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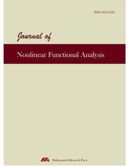

MULTIPLICITY OF SOLUTIONS FOR p-LAPLACIAN FRACTIONAL DIFFERENTIAL EQUATIONS WITH INSTANTANEOUS AND NON-INSTANTANEOUS IMPULSES: AN ADVANCED VARIATIONAL METHOD

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Abstract. This study investigates the multiplicity of solutions for a system of p-Laplacian fractional differential equations (FDEs) subjected to both instantaneous and non-instantaneous impulses. By employing a variant of Bonanno's local minimum theorem, we establish the existence of one or two solutions under appropriate algebraic conditions, including the classical Ambrosetti-Rabinowitz (AR) condition applied to the nonlinear term. Additionally, utilizing the critical point theorems proposed by Averna and Bonanno, we explore the existence of two and three solutions in a specific scenario of the system. The results contribute to a deeper understanding of the solution structure of impulsive FDEs and demonstrate the effectiveness of advanced variational techniques in addressing complex differential systems.

Keywords. Boundary value problem; Fractional differential equation; Instantaneous impulses; Multiple solutions; Variational methods.

1. INTRODUCTION

There are many different kind of tools in mathematics to investigate the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals), but fractional calculus is one of the valuable tools for this reasons. In particular, this field includes the notion and methods of solving of differential equations involving fractional derivatives of the unknown function (called fractional differential equations or in summary FDEs). In other words, FDEs are extension of ordinary differential equations and integration to arbitrary non-integer orders. FDEs are more useful compared to ordinary differential equations relatively, due to FDEs give a good template for biological and physical phenomena, mathematical modeling of engineering, and so on; see for details [14, 19, 22]. Recently, many authors studied

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the existence and multiplicity of solutions to nonlinear boundary value problems of FDEs by applying nonlinear analysis methods (fixed-point theorems, variational method etc.); see, e.g., [15, 17, 20, 26, 30, 31, 32] and the references therein. For example, author in [17], based on variational methods, estimated the existence of three distinct solutions for the following nonlinear perturbed fractional boundary value problem under suitable assumptions on the nonlinear term:

$$\begin{cases} \dfrac{d}{dt}\big({}_0D_t^{\alpha-1}({}_0^CD_t^{\alpha}w(t)) - {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha-1}w(t))\big) \\ +\lambda f(w(t)) + \mu g(w(t)) = 0, \quad t \in [0,T] \\ w(0) = w(T) = 0, \end{cases}$$

where $\alpha \in (\frac{1}{2}, 1]$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. The discussion of a coupled system of fractional order became an important field of research because this kind of system can often occur in many real-world problems; see, e.g., [1, 2, 5, 36] and the references therein. In [18] the existence of weak solutions for a class of fractional differential systems based on variational methods was examined.

Since real-world processes and phenomena can change suddenly and quickly and not stable, then using impulsive differential equations to describe such situations is more useful incomparison with differential equations or difference equations to a large extent. Studying on impulsive differential equations has attracted a special interest due to the various applications in the real world. Impulse effects can be widely seen in many realistic systems like signal processing systems, automatic control systems, flying abject motions and etc. For the general aspects of impulsive differential equations; see [7]. According to the duration of action, there exist two types of impulses: instantaneous and non-instantaneous impulses. Some dynamical processes contain not only instantaneous impulses but also non-instantaneous impulses such as intravenous injection. Actually, both of them occur at the same time. Since the drug enters the blood and the subsequent absorption of the body is a sudden and continuous process, this situation can be explained as an impulsive behavior. The impulse suddenly starts to jump at any fixed point in time (drug enters the blood) and continues to occur within a limited time interval (the body absorbs the drug). The study of boundary value problems for impulsive differential equations via variational approach was suggested by Tian–Ge [33] and Nieto–O'Regan [25]. There exist many studies using variational methods in order to solve non-instantaneous impulsive fractional differential equations.

Recently, due to the extensive progress in the theory of fractional calculus and impulsive differential equations as well as its application in several fields, some researchers studied the existence and multiplicity of solutions for fractional differential equations with impulses by employing the variational methods and fixed-point theorems; see, e.g., [6, 12, 16, 21, 23, 34, 37]. In [37], Zhang and Liu focused on the existence of solutions for the following fractional differential system include instantaneous and non-instantaneous impulses:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^CD_t^{\alpha}w(t)) = f_i(t, w(t)), \quad t \in (s_i, t_{i+1}], \quad i = 0, \cdots, n, \\ \Delta({}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}w))(t_i) = \mathscr{I}_i(w(t_i)), \quad i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}w)(t) = {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}w)(t_i^+), \quad t \in (s_i, t_i], \quad i = 1, 2, \cdots, n, \\ {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}w)(s_i^-) = {}_tD_T^{\alpha-1}({}_0^CD_t^{\alpha}w)(s_i^+), \quad i = 1, 2, \cdots, n, \\ w(0) = w(T) = 0, \end{cases}$$

where $\alpha \in (\frac{1}{2}, 1]$(other information of the system was mentioned in [37]). They established the existence result by using the variational method, and they present an example and demonstrates the main result.

The authors in [23] brought up the following system of p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses

$$\begin{cases} {}_tD_T^{\alpha}(\Theta_p({}_0^CD_t^{\alpha}w(t))) + a(t)\Theta_p(w(t)) = \lambda f_i(t,w(t)), \quad t \in (s_i, t_{i+1}], \ i = 0, \cdots, m, \\ \Delta_t D_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i))) = \mathscr{I}_i(w(t_i)), \quad i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t)) = {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^+))), \quad t \in (s_i, t_i], \ i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^-))) = {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^+))), \quad i = 1, 2, \cdots, m, \\ w(0) = w(T) = 0, \end{cases}$$

where $\alpha \in (\frac{1}{2}, 1]$ (other information of the system was mentioned in [23]). They established many solutions for the considered problem by applying a variational approches with Mountain Pass Lemma.

Motivated by the above results, in this work, we bring up the following p-Laplacian fractional differential system with instantaneous and non-instantaneous impulses

$$\begin{cases} {}_tD_T^{\alpha}(\Theta_p({}_0^CD_t^{\alpha}w(t))) + \gamma(t)\Theta_p(w(t)) = \lambda f_i(t,w(t)), \ t \in (s_i, t_{i+1}], \ i = 0, \cdots, m, \\ \Delta_t D_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i))) = \mu \mathscr{I}_i(w(t_i)), \ i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^-)) = {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^+))), \ t \in (t_i, s_i], \ i = 1, 2, \cdots, m, \\ {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^-))) = {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^+))), \quad i = 1, 2, \cdots, m, \\ w(0) = w(T) = 0, \end{cases} \qquad (D_{\lambda,\mu}^{\alpha,f})$$

where ${}_0^CD_t^{\alpha}, {}_tD_T^{\alpha}$ are the Caputo fractional derivative and the right Riemann-Liouville fractional derivative of order α, respectively, $1 < p < +\infty$, $\frac{1}{p} < \alpha \leq 1$, $\Theta_p(s) = |s|^{p-2}s(s \neq 0)$ is $p-$Laplacian operator, $\Theta_p(0) = 0$, $\lambda, \mu > 0$ are real parameters and $0 = s_0 < t_1 < s_1 < t_2 < \cdots < s_m < t_{m+1} = T$, $\gamma \in C([0,T], \mathbb{R})$ and there are two numbers γ_1 and γ_2 so that $0 < \gamma_1 \leq \gamma(t) \leq \gamma_2$, $\mathscr{I}_i \in C(\mathbb{R}, \mathbb{R})$ for $i = 1, \ldots, m$, $f_i \in C([s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$ for $i = 0, \ldots, m$ and

$$\Delta_t D_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i))) = {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^+))) - {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^-))),$$

$${}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^{\pm}))) = \lim_{t \to t_i^{\pm}} {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i))),$$

$${}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^{\pm}))) = \lim_{t \to s_i^{\pm}} {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i))).$$

We use variational methods and four local minimum theorems as the main approach and tools for differentiable functionals. By Bonanno's local minimum theorem, we investigate the existence of solutions for the problem under algebraic conditions with the classical AR condition on $f_0, f_1, \ldots, f_m$ (see [3]). In single parameter case (i.e., $\lambda = \mu$), we study the existence of two and there solutions for system $(D_{\lambda,\mu}^{\alpha,f})$ by using two critical point theorems, proposed by Averna and Bonanno, respectively. Compared to the previous results, we give some new assumptions to obtain the existence of at least one nontrivial weak solution of system $(D_{\lambda,\mu}^{\alpha,f})$.

2. PRELIMINARIES

We provide here some definitions, propositions, lemmas, and four theorems as the basic tools to demonstrate our results in the next sections. We refer to [13, 24, 28].

Definition 2.1. Let X be a real Banach space. The continuously Gâteaux differentiable functional $\mathscr{J} : X \to \mathbb{R}$ is said to satisfy the *Palais-Smale condition* (in short (PS)-condition) if any sequence $\{w_n\}$ such that $\{\mathscr{J}(w_n)\}$ is bounded and $\lim_{n\to\infty} \|\mathscr{J}'(w_n)\|_{X^*} = 0$ has a convergent subsequence.

Definition 2.2. Let $\mathscr{O}, \mathscr{Q} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set $\mathscr{J} = \mathscr{O} - \mathscr{Q}$ and fix $r_1, r_2 \in [-\infty, \infty]$ with $r_1 < r_2$. We recall that $\mathscr{J}$ satisfies the *Palais-Smale condition cut off lower at r_1 and upper at r_2* (in short ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition) if any sequence $\{w_n\}$ with this property that $\{\mathscr{J}(w_n)\}$ is bounded, $\lim_{n\to\infty} \|\mathscr{J}'(w_n)\|_{X^*} = 0$, and $r_1 < \mathscr{O}(w_n) < r_2$ for all $n \in \mathbb{N}$, has a convergent subsequence.

Remark 2.3. Obviously, if $r_1 = -\infty$ and $r_2 = \infty$, it coincides with the classical (PS)-condition. In addition, if $r_1 = -\infty$ and r_2 is a real number, it is marked by $(\mathrm{PS})^{[r_2]}$, while if $r_1 \in$ is a real number and $r_2 = \infty$, it is denoted by ${}^{[r_1]}(\mathrm{PS})$. Indeed, if $\mathscr{Q}$ and $\mathscr{O}$ are two continuously Gâteaux differentiable functionals defined on a real Banach space X and fixed $r \in \mathbb{R}$, the functional $\mathscr{J} = \mathscr{O} - \mathscr{Q}$ is said to verify the Palais-Smale condition cut off upper at r (in short $(\mathrm{PS})^{[r]}$) if any sequence $\{w_n\}_{n\in\mathbb{N}}$ in X with this property that $\{\mathscr{J}(w_n)\}$ is bounded, $\lim_{n\to\infty} \|\mathscr{J}'(w_n)\|_{X^*} = 0$, and $\mathscr{O}(w_n) < r$ for each $n \in \mathbb{N}$, has a convergent subsequence. Furthermore, if $\rho_1, \rho_2 \in [-\infty, \infty]$ with $r_1 \le \rho_1 < \rho_2 \le r_2$ and $\mathscr{J}$ satisfies ${}^{[r_1]}(\mathrm{PS})^{[r_2]}$-condition, then it has ${}^{[\rho_1]}(\mathrm{PS})^{[\rho_2]}$-condition. In particular, we deduce that if $\mathscr{J}$ satisfies the classical (PS)-condition, then it satisfies ${}^{[\rho_1]}(\mathrm{PS})^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, \infty]$ with $\rho_1 < \rho_2$.

We will use the following theorems to prove the main results.

Theorem 2.4. [11, Theorem 2.3] *Let X be a real Banach space and let $\mathscr{O}, \mathscr{Q} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with $\inf_{w\in X} \mathscr{O}(w) = \mathscr{O}(0) = \mathscr{Q}(0) = 0$. Assume that for one $\bar{w} \in X$ there exists $r > 0$ such that $0 < \mathscr{O}(\bar{w}) < r$ and*

- (a_1) $\frac{\sup_{\mathscr{O}(w)\le r} \mathscr{Q}(w)}{r} < \frac{\mathscr{Q}(\bar{w})}{\mathscr{O}(\bar{w})}$,
- (a_2) *for each $\lambda \in \Lambda_r := \left(\frac{\mathscr{O}(\bar{w})}{\mathscr{Q}(\bar{w})}, \frac{r}{\sup_{\mathscr{O}(w)\le r} \mathscr{Q}(w)}\right)$ the functional $\mathscr{J}_\lambda := \mathscr{O} - \lambda\mathscr{Q}$ satisfies $(\mathrm{PS})^{[r]}$-condition.*

Then, for each $\lambda \in \Lambda_r$, there exists $w_{0,\lambda} \in \mathscr{O}^{-1}(0, r)$ such that $\mathscr{J}_\lambda'(w_{0,\lambda}) \equiv 0_{X^*}$ and $\mathscr{J}_\lambda(w_{0,\lambda}) \le \mathscr{J}_\lambda(w)$ for all $w \in \mathscr{O}^{-1}(0, r)$.

Theorem 2.5. [11, Theorem 3.2] *Let X be a real Banach space, $\mathscr{O}, \mathscr{Q} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals so that $\mathscr{O}$ is below bounded and $\mathscr{O}(0) = \mathscr{Q}(0) = 0$. Put $r > 0$ and suppose that, for each*

$$\lambda \in \Gamma_r := \left(0, \frac{r}{\sup_{w\in\mathscr{O}^{-1}(-\infty,r)} \mathscr{Q}(w)}\right),$$

the functional $\mathscr{J}_\lambda := \mathscr{O} - \lambda\mathscr{Q}$ satisfies (PS)-condition and it is below unbounded. Then, for each $\lambda \in \Gamma_r$, the functional $\mathscr{J}_\lambda$ admits two distinct critical points.

Two following theorems are based on a variational principle that presented by Ricceri in [29]. Below we mean by $\overline{S}^w$ is the closure of S in the weak topology.

Theorem 2.6. [4, Theorem A] *Let $\mathscr{O}$ and $\mathscr{Q}$ be two real valued and continuously Gâteaux differentiable functional on real reflexive Banach space X such that $\mathscr{O}$ is sequentially weakly* *lower semicontinuous and Gâteaux derivative of it admits a continuous inverse on X^*, and Gâteaux derivative of $\mathscr{Q}$ is compact. Moreover, assume that*

(i) $\lim_{\|w\|\to\infty}(\mathscr{O}(w)+\lambda\mathscr{Q}(w))=\infty, \quad \forall\lambda\in[0,\infty)$;

(ii) *there is $r\in\mathbb{R}$ such that $\inf_X \mathscr{O} < r$ and $\varphi_1(r)<\varphi_2(r)$ where*

$$\varphi_1(r):=\inf_{w\in\mathscr{O}^{-1}(-\infty,r)} \frac{\mathscr{Q}(w)-\inf_{\overline{\mathscr{O}^{-1}(-\infty,r)}^{w}}\mathscr{Q}}{r-\mathscr{O}(w)}$$

and

$$\varphi_2(r):=\inf_{w\in\mathscr{O}^{-1}(-\infty,r)}\ \sup_{v\in\mathscr{O}^{-1}[r,\infty)} \frac{\mathscr{Q}(w)-\mathscr{Q}(v)}{\mathscr{O}(v)-\mathscr{O}(w)}.$$

Then, for each $\lambda\in\left(\frac{1}{\varphi_2(r)},\frac{1}{\varphi_1(r)}\right)$, the functional $\mathscr{O}+\lambda\mathscr{Q}$ has at least three critical points in X.

Remark 2.7. In Theorem 2.6, if $\varphi_1(r)=0$, then λ has no upper bound.

Theorem 2.8. [10, Theorem 1.1] *Let $\mathscr{O}$ and $\mathscr{Q}$ be two real valued, sequentially weakly lower semicontinuous and Gâteaux differentiable functions on real reflexive Banach space X. Assume that $\mathscr{O}$ is (strongly) continuous and satisfies $\lim_{\|w\|\to\infty}\mathscr{O}(w)=\infty$. Assume also that there exist two constants r_1 and r_2 such that*

(j) $\inf_X\mathscr{O}<r_1<r_2$;

(jj) $\varphi_1(r_1)<\varphi_2^*(r_1,r_2)$;

(jjj) $\varphi_1(r_2)<\varphi_2^*(r_1,r_2)$, *where φ_1 is defined as in Theorem 2.6 and*

$$\varphi_2^*(r_1,r_2):=\inf_{w\in\mathscr{O}^{-1}(-\infty,r_1)}\ \sup_{v\in\mathscr{O}^{-1}[r_1,r_2)} \frac{\mathscr{Q}(w)-\mathscr{Q}(v)}{\mathscr{O}(v)-\mathscr{O}(w)}.$$

Then, for each $\lambda\in\left(\frac{1}{\varphi_2^*(r_1,r_2)},\min\{\frac{1}{\varphi_1(r_1)},\frac{1}{\varphi_1(r_2)}\}\right)$, the functional $\mathscr{O}+\lambda\mathscr{Q}$ admits at least two critical points which lie in $\mathscr{O}^{-1}(-\infty,r_1]$ and $\mathscr{O}^{-1}[r_1,r_2)$ respectively.

We refer to [9], where Theorems 2.4 and 2.5 were successfully applied to prove the existence of at least one and two solutions for elliptic Dirichlet problems with variable exponent. We also refer the readers to [13] for situations of successful employments of results such as Theorems 2.6 and 2.8 to ensure the existence of at least one, two and three solutions for Kirchhoff-type second-order impulsive differential equations on the half-line.

This section contains notations, definitions, and preliminary facts which are utilized throughout this paper. In fact, the aim of this section is the convenience of the reader. For any fixed $t\in[0,T]$ and $1<p<\infty$,

$$\|w\|_\infty=\max_{t\in[0,T]}|w(t)|, \quad \forall w\in \mathrm{C}([0,T],\mathbb{R}^N),$$

$$\|w\|_{\mathrm{L}^p}=\left(\int_0^T|w(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}, \quad \forall w\in\mathrm{L}^p([0,T],\mathbb{R}^N),$$

$$\|w\|_{\mathrm{L}^p([0,T])}=\left(\int_0^t|w(s)|^p\mathrm{d}s\right)^{\frac{1}{p}}, \quad \forall w\in\mathrm{L}^p([0,T],\mathbb{R}^N),$$

and

$$\|w\|_{\mathrm{L}^p([s_i,t_{i+1}])}=\left(\int_{s_i}^{t_{i+1}}|w(t)|^p\mathrm{d}s\right)^{\frac{1}{p}},\ w\in\mathrm{L}^p([s_i,t_{i+1}],\mathbb{R}^N),\ i=0,\cdots,m.$$

Definition 2.9. [23] Let $\alpha \in \left(\frac{1}{p}, 1\right]$, $p \in (1, \infty)$, and the fractional derivative space $X_0^{\alpha,p}$ be defined as $X_0^{\alpha,p} = \overline{\mathrm{C}_0^\infty([0,T],\mathbb{R})}$ with the norm

$$\|w\|_{\alpha,p} = \Big(\int_0^T |w(t)|^p \mathrm{d}t + \int_0^T |{}_0^CD_t^\alpha w(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}}, \quad \forall w \in X_0^{\alpha,p}.$$

Proposition 2.10. [23] *Let* $0 < \alpha \leq 1$, $1 < p < \infty$*,*

$$X_0^{\alpha,p} = \{w : [0,T] \to \mathbb{R} |\ w, {}_0^CD_t^\alpha w \in \mathrm{L}^p([0,T],\mathbb{R}),\ w(0) = w(T) = 0\}.$$

Then $X_0^{\alpha,p}$ *is a reflexive and separable Banach space.*

Proposition 2.11. [23] *Let* $0 < \alpha \leq 1$ *and* $1 < p < \infty$*, for all* $w \in X_0^{\alpha,p}$*. Then*

$$\|w\|_{\mathrm{L}^p} \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\Big(\int_0^T |{}_0^CD_t^\alpha w(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}}.$$

Further, if $\alpha > \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then* $\|w\|_\infty \leq \dfrac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}\|{}_0^CD_t^\alpha w\|_{\mathrm{L}^p}$.

Lemma 2.12. [27, Lemma 2.2] *It is easy to check that if* $\gamma \in \mathrm{C}([0,T],\mathbb{R})$ *and* $0 < \gamma_1 \leq \gamma(t) \leq \gamma_2$*, then an equivalent norm in* $X_0^{\alpha,p}$ *is the following*

$$\|w\|_\gamma = \Big(\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \gamma(t)|w(t)|^p \mathrm{d}t + \int_0^T |{}_0^CD_t^\alpha w(t)|^p \mathrm{d}t\Big)^{\frac{1}{p}},$$

and $\|w\|_\infty \leq \Lambda\|w\|_\gamma$*, where*

$$\Lambda = \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)[q(\alpha-1)+1]^{\frac{1}{q}}}.$$

Moreover, for all $w \in X_0^{\alpha,p}$*,* $\Lambda_1\|w\|_{\alpha,p} \leq \|w\|_\gamma \leq \Lambda_2\|w\|_{\alpha,p}$*, where* $\Lambda_1 = \left[1 + \left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^p\right]^{\frac{-1}{p}}$ *and* $\Lambda_2 = 1$.

Definition 2.13. $w \in X_0^{\alpha,p}$ is called a *weak solution* to problem ($D_{\lambda,\mu}^{\alpha,f}$) if

$$\int_0^T \Theta_p({}_0^CD_t^\alpha w(t)){}_0^CD_t^\alpha y(t)\mathrm{d}t + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \gamma(t)|w(t)|^{p-2}w(t)y(t)\mathrm{d}t$$
$$-\lambda\sum_{i=0}^m \int_{s_i}^{t_{i+1}} f_i(t,w(t))y(t)dt + \mu\sum_{i=1}^m \mathscr{I}_i(w(t_i))y(t_i) = 0$$

holds for all $y \in X_0^{\alpha,p}$.

The condition we set on $\mathscr{I}_i$ for $i = 1,\dots,m$ is as follows.

(I) $\mathscr{I}_i$ is a Lipschitz continuous function with the Lipschitz constant $k > 0$; i.e.,

$$|\mathscr{I}_i(s_1) - \mathscr{I}_i(s_2)| \leq k|s_1 - s_2|^{p-1},\ i = 1,\dots,m \tag{2.1}$$

for all $s_1, s_2 \in \mathbb{R}$, satisfying $\mathscr{I}_i(0) = 0$.

Proposition 2.14. *Let* $\mathscr{L} : X_0^{\alpha,p} \to (X_0^{\alpha,p})^*$ *be defined by*

$$\mathscr{L}(w)(y) = \int_0^T \Theta_p({}_0^C D_t^\alpha w(t)) {}_0^C D_t^\alpha y(t)\mathrm{d}t + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \gamma(t)|w(t)|^{p-2}w(t)y(t)\mathrm{d}t$$

for every $w, y \in X_0^{\alpha,p}$*. Then* $\mathscr{L}$ *accepts a continuous inverse on* $(X_0^{\alpha,p})^*$.

Proof. For all $w \in X_0^{\alpha,p}$, we have

$$\mathscr{L}(w)(w) = \int_0^T \Theta_p({}_0^C D_t^\alpha w(t)) {}_0^C D_t^\alpha w(t)\mathrm{d}t + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \gamma(t)|w(t)|^{p-2}w(t)w(t)\mathrm{d}t = \frac{1}{p}\|w\|_\gamma^p.$$

Note that $\lim_{\|w\|_\gamma \to \infty} \|w\|_\gamma^{p-1} = \infty$. Thus $\mathscr{L}$ is coercive. According to our assumptions on the data, one has $\langle \mathscr{L}(w) - \mathscr{L}(y), w - y\rangle = N\|w-y\|_\gamma^p > 0$ for some $N > 0$ and for every $w, y \in X_0^{\alpha,p}$, which means that $\mathscr{L}$ is strictly monotone. By [35, Theorem 26.A(d)], $\mathscr{L}^{-1}$ exists and has continuous inverse on $(X_0^{\alpha,p})^*$. □

3. EXISTENCE OF ONE WEAK SOLUTION

In this section, we investigate the existence of one weak solution of problem $(D_{\lambda,\mu}^{\alpha,f})$. We set $\sigma := \sum_{i=0}^m (t_{i+1} - s_i)$.

Theorem 3.1. *Suppose that* d *and* c *are two positive constants with* $\Lambda^p|d|^p\gamma_2\sigma < c^p$ *and below conditions hold:*

(A$_1$) $\dfrac{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t}{c^p} < \dfrac{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t}{\Lambda^p|d|^p\gamma_2\sigma}$;

(A$_2$) $\limsup_{|\xi|\to\infty} \frac{F_i(t,\xi)}{|\xi|^p} \le 0$ *uniformly in* $\mathbb{R}$ *for* $i = 0, \cdots, m$.

Then, for each

$$\lambda \in \Omega = \left(\frac{|d|^p\gamma_2\sigma}{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t}, \frac{c^p}{\Lambda^p \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t} \right),$$

and for every functions $\mathscr{I}_1, \ldots, \mathscr{I}_m : \mathbb{R} \to \mathbb{R}$ *satisfying Lipschitz condition with the Lipschitz constant* $k > 0$ *and*

$$\limsup_{|\xi|\to\infty} \frac{\int_0^\xi \mathscr{I}_i(x)\mathrm{d}x}{|\xi|^p} < \infty \tag{3.1}$$

for $i = 1, \ldots, m$*, there exists* $\delta_\lambda > 0$ *given by*

$$\min\left\{ \frac{c^p - p\lambda\Lambda^p \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t}{p\Lambda^p k c^p}, \frac{p\lambda \sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t - |d|^p\gamma_2\sigma}{pk|d|^p} \right\}, \tag{3.2}$$

such that, for each $\mu \in [0, \delta_\lambda[$*, system* $(D_{\lambda,\mu}^{\alpha,f})$ *possesses at least one weak solution* w_λ *in* $X_0^{\alpha,p}$ *such that* $\max_{t\in[0,T]} |w_\lambda(t)| < c$.

Proof. In order to apply Theorem 2.4, we consider the space $X_0^{\alpha,p}$ with the norm defined in Lemma 2.12. We introduce the functionals $\mathscr{O}$ and $\mathscr{Q}$ for all $w \in X_0^{\alpha,p}$ as follows

$$\mathscr{O}(w) := \frac{1}{p}\|w\|_\gamma^p \tag{3.3}$$

and

$$\mathscr{Q}(w) := \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, w(t))\mathrm{d}t - \frac{\mu}{\lambda} \sum_{i=1}^{m} \int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s,$$

where $F_i(t,\xi) = \int_0^{\xi} f_i(t,s)\mathrm{d}s$ for $i = 0, \cdots, m$. It is easy to check that $\mathscr{O}$ is Gâteaux differentiable and sequentially weakly lower semicontinuous, and also its Gâteaux derivative is the functional $\mathscr{O}'(w) \in (X_0^{\alpha,p})^*$ given by

$$\mathscr{O}'(w)(y) = \int_0^T \Theta_p({}_0^C D_t^{\alpha} w(t))\, {}_0^C D_t^{\alpha} y(t)\mathrm{d}t + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \gamma(t)|w(t)|^{p-2} w(t) y(t)\mathrm{d}t$$

for every $y \in X_0^{\alpha,p}$. According to Proposition 2.14, we observe that $\mathscr{O}'$ accepts a continuous inverse on $(X_0^{\alpha,p})^*$. Further, $\mathscr{Q} \in \mathrm{C}^1(X_0^{\alpha,p}, \mathbb{R})$ and also it has compact derivative. Since $\mu < \delta_\lambda$ we can consider a constant number κ so that $\limsup_{|\xi|\to\infty} \frac{\int_0^{\xi} \mathscr{I}_i(\tau)\mathrm{d}\tau}{|\xi|^p} < \kappa$ for $i = 1, \ldots, m$ and $p\Lambda^p \mu m \kappa < 1$. So, there exists a positive constant ι such that

$$\int_0^{\xi} \mathscr{I}_i(\tau)\mathrm{d}\tau \leq \kappa \xi^p + \iota$$

for any $\xi \in \mathbb{R}$ and $i = 1, \ldots, m$. At the moment, we fix $0 < \varepsilon < \frac{1 - p\mu k \Lambda^p m}{p\lambda \Lambda^p \sigma}$. From assumption (A_2), there exists a function $\rho_\varepsilon \in \mathrm{L}^1([0,T])$ such that $F_i(t,\tau) \leq \varepsilon \tau^p + \rho_\varepsilon(t)$, for every $(t,\tau) \in [s_i, t_{i+1}] \times \mathbb{R}$ and $i = 0, \ldots, m$. Therefore, by using the above assumptions, for each $w \in X_0^{\alpha,p}$, we conclude

$$\begin{aligned}
\mathscr{O}(w) - \lambda \mathscr{Q}(w) &= \frac{1}{p}\|w\|_\gamma^p - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, w(t))\mathrm{d}t + \mu \sum_{i=1}^{m} \int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s \\
&\geq \frac{\|w\|_\gamma^p}{p} - \lambda\varepsilon \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} w^p(t)\mathrm{d}t - \lambda \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \rho_\varepsilon(t)\mathrm{d}t - \sum_{i=1}^{m} (\mu\kappa w^p(t_i) + \mu\iota) \\
&\geq \left(\frac{1}{p} - \lambda\Lambda^p \varepsilon\sigma - \mu\kappa\Lambda^p m\right)\|w\|_\gamma^p - \lambda \sum_{i=0}^{m} \|\rho_\varepsilon\|_{\mathrm{L}^1(s_i, t_{i+1})} - \mu\iota m
\end{aligned}$$

and thus $\lim_{\|w\|_\gamma \to \infty} (\mathscr{O}(w) - \lambda\mathscr{Q}(w)) = \infty$. Thus $\mathscr{J}_\lambda = \mathscr{O} - \lambda\mathscr{Q}$ is coercive. Accordingly, by [8, Proposition 2.1], $\mathscr{J}_\lambda = \mathscr{O} - \lambda\mathscr{Q}$ verifies $(\mathrm{PS})^{[r]}$-condition for each $r > 0$ and so condition (a_2) of Theorem 2.4 is vindicated. Taking $\lambda \in (0, \lambda^*)$, we have

$$p \frac{\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t - \frac{\mu}{\lambda} k|d|^p}{|d|^p \gamma_2 \sigma} > \frac{1}{\lambda}.$$

Denote $r = \dfrac{c^p}{p\Lambda^p}$. For each $w \in X_0^{\alpha,p}$ with $\|w\|_\gamma^p \leq pr$, by using Lemma 2.12 and (3.3), we see that

$$\frac{1}{p\Lambda^p}|w(t)|^p \leq \frac{1}{p\Lambda^p}\|w\|_\infty^p \leq \mathscr{O}(w) \leq r,$$

which means that $|w(t)| < c$ for every $t \in [0,T]$. Put $v(t) = d$ for all $t \in [0,T]$. Therefore, $v(t) \in X_0^{\alpha,p}$ and

$$\begin{aligned}\mathscr{O}(v) &= \frac{1}{p}\|v\|^p = \frac{1}{p}\left(\int_0^T |{}_0^C D_t^\alpha v(t)|^p \mathrm{d}t + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \gamma(t)|v(t)|^p \mathrm{d}t\right) \\ &= \frac{1}{p}\left(\int_0^T \left(\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} v' \mathrm{d}s\right)^p \mathrm{d}t + \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \gamma(t)|v(t)|^p \mathrm{d}t\right) \\ &\le \frac{|d|^p \gamma_2 \sigma}{p},\end{aligned}$$

which follows that

$$\frac{|d|^p\gamma_1\sigma}{p} \le \mathscr{O}(v) \le \frac{|d|^p\gamma_2\sigma}{p}. \tag{3.4}$$

On the other hand, since $\Lambda^p |d|^p \gamma_2 \sigma < c^p$, we conclude $0 < \mathscr{O}(v) < r$. Hence,

$$\mathscr{O}^{-1}(-\infty, r] = \{w \in X_0^{\alpha,p} : \mathscr{O}(w) \le r\} \subseteq \left\{w \in X_0^{\alpha,p} : |w(t)| \le c,\ \forall t \in [0,T]\right\}.$$

In addition, since $|\mathscr{I}_i(\xi)| < \frac{pk}{m}|\xi|^{p-1}$ for all $\xi \in \mathbb{R}$, we reach

$$-k\|w\|_\infty^p \le \sum_{i=1}^m \int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s \le k\|w\|_\infty^p. \tag{3.5}$$

Therefore, $\sup_{w\in\mathscr{O}^{-1}(-\infty,r)} \sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,w(t))\mathrm{d}t \le \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t$. Using the second inequality in (3.5), we arrive at

$$\sup_{w\in\mathscr{O}^{-1}(-\infty,r)} \sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,w(t))\mathrm{d}t - \frac{\mu}{\lambda}\sum_{i=1}^m \int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s \le \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t + \frac{\mu}{\lambda}kc^p$$

for each $w \in X_0^{\alpha,p}$ with $\mathscr{O}(w) < r$. Then $\sup_{\mathscr{O}(w)<r} \mathscr{Q}(w) \le \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t + \frac{\mu}{\lambda}kc^p$. Note that

$$\begin{aligned}\mathscr{Q}(v) &= \sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,v(t))\mathrm{d}t - \frac{\mu}{\lambda}\sum_{i=1}^m \int_0^{v(t_i)} \mathscr{I}_i(s)\mathrm{d}s \\ &\ge \sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t - \frac{\mu}{\lambda}k|d|^p.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\sup_{w\in\mathscr{O}^{-1}(-\infty,r]} \mathscr{Q}(w)}{r} &= \frac{\sup_{w\in\mathscr{O}^{-1}(-\infty,r]} \sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,w(t))\mathrm{d}t - \frac{\mu}{\lambda}\sum_{i=1}^m \int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s}{r} \\ &\le p\Lambda^p \frac{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t + \frac{\mu}{\lambda}kc^p}{c^p}\end{aligned}$$

and

$$\frac{\mathscr{Q}(v)}{\mathscr{O}(v)} \ge p\frac{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t - \frac{\mu}{\lambda}k|d|^p}{|d|^p\gamma_2\sigma}.$$

We obtain

$$p\Lambda^p \frac{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\le c} F_i(t,z)\mathrm{d}t + \frac{\mu}{\lambda}kc^p}{c^p} < \frac{1}{\lambda}.$$

Furthermore,

$$p\frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t-\frac{\mu}{\lambda}k|d|^p}{|d|^p\gamma_2\sigma}>\frac{1}{\lambda}.$$

It follows that

$$p\Lambda^p\frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\sup_{|z|\le c}F_i(t,z)\mathrm{d}t+\frac{\mu}{\lambda}kc^p}{c^p}<\frac{1}{\lambda}<p\frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} F_i(t,d)\mathrm{d}t-\frac{\mu}{\lambda}k|d|^p}{|d|^p\gamma_2\sigma}.$$

Since

$$\lambda\in\left(\frac{\mathscr{O}(v)}{\mathscr{Q}(v)},\frac{r}{\sup_{\mathscr{O}(w)\le r}\mathscr{Q}(w)}\right),$$

Theorem 2.4 with $\bar{w}=v$ guarantees the existence of a local minimum point w_λ for $\mathscr{J}_\lambda$ such that $0<\mathscr{O}(w_\lambda)<r$, so w_λ is a nontrivial weak solution to problem $(D^{\alpha,f}_{\lambda,\mu})$ with $\max_{t\in[0,T]}|w_\lambda(t)|<c$. This completes the proof. ☐

We show the applicability of Theorem 3.1 by giving the following example.

Example 3.2. Let $\alpha=0.75$, $T=1$, $p=2$, $m=1$, $\gamma(t)=1+t$ and $\mathscr{I}_1(t)=\sin t$ for all $t\in[0,1]$, $d=0.01$, $c=100$ and $0=s_0<t_1=\frac{1}{3}<\frac{2}{3}=s_1<t_2=1$. Thus $\gamma_1=1$, $\gamma_2=2$, $k=1$, $\Lambda=1$ and it is clear that $\Lambda^p|d|^p\gamma_2\sigma<c^p$ is fulfilled. Now we choose $f_0(t,\xi)=\sqrt{|\xi\sin t|}$ for every $(t,\xi)\in[0,\frac{1}{3}]\times\mathbb{R}$ and $f_1(t,\xi)=2t\xi^{-\frac{1}{2}}$ for all $(t,\xi)\in[\frac{2}{3},1]\times\mathbb{R}$. By the expression of f_0 and f_1, we have $F_0(t,\xi)=\frac{2}{3}\xi^{\frac{3}{2}}\sqrt{|\sin t|}$ for all $(t,\xi)\in[0,\frac{1}{3}]\times\mathbb{R}$ and $F_1(t,\xi)=4t\sqrt{\xi}$ for all $(t,\xi)\in[\frac{2}{3},1]\times\mathbb{R}$. Hence, we see that $\lim_{|\xi|\to\infty}\frac{F_0(t,\xi)}{|\xi|^2}=\lim_{|\xi|\to\infty}\frac{F_1(t,\xi)}{|\xi|^2}=0$ and then (A_1) and (A_2) hold. Therefore, all the assumptions in Theorem 2.4 are satisfied. So, it follows that, for every $\lambda\in\big(0.025,25\big)$ and for each $0\le\mu<\min\left\{\frac{900-31\lambda}{1800},\frac{2060\lambda-12}{18}\right\}$, the problem $(D^{\alpha,f}_{\lambda,\mu})$ in this case admits at least one weak solutions $w_\lambda\in X_0^{0.75,2}$ such that $\max_{t\in[0,T]}|w_\lambda(t)|<100$.

4. EXISTENCE OF TWO WEAK SOLUTIONS

Now, we prove the existence of two distinct weak solutions for the problem $(D^{\alpha,f}_{\lambda,\mu})$ where the assumption (A_2) is not required. To this aim, we utilize Theorem 2.5.

Theorem 4.1. *Consider two positive constants d and c with* $\Lambda^p|d|^p\gamma_2\sigma<c^p$ *and suppose that*

- (A_3) *there exists a constant* $\vartheta>p$ *such that* $\mathscr{I}_i(\xi)\xi\le\vartheta\int_0^\xi\mathscr{I}_i(s)\mathrm{d}s<0$ *for all* $\xi\in\mathbb{R}$, $R>0$ *and* $i=1,\dots,m$, *and*

$$0<\vartheta F_i(t,\xi)\le\xi f_i(t,\xi),\quad |\xi|\ge R \text{ and } t\in[s_i,t_{i+1}],\quad i=0,\cdots,m. \tag{4.1}$$

Then, for each

$$\lambda\in\left(0,\frac{c^p}{p\Lambda^p\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\sup_{|z|\le c}F_i(t,z)\mathrm{d}t}\right),$$

and for every function $\mathscr{I}_i:\mathbb{R}\to\mathbb{R}$, $i=1,\dots,m$, *satisfying condition* (I) *there is* $\delta_\lambda>0$ *given by* (3.2) *so that, for every* $\mu\in[0,\delta_\lambda[$, *the system* $(D^{\alpha,f}_{\lambda,\mu})$ *gets at least two weak solutions* w_1 *and* w_2 *in* $X_0^{\alpha,p}$ *that* $\max_{t\in[0,T]}|w_1(t)|<c$.

Proof. We apply Theorem 2.5 to prove this theorem and consider the space $X_0^{\alpha,p}$ with the norm is defined in Lemma 2.12 and $\mathscr{O}$ and $\mathscr{Q}$ as defined in the proof of Theorem 3.1. Indeed, suppose that $\{w_n\}_{n\in\mathbb{N}} \subset X_0^{\alpha,p}$ such that $\{\mathscr{J}_\lambda(w_n)\}_{n\in\mathbb{N}}$ is bounded and $\mathscr{J}'_\lambda(w_n) \to 0$ as $n\to\infty$. Then, there exists a positive constant θ_0 such that $|\mathscr{J}_\lambda(w_n)| \le \theta_0$ and $|\mathscr{J}'_\lambda(w_n)| \le \theta_0$, for all $n\in\mathbb{N}$. From the definition of $\mathscr{J}'_\lambda$ and assumptions (A_3), we see that

$$\begin{aligned}&\theta_0+\theta_1\|w_n\|_\gamma \ge \vartheta\mathscr{J}_\lambda(w_n)-\mathscr{J}'_\lambda(w_n)w_n\\&=(\frac{\vartheta}{p}-1)\|w_n\|_\gamma^p+\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}[f_i(t,w_n(t))(w_n(t))-\vartheta F_i(t,w_n(t))]\,\mathrm{d}t\\&\quad+\mu\sum_{i=1}^m\int_0^{w_n(t_i)}[\vartheta\mathscr{I}_i(s)-\mathscr{I}_i(s)s]\,\mathrm{d}s\ge(\frac{\vartheta}{p}-1)\|w_n\|_\gamma^p\end{aligned}$$

for some $\theta_1>0$. Since $\vartheta>p$, then (w_n) is bounded. Since $X_0^{\alpha,p}$ is a reflexive Banach space, we have, up to a subsequence, $w_n \rightharpoonup w$ in $X_0^{\alpha,p}$. By $\mathscr{J}'_\lambda(w_n)\to 0$ and $w_n\rightharpoonup w$ in $X_0^{\alpha,p}$, we obtain $\left(\mathscr{J}'_\lambda(w_n)-\mathscr{J}'_\lambda(w)\right)(w_n-w)\to 0$. From the continuity of f and $\mathscr{I}_i$, $i=1,\dots,m$, we have

$$\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\left(f_i(t,w_n(t))-f_i(t,w(t))\right)(w_n(t)-w(t))\mathrm{d}t\to 0,\quad \text{as } n\to\infty,$$

and $\sum_{i=1}^m\left(\mathscr{I}_i(w_n(t_i))-\mathscr{I}_i(w(t_i))\right)(w_n(t_i)-w(t_i))\to 0$ as $n\to\infty$. From [23, Theorem 1.1], we have

$$\begin{aligned}&\left(\mathscr{I}'_\lambda(w_n)-\mathscr{I}'_\lambda(w)\right)(w_n-w)=\int_0^T\Theta_p({}_0^CD_t^\alpha w_n(t)){}_0^CD_t^\alpha(w_n(t)-w(t))\mathrm{d}t\\&+\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\gamma(t)|w_n(t)|^{p-2}w(t)(w_n(t)-w(t))\mathrm{d}t-\int_0^T\Theta_p({}_0^CD_t^\alpha w(t)){}_0^CD_t^\alpha(w_n(t)-w(t))\mathrm{d}t\\&+\sum_{i=0}^m\int_{s_i}^{t_{i+1}}\gamma(t)|w(t)|^{p-2}w(t)(w_n(t)-w(t))\mathrm{d}t+\mu\sum_{i=1}^m(\mathscr{I}_i(w_n(t_i))-\mathscr{I}_i(w(t_i)))(w_n(t_i)-w(t_i))\\&-\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}(f_i(t,w_n(t))-f_i(t,w(t)))(w_n(t)-w(t))\mathrm{d}t\\&\ge K\|w_n-w\|_\gamma^p+\mu\sum_{i=1}^m(\mathscr{I}_i(w_n(t_i))-\mathscr{I}_i(w(t_i)))(w_n(t_i)-w(t_i))\\&-\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}(f_i(t,w_n(t))-f_i(t,w(t)))(w_n(t)-w(t))\mathrm{d}t\to 0,\end{aligned}$$

where K is a constant, introduced in [23, Theorem 1.1]. So, $w_n\to w$ strongly in $X_0^{\alpha,p}$. Hence, $\mathscr{J}_\lambda$ applies (PS)-condition. Moreover, by simplifying (4.1), we can obtain constants $a_1,a_2>0$ such that $F_i(t,x)\ge a_1|x|^\vartheta-a_2$ for all $t\in[s_i,t_{i+1}]$ and $x\in\mathbb{R}$ for $i=0,\cdots,m$. For any $w\in X_0^{\alpha,p}\setminus\{0\}$ and $\iota>0$, we have

$$\begin{aligned}\mathscr{J}_\lambda(\iota w)&\le\frac{\iota^p}{p}\|w\|_\gamma^p-\lambda\sum_{i=0}^m\int_{s_i}^{t_{i+1}}F_i(t,\iota w(t))\mathrm{d}t+k\iota\mu\|w\|_\infty^p\\&\le\frac{\iota^p}{p}\|w\|_\gamma^p-\lambda\iota^\vartheta a_1\sum_{i=0}^m\int_{s_i}^{t_{i+1}}|w(t)|^\vartheta\mathrm{d}t+\lambda a_2\sigma+k\mu\iota^p\Lambda^p\|w\|_\gamma^p.\end{aligned}$$

Since $\vartheta > p$, the above inequality shows that $\mathscr{J}_\lambda$ is below unbounded. Therefore, all the hypotheses of Theorem 2.5 are verified. So, for each

$$\lambda \in \left(0, \frac{c^p}{p\Lambda^p \sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\leq c} F_i(t,z)\mathrm{d}t}\right),$$

$\mathscr{J}_\lambda$ accepts two distinct critical points that are weak solutions to system $(D_{\lambda,\mu}^{\alpha,f})$. The proof is complete. □

Remark 4.2. In Theorem 2.4, we see that if there exists $i \in \{0,\ldots,m\}$ such that $f_i(t,0) \neq 0$ for $t \in [s_i,t_{i+1}]$, so Theorem 4.1 bonds the existence of two nonzero weak solutions for the system $(D_{\lambda,\mu}^{\alpha,f})$. If we remove the restriction $f_i(t,0) \neq 0$ for $t \in [s_i,t_{i+1}]$ for all $i \in \{0,\ldots,m\}$, the second solution w_2 of the System $(D_{\lambda,\mu}^{\alpha,f})$ may be zero, but it has at least a nonzero solution.

Remark 4.3. Following the proof in [11, Theorem 3.5], we can obtain the nonzeroness of the second weak solution given by Theorem 4.1 also in the case $f_i(t,0) = 0$ for all $i = 0,\cdots,m$ requiring the extra condition. There exist a non-empty open set $D \subseteq [s_i,t_{i+1}]$ and a set $B \subset D$ of positive Lebesgue measure such that, for each $i = 0,\cdots,m$,

$$\limsup_{\xi\to 0^+} \frac{ess\inf_{t\in B} F_i(t,\xi)}{|\xi|^2} = \infty \quad \text{and} \quad \liminf_{\xi\to 0^+} \frac{ess\inf_{t\in D} F_i(t,\xi)}{|\xi|^2} > -\infty.$$

See [13, Rimark 4.3] for more details.

To demonstrate Theorem 4.1, we preset the following example.

Example 4.4. Consider

$$\begin{cases} {}_tD_1^{0.75}({}_0^CD_t^{0.75}w(t)) + w(t) = \lambda e^{-t} g_0(w(t)), & t \in (0,\frac{1}{3}], \\ {}_tD_1^{0.75}({}_0^CD_t^{0.75}w(t)) + w(t) = \lambda e^{-t} g_1(w(t)), & t \in (\frac{2}{3},1], \\ \Delta_t D_1^{-0.25}({}_0^CD_t^{0.75}w(\frac{1}{3})) = \mathscr{I}_1(w(\frac{1}{3})), \\ {}_tD_1^{-0.25}({}_0^CD_t^{0.75}w(t)) = {}_tD_1^{-0.25}({}_0^CD_t^{0.75}w(\frac{1}{3}^+)), & t \in (\frac{1}{3},\frac{2}{3}], \\ {}_tD_1^{-0.25}({}_0^CD_t^{0.75}w(\frac{2}{3}^-)) = {}_tD_1^{-0.25}({}_0^CD_t^{0.75}w(\frac{2}{3}^+)), \\ w(0) = w(1) = 0. \end{cases} \tag{4.2}$$

Choosing $\mathscr{I}_1(\xi) = \arctan\xi$ for all $\xi \in \mathbb{R}$ and $k = 1$, we see that (2.1) is fulfilled. Now we choose g_0 and g_1 as $g_0(\xi) = g_1(\xi) = 1 + 3\xi^2$ for all $\xi \geq 0$ and $g_0(\xi) = g_1(\xi) = 1 - 3\xi^2$ for all $\xi < 0$. Fixing $p = 2$, $2 < \vartheta < 3$ and $R > 1$, we have $F_0(t,\xi) = F_1(t,\xi) = e^{-t}(\xi + |\xi|^3)$ for each $(t,\xi) \in [s_i,t_{i+1}] \times \mathbb{R}$. Due to $R > \max\left\{1, \sqrt{\frac{\vartheta-1}{3-\vartheta}}\right\}$, for each $t \in [s_i,t_{i+1}]$ and $|\xi| \geq R$, it results

$$F_0(t,\xi) = F_1(t,\xi) = e^{-t}\left(\xi + |\xi|^3\right) \geq e^{-t}\left(-|\xi| + |\xi|^3\right) \geq e^{-t}\left[R(-1+R^2)\right] > 0.$$

Then, for each $t \in [s_i, t_{i+1}]$ and $\xi < 0$, one has

$$\begin{aligned}&\xi f_0(t,\xi) - \vartheta F_0(t,\xi) = \xi f_1(t,\xi) - \vartheta F_1(t,\xi) = e^{-t}\left[\xi - 3\xi(-\xi)^2 - \vartheta(\xi + |\xi|^3)\right]\\ &= e^{-t}\left[(1-\vartheta)\xi + (3-\vartheta)|\xi|^3\right] = e^{-t}\left[(\vartheta - 1)|\xi| + (3-\vartheta)|\xi|^2\right] > 0.\end{aligned}$$

Moreover, by same argument as above, condition $R > \sqrt{\frac{\vartheta-1}{3-\vartheta}}$ ensures that $\xi f_0(t,\xi) - \vartheta F_0(t,\xi) > 0$ for each $t \in [s_i, t_{i+1}]$ and $\xi \geq 0$. Taking $c = 100, d = 0.01$, we obviously observe that all conditions of Theorem 4.1 are satisfied. Moreover, $\Lambda = \frac{2\sqrt{2}}{\sqrt{7}\Gamma(\frac{3}{4})} < c$ and $\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \sup_{|z|\leq c} F_i(t,z)\mathrm{d}t > 500$. Thus

$$\frac{c^p}{p\Lambda^p \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \sup_{|z|\leq c} F_i(t,z)\mathrm{d}t} > 0.001.$$

This follows that for each $\lambda \in (0, 0.001)$, the system (4.2) has at least two distinct non-negative weak solutions.

5. THE EXISTENCE RESULTS FOR A SINGLE PARAMETER CASE

This section is devoted to the existence of at least two and three weak solutions for the system ($D^{\alpha,f}_{\lambda,\mu}$) in a single parameter case (i.e., $\lambda = \mu$).

Theorem 5.1. *Let $\overline{c}$ and $\overline{d}$ be two positive constants such that*

$$\Lambda^p |\overline{d}|^p \gamma_1 \sigma > \overline{c}^p \tag{5.1}$$

and assume that (A_2) and the following conditions hold:

(A$_4$) $\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t,\overline{d})\mathrm{d}t \geq 0$;

(A$_5$) $\Lambda^p \frac{\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \sup_{|z|\leq \overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p}{\overline{c}^p} < \frac{k(\overline{c}^p + |\overline{d}|^p) - \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \sup_{|z|\leq \overline{c}} F_i(t,z)\mathrm{d}t}{|\overline{d}|^p \gamma_2 \sigma}$.

Then, for each

$$\lambda \in \Big(\frac{|\overline{d}|^p \gamma_2 \sigma}{pk(\overline{c} + |\overline{d}|^p) - p\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \sup_{|z|\leq \overline{c}} F_i(t,z)\mathrm{d}t}, \frac{\overline{c}^p}{p\Lambda^p \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \sup_{|z|\leq \overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p}\Big)$$

and for every function $\mathscr{I}_i : \mathbb{R} \to \mathbb{R}$ satisfying the conditions (I) and (3.1) for $i = 1, \ldots, m$, the system ($D^{\alpha,f}_{\lambda,\mu}$) in single parameter case (i.e. $\lambda = \mu$) admits at least three weak solutions in $X_0^{\alpha,p}$.

Proof. Take $\mathscr{J}_\lambda = \mathscr{O} + \lambda \mathscr{Q}$, where $\mathscr{O}(w) := \frac{1}{p}\|w\|^p$ and

$$\mathscr{Q}(w) := -\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t, w(t))\mathrm{d}t + \sum_{i=1}^{m} \int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s,$$

for all $w \in X_0^{\alpha,p}$. Clearly, $\mathscr{O}$ and $\mathscr{Q}$ are Gâteaux differentiable functionals whose Gâteaux derivatives at the point $w \in X_0^{\alpha,p}$ are given by

$$\mathscr{O}'(w)(y) = \int_0^T \Theta_p({}_0^C D_t^\alpha w(t)) {}_0^C D_t^\alpha y(t)\mathrm{d}t + \sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} \gamma(t)|w(t)|^{p-2} w(t) y(t)\mathrm{d}t$$

and

$$\mathscr{Q}'(w)(y)=-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}f_i(t,w(t))y(t)\mathrm{d}t+\sum_{i=1}^{m}\mathscr{I}_i(w(t_i))y(t_i)$$

for all $w,y\in X_0^{\alpha,p}$, respectively. So, a critical point of $\mathscr{O}+\lambda\mathscr{Q}$ gives us a weak solution of $(D^{\alpha,f}_{\lambda,\mu})$ in the single parameter case. The idea of the proof consists in using Theorem 2.6 to $\mathscr{O}$ and $\mathscr{Q}$. Moreover, the weakly lower semicontinuity of $\mathscr{O}$ can be proved in a standard way. From Proposition 2.14, $\mathscr{O}$ is continuously Gâteaux differentiable and its Gâteaux derivative accepts a continuous inverse on $(X_0^{\alpha,p})^*$. $\mathscr{Q}:X_0^{\alpha,p}\to\mathbb{R}$ is continuously Gâteaux differentiable and whose Gâteaux derivative is compact. Therefore, it is adequate to illustrate that $\mathscr{O}$ and $\mathscr{Q}$ satisfy (i) and (ii) in Theorem 2.6. In addition, we can fix κ such that $\limsup_{|\xi|\to\infty}\frac{\int_0^\xi\mathscr{I}_i(x)\mathrm{d}x}{|\xi|^p}<\kappa$ for $i=1,\dots,m$ and $p\Lambda^p\lambda m\kappa<1$. So, there exists a positive constant ι such that $\int_0^\xi\mathscr{I}_i(x)\mathrm{d}x\le\kappa\xi^p+\iota$ for every $\xi\in\mathbb{R}$ and $i=1,\dots,m$. Now, we fix $0<\varepsilon<\frac{1-p\lambda k\Lambda^p m}{p\lambda\Lambda^p\sigma}$. In view of (A_2), we see that there exits a function $\rho_\varepsilon\in\mathrm{L}^1([0,T])$ such that $F_i(t,x)\le\varepsilon x^p+\rho_\varepsilon(t)$, for each $(t,x)\in[s_i,t_{i+1}]\times\mathbb{R}$. Thus, for each $w\in X_0^{\alpha,p}$, we obtain

$$\begin{aligned}\mathscr{O}(w)+\lambda\mathscr{Q}(w)&\ge\frac{\|w\|_\gamma^p}{p}-\lambda\varepsilon\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}w^p(t)\mathrm{d}t-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\rho_\varepsilon(t)\mathrm{d}t-\sum_{i=1}^{m}(\lambda\kappa w^p(t_i)+\mu\iota)\\&\ge\frac{\|w\|_\gamma^p}{p}-\lambda\varepsilon\|w\|_\infty^p\sigma-\lambda\sum_{i=0}^{m}\|\rho_\varepsilon\|_{\mathrm{L}^1(s_i,t_{i+1})}-\lambda\kappa\|w\|_\infty^p m-\lambda\iota m\\&\ge\Big(\frac{1}{p}-\lambda\Lambda^p\varepsilon\sigma-\lambda\kappa\Lambda^p m\Big)\|w\|_\gamma^p-\lambda\sum_{i=0}^{m}\|\rho_\varepsilon\|_{\mathrm{L}^1(s_i,t_{i+1})}-\lambda\iota m.\end{aligned}$$

Hence $\lim_{\|w\|_\gamma\to\infty}(\mathscr{O}(w)+\lambda\mathscr{Q}(w))=\infty$, which asserts that $\mathscr{O}+\lambda\mathscr{Q}$ is coercive. So, it is sufficient to display (ii) of Theorem 2.6. Put $\bar r:=\frac{\bar c^p}{p\Lambda^p}$ and $\nu(t)=\bar d$ for all $t\in[0,T]$. One has $\nu\in X_0^{\alpha,p}$. From (3.4) and (5.1), one obtains $\mathscr{O}(\nu)>\bar r$. Moreover, an application of (A_4) yields

$$\begin{aligned}\mathscr{Q}(\nu):&=-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_i(t,\nu(t))\mathrm{d}t+\sum_{i=1}^{m}\int_0^{\nu(t_i)}\mathscr{I}_i(s)ds,\\&\le-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_i(t,\bar d)\mathrm{d}t+k|\bar d|^p\le k|\bar d|^p.\end{aligned}$$

Taking Lemma 2.12 into account, for every $w\in X_0^{\alpha,p}$ such that $\mathscr{O}(w)<\bar r$, we have $\sup_{t\in[0,T]}|w(t)|\le\bar c$. Applying the second inequality in (3.5) guarantees

$$\begin{aligned}\sup_{w\in\mathscr{O}^{-1}(-\infty,\bar r)}\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_i(t,w(t))\mathrm{d}t-\sum_{i=1}^{m}\int_0^{w(t_i)}\mathscr{I}_i(s)\mathrm{d}s&\le\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\sup_{|z|\le\bar c}F_i(t,z)\mathrm{d}t+k\|w\|_\infty^p\\&\le\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\sup_{|z|\le\bar c}F_i(t,z)\mathrm{d}t+k\bar c^p\end{aligned}\tag{5.2}$$

for every $w\in X_0^{\alpha,p}$ with $\mathscr{O}(w)<\bar r$. Thus $\sup_{\mathscr{O}(w)<\bar r}\mathscr{Q}(w)\le\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}\sup_{|z|\le\bar c}F_i(t,z)\mathrm{d}t+k\bar c^p$. Since $\mathscr{O}(0)=\mathscr{Q}(0)=0$ and $\overline{\mathscr{O}^{-1}(-\infty,\bar r)}^{w}=\mathscr{O}^{-1}(-\infty,\bar r)$, one sees from the definition of $\varphi(\bar r)$

that

$$\begin{aligned}\varphi_1(\bar{r}) = \inf_{w\in\mathscr{O}^{-1}(]-\infty,\bar{r}[)} \frac{\mathscr{Q}(w) - \inf_{\overline{\mathscr{O}^{-1}(-\infty,\bar{r})}^{w}} \mathscr{Q}}{\bar{r}-\mathscr{O}(w)} &\leq \frac{-\inf_{\overline{\mathscr{O}^{-1}(-\infty,\bar{r})}^{w}} \mathscr{Q}}{\bar{r}} \\ \leq p\Lambda^p \frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p}{\overline{c}^p}. \end{aligned}$$

It follows from (5.2) that

$$\begin{aligned}\varphi_2(\bar{r}) &= \inf_{w\in\mathscr{O}^{-1}(-\infty,\bar{r})} \sup_{\omega\in\mathscr{O}^{-1}[\bar{r},\infty)} \frac{\mathscr{Q}(w)-\Psi(\omega)}{\mathscr{O}(w)-\mathscr{O}(\omega)} \geq \inf_{w\in\mathscr{O}^{-1}(-\infty,\bar{r})} \frac{\mathscr{Q}(w)-\mathscr{Q}(\nu)}{\mathscr{O}(\nu)} \\ &\geq \frac{\inf_{w\in\mathscr{O}^{-1}(-\infty,\bar{r})} \mathscr{Q}(w)-\mathscr{Q}(\nu)}{\mathscr{O}(\nu)} \\ &\geq \frac{-\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p + k|\overline{d}|^p}{\mathscr{O}(\nu)} \\ &\geq p\frac{k(\overline{c}^p+|\overline{d}|^p) - \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t}{|\overline{d}|^p\gamma_2\sigma}. \end{aligned}$$

From (A_5), one also has $\varphi_1(\bar{r}) < \varphi_2(\bar{r})$. Taking also into account that

$$\frac{1}{\varphi_2(\bar{r})} \leq \frac{|\overline{d}|^p\gamma_2\sigma}{pk(\overline{c}^p+|\overline{d}|^p) - p\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t}$$

and

$$\frac{1}{\varphi_1(\bar{r})} \geq \frac{\overline{c}^p}{p\Lambda^p\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p},$$

one can obtain the desired conclusion immediately. □

Remark 5.2. It is not hard to see that

(A$_6$) $\Lambda^p\frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p}{\overline{c}^p} < \frac{k}{\gamma_2\sigma}$

implies $(A5)$ of Theorem 5.1. So, if assumptions (5.1), (A_4), and (A_6) hold, then, for each

$$\lambda \in \left(\frac{\gamma_2\sigma}{kp}, \frac{\overline{c}^p}{p\Lambda^p\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\leq\overline{c}} F_i(t,z)\mathrm{d}t + k\overline{c}^p}\right),$$

the system $(D_{\lambda,\mu}^{\alpha,f})$ in single parameter case (i.e. $\lambda=\mu$) accepts at least three weak solutions.

Now we present Theorem 5.1 with the following example.

Example 5.3. Consider

$$\begin{cases} {}_tD_1^{0.8}({}_0^CD_t^{0.8}w(t)) + (0.1+t)w(t) = \lambda f_0(t, w(t)), \quad t \in (0, \frac{1}{3}] \\ {}_tD_1^{0.8}({}_0^CD_t^{0.8}w(t)) + (0.1+t)w(t) = \lambda f_1(t, w(t)), \quad t \in (\frac{2}{3}, 1], \\ \Delta_tD_1^{-0.2}({}_0^CD_t^{0.8}w(\frac{1}{3})) = \lambda \sin(u(\frac{1}{3})), \\ {}_tD_1^{-0.2}({}_0^CD_t^{0.8}w((\frac{1}{3})^-)) = {}_tD_1^{-0.2}(({}_0^CD_t^{0.8}w((\frac{1}{3})^+))), \\ {}_tD_1^{-0.2}(({}_0^CD_t^{0.8}w((\frac{2}{3})^-))) = {}_tD_T^{-0.2}(({}_0^CD_t^{0.8}w((\frac{2}{3})^+))), \\ w(0) = w(1) = 0. \end{cases} \tag{5.3}$$

Let $\bar{d} = 10^4, \bar{c} = 10^2$, and $0 = s_0 < t_1 = \frac{1}{3} < s_1 = \frac{2}{3} < t_2 = \frac{3}{3} = 1$. Thus $\gamma_1 = 0.01$, $\gamma_2 = 1.1$, $k = 1$, $\Lambda = 1.109$ and $\Lambda^p|\bar{d}|^p\gamma_1\sigma = 819.33 \times 10^3 > 10^4 = \bar{c}^p$ is satisfied. Now we choose $f_0(t,\xi) = (2t+1)\xi^{\frac{4}{5}}$ for all $(t,\xi) \in [0, \frac{1}{3}] \times \mathbb{R}$ and $f_1(t,\xi) = \ln t\sqrt{|\xi|}$ for all $(t,\xi) \in [\frac{2}{3}, 1] \times \mathbb{R}$. From the expression of f_0 and f_1, we have $F_0(t,\xi) = \frac{5}{9}\xi^{\frac{9}{5}}(2t+1)$ for all $(t,\xi) \in [0, \frac{1}{3}] \times \mathbb{R}$ and $F_1(t,\xi) = \frac{2}{3}\xi^{\frac{3}{2}}\ln t$ for all $(t,\xi) \in [\frac{2}{3}, 1] \times \mathbb{R}$. According to the following calculations

$$\limsup_{|\xi|\to\infty} \frac{F_0(t,\xi)}{|\xi|^2} = \frac{5}{9}(2t+1)\limsup_{|\xi|\to\infty} \frac{|\xi|^{\frac{9}{5}}}{|\xi|^2} = 0$$

and

$$\limsup_{|\xi|\to\infty} \frac{F_1(t,\xi)}{|\xi|^2} = \frac{2}{3}\ln t \limsup_{|\xi|\to\infty} \frac{|\xi|^{\frac{3}{2}}}{|\xi|^2} = 0.$$

condition (A_2) is verified. Also, (A_4) holds due to

$$\sum_{i=0}^{m} \int_{s_i}^{t_{i+1}} F_i(t,\bar{d})\mathrm{d}t = \int_0^{\frac{1}{3}} F_0(t,\bar{d})\mathrm{d}t + \int_{\frac{2}{3}}^{1} F_1(t,\bar{d})\mathrm{d}t = \frac{20}{81}(10^4)^{\frac{9}{5}} - (44 \times 10^3) \geq 0.$$

Finally, $p\Lambda^p \frac{\sum_{i=0}^m \int_{s_i}^{t_{i+1}} \sup_{|z|\leq\bar{c}} F_i(t,z)\mathrm{d}t + k\bar{c}^p}{\bar{c}^p} = 2.69 < \frac{kp}{a_2\sigma} = 2.72$ shows that assumption (A_6) holds. Therefore, by considering Theorem 2.6 and Remark 5.2 in to account for every

$$\lambda \in \left(\frac{11}{30}, \frac{10^4}{2.46(\frac{20}{81}(100)^{\frac{9}{5}} - 44 + 10^4)}\right) \approx \left(0.366, 0.3716\right),$$

system (5.3) accepts at least three weak solutions.

Now, we investigate multiple solutions for $(D_{\lambda,\mu}^{\alpha,f})$ in the case $\lambda = \mu$, while we do not require condition (A_2). For this purpose, we apply Theorem 2.8.

Theorem 5.4. *Consider three positive constants $\bar{c}_1$, $\bar{d}$, and $\bar{c}_2$ with*

$$\frac{\bar{c}_1^p}{\Lambda^p\gamma_1\sigma} < |\bar{d}|^p < \frac{\bar{c}_2^p}{\Lambda^p\gamma_2\sigma} \tag{5.4}$$

and let the assumption (A_4) of Theorem 5.1 be satisfied and

(A_7)

$$p\Lambda^p \max\left\{ \frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\le \bar c_1} F_i(t,z)\mathrm{d}t + k\bar c_1^p}{\bar c_1^p}, \right.$$
$$\left. \frac{\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\le \bar c_2} F_i(t,z)\mathrm{d}t + k\bar c_2^p}{\bar c_2^p} \right\} < \frac{pk}{\gamma_2\sigma}.$$

Then, for each

$$\lambda \in \Lambda := \left(\frac{\gamma_2\sigma}{pk}, \min\left\{ \frac{\bar c_1^p}{p\Lambda^p \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\le \bar c_1} F_i(t,z)\mathrm{d}t + k\bar c_1^p}, \right.\right.$$
$$\left.\left. \frac{\bar c_2^p}{p\Lambda^p \sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\le \bar c_2} F_i(t,z)\mathrm{d}t + k\bar c_2^p} \right\}\right),$$

system ($D_{\lambda,\mu}^{\alpha,f}$) *in single parameter case (i.e.* $\lambda=\mu$*) adopts at least two weak solutions* $w_{1,\lambda}$ *and* $w_{2,\lambda}$ *such that* $\max_{t\in[0,T]}|w_{1,\lambda}(t)| < \bar c_1$ *and* $\max_{t\in[0,T)}|w_{2,\lambda}(t)| < \bar c_2$.

Proof. For $i=0,\cdots,m$, we consider

$$\overline{f}_i(t,x) = \begin{cases} f_i(t,-\bar c_2) & \text{if } (t,x)\in[s_i,t_{i+1}]\times(-\infty,\bar c_2) \\ f_i(t,x) & \text{if } (t,x)\in[s_i,t_{i+1}]\times[-\bar c_2,\bar c_2] \\ f_i(t,\bar c_2) & \text{if } (t,x)\in[s_i,t_{i+1}]\times(\bar c_2,\infty). \end{cases}$$

It is obvious that, $\overline{f}_0,\ldots,\overline{f}_m : [s_i,t_{i+1}]\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions. Now, we put $\overline{F}_i(t,\xi) = \int_0^{\xi}\overline{f}_i(t,x)\mathrm{d}x$ for all $(t,\xi)\in[s_i,t_{i+1}]\times\mathbb{R}$, $i=0,\ldots,m$, and set $\mathscr{O}(w)=\frac{1}{p}\|w\|^p$ and

$$\mathscr{Q}(w) := -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \bar F_i(t,w(t))\mathrm{d}t + \sum_{i=1}^{m}\int_0^{w(t_i)} \mathscr{I}_i(s)\mathrm{d}s,$$

for all $w\in X_0^{\alpha,p}$. To get the desired result, we use Theorem 2.8 to $\mathscr{O}$ and $\mathscr{Q}$. It is clear that $\lim_{\|w\|_\gamma\to\infty}\mathscr{O}(w)=\infty$ and $\mathscr{Q}$ is a differentiable functional whose differential at the point $w\in X_0^{\alpha,p}$ is

$$\mathscr{Q}'(w)(y) = -\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \bar f_i(t,w(t))y(t)\mathrm{d}t + \sum_{i=1}^{m}\mathscr{I}_i(w(t_i))y(t_i)$$

for every $y\in X_0^{\alpha,p}$ that is sequentially weakly lower semicontinuous. Further, $\mathscr{Q}' : X_0^{\alpha,p}\to (X_0^{\alpha,p})^*$ is a compact operator. In order to get the conclusions, we need to take $\bar r_1 := \frac{\bar c_1^p}{p\Lambda^p}$, $\bar r_2 := \frac{\bar c_2^p}{p\Lambda^p}$, and $\nu(t)=\overline{d}$ for all $t\in[0,T]$. Bearing in mind the properties (3.4) and (5.4), we have $\nu\in X_0^{\alpha,p}$, $\bar r_1 < \mathscr{O}(\nu) < \bar r_2$, and $\inf_X \mathscr{O} < \bar r_1 < \bar r_2$. Moreover, using the proof of Theorem 5.1 and taking also into account Remark 5.2, we obtain

$$\varphi_1(\bar r_1) \le \frac{p\Lambda^p\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\le \bar c_1} F_i(t,z)\mathrm{d}t + k\bar c_1^p}{\bar c_1^p},$$

$$\varphi_1(\bar r_2) \le \frac{p\Lambda^p\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}} \sup_{|z|\le \bar c_2} F_i(t,z)\mathrm{d}t + k\bar c_2^p}{\bar c_2^p},$$

and $\varphi_2^*(\bar{r}_1,\bar{r}_2)\geq\frac{pk}{\gamma_2\sigma}$. From (A_4) and (A_7), the assumptions (jj) and (jjj) of Theorem 2.8 hold. Therefore, from Theorem 2.8 we conclude that, for each $\lambda\in\Lambda$, the system

$$\begin{cases} {}_tD_T^{\alpha}(\Theta_p({}_0^CD_t^{\alpha}w(t)))+\gamma(t)\Theta_p(w(t))=\lambda\bar{f}_i(t,w(t)),\ t\in(s_i,t_{i+1}],\ i=0,\cdots,m,\\ \Delta_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i)))=\lambda\mathscr{I}_i(w(t_i)),\quad i=1,2,\cdots,m,\\ {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t))={}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_i^+))),\quad t\in(t_i,s_i],\ i=1,2,\cdots,m,\\ {}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^-)))={}_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(s_i^+))),\quad i=1,2,\cdots,m,\\ w(0)=w(T)=0. \end{cases}$$

admits at least two weak solutions $w_{1,\lambda}$ and $w_{2,\lambda}$ such that

$$\max_{t\in[0,T]}|w_{1,\lambda}(t)|<\bar{c}_1\quad\text{and}\quad\max_{t\in[0,T]}|w_{2,\lambda}(t)|<\bar{c}_2.$$

Note that these solutions are also solutions to $(D_{\lambda,\mu}^{\alpha,f})$ in single parameter case (i.e. $\lambda=\mu$). □

Remark 5.5. In Theorems 5.1 and 5.4, we searched the critical points of $\mathscr{J}_\lambda$ naturally affiliated with the problem $(D_{\lambda,\mu}^{\alpha,f})$ in single parameter case (i.e. $\lambda=\mu$). Generally, $\mathscr{J}_\lambda$ can be below unbounded in $X_0^{\alpha,p}$. Actually, for the case $f_i(\xi)=i+|\xi|^{\gamma-p^+}\xi^{p^+-1}$ for all $\xi\in\mathbb{R}$ and $i=1,\dots,m$ with $b>p^+$, for any fixed $w\in X_0^{\alpha,p}\backslash\{0\}$ and $\iota\in\mathbb{R}$, we take

$$\begin{aligned}\mathscr{J}_\lambda(\iota w)&\leq\frac{\iota^p}{p}\|w\|_\gamma^p-\lambda\sum_{i=0}^{m}\int_{s_i}^{t_{i+1}}F_i(t,\iota w(t))\mathrm{d}t+\lambda k\iota^p\|w\|_\infty^p\\ &\leq\frac{\iota^p}{p}\|w\|_\gamma^p-\lambda\iota\sum_{i=0}^{m}\|w\|_{\mathrm{L}^1([s_i,t_{i+1}])}-\lambda\frac{\iota^b}{b}\sum_{i=0}^{m}\|w\|_{\mathrm{L}^b([s_i,t_{i+1}])}^b\\ &\quad+\lambda k\iota^p\Lambda^p\|w\|_\gamma^p\to-\infty\end{aligned}$$

as $\iota\to\infty$. So, we cannot utilize direct minimization.

Remark 5.6. If $f_i(t)>0$ for all $t\in[s_i,t_{i+1}]$ and $i=0,\dots,m$, and $\mathscr{I}_i(t)<0$ for all $t\in[s_i,t_{i+1}]$ and $i=1,\dots,m$, Theorem 5.4 is a bifurcation case in the sense that $(0,0)\in\overline{\mathscr{A}}$ (closer of $\mathscr{A}$), where $\mathscr{A}$ is the set of all pairs $(w_\lambda,\lambda)\in X_0^{\alpha,p}\times(0,\infty)$ that w_λ is a nontrivial weak solution to $(D_{\lambda,\mu}^{\alpha,f})$ in the same parameter case (i.e. $\lambda=\mu$). Actually, by Theorem 5.4, we see that $\|w_\lambda\|\to0$ as $\lambda\to0$. Thus there exist two sequences $\{w_k\}$ in $X_0^{\alpha,p}$ and $\{\lambda_k\}$ in $\mathbb{R}^+$ (here $w_k=w_{\lambda_k}$) such that $\lambda_k\to0^+$ and $\|w_k\|\to0$, as $k\to\infty$. In addition, since $f_i(t)>0$ for all $t\in[s_i,t_{i+1}]$ and $i=0,\dots,m$, and $\mathscr{I}_i(t)<0$ for all $t\in[s_i,t_{i+1}]$ and $i=1,\dots,m$, $\mathscr{Q}(w)<0$ for all $w\in X_0^{\alpha,p}$ and thus $\lambda\mapsto I_\lambda(w_\lambda)$ with $\lambda\in(0,\lambda^*)$ is strictly decreasing. Hence, for every $\lambda_1,\lambda_2\in(0,\lambda^*)$, with $\lambda_1\neq\lambda_2$, the weak solutions w_{λ_1} and w_{λ_2} guaranteed by Theorem 2.8 are different.

Remark 5.7. If $f_i(t)>0$ for all $t\in[s_i,t_{i+1}]$ and $i=0,\dots,m$, and $\mathscr{I}_i(t)<0$ for all $t\in[s_i,t_{i+1}]$ and $i=1,\dots,m$, the weak solution in Theorems 5.1 and 5.4 are non-negative (see [13, Remark 5.4]).

6. SINGLE INSTANTANEOUS IMPULSE

We consider the following problem

$$\begin{cases} {}_tD_T^{\alpha}(\Theta_p({}_0^CD_t^{\alpha}w(t)))+\gamma(t)\Theta_p(w(t))=\lambda\theta(t)h(w(t)),\ t\in[0,T],\\ \Delta_tD_T^{\alpha-1}(\Theta_p({}_0^CD_t^{\alpha}w(t_1)))=\lambda\mathscr{I}_1(w(t_1)),\\ w(0)=w(T)=0.\end{cases}\tag{$D_{h,\theta}^{\alpha}$}$$

where $0 < t_1 < T$, $\theta : \Omega \to \mathbb{R}$ is a non-negative and nonzero function such that $\theta \in \mathrm{L}^1([0,T])$, $h : \mathbb{R} \to \mathbb{R}$ is a non-negative and continuous function, and $\mathscr{I}_1 \in \mathrm{C}(\mathbb{R},\mathbb{R})$ satisfies assumption (I). Put $H(\xi) = \int_0^\xi h(x)\mathrm{d}x$ for all $\xi \in \mathbb{R}$. The following results are directly obtained from Theorems 5.1 and 5.4, respectively. Setting $f(t,x) = \theta(t)h(x)$ for every $(t,x) \in [s_i,t_{i+1}] \times \mathbb{R}$, we have the following two theorems.

Theorem 6.1. *Assume that there exist positive constants $\bar{c}$ and $\bar{d}$ with $\Lambda^p \gamma_1 |\bar{d}|^p t_1 > \bar{c}^p$ and $\limsup_{|\xi|\to\infty} \frac{h(\xi)}{|\xi|^{p-1}} \leq 0$ uniformly in $\mathbb{R}$. Moreover let*

(A$_8$) $\Lambda^p \frac{\|\theta\|_{\mathrm{L}^1[0,T]} H(\bar{c}) + k\bar{c}^p}{\bar{c}^p} < \frac{k}{\gamma_2 t_1}$.

Then, for each

$$\lambda \in \left(\frac{\gamma_2 t_1}{pk}, \frac{\bar{c}^p}{p\Lambda^p \|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}) + k\bar{c}^p} \right),$$

problem $(D^{\alpha}_{h,\theta})$ accepts at least three weak solutions in $X_0^{\alpha,p}$.

Theorem 6.2. *Consider three positive constants $\bar{c}_1$, $\bar{d}$ and $\bar{c}_2$ with*

$$\frac{\bar{c}_1^p}{\Lambda^p \gamma_1 t_1} < |\bar{d}|^p < \frac{\bar{c}_2^p}{\Lambda^p \gamma_2 t_1}$$

and

(A$_9$) $\Lambda^p \max\left\{ \frac{\|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}_1) + k\bar{c}_1^p}{\bar{c}_1^p}, \frac{\|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}_2) + k\bar{c}_2^p}{\bar{c}_2^p} \right\} < \frac{k}{\gamma_2 t_1}$.

Then, for each

$$\lambda \in \left(\frac{\gamma_2 t_1}{pk}, \min\left\{ \frac{\bar{c}_1^p}{p\Lambda^p \|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}_1) + k\bar{c}_1^p}, \frac{\bar{c}_2^p}{p\Lambda^p \|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}_2) + k\bar{c}_2^p} \right\} \right),$$

problem $(D^{\alpha}_{h,\theta})$ accepts at least two weak solutions $w_{1,\lambda}$ and $w_{2,\lambda}$ such that

$$\max_{t\in[0,T]} |w_{1,\lambda}(t)| < \bar{c}_1 \quad \text{and} \quad \max_{t\in[0,T]} |w_{2,\lambda}(t)| < \bar{c}_2.$$

Here, we mention a special case of Theorem 6.1.

Theorem 6.3. *Assume that*

$$\lim_{\xi\to 0^+} \frac{h(\xi)}{\xi^{p-1}} = \lim_{|\xi|\to\infty} \frac{h(\xi)}{|\xi|^{p-1}} = 0, \tag{6.1}$$

there exists a positive constant $\bar{d}$ such that $H(\bar{d}) > 0$ and $\limsup_{|\xi|\to\infty} \frac{\mathscr{I}_1(\xi)}{|\xi|^{p-1}} < \infty$. Then, for each $\lambda > \frac{\gamma_2 t_1}{pk}$, the problem $(D^{\alpha}_{h,\theta})$ accepts at least one non-negative and one nonzero weak solution in $X_0^{\alpha,p}$.

Proof. Let $\lambda > \frac{\gamma_2 t_1}{pk}$. By (6.1), we obtain

$$\lim_{\xi\to 0^+} \frac{\sup_{|z|\leq\xi} h(z)}{\xi^{p-1}} = \lim_{\xi\to\infty} \frac{\sup_{|z|\leq\xi} h(z)}{\xi^{p-1}} = 0.$$

Hence, one can fix $\bar{c}_1$ and $\bar{c}_2$ such that $\bar{c}_1^p < \Lambda^p \gamma_1 |\bar{d}|^p t_1 < \bar{c}_2^p$,

$$\frac{\sup_{|\xi|\leq\bar{c}_1} h(\xi)}{\bar{c}_1} < \frac{\bar{c}_1^p}{p\Lambda^p \|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}_1) + k\bar{c}_1^p}$$

and

$$\frac{\sup_{|\xi|\leq \bar{c}_2} h(\xi)}{\bar{c}_2} < \frac{\bar{c}_2^p}{p\Lambda^p \|\theta\|_{\mathrm{L}^1([0,T])} H(\bar{c}_2) + k\bar{c}_2^p}.$$

Hence, from Theorem 6.2 we obtain the desired conclusion. □

By employing Theorem 6.3, we can obtain the below example.

Example 6.4. Consider the problem

$$\begin{cases} {}_tD_T^{0.6}({}_0^CD_t^{0.6}w(t)) + w(t) = \lambda e^{-t}\frac{w^4(t)}{5+w^4(t)}, \; t \in [0,1], \\ \Delta_t D_T^{-0.4}({}_0^CD_t^{0.6}w(0.5) = \lambda w(0.5), \\ w(0) = w(1) = 0. \end{cases}$$

One can see that

$$\lim_{\xi\to 0^+} \frac{h(\xi)}{\xi} = \lim_{\xi\to 0^+} \frac{\xi^4}{\xi(5+\xi^4)} = 0 \quad \mathit{and} \quad \lim_{|\xi|\to\infty} \frac{\xi^4}{|\xi|(5+\xi^4)} = \lim_{|\xi|\to\infty} \frac{h(\xi)}{|\xi|} = 0.$$

Thus (6.4), for every $\lambda > 0.25$, accepts at least one non-negative and one nonzero weak solution in $X_0^{0.6,2}$.

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