \leqslant C, \quad \sup_{t\in[0,T]} \|\varphi(t)\|_\infty \leqslant C, \quad \sup_{t\in[0,T]} \{\|\Upsilon_1\varphi(t)\|_\infty, \|\Upsilon_2\varphi(t)\|_\infty\} \leqslant C, \tag{3.15}$$

*where $\Upsilon_j\varphi := \int_0^1 f^{(j)}(\tau\varphi_1 + (1-\tau)\varphi_2)\mathrm{d}\tau$,* for $j \geqslant 1$, *the positive constant $C$ depends only on $\nu_1, \kappa_1, \eta_2, T$ and the initial data.*

Now we are in a position to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Based on the *a priori* estimates in Propositions 3.1, 3.2 and 3.3, the rigorous proof of these estimates can be achieved by introducing a smooth-out version of (1.3)-(1.4) by means of artificial viscosity, magnetic diffusivity and mobility. More precisely, let $\alpha > 0$ and $\delta > 0$ be two small parameters and consider a family of solutions $(u_{\alpha,\delta}, \varphi_{\alpha,\delta}, B_{\alpha,\delta})$ satisfying

$$\begin{cases} u_t + u\cdot\nabla u + \nabla p = \nu_1 u_{xx} + \alpha\Delta u + B\cdot\nabla B - \lambda\varphi\nabla\mu, \\ B_t + u\cdot\nabla B - B\cdot\nabla u = \eta_2 B_{yy} + \alpha\Delta B, \\ \varphi_t + u\cdot\nabla\varphi = \kappa_1\mu_{xx} + \alpha\Delta\mu, \\ \mu = -\Delta\varphi + f(\varphi), \\ \operatorname{div} u = 0, \operatorname{div} B = 0, \end{cases} \tag{3.16}$$

subject to the following set of initial conditions

$$u_{\alpha,\delta}(0) = \psi_\delta * u_0, \quad \varphi_{\alpha,\delta}(0) = \psi_\delta * \varphi_0, \quad B_{\alpha,\delta}(0) = \psi_\delta * B_0, \tag{3.17}$$

where $\psi_\delta(x) = \delta^{-2}\psi(x/\delta)$ for a smooth function $\psi \geqslant 0$, $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\|\psi\|_1 = 1$. Since $u_{\alpha,\delta}(0), B_{\alpha,\delta}(0) \in H^2_{\text{div}}$ and $\varphi_{\alpha,\delta}(0) \in H^4$ are smooth, the standard theory for the regularized system (3.16)-(3.17) (see, e.g., [31]) guarantees that there exists a smooth solution

$$(u_{\alpha,\delta}, \varphi_{\alpha,\delta}, B_{\alpha,\delta}) \in L^\infty(\mathbb{R}_+; H^2_{\text{div}} \times H^4 \times H^2_{\text{div}}).$$

Actually, using bootstrap arguments, one can check that this regularity controls higher Sobolev norms. As the initial data $(u_{\alpha,\delta}(0), \varphi_{\alpha,\delta}(0), B_{\alpha,\delta}(0))$ belongs to $H^k_{\text{div}} \times H^{k+2} \times H^k_{\text{div}}$, for all $k \geqslant 0$, the solution $(u_{\alpha,\delta}, \varphi_{\alpha,\delta}, B_{\alpha,\delta})$ thus belongs to $H^k_{\text{div}} \times H^{k+2} \times H^k_{\text{div}}$, for all $k \geqslant 0$, which will enable us to make all the previous computations rigorous. Furthermore, it is easy to see that $(u_{\alpha,\delta}, \varphi_{\alpha,\delta}, B_{\alpha,\delta})$ obeys the *a priori* estimates in Propositions 3.1, 3.2 and 3.3 uniformly in $(\alpha,\delta)$. Hence, we can pass to the limit as $\alpha \to 0^+$ and $\delta \to 0^+$ in (3.16)-(3.17) and deduce that $(u,\varphi,B)$ is indeed a solution of (1.3)-(1.4) that satisfies properties (1.1).

As a final step, it remains to show that $(u,\varphi,B)$ is also uniquely determined by the initial data. To this end, let us set $\psi := \varphi - \overline{\varphi}, \chi := B - \overline{B}, v := u - \overline{u}$ and $\mu := \mu_1 - \mu_2$, where $(u,\varphi,B)$

and $(\overline{u},\overline{\varphi},\overline{B})$ are any two solutions of the Cauchy problem (1.3)-(1.4). Then we easily realize that $(v,\varphi,\chi)$ solves the system

$$\begin{cases}\partial_t v+\overline{u}\cdot\nabla v+v\cdot\nabla u+\nabla\widetilde{p}=\chi\cdot\nabla B+\bar{B}\cdot\nabla\chi+\nu_1 v_{xx}-\Delta\psi\nabla\varphi-\Delta\overline{\varphi}\nabla\psi,\\ \partial_t\chi+v\cdot\nabla B+\bar{u}\cdot\nabla\chi=\eta_2\chi_{yy}+\chi\cdot\nabla u+\bar{B}\cdot\nabla v,\\ \partial_t\psi+\overline{u}\cdot\nabla\psi+v\cdot\nabla\varphi=\kappa_1\mu_{xx},\\ \mu=-\Delta\psi+f(\varphi)-f(\bar{\varphi}),\\ \mathrm{div}v=0,\quad \mathrm{div}\chi=0,\\ v_{|t=0}=0,\quad \psi_{|t=0}=0,\quad \chi_{|t=0}=0.\end{cases}\tag{3.18}$$

Taking the inner product in $L^2(\mathbb{R}^2)$ of $(3.18)_1$ with $v$, $(3.18)_2$ with $\chi$, $(3.18)_3$ with $-\Delta\psi,\psi$ and $(3.18)_4$ with $-\kappa_1\mu_{xx}$, respectively, we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v(t)\|_2^2+\|\chi(t)\|_2^2+\|\psi(t)\|_{H^1}^2)+\nu_1\|v_x\|_2^2+\kappa_1\|\mu_x\|_2^2+\eta_2\|\chi_y\|_2^2\\ &\qquad=-\kappa_1\left\langle\mu_x,\psi_x\right\rangle_2-\langle v\cdot\nabla u,v\rangle_2+\langle-\Delta\overline{\varphi}\nabla\psi,v\rangle_2+\langle\overline{u}\cdot\nabla\psi,\Delta\psi\rangle_2\\ &\qquad\quad-\langle v\cdot\nabla\varphi,\psi\rangle_2+\kappa_1\left\langle\partial_x(f(\varphi)-f(\overline{\varphi})),\mu_x\right\rangle_2+\langle\chi\cdot\nabla u,\chi\rangle_2\\ &\qquad\triangleq J_1+J_2+J_3+J_4+J_5+J_6+J_7.\end{aligned}\tag{3.19}$$

Obviously,

$$J_1=\kappa_1\langle\mu_x,\psi_x\rangle_2\leqslant\frac{\kappa_1}{2}\|\mu_x\|_2^2+C\|\psi\|_{H^1}^2.$$

In order to bound $J_2$, we write it explicitly and split it up in three integrals as follows:

$$\begin{aligned}J_2&=\int\int(v_1)^2u_{1x}\mathrm{d}x\,\mathrm{d}y+\int\int v_1v_2u_{2x}\mathrm{d}x\,\mathrm{d}y+\int\int(v_2)^2u_{2y}\mathrm{d}x\,\mathrm{d}y\\ &\triangleq J_{21}+J_{22}+J_{23}.\end{aligned}$$

Using (2.2), the incompressible condition $\partial_x v_1+\partial_y v_2=0$ and the energy estimate (3.10), we deduce that

$$\begin{aligned}J_{21}&=-2\int\int v_1v_{1x}u_1\mathrm{d}x\,\mathrm{d}y\\ &\leqslant C\|v_{1x}\|_2\|v_1\|_2^{1/2}\|v_{1x}\|_2^{1/2}\|u_1\|_2^{1/2}\|u_{1y}\|_2^{1/2}\\ &\leqslant C\|v\|_2^{1/2}\|v_x\|_2^{3/2}\\ &\leqslant\frac{\nu_1}{16}\|v_x\|_2^2+C\|v\|_2^2.\end{aligned}$$

Similarly, for $J_{22}$ and $J_{23}$, we have

$$J_2\leqslant\frac{3\nu_1}{16}\|v_x\|_2^2+C\|v\|_2^2.\tag{3.20}$$

Similarly,

$$J_7\leqslant\frac{3\nu_1}{16}\|v_x\|_2^2+C\|\chi\|_2^2.\tag{3.21}$$

In view of (2.1), (2.4), and (3.10) and the incompressible condition $\partial_x v_1 + \partial_y v_2 = 0$, we find from Young's inequality that

$$\begin{aligned} J_3 &\leqslant C\|\Delta\overline{\varphi}\|_2\|\psi_x\|_\infty\|v\|_2 + C\|\Delta\overline{\varphi}\|_2\|\psi_y\|_2^{1/2}\|\psi_{xy}\|_2^{1/2}\|v_y\|_2^{1/2}\|v\|_2^{1/2} \\ &\leqslant C\|v\|_2^2 + \eta_2(\|\nabla\psi\|_2^2 + \|\Delta\psi_x\|_2^2) \\ &\quad + \frac{\nu_1}{8}\|v_x\|_2^2 + C\|v\|_2^{2/3}\|\nabla\psi\|_2^{2/3}\|\nabla\psi_x\|_2^{2/3} \\ &\leqslant \frac{\nu_1}{8}\|v_x\|_2^2 + C(\|v\|_2^2 + \|\nabla\psi\|_2^2) + \eta_2(\|\Delta\psi_x\|_2^2 + \|\nabla\psi_x\|_2^2), \end{aligned}$$

for every $\eta \in (0, \kappa/2)$. Collecting the above estimates, we see that

$$J_3 \leqslant \frac{\nu_1}{2}\|v_x\|_2^2 + C(\|v\|_2^2 + \|\nabla\psi\|_2^2) + \eta_2\|(\Delta\psi)_x\|_2^2. \tag{3.22}$$

In order to properly estimate $J_4 + J_5$, one proceeds exactly as for the above integral $J_3$. Indeed, similar computations give without too much difficulty that

$$J_4 + J_5 \leqslant \frac{\nu_1}{2}\|v_x\|_2^2 + C(\|v\|_2^2 + \|\nabla\psi\|_2^2) + \eta_2\|(\Delta\psi)_x\|_2^2, \tag{3.23}$$

for some sufficiently small $\eta_2 < \kappa_1/2$. For $J_6$, applying (3.15) yields

$$J_6 \leqslant \frac{\kappa_1}{4}\|\mu_x\|_2^2 + C\|\partial_x(f(\varphi) - f(\overline{\varphi}))\|_2^2 \leqslant \frac{\kappa_1}{4}\|\mu_x\|_2^2 + C(\|\nabla\psi\|_2^2 + \|\psi\|_2^2). \tag{3.24}$$

Finally, it follows from $(3.18)_4$ and (3.15), that

$$\|j_x\|_2^2 \leqslant \|\mu_x\|_2^2 - \|\partial_x(f(\varphi) - f(\overline{\varphi}))\|_2^2 \leqslant \|\mu_x\|_2^2 + C(\|\nabla\psi\|_2^2 + \|\psi\|_2^2). \tag{3.25}$$

Plugging (3.20), (3.21), (3.22), (3.23), (3.24), and (3.25) into (3.19) yields that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|v(t)\|_2^2 + \|\chi(t)\|_2^2 + \|\psi(t)\|_{H^1}^2) + \nu_1\|v_x\|_2^2 + \kappa_1\|\mu_x\|_2^2 + \eta_2\|\chi_y\|_2^2 \\ &\qquad \leqslant C(\|v(t)\|_2^2 + \|\chi(t)\|_2^2 + \|\psi(t)\|_{H^1}^2), \end{aligned}$$

which together with Gronwall's inequality concludes the uniqueness holds.

The purpose of this section is to exhibit the proof of Theorem 1.2. We divide it into the following three propositions.

**Proposition 3.5.** *Under the assumptions of Theorem 1.2, let $(u, \varphi, B)$ be a smooth enough solution of the Cauchy problem (1.3) -(1.4). Then*

$$\begin{aligned} \|u(t)\|_2^2 + \|\varphi(t)\|_{H^1}^2 + \|B(t)\|_2^2 + \int\int_{\mathbb{R}^2} F(\varphi)\mathrm{d}x\,\mathrm{d}y + \int_0^t (\nu_2\|u_y(s)\|_2^2 + \kappa_2\|\mu_x(y)\|_2^2 + \eta_1\|B_x\|_2^2)\mathrm{d}s \\ \leq Q_T\big(\|u_0\|_2 + \|\varphi_0\|_{H^1} + \|B_0\|_2\big), \end{aligned} \tag{3.26}$$

*for all $t \in [0, T]$, where the positive monotone function $Q_T$ depends only on $\nu_2, \kappa_2$, and $\eta_1$.*

**Proof.** The derivation is similar to Proposition 3.1, we omit it here.

**Proposition 3.6.** *Under the assumptions of Theorem 1.2, let $(u, \varphi, B)$ be a smooth enough solution of the Cauchy problem (1.3)-(1.4). Then, for all $t \in [0, T]$,*

$$\int_0^t \|\Delta\varphi_y(s)\|_2^2\mathrm{d}s \leqslant C, \tag{3.27}$$

*where the positive constant $C$ depends only on $\nu_2, \kappa_2, \eta_1, T$, and the initial data.*

**Proof.** First, multiplying (1.3)$_4$ by $\varphi_{yy}$, integrating over $\mathbb{R}^2$, and employing the assumption (ii), Hölder's and Young's inequalities, thanks to (2.2), we conclude that

$$\begin{aligned}\|\nabla\varphi_y\|_2^2 &\leqslant C\big(\|\varphi_y\mu_y\|_1 + \|(f(\varphi))_y\varphi_y\|_1\big)\\ &\leqslant C\big(\|\mu_y\|_2^2 + \|\varphi_y\|_2^2 + \|\varphi\|_{2(p+1)}^{(p+1)}\|\varphi_y\|_4^2\big)\\ &\leqslant C\big(\|\mu_y\|_2^2 + \|\varphi_y\|_2^2 + \|\varphi\|_{2(p+1)}^{2(p+1)} + \tilde{\delta}\|\varphi_y\|_4^4\big)\\ &\leqslant C\big(\|\mu_y\|_2^2 + \|\varphi_y\|_2^2 + \|\varphi\|_{2(p+1)}^{2(p+1)}\big) + C\tilde{\delta}\|\varphi_y\|_2^2\|\varphi_{yy}\|_2\|\varphi_{xy}\|_2.\end{aligned}$$

Taking a sufficiently small positive $\tilde{\delta}$ and using $H^1 \subset L^s(s \geqslant 2)$ and (3.26) implies that

$$\nabla\varphi_y \in L^2([0,T]; L^2(\mathbb{R}^2)). \tag{3.28}$$

In order to deduce a proper upper bound for the $L^2$-norm of $\Delta\varphi_y$, we exploit equation (1.3)$_4$ once more. Using (3.26) and (2.2), and recalling the fact that $f$ satisfies bound (ii), we have

$$\begin{aligned}\|\Delta\varphi_y\|_2^2 &\leqslant C(\|\mu_y\|_2^2 + \|(f(\varphi))_y\|_2^2)\\ &\leqslant C\|\mu_y\|_2^2 + C\|\varphi\|_{4(p+1)}^{2(p+1)}\|\varphi_y\|_4^2\\ &\leqslant C\|\mu_y\|_2^2 + C\big(1 + \|\varphi_{yy}\|_2^2 + \|\varphi_{xy}\|_2^2\big),\end{aligned}$$

which together with (3.26) and (3.28) implies that (3.27) holds.

**Proposition 3.7.** *Under the assumptions of Theorem 1.2, let $(u, \varphi, B)$ be a smooth enough solution of the Cauchy problem (1.3)-(1.4). Then*

$$\|\omega(t)\|_2^2 + \|\varphi(t)\|_{H^2}^2 + \|h(t)\|_2^2 + \int_0^t (\nu_2\|\omega_y(s)\|_2^2 + \kappa_2\|\mu_{xx}(s)\|_2^2 + \eta_1\|h_x(s)\|_2^2)\mathrm{d}s \leqslant C, \tag{3.29}$$

*for all $t \in [0,T]$, for some positive constant $C$ which depends only on $\nu_2, \kappa_2, \eta_1, T$ and the initial data.*

*Proof.* Note that $j = \Delta\varphi$ and $h = \nabla \times B$ solves the following system

$$\begin{cases}\partial_t\omega + u\cdot\nabla\omega = \nu_2\omega_{yy} - d\cdot\nabla j + B\cdot\nabla h,\\ \partial_t h + u\cdot\nabla h = \eta_1 h_{xx} + B\cdot\nabla\omega + 2\partial_x B_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x B_2 + \partial_y B_1),\\ \partial_t j + \kappa_2\Delta j_{yy} + u\cdot\nabla j + d\cdot\nabla\omega = -2\nabla u_1\cdot\nabla\varphi_x - 2\nabla u_2\cdot\nabla\varphi_y + \kappa_2\Delta(f(\varphi))_{yy}.\end{cases} \tag{3.30}$$

Set

$$\tilde{\mathscr{E}}(t) := \|\omega(t)\|_2^2 + \|j(t)\|_2^2 + \|h(t)\|_2^2, \quad \tilde{\mathscr{R}}(t) := C(1 + \|j_y\|_2^2 + \|u_y\|_2^2 + \|\nabla\varphi_y\|_2^2).$$

First, take the inner products in $L^2$ of (3.30)$_1$ with $\omega$, (3.30)$_2$ with $h$ and (3.30)$_3$ with $j$ respectively, integrating by parts, and then adding the resulting equalities, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathscr{E}}(t)+2\nu_2\|\omega_y\|_2^2+2\kappa_2\|\nabla j_y\|_2^2+2\eta_1\|h_x\|_2^2\\
&\quad=-2\int\int(\nabla u_2\cdot\nabla\varphi_y)j\,\mathrm{d}x\,\mathrm{d}y-2\int\int(\nabla u_1\cdot\nabla\varphi_x)j\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+2\kappa_2\int\int\Delta(f(\varphi))_{yy}j\,\mathrm{d}x\,\mathrm{d}y+2\int[\partial_xB_1(\partial_xu_2+\partial_yu_1)-\partial_xu_1(\partial_xB_2+\partial_yB_1)]h\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\triangleq K_1+K_2+K_3+K_4.
\end{aligned}\tag{3.31}
$$

We bound term by term above in what follows. To bound $K_3$, we integrate by parts twice and divide $K_3$ further into two terms:

$$
\begin{aligned}
K_3&=2\kappa_2\int\int f^{'}(\varphi)\nabla\varphi_y\cdot\nabla j_y\mathrm{d}x\,\mathrm{d}y+2\kappa_2\int\int f^{''}(\varphi)\varphi_y\nabla\varphi\cdot\nabla j_y\mathrm{d}x\,\mathrm{d}y\\
&\triangleq K_{31}+K_{32}.
\end{aligned}
$$

Employing (2.1), (2.2), the assumption (ii) on $f$ and Young's inequality, $H^1\subset L^s(s\geqslant 2)$, and (3.26), we have

$$
\begin{aligned}
|K_{31}|&\leqslant C\|\nabla j_y\|_2\|f^{'}(\varphi)\|_2^{1/2}\|f^{''}(\varphi)\varphi_y\|_2^{1/2}\|\nabla\varphi_y\|_2^{1/2}\|\nabla\varphi_{yy}\|_2^{1/2}\\
&\leqslant\frac{\kappa_2}{8}\|\nabla j_y\|_2^2+C\|f^{'}(\varphi)\|_2\|f^{''}(\phi)\varphi_y\|_2\|\nabla\varphi_y\|_2\|\nabla\varphi_{yy}\|_2\\
&\leqslant\frac{\kappa_2}{8}\|\nabla j_x\|_2^2+C(1+\|\varphi\|_{2(p+1)}^{p+1})\\
&\quad\times\left(C\|\varphi\|_{4p}^{p1}\|\varphi_y\|_2^{1/4}\|\varphi_{xy}\|_2^{1/4}\|\varphi_{yy}\|_2^{1/4}\right)\|\nabla\varphi_y\|_2\|\nabla\varphi_{yy}\|_2\\
&\leqslant\frac{\kappa_2}{8}\|\nabla j_y\|_2^2+C\|j\|_2^{3/2}\|\Delta\varphi_y\|_2.
\end{aligned}
$$

Applying (2.1), (2.2), (2.3), the assumption (ii) on $f$, $H^1\subset L^s(s\geqslant 2)$, and (3.26), we infer that

$$
\begin{aligned}
|K_{32}|&\leqslant C(\|\nabla\varphi\|_4\|\varphi_y\|_2\|\nabla j_y\|_2+\|\phi\|_{12(2p+1)}^{2p+1}\|\nabla\varphi\|_4\|\varphi_y\|_6\|\nabla j_y\|_2)\\
&\leqslant C\|\nabla\varphi\|_2^{1/2}\|\nabla\varphi_x\|_2^{1/4}\|\nabla\varphi_y\|_2^{1/4}\|\nabla j_y\|_2\\
&\quad+C\|\varphi\|_{12(2p+1)}^{2p+1}\left(\|\nabla\varphi\|_2^{1/2}\|\nabla\varphi_x\|_2^{1/4}\|\nabla\varphi_y\|_2^{1/4}\|\varphi_y\|_2^{5/12}\|\varphi_{xy}\|_2^{1/4}\|\varphi_{yy}\|_2^{1/4}\right)\\
&\quad\times\|\varphi_{xxy}\|_2^{1/48}\|\varphi_{yyy}\|_2^{1/48}\|\varphi_{xyy}\|_2^{1/24}\|\nabla j_y\|_2\\
&\leqslant C\|\nabla j_y\|_2\|j\|_2^{1/2}+C\|j\|_2\|\Delta\varphi_y\|_2^{1/12}\\
&\leqslant\frac{\kappa_2}{8}\|\nabla j_y\|_2^2+C\|j\|_2++C\|j\|_2\|\Delta\varphi_y\|_2^{1/12}.
\end{aligned}
$$

For $K_1$, we split it into two parts:

$$
K_1=-2\int\int u_{2x}\varphi_{xy}j\,\mathrm{d}x\,\mathrm{d}y-2\int\int u_{2y}\varphi_{yy}j\,\mathrm{d}x\,\mathrm{d}y\triangleq K_{11}+K_{12}.
$$

Applying (2.1) to $K_{11}$ and $K_{12}$, and then using Young's inequality, we have

$$
\begin{aligned}
|K_{11}| &\leqslant C\|j\|_2\|u_{2x}\|_2^{1/2}\|u_{2xy}\|_2^{1/2}\|\varphi_{xy}\|_2^{1/2}\|\varphi_{xxy}\|_2^{1/2} \\
&\leqslant C\|j\|_2\|\omega\|_2^{1/2}\|\omega_y\|_2^{1/2}\|\varphi_{xy}\|_2^{1/2}\|\Delta\varphi_x\|_2^{1/2} \\
&\leqslant \frac{\nu_2}{8}\|\omega_y\|_2^2 + C\|j\|_2^{4/3}\|\omega\|_2^{2/3}\|\varphi_{xy}\|_2^{2/3}\|\Delta\varphi_y\|_2^{2/3},
\end{aligned}
$$

and

$$
\begin{aligned}
|K_{12}| &\leqslant C\|j\|_2\|u_{2y}\|_2^{1/2}\|u_{2yy}\|_2^{1/2}\|\varphi_{yy}\|_2^{1/2}\|\varphi_{xyy}\|_2^{1/2} \\
&\leqslant C\|j\|_2^{3/2}\|u_{2y}\|_2^{1/2}\|\omega_y\|_2^{1/2}\|\Delta\varphi_y\|_2^{1/2} \\
&\leqslant \frac{\nu_2}{8}\|\omega_y\|_2^2 + C\|j\|_2^2\|u_{2y}\|_2^{2/3}\|\Delta\varphi_y\|_2^{2/3}.
\end{aligned}
$$

For $K_2$, we divide $K_2 = K_{21} + K_{22}$ as follows

$$
K_{21} := -2\int\int u_{1x}\varphi_{xx} j \,\mathrm{d}x\,\mathrm{d}y, \quad K_{22} := -2\int\int u_{1y}\varphi_{xy} j \,\mathrm{d}x\,\mathrm{d}y.
$$

By virtue of the incompressible condition $\partial_x u_1 + \partial_y u_2 = 0$, (2.1) and (3.26), we get

$$
\begin{aligned}
|K_{21}| &\leqslant C\|j\|_2\|u_{2y}\|_2^{1/2}\|u_{2xy}\|_2^{1/2}\|\varphi_{xx}\|_2^{1/2}\|\varphi_{xxy}\|_2^{1/2} \\
&\leqslant C\|j\|_2^{3/2}\|u_{2y}\|_2^{1/2}\|\omega_y\|_2^{1/2}\|\Delta\varphi_y\|_2^{1/2} \\
&\leqslant \frac{\nu_2}{8}\|\omega_y\|_2^2 + C\|j\|_2^2(1 + \|u_{2y}\|_2^2 + \|\Delta\varphi_y\|_2^2).
\end{aligned}
$$

In order to estimate $K_{22}$, we split the integral into two parts and integrate by parts twice. More precisely, we have

$$
\begin{aligned}
K_{22} &= 2\int\int (u_{1y}j)_x\varphi_y \mathrm{d}x\,\mathrm{d}y = 2\int\int u_{1xy}j\varphi_y \mathrm{d}x\,\mathrm{d}y + 2\int\int u_{1y}j_x\varphi_y \mathrm{d}x\,\mathrm{d}y \\
&= 2\int\int u_{1xy}j\varphi_y \mathrm{d}x\,\mathrm{d}y - 2\int\int u_1 j_{xy}\varphi_y \mathrm{d}x\,\mathrm{d}y - 2\int\int u_1 j_x\varphi_{yy} \mathrm{d}x\,\mathrm{d}y \\
&\triangleq K_{221} + K_{222} + K_{223}.
\end{aligned}
$$

It follows from (2.1) and (3.26), that

$$
\begin{aligned}
|K_{221}| &\leqslant C\|u_{1xy}\|_2\|j\|_2^{1/2}\|j_y\|_2^{1/2}\|\varphi_y\|_2^{1/2}\|\varphi_{xy}\|_2^{1/2} \\
&\leqslant C\|\omega_y\|_2\|j\|_2^{1/2}\|\Delta\varphi_y\|_2^{1/2}\|\nabla\varphi_y\|_2^{1/2} \\
&\leqslant \frac{\nu_2}{4}\|\omega_y\|_2^2 + C\|j\|_2\|\Delta\varphi_x\|_2\|\nabla\varphi_y\|_2 \\
&\leqslant \frac{\nu_2}{8}\|\omega_y\|_2^2 + C\|j\|_2^2\|\Delta\varphi_x\|_2^2 + \|\nabla\varphi_y\|_2^2.
\end{aligned}
$$

Using (2.1) and (3.26), thanks to the incompressible condition $\partial_x u_1 + \partial_y u_2 = 0$, we have

$$
\begin{aligned}
|K_{222}| &\leqslant C\|j_{xy}\|_2\|u_1\|_2^{1/2}\|u_{1x}\|_2^{1/2}\|\varphi_y\|_2^{1/2}\|\varphi_{yy}\|_2^{1/2} \\
&\leqslant C\|\nabla j_y\|_2\|u_{1x}\|_2^{1/2}\|j\|_2^{1/2} \\
&\leqslant \frac{\kappa_2}{8}\|\nabla j_y\|_2^2 + C(\|j\|_2^2 + \|u_{2y}\|_2^2),
\end{aligned}
$$

and

$$\begin{aligned}|K_{223}| &\leqslant C\|\varphi_{yy}\|_2\|u_1\|_2^{1/2}\|u_{1x}\|_2^{1/2}\|j_x\|_2^{1/2}\|j_{xy}\|_2^{1/2}\\ &\leqslant C\|j\|_2\|u_{1x}\|_2^{1/2}\|j_x\|_2^{1/2}\|\nabla j_y\|_2^{1/2}\\ &\leqslant \frac{\kappa_2}{8}\|\nabla j_y\|_2^2+C\|j\|_2^2(\|u_{2y}\|_2^2+\|j_x\|_2^2).\end{aligned}$$

For $K_4$, we rewrite it as follows

$$\begin{aligned}K_4 &= \int \partial_x B_1\partial_x u_2 h\mathrm{d}x\,\mathrm{d}y+\int \partial_x B_1\partial_y u_1 h\mathrm{d}x\,\mathrm{d}y+\int \partial_x u_1\partial_x B_2 h\mathrm{d}x\,\mathrm{d}y+\int \partial_x u_1\partial_y B_1 h\mathrm{d}x\,\mathrm{d}y\\ &\triangleq K_{41}+K_{42}+K_{43}+K_{44}.\end{aligned}$$

It follows from (2.1) and Young's inequality, that

$$\begin{aligned}|K_{41}| &= \int |\partial_x B_1||\partial_x u_2||h|\mathrm{d}x\,\mathrm{d}y\\ &\leqslant C\|\partial_x u_2\|_2^{1/2}\|\partial_{xy}u_2\|_2^{1/2}\|h\|_2^{1/2}\|h_x\|_2^{1/2}\|\partial_x B_1\|_2\\ &\leqslant \frac{\nu_2}{8}\|\partial_{xy}u_2\|_2^2+\frac{\eta_1}{8}\|h_x\|_2^2+C\|\partial_x u_2\|_2\|\partial_x B_1\|^2\|h\|_2\\ &\leqslant \frac{\nu_2}{8}\|\omega_y\|_2^2+\frac{\eta_1}{8}\|h_x\|_2^2+C\|\omega\|_2\|\partial_x B_1\|_2^2\|h\|_2\end{aligned}$$

and

$$\begin{aligned}|K_{42}| &= \int |\partial_x B_1||\partial_y u_1||h|\mathrm{d}x\,\mathrm{d}y\\ &\leqslant C\|\partial_x B_1\|_2^{\frac12}\|\partial_{xx}B_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_{yy}u_1\|_2^{\frac12}\|h\|_2\\ &\leqslant \frac{\nu_2}{8}\|\partial_{yy}u_1\|_2^2+\frac{\eta_1}{8}\|\partial_{xx}B_1\|_2^2+C\|\partial_x B_1\|\|\partial_y u_1\|_2\|h\|_2^2\\ &\leqslant \frac{\nu_2}{8}\|\omega_y\|_2^2+\frac{\eta_1}{8}\|h_x\|_2^2+C\left(\|\partial_x B_1\|_2^2+\|\partial_y u_1\|_2^2\right)\|h\|_2^2.\end{aligned}$$

Integrating by parts and then using (2.1) and Young's inequality yield

$$\begin{aligned}|K_{43}| &= \left|\int \partial_x u_1\partial_x B_2 h\mathrm{d}x\,\mathrm{d}y\right|\\ &= \left|\int (u_1\partial_{xx}B_2 h+u_1\partial_x B_2 h_x)\mathrm{d}x\,\mathrm{d}y\right|\\ &\leqslant C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|h\|_2^{\frac12}\|h_x\|_2^{\frac12}\|\partial_{xx}B_2\|_2\\ &\quad +C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_x B_2\|_2^{\frac12}\|\partial_{xx}B_2\|_2^{\frac12}\|h_x\|_2\\ &\leqslant C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|h\|_2^{\frac12}\|h_x\|_2^{\frac32}+C\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_x B_2\|_2^{\frac12}\|h_x\|_2^{\frac32}\\ &\leqslant \frac{\eta_1}{4}\|h_x\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|h\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\partial_x B_2\|_2^2\\ &\leqslant \frac{\eta_1}{4}\|h_x\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|h\|_2^2,\end{aligned}$$

and

$$
\begin{aligned}
|K_{44}| &= \left| \int \partial_x u_1 \partial_y B_1 h \mathrm{d}x\, \mathrm{d}y \right| \\
&\leqslant \left| \int (u_1 \partial_{xy} B_1 h + u_1 \partial_y B_1 h_x) \mathrm{d}x\, \mathrm{d}y \right| \\
&\leqslant C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{1}{2}} \|\partial_{xy} B_1\|_2 \\
&\quad + C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_y B_1\|_2^{\frac{1}{2}} \|\partial_{xy} B_1\|_2^{\frac{1}{2}} \|h_x\|_2 \\
&\leqslant C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{3}{2}} + C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_y B_1\|_2^{\frac{1}{2}} \|h_x\|_2^{\frac{3}{2}} \\
&\leqslant \frac{\eta_1}{4} \|h_x\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|h\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|\partial_y B_1\|_2^2 \\
&\leqslant \frac{\eta_1}{4} \|h_x\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|h\|_2^2.
\end{aligned}
$$

Inserting all the above estimates into the right-hand side of (3.31), we finally conclude, for all $t \geqslant 0$, that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{E}(t) + \nu_2 \|\omega_y(t)\|_2^2 + \kappa_2 \|\nabla j_y\|_2^2 + \eta_1 \|h_x\|_2^2 \leqslant C\tilde{\mathscr{R}}(t)\tilde{\mathscr{E}}(t) + \tilde{\mathscr{R}}(t). \tag{3.32}
$$

According to estimates (3.26), (3.27), and (3.28), we easily deduce

$$
\int_0^T \mathscr{R}(s)\mathrm{d}s \leqslant C, \quad \forall T > 0. \tag{3.33}
$$

Applying Gronwall's inequality to (3.32) and taking (3.33) into account imply that (3.29) holds for all $t \in [0, T]$. □

Now we are in a position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2** With the a priori bounds in Propositions 3.5, 3.6 and 3.7 at our disposal, the proof of the remaining Theorem 1.2 can be achieved by the similar method in Theorem 1.1, we omit it here. Following a similar derivation process of Theorems 1.1-1.2, we also obtain the proof of Theorems 1.3-1.4, and omit it here.

**Acknowledgements**

This paper was supported by the National Natural Science Foundation of China (12326430) and the Natural Science Foundation of Shandong Province (ZR2021MA017).

## References

[1] H. Abels, On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities, Arch. Ration. Mech. Anal. 194 (2009) 463–506.
[2] D.M. Anderson, G.B. McFadden, A.A. Wheeler, Diffuse-interface methods in fluid mechanics, Ann. Rev. Fluid Mech. 30 (1998) 139–165.
[3] N.D. Alikakos, P.W. Bates, X. Chen, Convergence of the Cahn-Hilliard equation to the Hele-Shaw model, Arch. Ration. Mech. Anal. 128 (1994) 165–205.
[4] C. Cao, C.G. Gal, Global solutions for the 2D NSCH model for a two-phase flow of viscous, incompressible fluids with mixed partial viscosity and mobility, Nonlinearity, 25 (2012) 3211–3234.
[5] H. Cabannes, Theoretical Magnetofluid Dynamics, Academic Press, New York, 1970.
[6] T. G. Cowling, Magnetohydrodynamics, 2nd ed., Adam Hilger, Bristol, 1976.

[7] C. Cao, J. Wu, Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation, Arch. Rational Mech. Anal. 208 (2013) 985–1004.
[8] C. Cao, D. Regmi, J. Wu, The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, J. Differential Equations 254 (2013) 2661–2681.
[9] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Ration. Mech. Anal. 92 (1986) 205–245.
[10] C. Cao, J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math. 226 (2011) 1803–1822.
[11] J. Cen, S. Migórski, C. Vetro, S. Zeng, Stability analysis for a contaminant convection-reaction-diffusion model of recovered fracturing fluid, J. Nonlinear Var. Anal. 8 (2024) 581-600.
[12] R. Danchin, M. Paicu, Global existence results for the anisotropic Boussinesq system in dimension two, Math. Models Meth. Appl. Sci. 21 (2011) 421–457.
[13] P. A. Davidson, E. V. Belova, An Introduction to Magnetohydrodynamics, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2010.
[14] A.E. Diegel, X.H. Feng, S.M. Wise, Analysis of a mixed finite element method for a Cahn-Hilliard-Darcy-Stokes system, SIAM J. Numer. Anal. 53 (2015) 127–152.
[15] P.A. Davidson, An Introduction To Magnetohydrodynamics, in: Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, Cambridge, 2001.
[16] X. Feng, Fully discrete finite element approximations of the Navier-Stokes-Cahn-Hilliard diffuse interface model for two-phase fluid flows, SIAM J. Numer. Anal. 44 (2006) 1049–1072.
[17] P.C. Fife, Models for phase separation and their mathematics, Electron. J. Differ. Equ. 48 (2000) 1–26.
[18] S. Frigeri, C. G. Gal, M. Grasselli, Two-dimensional nonlocal Cahn-Hilliard-Navier-Stokes systems with variable viscosity, degenerate mobility and singular potential, Nonlinearity, 32 (2019) 678.
[19] C. G. Gal, M. Grasselli, Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D, Ann. Inst. Henri Poincaré Anal. Non Linéaire, 27 (2010) 401–36.
[20] C. G. Gal, M. Grasselli, Trajectory attractors for binary fluid mixtures in 3D, Chin. Ann. Math. Ser. B, 31 (2010) 655–78.
[21] C.G. Gal, M. Grasselli, A. Miranville, Cahn-Hilliard-Navier-Stokes systems with moving contact lines, Calc. Var. Partial Differ. Equ. 55 (50) (2016) 1–47.
[22] J. F. Gerbeau, B. C. Le, T. Lelièvre, Mathematical methods for the magnetohydrodynamics of liquid metals, Oxford University Press, Oxford, 2006.
[23] D. Jacqmin, Calculation of two-phase Navier-Stokes flows using phase-field modeling, J. Comput. Phys. 155 (1999) 96–127.
[24] J. Kim, K. Kang, J. Lowengrub, Conservative multigrid methods for Cahn-Hilliard fluids, J. Comput. Phys. 193 (2004), 511–543.
[25] J. Liu, L. Dedè, J. A. Evans, M. J. Borden, T. J.R. Hughes, Isogeometric analysis of the advective Cahn-Hilliard equation: Spinodal decomposition under shear flow, J. Comput. Phys. 242 (2013) 321–350.
[26] C. Liu, J. Shen, A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method, Physica D. 179 (2003) 211–228.
[27] N.B. Morley, S. Smolentsev, L. Barleon, I. R. Kirillov, M. Takahashi, Liquid magnetohydrodynamics-recent progress and future directions for fusion, Fusion Eng. Des. 51 (2000) 701–713.
[28] M.S. Malashetty, J.C. Umavathi, J. Prathap Kumar, Convective magnetohydrodynamic two fluid flow and heat transfer in an inclined channel, Heat Mass Transf. 37 (2001) 259–264.
[29] A. Onuki, Phase transitions of fluids in shear flow, J. Phys.: Condens. Matter, 9 (1997) 6119–6157.
[30] R. Shail, On laminar two-phase flows in magnetohydrodynamics, Internat. J. Engrg. Sci. 11 (1973) 1103–1108.
[31] V. N. Starovoitov, The dynamics of a two-component fluid in the presence of capillary forces, Math. Notes, 62 (1997) 244–54.
[32] H. Su, G. Zhang, Highly efficient and energy stable schemes for the 2D/3D diffuse interface model of two-phase magnetohydrodynamics, J. Sci. Comput. 90 (2022) 63.
[33] J. Wu, Regularity criteria for the generalized MHD equations, Commun. Partial Differ. Equ., 33 (2008) , 285–306.

[34] C. Wang, J. Wang, S. Wise, Z. Xia, L. Xu, Convergence analysis of a temporally second-order accurate finite element scheme for the Cahn-Hilliard-Magnetohydrodynamics system of equations, J. Comput. Appl. Math. 436 (2024) 115409.

[35] D. Wang, Z. Wang, C. Zhang, H. Jia, J. Zhang, Error analysis of fully decoupled sav scheme for two phase magnetohydrodynamic diffuse interface model, Comput. Appl. Math. 43( (2024), 374.

[36] F. Xu, X. Li, Y. Cui, Y. Wu, A scaling invariant regularity criterion for the 3D incompressible magneto-hydrodynamics equations, Z. Angew. Math. Phys. 68 (2017) 1–8.

[37] P. Yue, J. Feng, C. Liu, J. Shen, A diffuse-interface method for simulating two-phase flows of complex fluids, J. Fluid Mech. 515 (2004) 293–317.