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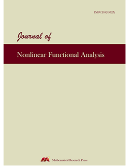

A RECTANGULAR MFE COMBINED WITH VARIATIONAL DISCRETIZATION FOR ELLIPTIC OPTIMIZATION PROBLEMS WITH INTEGRAL CONSTRAINTS

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Abstract. This paper investigates the use of a rectangular mixed finite element combined with variational discretization to solve elliptic optimization problems with integral constraints. The state and co-state variables are discretized by using the $Q_{k-1,k} \times Q_{k,k-1} - Q_{k,k}$ mixed finite element. The control variable is obtained via a variational discretization technique. Under appropriate regularity assumptions, convergence and superconvergence results are rigorously derived by introducing some auxiliary variables and projection operators. Some examples are given to confirm the results of the theoretical analysis.

Keywords. Convergence and superconvergence; Elliptic optimization problems; Integral constraints; Mixed finite element.

1. INTRODUCTION

Elliptic optimization problems (EOPs) with various constraints are becoming increasingly important in engineering physics, the social sciences, electrical machine design, pollution control, temperature control, flow control and so on; see, e.g., [1–3]. The numerical treatment of EOPs has been studied extensively. A systematic introduction can be found in [4–6]. Numerous numerical approaches including the least-squares method [7,8], the finite difference method [9], the multigrid method [10, 11], the finite element method [12–16], the mixed finite element (MFE) method [17], the spectral method [18, 19], the finite volume element method [20, 21] and the virtual element method [22–24] have been successfully applied to solve EOPs with different constraints.

The MFE is the best choice for solving optimization problems where the objective functional contains the desired state variable and its gradient. However, conventional MFEs [25–27] must satisfy the Ladyženskaja-Babuška-Brezzi (LBB) condition between the approximating subspace pairs. This severely limits the choice of discrete subspace pairs. To overcome this drawback,

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Pani [28] presented an H^1-Galerkin MFE for parabolic partial differential equations, Yang [29] developed a splitting positive definite MFE for miscible displacement in compressible flow problems in porous media, and Chen et al. [30] proposed a $P_{k-1}^2 - P_k$ MFE based on a new mixed weak form and triangular meshes for elliptic partial differential equations. These have recently been used to solve various partial differential equations [31–33] and optimization problems [34–36]. In [30], the authors also developed a $Q_{k-1,k} \times Q_{k,k-1} - Q_{k,k}$ MFE based on the new mixed weak form and rectangular meshes for solving elliptic partial differential equations. In this case, the gradient of the state variable just belongs to the square-integrable space $(L^2(\Omega))^d$ instead of the classical Hilbert space $H(\mathrm{div};\Omega)$.

The primary contributions of this paper are fourfold: (1) the design of the $Q_{k-1,k} \times Q_{k,k-1} - Q_{k,k}$ MFE combined with variational discretization (VD) approximation for EOPs with integral constraints based on a new mixed optimality condition and rectangular meshes; (2) optimal convergence results are obtained for the L^2-norm of the control, state and co-state; (3) the derivation of superconvergence between the derivatives of the state and co-state and their respective elliptic projections; (4) the verification of the theoretical analysis results through numerical examples.

We focus on the below EOP with integral constraints:

$$
\begin{cases}
\min\limits_{u\in K} \frac{1}{2}\left(\|u\|^2 + \|\boldsymbol{q}-\boldsymbol{q}_d\|^2 + \|y-y_d\|^2\right), & (1.1)\\
\boldsymbol{q} = -\nabla y, \quad \text{in } \Omega, & (1.2)\\
\mathrm{div}\boldsymbol{q} = u + f, \quad \text{in } \Omega, & (1.3)\\
y(\boldsymbol{x}) = 0, \quad \text{on } \partial\Omega, & (1.4)
\end{cases}
$$

where $\Omega \subset \mathbf{R}^2$ is a rectangle. Let $\boldsymbol{q}_d \in (U,U)$, $y_d, f \in U$ with $U = L^2(\Omega)$. Let K be defined by

$$
K = \left\{\kappa \in U : \int_\Omega \kappa \geq 0\right\}. \tag{1.5}
$$

Throughout the paper, we denote the standard Sobolev spaces on Ω by $W^{m,p}(\Omega)$, which are equipped with a norm $\|\varphi\|_{m,p}$ and a semi-norm $|\varphi|_{m,p}$. For $p=2$, we set $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^1(\Omega) = \{\varphi \in H^1(\Omega) : \varphi|_{\partial\Omega} = 0\}$, $\|\cdot\| = \|\cdot\|_{0,2}$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$. In addition, $C > 0$ denotes a generic constant.

The outline of this paper is as follows: Section 2 presents the $Q_{k-1,k} \times Q_{k,k-1} - Q_{k,k}$ MFE combined with VD approximation of the EOPs with integral constraints (1.1)-(1.5). Section 3 rigorously derives convergence and superconvergence results. Section 4 provides two numerical examples to confirm the theoretical analysis from the previous section.

2. $Q_{k-1,k} \times Q_{k,k-1} - Q_{k,k}$ MFE COMBINED WITH VD APPROXIMATION OF EOPS

In this section, we consider the $Q_{k-1,k} \times Q_{k,k-1} - Q_{k,k}$ MFE combined with VD approximation of (1.1)-(1.5). For clarity, we set $\boldsymbol{W} = (U,U)$ and $V = H_0^1(\Omega)$. As in [30], the weak formulation of (1.2)-(1.3) is as follows:

$$
\begin{cases}
(\boldsymbol{q},\boldsymbol{w}) + (\nabla y, \boldsymbol{w}) = 0, \quad \forall \boldsymbol{w} \in \boldsymbol{W}, & (2.1)\\
-(\boldsymbol{q}, \nabla v) = (u+f, v), \quad \forall v \in V & (2.2)
\end{cases}
$$

and the EOP with integral constraints (1.1)-(1.5) can be rewritten as the below new mixed weak formulation:

$$\begin{cases} \min\limits_{u\in K} \frac{1}{2}\left(\|u\|^2+\|\boldsymbol{q}-\boldsymbol{q}_d\|^2+\|y-y_d\|^2\right) & (2.3)\\ (\boldsymbol{q},\boldsymbol{w})+(\nabla y,\boldsymbol{w})=0, \quad \forall \boldsymbol{w}\in \boldsymbol{W}, & (2.4)\\ -(\boldsymbol{q},\nabla v)=(u+f,v), \quad \forall v\in V. & (2.5)\end{cases}$$

It follows from [4] that the EOP with integral constraints (2.3)-(2.5) possesses a unique solution $(u,\boldsymbol{q},y)$, and that $(u,\boldsymbol{q},y)\in K\times\boldsymbol{W}\times V$ fulfills (2.3)-(2.5) if and only if there exists a co-state $(\boldsymbol{p},z)\in\boldsymbol{W}\times V$ such that $(u,\boldsymbol{q},y,\boldsymbol{p},z)$ fulfills:

$$(\boldsymbol{q},\boldsymbol{w})+(\nabla y,\boldsymbol{w})=0, \quad \forall \boldsymbol{w}\in\boldsymbol{W}, \tag{2.6}$$
$$-(\boldsymbol{q},\nabla v)=(u+f,v), \quad \forall v\in V, \tag{2.7}$$
$$(\boldsymbol{p},\boldsymbol{w})+(\nabla z,\boldsymbol{w})=-(\boldsymbol{q}-\boldsymbol{q}_d,\boldsymbol{w}), \quad \forall \boldsymbol{w}\in\boldsymbol{W}, \tag{2.8}$$
$$-(\boldsymbol{p},\nabla v)=(y-y_d,v), \quad \forall v\in V, \tag{2.9}$$
$$(u+z,\kappa-u)\geq 0, \quad \forall \kappa\in K. \tag{2.10}$$

Like [16, 18], the variational inequality (2.10) equals to

$$u=\max\{0,\bar{z}\}-z, \tag{2.11}$$

where $\bar{v}=\int_\Omega v/\int_\Omega 1$ denotes the integral average on Ω of the function v.

Let $\mathscr{T}_h$ be a rectangular subdivision of Ω and $h=\max_{E\in\mathscr{T}_h}\{h_E\}$ with $h_E=\mathrm{diam}(E)$. $Q_{n,s}(E)$ denotes the space of polynomials of degree at most n and s in x_1 and x_2 on E, respectively. For integer $k\geq 1$, $\boldsymbol{W}_h^k\times V_h^k\subset\boldsymbol{W}\times V$ denotes the new rectangular MFE approximation spaces [30] on $\mathscr{T}_h$, namely,

$$\boldsymbol{W}_h^k:=\{\boldsymbol{w}_h\in\boldsymbol{W}:\boldsymbol{w}_h|_E\in Q_{k-1,k}(E)\times Q_{k,k-1}(E),\forall E\in\mathscr{T}_h\},$$
$$V_h^k:=\{v_h\in V:v_h|_E\in Q_{k,k}(E),\forall E\in\mathscr{T}_h\}.$$

Then the $Q_{k-1,k}\times Q_{k,k-1}-Q_{k,k}$ MFE combined with VD approximation of (2.3)-(2.5) is as follows:

$$\begin{cases} \min\limits_{u_h\in K} \frac{1}{2}\left(\|u_h\|^2+\|\boldsymbol{q}_h-\boldsymbol{q}_d\|^2+\|y_h-y_d\|^2\right), & (2.12)\\ (\boldsymbol{q}_h,\boldsymbol{w}_h)+(\nabla y_h,\boldsymbol{w}_h)=0, \quad \forall \boldsymbol{w}_h\in \boldsymbol{W}_h^k, & (2.13)\\ -(\boldsymbol{q}_h,\nabla v_h)=(u_h+f,v_h), \quad \forall v_h\in V_h^k. & (2.14)\end{cases}$$

As [6], the EOP (2.12)-(2.14) possesses a unique solution $(u_h,\boldsymbol{q}_h,y_h)$ and $(u_h,\boldsymbol{q}_h,y_h)\in K\times\boldsymbol{W}_h^k\times V_h^k$ fulfills (2.12)-(2.14) if and only if there exists a co-state $(\boldsymbol{p}_h,z_h)\in\boldsymbol{W}_h^k\times V_h^k$ such that $(u_h,\boldsymbol{q}_h,y_h,\boldsymbol{p}_h,z_h)$ fulfills

$$(\boldsymbol{q}_h,\boldsymbol{w}_h)+(\nabla y_h,\boldsymbol{w}_h)=0, \quad \forall \boldsymbol{w}_h\in\boldsymbol{W}_h^k, \tag{2.15}$$
$$-(\boldsymbol{q}_h,\nabla v_h)=(u_h+f,v_h), \quad \forall v_h\in V_h^k, \tag{2.16}$$
$$(\boldsymbol{p}_h,\boldsymbol{w}_h)+(\nabla z_h,\boldsymbol{w}_h)=-(\boldsymbol{q}_h-\boldsymbol{q}_d,\boldsymbol{w}_h), \quad \forall \boldsymbol{w}_h\in\boldsymbol{W}_h^k, \tag{2.17}$$
$$-(\boldsymbol{p}_h,\nabla v_h)=(y_h-y_d,v_h), \quad \forall v_h\in V_h^k, \tag{2.18}$$
$$(u_h+z_h,\tilde{\kappa}-u_h)\geq 0, \quad \forall \tilde{\kappa}\in K. \tag{2.19}$$

Similarly to (2.11), inequality (2.19) equals to

$$u_h = \max\{0, \bar{z}_h\} - z_h. \tag{2.20}$$

3. CONVERGENCE AND SUPERCONVERGENCE ANALYSIS

In this section, we strictly derive the convergence and superconvergence of approximation scheme (2.15)-(2.20).

3.1. **Convergence analysis.** The following projection operators are used later in the convergence analysis. First, we define the following projections $P_h : V \to V_h^k$ [26] with

$$(\nabla(P_h v - v), \nabla v_h) = 0, \quad \forall v_h \in V_h^k, v \in V, \tag{3.1}$$

$$\|v - P_h v\| + h\|\nabla(v - P_h v)\| \leq Ch^{k+1}\|v\|_{k+1}, \quad \forall v \in H^{k+1}(\Omega) \tag{3.2}$$

and $\Pi_h : \boldsymbol{W} \to \boldsymbol{W}_h^m$ [32] with

$$(\Pi_h \boldsymbol{w} - \boldsymbol{w}, \boldsymbol{w}_h) = 0, \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \boldsymbol{w} \in \boldsymbol{W}, \tag{3.3}$$

$$\|\boldsymbol{w} - \Pi_h \boldsymbol{w}\| \leq Ch^s \|\boldsymbol{w}\|_s, \quad \forall \boldsymbol{w} \in (H^s(\Omega))^2, \quad 0 \leq s \leq k+1. \tag{3.4}$$

Next, for any $\hat{\kappa} \in K$, we set auxiliary variables $(\boldsymbol{q}_h(\hat{\kappa}), y_h(\hat{\kappa}), \boldsymbol{p}_h(\hat{\kappa}), z_h(\hat{\kappa})) \in (\boldsymbol{W}_h^k \times V_h^k)^2$ satisfies

$$(\boldsymbol{q}_h(\hat{\kappa}), \boldsymbol{w}_h) + (\nabla y_h(\hat{\kappa}), \boldsymbol{w}_h) = 0, \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.5}$$

$$-(\boldsymbol{q}_h(\hat{\kappa}), \nabla v_h) = (\hat{\kappa} + f, v_h), \quad \forall v_h \in V_h^k, \tag{3.6}$$

$$(\boldsymbol{p}_h(\hat{\kappa}), \boldsymbol{w}_h) + (\nabla z_h(\hat{\kappa}), \boldsymbol{w}_h) = -(\boldsymbol{q}_h(\hat{\kappa}) - \boldsymbol{q}_d, \boldsymbol{w}_h), \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.7}$$

$$-(\boldsymbol{p}_h(\hat{\kappa}), \nabla v_h) = (y_h(\hat{\kappa}) - y_d, v_h), \quad \forall v_h \in V_h^k. \tag{3.8}$$

Lemma 3.1. *([37]) For any $F \in L^2(\Omega)$, the solution ψ of*

$$-\mathrm{div}(\nabla\psi) = F \quad \text{in } \Omega, \ \psi|_{\partial\Omega} = 0, \tag{3.9}$$

belongs to $V \cap H^2(\Omega)$. Moreover,

$$\|\psi\|_2 \leq C\|F\|. \tag{3.10}$$

Lemma 3.2. *Let $(u_h, \boldsymbol{q}_h, y_h, \boldsymbol{p}_h, z_h)$ and $(\boldsymbol{q}_h(u), y_h(u), \boldsymbol{p}_h(u), z_h(u))$ be the solutions of (2.15)-(2.19) and (3.5)-(3.8) with $\hat{\kappa} = u$, respectively. Then*

$$\|\nabla(y_h - y_h(u))\| + \|\nabla(z_h - z_h(u))\| \leq C\|u_h - u\|, \tag{3.11}$$

$$\|\boldsymbol{q}_h - \boldsymbol{q}_h(u)\| + \|\boldsymbol{p}_h - \boldsymbol{p}_h(u)\| \leq C\|u_h - u\|. \tag{3.12}$$

Proof. For simplicity, we define

$$\eta_y = y_h - y_h(u), \eta_z = z_h - z_h(u),$$
$$\boldsymbol{\theta}_{\boldsymbol{q}} = \boldsymbol{q}_h - \boldsymbol{q}_h(u), \boldsymbol{\theta}_{\boldsymbol{p}} = \boldsymbol{p}_h - \boldsymbol{p}_h(u).$$

From (2.15)-(2.16) subtracting (3.5)-(3.6), we have

$$\left(\boldsymbol{\theta}_{\boldsymbol{q}}, \boldsymbol{w}_h\right) + (\nabla\eta_y, \boldsymbol{w}_h) = 0, \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.13}$$

$$-\left(\boldsymbol{\theta}_{\boldsymbol{q}}, \nabla v_h\right) = (u_h - u, v_h), \quad \forall v_h \in V_h^k. \tag{3.14}$$

Taking $\boldsymbol{w}_h = \nabla \eta_y$ in (3.13) and $v_h = \eta_y$ in (3.14) and substituting (3.14) into (3.13) yields

$$(\nabla \eta_y, \nabla \eta_y) = (u_h - u, \eta_y). \tag{3.15}$$

According to (3.15), Cauchy inequality, and Poincaré inequality, we derive

$$\|\nabla \eta_y\|^2 \le \|u_h - u\| \|\eta_y\| \le C\|u_h - u\| \|\nabla \eta_y\|. \tag{3.16}$$

Choosing $\boldsymbol{w}_h = \boldsymbol{\theta}_{\boldsymbol{q}}$ in (3.13) and utilizing ε-Cauchy inequality, we obtain

$$\|\boldsymbol{\theta}_{\boldsymbol{q}}\|^2 \le C(\varepsilon)\|\nabla \eta_y\|^2 + \varepsilon\|\boldsymbol{\theta}_{\boldsymbol{q}}\|^2. \tag{3.17}$$

From (2.17)-(2.18) and (3.7)-(3.8), we have

$$(\boldsymbol{\theta}_{\boldsymbol{p}}, \boldsymbol{w}_h) + (\nabla \eta_z, \boldsymbol{w}_h) = -\left(\boldsymbol{\theta}_{\boldsymbol{q}}, \boldsymbol{w}_h\right), \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.18}$$

$$-(\boldsymbol{\theta}_{\boldsymbol{p}}, \nabla v_h) = (\eta_y, v_h), \quad \forall v_h \in V_h^k. \tag{3.19}$$

Selecting $\boldsymbol{w}_h = \nabla \eta_z$ in (3.18) and $v_h = \eta_z$ in (3.19), then substituting (3.19) into (3.18), we obtain

$$(\nabla \eta_z, \nabla \eta_z) = -\left(\boldsymbol{\theta}_{\boldsymbol{q}}, \nabla \eta_z\right) + (\eta_y, \eta_z). \tag{3.20}$$

It follows from (3.20), ε-Cauchy inequality, and Poincaré inequality that

$$\begin{aligned} \|\nabla \eta_z\|^2 \le & C(\varepsilon)(\|\boldsymbol{\theta}_{\boldsymbol{q}}\|^2 + \|\eta_y\|^2) + \varepsilon(\|\nabla \eta_z\|^2 + \|\eta_z\|^2) \\ \le & C(\varepsilon)(\|\boldsymbol{\theta}_{\boldsymbol{q}}\|^2 + C\|\nabla \eta_y\|^2) + \varepsilon(1+C)\|\nabla \eta_z\|^2. \end{aligned} \tag{3.21}$$

Taking $\boldsymbol{w}_h = \boldsymbol{\theta}_{\boldsymbol{p}}$ in (3.18), and then utilizing ε-Cauchy inequality, we see that

$$\|\boldsymbol{\theta}_{\boldsymbol{p}}\|^2 \le C(\varepsilon)(\|\nabla \eta_z\|^2 + \|\boldsymbol{\theta}_{\boldsymbol{q}}\|^2) + 2\varepsilon\|\boldsymbol{\theta}_{\boldsymbol{p}}\|^2. \tag{3.22}$$

Let ε be small enough. It is easy to arrive at (3.11)-(3.12) from (3.16)-(3.17) and (3.21)-(3.22). □

Lemma 3.3. *Let* $(u, \boldsymbol{q}, y, \boldsymbol{p}, z)$ *and* $(\boldsymbol{q}_h(u), y_h(u), \boldsymbol{p}_h(u), z_h(u))$ *be the solutions of (2.6)-(2.10) and (3.5)-(3.8) with* $\hat{\kappa} = u$*, respectively. Assume that* $\boldsymbol{q}, \boldsymbol{p} \in (H^k(\Omega))^2$*,* $y, z \in H^{k+1}(\Omega)$*. Then*

$$\|\nabla(y - y_h(u))\| + \|\nabla(z - z_h(u))\| \le Ch^k, \tag{3.23}$$

$$\|\boldsymbol{q} - \boldsymbol{q}_h(u)\| + \|\boldsymbol{p} - \boldsymbol{p}_h(u)\| \le Ch^k, \tag{3.24}$$

$$\|y - y_h(u)\| + \|z - z_h(u)\| \le Ch^{k+1}. \tag{3.25}$$

Proof. For convenience, we set

$$\begin{aligned} &\phi_y = y - P_h y, \varphi_y = P_h y - y_h(u), \boldsymbol{\psi}_{\boldsymbol{q}} = \boldsymbol{q} - \Pi_h \boldsymbol{q}, \boldsymbol{\chi}_{\boldsymbol{q}} = \Pi_h \boldsymbol{q} - \boldsymbol{q}_h(u), \\ &\phi_z = z - P_h z, \varphi_z = P_h z - z_h(u), \boldsymbol{\psi}_{\boldsymbol{p}} = \boldsymbol{p} - \Pi_h \boldsymbol{p}, \boldsymbol{\chi}_{\boldsymbol{p}} = \Pi_h \boldsymbol{p} - \boldsymbol{p}_h(u). \end{aligned}$$

It follows from (2.6)-(2.7), (3.5)-(3.6) that

$$(\boldsymbol{\chi}_{\boldsymbol{q}}, \boldsymbol{w}_h) + (\nabla \varphi_y, \boldsymbol{w}_h) = -(\nabla \phi_y, \boldsymbol{w}_h), \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.26}$$

$$-(\boldsymbol{\chi}_{\boldsymbol{q}}, \nabla v_h) = (\boldsymbol{\psi}_{\boldsymbol{q}}, \nabla v_h), \quad \forall v_h \in V_h^k. \tag{3.27}$$

Choose $\boldsymbol{w}_h = \nabla \varphi_y$ and $v_h = \varphi_y$ in (3.26) and (3.27), respectively. Substituting (3.27) into (3.26), and using the definition of P_h, we obtain

$$(\nabla \varphi_y, \nabla \varphi_y) = (\boldsymbol{\psi}_{\boldsymbol{q}}, \nabla \varphi_y). \tag{3.28}$$

From (3.4), (3.28) and ε-Cauchy inequality, we have

$$\begin{aligned}\|\nabla\varphi_y\|^2 = (\boldsymbol{q}-\Pi_h\boldsymbol{q},\nabla\varphi_y) &\le C(\varepsilon)\|\boldsymbol{q}-\Pi_h\boldsymbol{q}\|^2+\varepsilon\|\nabla\varphi_y\|^2\\ &\le C(\varepsilon)h^{2k}\|\boldsymbol{q}\|_k^2+\varepsilon\|\nabla\varphi_y\|^2.\end{aligned} \tag{3.29}$$

Taking $\boldsymbol{w}_h=\boldsymbol{\chi_q}$ in (3.26) yields

$$(\boldsymbol{\chi_q},\boldsymbol{\chi_q})+(\nabla\varphi_y,\boldsymbol{\chi_q})=-(\nabla\phi_y,\boldsymbol{\chi_q}). \tag{3.30}$$

From (3.2), (3.29)-(3.30), Poincaré inequality and ε-Cauchy inequality, we arrive at

$$\begin{aligned}\|\boldsymbol{\chi_q}\|^2 \le& C(\varepsilon)(\|\nabla\varphi_y\|^2+\|\nabla\phi_y\|^2)+2\varepsilon\|\boldsymbol{\chi_q}\|\\ \le& C(\varepsilon)(h^{2k}\|\boldsymbol{q}\|_k^2+h^{2k}\|y\|_{k+1}^2)+2\varepsilon\|\boldsymbol{\chi_q}\|.\end{aligned} \tag{3.31}$$

According to (2.8)-(2.9), (3.7)-(3.8) and the definitions of P_h and Π_h, we have

$$(\boldsymbol{\chi_p},\boldsymbol{w}_h)+(\nabla\varphi_z,\boldsymbol{w}_h)=(\nabla\phi_z,\boldsymbol{w}_h)-(\boldsymbol{\chi_q},\boldsymbol{w}_h),\quad \forall \boldsymbol{w}_h\in\boldsymbol{W}_h^k, \tag{3.32}$$

$$-(\boldsymbol{\chi_p},\nabla v_h)=(\boldsymbol{\psi_p},\nabla v_h)+(\phi_y,v_h)+(\varphi_y,v_h),\quad \forall v_h\in V_h^k. \tag{3.33}$$

Select $\boldsymbol{w}_h=\nabla\varphi_z$ in (3.32), $v_h=\varphi_z$ in (3.27) and (3.33). Substituting (3.27) and (3.33) into (3.32) and then combing (3.2), (3.28), ε-Cauchy inequality, and Poincaré inequality, we obtain

$$\begin{aligned}\|\nabla\varphi_z\|^2 =&(\boldsymbol{\psi_p},\nabla\varphi_z)+(\phi_y,\varphi_z)+(\varphi_y,\varphi_z)+(\boldsymbol{\psi_q},\nabla\varphi_z)\\ \le& C(\varepsilon)(\|\boldsymbol{\psi_p}\|^2+\|\phi_y\|^2+\|\varphi_y\|^2+\|\boldsymbol{\psi_q}\|^2)+2\varepsilon(\|\varphi_z\|^2+\|\nabla\varphi_z\|^2)\\ \le& C(\varepsilon)(h^{2k}\|\boldsymbol{p}\|_k^2+h^{2(k+1)}\|y\|_{k+1}^2+h^{2k}\|\boldsymbol{q}\|_k^2)+2\varepsilon(C+1)\|\nabla\varphi_z\|^2.\end{aligned} \tag{3.34}$$

Taking $\boldsymbol{w}_h=\boldsymbol{\chi_p}$ in (3.32) and then using (3.2), (3.4) and ε-Cauchy inequality, we find that

$$\begin{aligned}\|\boldsymbol{\chi_p}\|^2 =&-(\nabla\phi_z,\boldsymbol{\chi_p})-(\boldsymbol{\chi_q},\boldsymbol{\chi_p})+(\nabla\varphi_z,\boldsymbol{\chi_p})\\ \le& C(\varepsilon)(\|\nabla\phi_z\|^2+\|\boldsymbol{\chi_q}\|^2+\|\nabla\varphi_z\|^2)+3\varepsilon\|\boldsymbol{\chi_p}\|^2\\ \le& C(\varepsilon)(h^{2(k+1)}\|z\|_{k+1}^2+\|\boldsymbol{\chi_q}\|^2+\|\nabla\varphi_z\|^2)+3\varepsilon\|\boldsymbol{\chi_p}\|^2.\end{aligned} \tag{3.35}$$

Combing (3.2), (3.4), (3.31), (3.34)-(3.35), Poincaré inequality, and triangle inequality, we arrive at (3.23)-(3.24) for sufficiently small ε. From Lemma 3.1, Aubin-Nitsche technique and (3.23)-(3.24), it is easy to derive (3.25). □

Theorem 3.4. *Let* $(u,\boldsymbol{q},y,\boldsymbol{p},z)$ *and* $(u_h,\boldsymbol{q}_h,y_h,\boldsymbol{p}_h,z_h)$ *be the solutions of* (2.6)-(2.10) *and* (2.15)-(2.19)*, respectively. Suppose that* $\boldsymbol{q},\boldsymbol{p}\in(H^k(\Omega))^2$, $y,z\in H^{k+1}(\Omega)$*. Then*

$$\|u-u_h\|\le Ch^{k+1}, \tag{3.36}$$

$$\|y-y_h\|+\|z-z_h\|\le Ch^{k+1}, \tag{3.37}$$

$$\|\nabla(y-y_h)\|+\|\nabla(z-z_h)\|\le Ch^{k}, \tag{3.38}$$

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|+\|\boldsymbol{p}-\boldsymbol{p}_h\|\le Ch^{k}. \tag{3.39}$$

Proof. Choosing $\kappa=u_h$ in (2.10) and $\tilde{\kappa}=u$ in (2.19), we have

$$(u+z,u_h-u)\ge 0, \tag{3.40}$$

$$(u_h+z_h,u-u_h)\ge 0. \tag{3.41}$$

It follows from (3.40)-(3.41) that

$$\begin{aligned}\|u-u_h\|^2 =&(u-u_h,u-u_h) \leq (z_h-z,u-u_h)\\ =&(z_h(u)-z,u-u_h)+(z_h-z_h(u),u-u_h).\end{aligned} \tag{3.42}$$

According to (2.15)-(2.18) and (3.5)-(3.8), we get

$$(z_h-z_h(u),u-u_h) = -\|y_h-y_h(u)\|^2-\|\nabla(y_h-y_h(u))\|^2 \leq 0. \tag{3.43}$$

From ε-Cauchy inequality, (3.25) and Poincaré inequality, we find that

$$(z_h(u)-z,u-u_h) \leq C(\varepsilon)\|z_h(u)-z\|^2+\varepsilon\|u-u_h\|^2 \leq C(\varepsilon)h^{2(k+1)}+\varepsilon\|u-u_h\|^2. \tag{3.44}$$

Collecting (3.42)-(3.44), we obtain (3.36) immediately. Then (3.37) follows from (3.11)-(3.12), (3.25), (3.36), Poincaré inequality, and triangle inequality. It is easy to see (3.38)-(3.39) from (3.36), Lemmas 3.2-3.3, and the triangle inequality. □

3.2. **Superconvergence analysis.** We analyze the superconvergence of the state and co-state variables in the following.

Theorem 3.5. *Let $(u,\boldsymbol{q},y,\boldsymbol{p},z)$ and $(u_h,\boldsymbol{q}_h,y_h,\boldsymbol{p}_h,z_h)$ be the solutions of (2.6)-(2.10) and (2.15)-(2.19), respectively. Suppose that $\boldsymbol{q},\boldsymbol{p} \in (H^{k+1}(\Omega))^2$ and all the conditions of Theorem 3.4 are valid. Thne*

$$\|\nabla(P_hy-y_h)\|+\|\nabla(P_hz-z_h)\| \leq Ch^{k+1}. \tag{3.45}$$

Proof. From (2.6)-(2.9), (2.15)-(2.18), the definitions P_h and Π_h, we have

$$(\Pi_h\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{w}_h)+(\nabla(P_hy-y_h),\boldsymbol{w}_h) = -(\nabla\phi_y,\boldsymbol{w}_h), \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.46}$$

$$-(\Pi_h\boldsymbol{q}-\boldsymbol{q}_h,\nabla v_h) = (\boldsymbol{\psi}_{\boldsymbol{q}},\nabla v_h)+(u-u_h,v_h), \quad \forall v_h \in V_h^k, \tag{3.47}$$

$$(\Pi_h\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{w}_h)+(\nabla(P_hz-z_h),\boldsymbol{w}_h) = -(\nabla\phi_z,\boldsymbol{w}_h)+(\Pi_h\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{w}_h), \quad \forall \boldsymbol{w}_h \in \boldsymbol{W}_h^k, \tag{3.48}$$

$$-(\Pi_h\boldsymbol{p}-\boldsymbol{p}_h,\nabla v_h) = (\boldsymbol{\psi}_{\boldsymbol{p}},\nabla v_h)+(y-y_h,v_h), \quad \forall v_h \in V_h^k. \tag{3.49}$$

Taking $\boldsymbol{w}_h = \nabla(P_hy-y_h)$ and $v_h = P_hy-y_h$ in (3.46) and (3.47), respectively. Substituting (3.47) into (3.46) then combing ε-Cauchy inequality, Poincaré inequality, (3.4) and (3.36), we obtain

$$\begin{aligned}\|\nabla(P_hy-y_h)\|^2 =&(u-u_h,P_hy-y_h)+(\boldsymbol{q}-\Pi_h\boldsymbol{q},\nabla(P_hy-y_h))\\ \leq& C(\varepsilon)(\|u-u_h\|^2+\|\boldsymbol{q}-\Pi_h\boldsymbol{q}\|^2)+\varepsilon(\|P_hy-y_h\|^2+\|\nabla(P_hy-y_h)\|^2)\\ \leq& C(\varepsilon)h^{2(k+1)}(1+\|\boldsymbol{q}\|_{k+1}^2)+\varepsilon(C+1)\|\nabla(P_hy-y_h)\|^2.\end{aligned} \tag{3.50}$$

Select $v_h = P_hz-z_h$ in (3.47), $\boldsymbol{w}_h = \nabla(P_hz-z_h)$ in (3.48) and $v_h = P_hz-z_h$ in (3.49). Collecting (3.47)-(3.49) and then using ε-Cauchy inequality, Poincaré inequality, (3.4) and (3.36)-(3.37), we arrive at

$$\begin{aligned}\|\nabla(P_hz-z_h)\|^2 =&(y-y_h,P_hz-z_h)+(\boldsymbol{p}-\Pi_h\boldsymbol{p},\nabla(P_hz-z_h))\\ &-(u-u_h,P_hz-z_h)+(\boldsymbol{q}-\Pi_h\boldsymbol{q},\nabla(P_hz-z_h))\\ \leq& C(\varepsilon)(\|y-y_h\|^2+\|\boldsymbol{p}-\Pi_h\boldsymbol{p}\|^2+\|u-u_h\|^2+\|\boldsymbol{q}-\Pi_h\boldsymbol{q}\|^2)\\ &+2\varepsilon(\|P_hz-z_h\|^2+\|\nabla(P_hz-z_h)\|^2)\\ \leq& C(\varepsilon)h^{2(k+1)}(2+\|\boldsymbol{p}\|_{k+1}^2+\|\boldsymbol{q}\|_{k+1}^2)+2\varepsilon(C+1)\|\nabla(P_hz-z_h)\|^2.\end{aligned} \tag{3.51}$$

Let ε be small enough. It is easy to get (3.45) from (3.50) and (3.51). □

4. NUMERICAL EXPERIMENTS

In this section, we present a numerical algorithm for the $Q_{k-1,k}\times Q_{k,k-1}-Q_{k,k}$ MFE combined with VD approximation of the EOPs subject to integral constraints (1.1)-(1.5) and conduct some experiments to validate the previous theoretical results. Similar to [38], we propose the following numerical algorithm for an acceptable error Tol by using the $Q_{k-1,k}\times Q_{k,k-1}-Q_{k,k}$ MFE combined with the VD approximation (2.15)-(2.20) of the EOP with integral constraints (1.1)-(1.5).

Algorithm 4.1. Step 1. Initialize $n=0$ and $u_h^{(n)}$.
Step 2. Solve $\left(u_h^{(n+1)},\boldsymbol{q}_h^{(n+1)},y_h^{(n+1)},\boldsymbol{p}_h^{(n+1)},z_h^{(n+1)}\right)$ such that:

$$\begin{cases}\left(\boldsymbol{q}_h^{(n+1)},\boldsymbol{w}_h\right)+\left(\nabla y_h^{(n+1)},\boldsymbol{w}_h\right)=0, \quad \forall \boldsymbol{w}_h\in\boldsymbol{W}_h^k,\\ -\left(\boldsymbol{q}_h^{(n+1)},\nabla v_h\right)=\left(u_h^{(n)}+f,v_h\right), \quad \forall v_h\in V_h^k,\\ \left(\boldsymbol{p}_h^{(n+1)},\boldsymbol{w}_h\right)+\left(\nabla z_h^{(n+1)},\boldsymbol{w}_h\right)=-\left(\boldsymbol{q}_h^{(n+1)}-\boldsymbol{q}_d,\boldsymbol{w}_h\right), \quad \forall \boldsymbol{w}_h\in\boldsymbol{W}_h^k,\\ -\left(\boldsymbol{p}_h^{(n+1)},\nabla v_h\right)=\left(y_h^{(n+1)}-y_d,v_h\right), \quad \forall v_h\in V_h^k,\\ u_h^{(n+1)}=\max\left\{0,\bar{z}_h^{(n+1)}\right\}-z_h^{(n+1)}.\end{cases}$$

Step 3. Compute error: $E_n=\left\|u_h^{(n+1)}-u_h^{(n)}\right\|$. If $E_n\le Tol$, stop; else set $n:=n+1$ go to 2.

Let $\Omega=(0,1)\times(0,1)$. We solve the following numerical examples by Algorithm 4.1 based on AFEPack [38]. The convergence rate:

$$rate=\frac{\log_{10}(e_{i+1})-\log_{10}(e_i)}{\log_{10}(h_{i+1})-\log_{10}(h_i)},$$

where e_i and e_{i+1} denote errors when mesh size $h=h_i$ and $h=h_{i+1}$, respectively.

Example 1. The data to be tested is given by:

$$\begin{cases}y(\boldsymbol{x})=(x_2-x_2^2)(x_1-x_1^2),\\ \boldsymbol{q}(\boldsymbol{x})=-\left((x_2-x_2^2)(1-2x_1),(x_1-x_1^2)(1-2x_2)\right),\\ z(\boldsymbol{x})=(x_2-x_2^2)(x_1-x_1^2)\\ \boldsymbol{p}(\boldsymbol{x})=\left((x_2-x_2^2)(1-2x_1),(1-2x_2)(x_1-x_1^2)\right),\\ u(\boldsymbol{x})=\max\{0,\bar{z}(\boldsymbol{x})\}-z(\boldsymbol{x}),\\ f=\mathrm{div}\boldsymbol{q}-u,\boldsymbol{q}_d=\boldsymbol{q}+\boldsymbol{p}+\nabla z,y_d=-\mathrm{div}\boldsymbol{p}+y.\end{cases}$$

In Example 1, we set $k=1$. Errors $\|u-u_h\|$, $\|\boldsymbol{q}-\boldsymbol{q}_h\|$, $\|y-y_h\|$, $\|\boldsymbol{p}-\boldsymbol{p}_h\|$, $\|z-z_h\|$, $\|\nabla(P_hy-y_h)\|$, $\|\nabla(P_hz-z_h)\|$, $\|\nabla(y-y_h)\|$, $\|\nabla(z-z_h)\|$ based on $h=\frac{1}{10},\frac{1}{20},\frac{1}{40},\frac{1}{80}$ are presented in Table 1, and convergent rates are reported in Figure 1.

TABLE 1. Errors of Example 1.

h	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\|u-u_h\|$	1.2537e-02	3.1405e-03	7.8356e-04	1.9589e-04
$\|y-y_h\|$	1.6483e-02	4.1218e-03	1.0302e-03	2.5755e-04
$\|z-z_h\|$	1.6756e-02	4.1902e-03	1.0473e-03	2.6181e-04
$\|\nabla(P_hy-y_h)\|$	3.0857e-02	7.7203e-03	1.9286e-03	4.8214e-04
$\|\nabla(P_hz-z_h)\|$	3.2185e-02	8.1463e-03	2.0116e-03	5.0289e-04
$\|\nabla(y-y_h)\|$	4.4579e-02	2.2301e-02	1.1145e-02	5.5724e-03
$\|\nabla(z-z_h)\|$	4.6508e-02	2.3261e-02	1.1628e-02	5.8135e-03
$\|\boldsymbol{p}-\boldsymbol{p}_h\|$	6.5217e-02	3.2611e-02	1.6305e-02	8.1521e-03
$\|\boldsymbol{q}-\boldsymbol{q}_h\|$	6.8072e-02	3.4125e-02	1.7108e-02	8.5091e-03

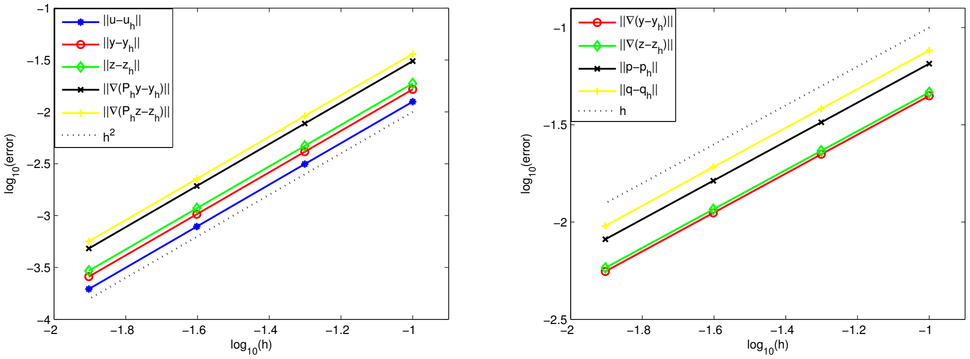

FIGURE 1. Convergence rates $\mathscr{O}(h^2)$ (left) and $\mathscr{O}(h)$ (right) of Example 1.

Example 2. The data to be tested is given by:

$$\begin{cases} y(\boldsymbol{x}) = \sin(\pi x_2)\sin(2\pi x_1), \\ \boldsymbol{q}(\boldsymbol{x}) = -\left(2\pi\sin(\pi x_2)\cos(2\pi x_1), \pi\cos(\pi x_2)\sin(2\pi x_1)\right), \\ z(\boldsymbol{x}) = \sin(2\pi x_2)\sin(\pi x_1), \\ \boldsymbol{p}(\boldsymbol{x}) = \left(\pi\sin(2\pi x_2)\cos(\pi x_1), 2\pi\cos(2\pi x_2)\sin(\pi x_1)\right), \\ u(\boldsymbol{x}) = \max\{0, \bar{z}(\boldsymbol{x})\} - z(\boldsymbol{x}), \\ f = \operatorname{div}\boldsymbol{q} - u, \boldsymbol{q}_d = \boldsymbol{q} + \boldsymbol{p} + \nabla z, y_d = -\operatorname{div}\boldsymbol{p} + y. \end{cases}$$

In Example 2, we choose $k = 2$. Numerical results based on $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$ are given in Table 2 and Figure 2. They are consistent with the previous theoretical analysis results.

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TABLE 2. Errors of Example 2.

h	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{80}$
$\lVert u-u_h\rVert$	3.1065e-02	3.8831e-03	4.8539e-04	6.0674e-05
$\lVert y-y_h\rVert$	3.4852e-02	4.3565e-03	5.4456e-04	6.8070e-05
$\lVert z-z_h\rVert$	3.6047e-02	4.5504e-03	5.6881e-04	7.1101e-05
$\lVert\nabla(P_hy-y_h)\rVert$	7.0125e-02	8.7815e-03	1.0977e-03	1.3721e-04
$\lVert\nabla(P_hz-z_h)\rVert$	7.2408e-02	9.0561e-03	1.1320e-03	1.4151e-04
$\lVert\nabla(y-y_h)\rVert$	8.4513e-02	2.1128e-02	5.2821e-03	1.3205e-03
$\lVert\nabla(z-z_h)\rVert$	8.7275e-02	2.1819e-02	5.4745e-03	1.3686e-03
$\lVert\boldsymbol{p}-\boldsymbol{p}_h\rVert$	1.2415e-01	3.1038e-02	7.7595e-03	1.9398e-03
$\lVert\boldsymbol{q}-\boldsymbol{q}_h\rVert$	1.2408e-01	3.1021e-02	7.7553e-03	1.9388e-03

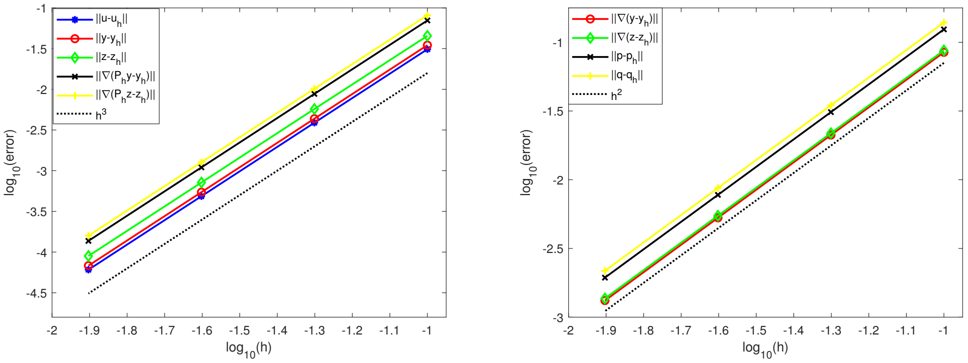

FIGURE 2. Convergence rates $\mathscr{O}(h^3)$ (left) and $\mathscr{O}(h^2)$ (right) of Example 2.

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