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ITERATIVE APPROACHES FOR SOLVING SPLIT GENERAL VARIATIONAL INEQUALITIES SYSTEMS WITH APPLICATIONS

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Abstract. This paper proposes a novel iterative method for addressing the split general system of variational inequalities and related fixed point problems within the framework of real Hilbert spaces. The algorithm is constructed by combining Halpern-type iterations with nonexpansive mappings, and its design guarantees strong convergence under appropriate conditions. The analytical results demonstrate that the proposed method not only unifies but also generalizes several classical schemes, including the split feasibility problem and the split variational inequality problem. To substantiate the theoretical findings, numerical experiments are provided, illustrating the robustness and efficiency of the algorithm in solving complex optimization and equilibrium problems.

Keywords. Common fixed point problems; Iterative methods; Nonexpansive mappings; Split general system of variational inequalities; Strong convergence.

1. Introduction

Optimization problems formulated in real Hilbert spaces represent a central theme in both theoretical investigations and applied research. These problems frequently emerge in diverse domains such as engineering, economics, and computational science, where they typically involve the determination of fixed points of some nonlinear operator; see, e.g., [8, 9, 12, 17, 20] and the references therein. Among the advanced frameworks developed to address such problems, the split general system of variational inequalities (SGSV) stands out for its generality. This framework extends classical variational inequalities, which provide a flexible and unified approach for modeling and solving a wide range of complex problems arising in real-world applications.

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A foundational concept in this field is the variational inequality problem (VIP). Let H be a real Hilbert space endowed with inner product $\langle \cdot,\cdot \rangle$ and associated norm $\|\cdot\|$. Suppose that $C \subset H$ is a closed, convex, and nonempty set. Given a mapping $A : H \to H$, the variational inequality problem seeks to determine a point $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \text{for all } v \in C.$$

The solution set of this problem is denoted by $VI(C,A)$. Originally introduced by Lions and Stampacchia in [10], the VIP has attracted considerable attention due to its wide-ranging applications in economic equilibrium models and the analysis of physical systems.

Censor and Elfving [7] considered the classical variational inequality framework by introducing the split feasibility problem (SFP), which aims to find a point $x \in C$ such that $Dx \in Q$, where $D : H_1 \to H_2$ is a bounded and linear operator, and $C \subset H_1$ and $Q \subset H_2$ are nonempty, convex and closed subsets of a real Hilbert space. The SFP arises naturally in numerous applications, including signal processing, image reconstruction, and computed tomography, where it is employed to recover signals or images from incomplete or corrupted data. The solution set of the SFP is denoted by $\Gamma = \{x \in C : Dx \in Q\}$.

To address the SFP, Byrne [2] proposed the CQ algorithm, a projection-type iterative method that alternates between the sets C and Q while incorporating the action of the operator D. The iterative procedure is defined by

$$x_{n+1} = P_C(x_n - \gamma D^*(I-P_Q)Dx_n),$$

where $\gamma > 0$ is a step size and D^* denotes the adjoint of D. Convergence was obtained provided that Γ is nonempty.

Building on this model, Censor and Segal [5] introduced the split common fixed point problem (SCFP), which generalizes the SFP by seeking a point $x \in \mathrm{Fix}(S)$ such that $Dx \in \mathrm{Fix}(T)$, where $S : C \to C$ and $T : Q \to Q$ are nonexpansive mappings. This framework accommodates more intricate models involving multiple mappings and constraint sets.

Subsequently, Censor et al. [6] further generalized these ideas through the split variational inequality problem (SVIP). The SVIP consists of finding a point $x^* \in C$ such that

$$\langle f(x^*), x-x^* \rangle \geq 0, \quad \text{for all } x \in C$$

and

$$\langle g(y^*), y-y^* \rangle \geq 0, \quad \text{for all } y \in Q,$$

where $y^* = Ax^* \in Q$, $A : H_1 \to H_2$ is a bounded linear operator, and $f : H_1 \to H_1$, $g : H_2 \to H_2$ are monotone operators. The SVIP encompasses the SFP as a special case, underscoring its generality and wide applicability in solving constrained variational problems.

In 2008, Ceng, Wang, and Yao [3] introduced a general system of variational inequalities, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \text{for all } x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \text{for all } x \in C, \end{cases} \tag{1.1}$$

where $A, B : C \to H$ are two mappings and $\lambda, \mu > 0$ are two positive constants.

The split general system of variational inequalities (SGSV), introduced by Siriyan and Kangtunyakarn in 2018 [15], extends foundational concepts to address complex systems of variational inequalities and fixed point problems in split settings. The SGSV seeks to find pairs

$(x^*, y^*) \in C \times C$ that satisfy

$$\begin{cases} \langle \lambda A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \text{for all } x \in C, \\ \langle \mu B x^* + y^* - x^*, x - y^* \rangle \geq 0, & \text{for all } x \in C, \end{cases} \tag{1.2}$$

where $A, B : C \to H_1$ and $\lambda, \mu > 0$ are constants. Additionally, the SGSV problem may also seek to find $(\bar{x}^* = Dx^*, \bar{y}^* = Dy^*) \in Q \times Q$ satisfying:

$$\begin{cases} \langle \alpha \bar{A} \bar{y}^* + \bar{x}^* - \bar{y}^*, \bar{x} - \bar{x}^* \rangle \geq 0, & \text{for all } \bar{x} \in Q, \\ \langle \gamma \bar{B} \bar{x}^* + \bar{y}^* - \bar{x}^*, \bar{x} - \bar{y}^* \rangle \geq 0, & \text{for all } \bar{x} \in Q, \end{cases} \tag{1.3}$$

where $\bar{A}, \bar{B} : Q \to H_2$ and $\alpha, \gamma > 0$ are constants. Let $\Omega_{A,B}$ and $\Omega_{\bar{A},\bar{B}}$ denote the solution sets of (1.2) and (1.3), respectively. The set of all solutions to the SGSV is given by:

$$\Omega^{A,B}_{\bar{A},\bar{B}} = \{(x^*, y^*) \in \Omega_{A,B} : (\bar{x}^*, \bar{y}^*) \in \Omega_{\bar{A},\bar{B}}\},$$

where $\bar{x}^* = Dx^*$ and $\bar{y}^* = Dy^*$.

Remark 1.1. *The SGSV framework described in (1.2) and (1.3) simplifies to (1.1) under the conditions $A \equiv B \equiv \bar{A} \equiv \bar{B} \equiv 0$, $\bar{x} = \bar{y}$, and $x^* = y^*$. Moreover, it reduces to the SVIP when $x^* = y^*$ and $\bar{x}^* = \bar{y}^*$, with appropriately chosen parameters λ, μ, α, and γ.*

This equivalence highlights the flexibility of the SGSV framework in addressing a broader class of problems, including those solvable by the SVIP. Additionally, the SGSV framework extends the system of variational inequalities, introduced by Ceng, Wang, and Yao [3], showcasing its broad applicability and effectiveness as a unified approach for solving complex optimization and variational inequality problems. Moreover, Siriyan and Kangtunyakarn [15] established a strong convergence theorem that identifies a common element between the solution set of the SGSV problem and the fixed point set of a nonexpansive mapping, marking a significant milestone in this research. Their work has been extensively studied by many researchers, demonstrating its profound impact on the field of optimization and variational inequalities. For instance, Abass and Osilike [1] proposed an inertial forward-backward splitting method based on the SGSV framework developed by Siriyan and Kangtunyakarn. This method incorporates inertial terms to address more complex optimization and variational inequality problems, significantly enhancing both convergence rate and computational efficiency. This example underscores the adaptability and influence of the SGSV framework in advancing modern optimization methods.

In this paper, we propose a new iterative algorithm for solving the SGSV problem under appropriate conditions, with the aim of improving convergence efficiency and extending its applicability to complex problems in optimization and variational inequalities. The method is developed based on modern iterative techniques and is supported by a strong convergence theorem, which guarantees the identification of a common element between the set of fixed points of a nonexpansive mapping and the solution set of the SGSV problem. Rigorous theoretical analysis establishes a firm mathematical foundation, while numerical experiments confirm the method's effectiveness. These results demonstrate the potential of the proposed approach as a reliable framework for addressing challenges in optimization and computational mathematics.

2. Preliminaries

This section collects basic definitions, properties, and lemmas that are used in the convergence analysis of our proposed method.

Let H be a real Hilbert space equipped with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. A subset $C \subseteq H$ is called convex if, for any $x,y \in C$ and $\lambda \in [0,1]$, $\lambda x+(1-\lambda)y \in C$. For any $x \in H$, the nearest point projection onto C, denoted by $P_C x$, satisfies $\|x-P_C y\| \leq \|x-y\|$ for all $y \in C$. The operator P_C is also known as the metric projection of H onto C. Moreover, P_C is firmly nonexpansive, i.e., for all $x,y \in H$, $\|P_C x-P_C y\|^2 \leq \langle P_C x-P_C y, x-y\rangle$, which implies that

$$\|(x-y)-(P_C x-P_C y)\|^2 \leq \|x-y\|^2 - \|P_C x-P_C y\|^2, \text{ for all } x,y \in H.$$

In this paper, we use the notations $x_n \to x$ and $x_n \rightharpoonup x$ to denote the strong and weak convergence of the sequence $\{x_n\}$ to x, respectively.

In a real Hilbert space H, the following properties hold

(i) $\|x+y\|^2 \leq \|x\|^2+2\langle y,x+y\rangle$, for all $x,y \in H$.
(ii) For all $x,y \in H$ and $\alpha \in [0,1]$, $\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2$.
(iii) For all $x,y \in H$ and $s,t \in \mathbb{R}$, $\|tx+sy\|^2 = t(t+s)\|x\|^2+s(t+s)\|y\|^2-st\|x-y\|^2$.

An operator $A : H \to H$ is called strongly positive if there exists a constant $\alpha > 0$ such that, for all x in H, $\langle Ax,x\rangle \geq \alpha\|x\|^2$. A mapping $A : C \to H$ is called γ-inverse strongly monotone if there exists a constant $\gamma > 0$ such that, for all $x,y \in C$, $\langle Ax-Ay, x-y\rangle \geq \gamma\|Ax-Ay\|^2$. A mapping $T : C \to C$ is called nonexpansive if, for all $x,y \in C$, $\|Tx-Ty\| \leq \|x-y\|$. The set of fixed points of T, denoted by $F(T)$, is $F(T) = \{x \in C : Tx = x\}$. Moreover, if T is a nonexpansive mapping of H into itself, then

$$\langle Ty-Tx, (I-T)x-(I-T)y\rangle \leq \frac{1}{2}\|(I-T)x-(I-T)y\|^2, \text{ for all } x,y \in H.$$

Every Hilbert space H also satisfies Opial's condition [14], which asserts that, for any sequence $x_n \subseteq H$ with $x_n \rightharpoonup x$,

$$\liminf_{n\to\infty}\|x_n-x\| < \liminf_{n\to\infty}\|x_n-y\|, \text{ for all } y \neq x.$$

The following lemmas are used in our convergence analysis.

Lemma 2.1. [16] For $x \in H$ and $y \in C$, $P_C x = y$ if and only if, for all z in Cm $\langle x-y, y-z\rangle \geq 0$.

Lemma 2.2. [18] Assume that $s_{n+1} \leq (1-\alpha_n)s_n+\beta_n$ for all $n \geq 0$, where real sequences $\{s_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ satisfy the conditions

(i) $\limsup_{n\to\infty}\frac{\beta_n}{\alpha_n} \leq 0$ or $\sum_{n=1}^{\infty}|\beta_n| < \infty$,
(ii) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then $\lim_{n\to\infty}s_n = 0$.

Lemma 2.3. [13] Assume that $\alpha_{n+1} \leq (1-\delta_n)\alpha_n+\beta_n+\gamma_n$ for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are real sequences such that $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\beta_n\}$ is a real sequence, and $\sum_{n=1}^{\infty}\gamma_n < \infty$. Then, the following results hold:

(i) If $\beta_n \leq \delta_n M$ for some $M \geq 0$, then $\{\alpha_n\}$ is a bounded sequence.
(ii) If $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\frac{\beta_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.4. [14] Let E be a uniformly convex Banach space, C be a nonempty, closed, and convex subset of E, and $S : C \to C$ be a nonexpansive mapping with a fixed point. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x$ and $(I-S)x_n \to 0$. Then $Sx = x$.

Lemma 2.5. [15] Let C, Q be nonempty subsets of H_1, H_2, respectively. Let $A, B : C \to H_1$ be a, b-inverse strongly monotone operators with $\lambda, \mu \in (0, 2\bar{d})$, where $\bar{d} = \min\{a, b\}$. Let $\bar{A}, \bar{B} : Q \to H_2$ be $\bar{a}, \bar{b}$-inverse strongly monotone operators with $\alpha, \gamma \in (0, 2\hat{d})$, where $\hat{d} = \min\{\bar{a}, \bar{b}\}$. Let $D : H_1 \to H_2$ be a linear operator with adjoint D^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator D^*D. Define $G_C : C \to C$ by, for all $x \in C$, $G_C(x) = P_C(I-\lambda A)P_C(I-\mu B)x$ and define $G_Q : Q \to Q$ by, for all $\hat{x} \in Q$, $G_Q(\hat{x}) = P_Q(I-\alpha\bar{A})P_Q(I-\gamma\bar{B})\hat{x}$. If $\Omega_{\bar{A},\bar{B}}^{A,B}$ is nonempty, where $\Omega_{\bar{A},\bar{B}}^{A,B} = \{(x^*, y^*) \in \Omega_{A,B} : (\bar{x}^*, \bar{y}^*) \in \Omega_{\bar{A},\bar{B}}\}$, then $(x^*, y^*) \in \Omega_{\bar{A},\bar{B}}^{A,B}$ if and only if $x^* = G_C(x^* - \eta D^*(I-G_Q)Dx^*)$, where $y^* = P_C(I-\mu B)x^*$ and $\bar{y}^* = P_Q(I-\gamma\bar{B})\bar{x}^*$ with $\bar{x}^* = Dx^*$ and $\bar{y}^* = Dy^*$.

3. THE CONVERGENCE THEOREM

This section establishes a strong convergence result for an iterative scheme designed to solve the split general system of variational inequalities (SGSV) and to find common fixed points of certain nonexpansive mappings under suitable assumptions.

Theorem 3.1. *Let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed, and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A, B : C \to H_1$ be a, b–inverse strongly monotone mappings with $d = \min\{a, b\}$. Let $\bar{A}, \bar{B} : Q \to H_2$ be $\bar{a}, \bar{b}$–inverse strongly monotone mappings with $\bar{d} = \min\{\bar{a}, \bar{b}\}$. Let $D : H_1 \to H_2$ be a bounded and linear operator with adjoint D^*. Let $G_C : C \to C$ and $G_Q : Q \to Q$ be defined as in Lemma 2.5, and $G : C \to C$ be defined by $G(x) = G_C(x - \eta D^*(I-G_Q)Dx)$ for all x in C. Let $T : C \to C$ and $S : C \to C$ be nonexpansive mappings. Assume that the solution set $\mathfrak{I} = F(G) \cap F(T) \cap F(S)$ is nonempty. For given $u, x_1 \in C$, define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = G_C(x_n - \eta D^*(I-G_Q)Dx_n), \\ x_{n+1} = \alpha_n u + \beta_n T(c_n y_n + (1-c_n)Sy_n), \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{c_n\} \subset [0,1]$, $\lambda, \mu \in (0, 2d)$, $\alpha, \gamma \in (0, 2\bar{d})$, and $\eta \in \left(0, \frac{1}{L}\right)$ with L denoting the spectral radius of D^*D. Let the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$,
(iv) $\alpha_n + \beta_n \leq 1$ *for all n.*

Then the sequence $\{x_n\}$ defined above converges strongly to $x_0 = P_{\mathfrak{I}}u$, where $(x_0, y_0) \in \Omega_{\bar{A},\bar{B}}^{A,B}$, $y_0 = P_C(I-\mu B)x_0$, and $\bar{y}_0 = P_Q(I-\gamma\bar{B})\bar{x}_0$ with $\bar{x}_0 = Dx_0$ and $\bar{y}_0 = Dy_0$.

Proof. Before proceeding to the main steps of the proof, we begin by establishing a key identity that confirms the consistency of the reference point $z \in \mathfrak{I}$ with both the fixed point formulation and the associated variational inequality system under the given mappings. Let $z \in \mathfrak{I}$. It follows from the definition of G that $z = G(z) = G_C(z - \eta D^*(I-G_Q)Dz)$. By Lemma 2.5, it follows that $(z,z_0) \in \Omega^{A,B}_{\bar{A},\bar{B}}$, where

$$z_0 = P_C(I-\mu B)z \quad \text{and} \quad \bar{z}_0 = P_Q(I-\gamma\bar{B})\bar{z}, \quad \text{with } \bar{z} = Dz \text{ and } \bar{z}_0 = Dz_0.$$

Since $(z,z_0) \in \Omega^{A,B}_{\bar{A},\bar{B}}$, we have $(z,z_0) \in \Omega_{A,B}$ and $(Dz,Dz_0) \in \Omega_{\bar{A},\bar{B}}$. Thus, the following inequalities hold

$$\langle \alpha\bar{A}Dz_0 + Dz - Dz_0, \bar{x} - Dz\rangle \geq 0, \quad \text{for all } \bar{x} \in Q,$$
$$\langle \gamma\bar{B}Dz + Dz_0 - Dz, \bar{x} - Dz_0\rangle \geq 0, \quad \text{for all } \bar{x} \in Q.$$

It follows that $Dz = P_Q(I-\alpha\bar{A})Dz_0$ and $Dz_0 = P_Q(I-\gamma\bar{B})Dz$. Substituting the second into the first yields $Dz = P_Q(I-\alpha\bar{A})P_Q(I-\gamma\bar{B})Dz = G_QDz$. Finally, we recall the convex combination $M_n := c_ny_n + (1-c_n)Sy_n$, which allows us to express the iterative step as $x_{n+1} = \alpha_nu + \beta_nTM_n$.

Step 1. Prove that $\{x_n\}$ is bounded.

We first demonstrate that $\{x_n\}$ is bounded. To this end, we consider the definition of y_n and the reference point $z \in \Im$. Observe that

$$\begin{aligned}
\|y_n - z\|^2 &\leq \|x_n - z - \eta D^*(I-G_Q)(Dx_n - Dz)\|^2 \\
&= \|x_n - z\|^2 - 2\eta\langle D(x_n - z), (I-G_Q)Dx_n\rangle + \eta^2\|D^*(I-G_Q)Dx_n\|^2 \\
&= \|x_n - z\|^2 + 2\eta\langle Dz - G_QDx_n, (I-G_Q)Dx_n\rangle - 2\eta\|(I-G_Q)Dx_n\|^2 \\
&\quad + \eta^2\|D^*(I-G_Q)Dx_n\|^2 \\
&\leq \|x_n - z\|^2 - \eta(1-\eta L)\|(I-G_Q)Dx_n\|^2 \\
&\leq \|x_n - z\|^2.
\end{aligned}$$

Therefore, we obtain $\|y_n - z\| \leq \|x_n - z\|$. Next, we analyze the recursive formula of $\{x_n\}$

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n(u-z) + \beta_n(TM_n - z) - (1-\alpha_n-\beta_n)z\| \\
&\leq \alpha_n\|u-z\| + (1-\alpha_n-\beta_n)\|z\| + \beta_n\|TM_n - z\| \\
&\leq \alpha_n\|u-z\| + (1-\alpha_n-\beta_n)\|z\| \\
&\quad + \beta_n\|c_n(y_n - z) + (1-c_n)(Sy_n - z)\| \\
&\leq \alpha_n\|u-z\| + (1-\alpha_n-\beta_n)\|z\| + \beta_n\|y_n - z\| \\
&\leq \alpha_n\|u-z\| + (1-\alpha_n-\beta_n)\|z\| + \beta_n\|x_n - z\| \\
&\leq \alpha_n\|u-z\| + (1-\alpha_n-\beta_n)\|z\| + (1-\alpha_n)\|x_n - z\|.
\end{aligned}$$

Since $\{\alpha_n\}$ and $\{\beta_n\}$ lie in $[0,1]$ and satisfy the summability conditions stated in the theorem, and $\|x_1 - z\|$ is finite, we see from Lemma 2.3 that $\{x_n\}$ remains bounded for all $n \in \mathbb{N}$.

Step 2. Prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

From the definition of the sequence $\{y_n\}$, we first estimate

$$\begin{aligned}
\|y_n - y_{n-1}\|^2 &= \|G_C(x_n - \eta D^*(I-G_Q)Dx_n) - G_C(x_{n-1} - \eta D^*(I-G_Q)Dx_{n-1})\|^2 \\
&\leq \|(x_n - \eta D^*(I-G_Q)Dx_n) - (x_{n-1} - \eta D^*(I-G_Q)Dx_{n-1})\|^2 \\
&= \|(x_n - x_{n-1}) - \eta\left(D^*(I-G_Q)Dx_n - D^*(I-G_Q)Dx_{n-1}\right)\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\eta\langle x_n - x_{n-1}, D^*((I-G_Q)Dx_n - (I-G_Q)Dx_{n-1})\rangle \\
&\quad + \eta^2\|D^*((I-G_Q)Dx_n - (I-G_Q)Dx_{n-1})\|^2.
\end{aligned}$$

Using the linearity of D and the identity $\langle x, D^*y\rangle = \langle Dx, y\rangle$, we obtain

$$\begin{aligned}\|y_n - y_{n-1}\|^2 &\le \|x_n - x_{n-1}\|^2 - 2\eta\,\langle Dx_n - Dx_{n-1}, (I-G_Q)Dx_n - (I-G_Q)Dx_{n-1}\rangle \\ &\quad + \eta^2 L\,\|(I-G_Q)Dx_n - (I-G_Q)Dx_{n-1}\|^2 \\ &= \|x_n - x_{n-1}\|^2 - \eta(1-\eta L)\,\|(I-G_Q)Dx_n - (I-G_Q)Dx_{n-1}\|^2 \\ &\le \|x_n - x_{n-1}\|^2,\end{aligned}$$

which indicates $\|y_n - y_{n-1}\| \le \|x_n - x_{n-1}\|$. Now, we estimate the difference $\|x_{n+1} - x_n\|$ by using condition (iv)

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(\alpha_n - \alpha_{n-1})u + (\beta_n - \beta_{n-1})TM_{n-1} + \beta_n(TM_n - TM_{n-1})\| \\ &\le |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|TM_{n-1}\| + \beta_n\|M_n - M_{n-1}\|.\end{aligned}$$

Note that $M_n = c_n y_n + (1-c_n)Sy_n$, which yields

$$\begin{aligned}\|M_n - M_{n-1}\| &\le c_n\|y_n - y_{n-1}\| + (1-c_n)\|Sy_n - Sy_{n-1}\| + |c_n - c_{n-1}|(\|y_{n-1}\| + \|Sy_{n-1}\|) \\ &\le \|y_n - y_{n-1}\| + |c_n - c_{n-1}|(\|y_{n-1}\| + \|Sy_{n-1}\|).\end{aligned}$$

Substituting this into the previous bound, we arrive at

$$\begin{aligned}\|x_{n+1} - x_n\| &\le |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|TM_{n-1}\| \\ &\quad + \beta_n\|x_n - x_{n-1}\| + \beta_n|c_n - c_{n-1}|(\|y_{n-1}\| + \|Sy_{n-1}\|) \\ &\le |\alpha_n - \alpha_{n-1}|\|u\| + |\beta_n - \beta_{n-1}|\|TM_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &\quad + \beta_n|c_n - c_{n-1}|(\|y_{n-1}\| + \|Sy_{n-1}\|). \qquad (3.1)\end{aligned}$$

Finally, since $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{TM_n\}$ are all bounded and satisfy the conditions of Lemma 2.2, together with conditions (i) and (iii), we deduce from (3.1) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \qquad (3.2)$$

Step 3. Prove that $\lim_{n\to\infty} \|y_n - Sy_n\| = \lim_{n\to\infty} \|(I-G_Q)Dx_n\| = 0$.

We begin by analyzing the squared norm $\|x_{n+1} - z\|^2$, where $z \in \mathfrak{F} := F(G_C) \cap F(G_Q) \cap F(T) \cap F(S)$. From the iterative rule, we have $x_{n+1} = \alpha_n u + \beta_n TM_n$, which implies

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n(u-z) + \beta_n(TM_n - z) - (1-\alpha_n-\beta_n)z\|^2 \\ &\le \|\alpha_n(u-z) + \beta_n(TM_n - z)\|^2 - 2\langle(1-\alpha_n-\beta_n)z, x_{n+1} - z\rangle \\ &\le \alpha_n(\alpha_n+\beta_n)\|u-z\|^2 + \beta_n(\alpha_n+\beta_n)\|TM_n - z\|^2 \\ &\quad - \alpha_n\beta_n\|u - TM_n\|^2 + 2(1-\alpha_n-\beta_n)\|z\|\cdot\|x_{n+1} - z\|. \qquad (3.3)\end{aligned}$$

Since $M_n = c_n y_n + (1-c_n)Sy_n$, we have

$$\|TM_n - z\|^2 \le \|M_n - z\|^2 = c_n\|y_n - z\|^2 + (1-c_n)\|Sy_n - z\|^2 - c_n(1-c_n)\|y_n - Sy_n\|^2.$$

Substituting into (3.3), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \alpha_n(\alpha_n+\beta_n)\|u-z\|^2 + \beta_n(\alpha_n+\beta_n)\left[\|y_n - z\|^2 - c_n(1-c_n)\|y_n - Sy_n\|^2\right] \\ &\quad - \alpha_n\beta_n\|u - TM_n\|^2 + 2(1-\alpha_n-\beta_n)\|z\|\cdot\|x_{n+1} - z\|. \qquad (3.4)\end{aligned}$$

Note that

$$\begin{aligned}&\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\&\le \alpha_n(\alpha_n+\beta_n)\|u-z\|^2+\|y_n-z\|^2-\|x_{n+1}-z\|^2\\&\quad+2(1-\alpha_n-\beta_n)\|z\|\cdot\|x_{n+1}-z\|-\alpha_n\beta_n\|u-TM_n\|^2.\end{aligned}$$

Since $\|y_n-z\|\le\|x_n-z\|$, we further estimate

$$\begin{aligned}&\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\&\le \alpha_n(\alpha_n+\beta_n)\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\&\quad+2(1-\alpha_n-\beta_n)\|z\|\cdot\|x_{n+1}-z\|-\alpha_n\beta_n\|u-TM_n\|^2.\end{aligned}$$

Thus

$$\lim_{n\to\infty}\|y_n-Sy_n\|=0. \tag{3.5}$$

In view of Step 1, we have

$$\|y_n-z\|^2\le\|x_n-z\|^2-\eta(1-\eta L)\|(I-G_Q)Dx_n\|^2,$$

which together with (3.4) yields

$$\begin{aligned}\eta(1-\eta L)\|(I-G_Q)Dx_n\|^2\le{}&\alpha_n(\alpha_n+\beta_n)\|u-z\|^2-\alpha_n\beta_n\|u-TM_n\|^2\\&+2(1-\alpha_n-\beta_n)\|z\|\cdot\|x_{n+1}-z\|+\|x_n-z\|^2-\|x_{n+1}-z\|^2.\end{aligned}$$

In view of Step 2, we obtain

$$\lim_{n\to\infty}\|(I-G_Q)Dx_n\|=0. \tag{3.6}$$

Step 4. Prove that $\lim_{n\to\infty}\|TM_n-M_n\|=\lim_{n\to\infty}\|x_n-M_n\|=0$.

From the definition of x_n, we see

$$x_{n+1}-x_n=\alpha_n(u-x_n)-(1-\alpha_n-\beta_n)x_n+\beta_n(TM_n-x_n).$$

By (3.2) and condition (i), we conclude $\lim_{n\to\infty}\|TM_n-x_n\|=0$. From $M_n=c_ny_n+(1-c_n)Sy_n$, it follows that $\|M_n-y_n\|=\|c_ny_n+(1-c_n)Sy_n-y_n\|=(1-c_n)\|Sy_n-y_n\|$. By (3.5), we have

$$\lim_{n\to\infty}\|M_n-y_n\|=0. \tag{3.7}$$

It follows that

$$\|x_n-Ty_n\|\le\|x_n-TM_n\|+\|TM_n-Ty_n\|\le\|x_n-TM_n\|+\|M_n-y_n\|,$$

which implies that

$$\lim_{n\to\infty}\|x_n-Ty_n\|=0. \tag{3.8}$$

Note that $y_n=P_C(I-\lambda A)h_n$ and $z=P_C(I-\lambda A)h^*$, where

$$h_n=P_C(I-\mu B)\left(x_n-\eta D^*(I-G_Q)Dx_n\right),\quad h^*=P_C(I-\mu B)\left(z-\eta D^*(I-G_Q)Dz\right).$$

To estimate the convergence of y_n to z, we first bound the difference $\|h_n-h^*\|$. Using the nonexpansiveness of P_C, we obtain

$$\begin{aligned}\|h_n-h^*\|^2&\le\|x_n-\eta D^*(I-G_Q)Dx_n-z+\eta D^*(I-G_Q)Dz\|^2\\&=\|x_n-z\|^2+\eta^2\|D^*(I-G_Q)(Dx_n-Dz)\|^2-2\eta\langle x_n-z,D^*(I-G_Q)(Dx_n-Dz)\rangle.\end{aligned}$$

By the adjoint property $\langle x, D^*y\rangle = \langle Dx, y\rangle$, we have

$$\langle x_n - z, D^*(I-G_Q)(Dx_n - Dz)\rangle = \langle Dx_n - Dz, (I-G_Q)(Dx_n - Dz)\rangle.$$

Hence,

$$\|h_n - h^*\|^2 \le \|x_n - z\|^2 + \eta^2\|D^*(I-G_Q)(Dx_n - Dz)\|^2 - 2\eta\|(I-G_Q)(Dx_n - Dz)\|^2.$$

If $\|D^*\|^2 \le L$, we estimate $\|D^*(I-G_Q)(Dx_n - Dz)\|^2 \le L\|(I-G_Q)(Dx_n - Dz)\|^2$. Thus

$$\|h_n - h^*\|^2 \le \|x_n - z\|^2 + (\eta^2 L - \eta)\|(I-G_Q)(Dx_n - Dz)\|^2.$$

If $\eta \le \frac{1}{L}$, then the last term is nonpositive, and we conclude that

$$\|h_n - h^*\|^2 \le \|x_n - z\|^2. \tag{3.9}$$

Next, we define

$$k_n = x_n - \eta D^*(I-G_Q)Dx_n, \quad k^* = z - \eta D^*(I-G_Q)Dz.$$

With this notation, the sequence y_n can be expressed as $y_n = P_C(I-\lambda A)P_C(I-\mu B)k_n$. Similarly, the limit point satisfies $z = P_C(I-\lambda A)P_C(I-\mu B)k^*$. Applying the same arguments as in the previous part, and using the nonexpansiveness of both metric projections, we obtain

$$\|k_n - k^*\|^2 \le \|x_n - z\|^2. \tag{3.10}$$

From (3.4), (3.5), and (3.9), we aim to estimate the convergence of $\|Ah_n - Ah^*\|$. Since $y_n = P_C(I-\lambda A)h_n$ and $z = P_C(I-\lambda A)h^*$, the nonexpansiveness of P_C and $(I-\lambda A)$ yields

$$\begin{aligned}\|y_n - z\|^2 &= \|h_n - h^* - \lambda(Ah_n - Ah^*)\|^2\\ &= \|h_n - h^*\|^2 - 2\lambda\langle h_n - h^*, Ah_n - Ah^*\rangle + \lambda^2\|Ah_n - Ah^*\|^2.\end{aligned}$$

Since A is a-inverse strongly monotone, we have $\langle h_n - h^*, Ah_n - Ah^*\rangle \ge a\|Ah_n - Ah^*\|^2$. Thus

$$\|y_n - z\|^2 \le \|h_n - h^*\|^2 - \lambda(2a - \lambda)\|Ah_n - Ah^*\|^2.$$

Substituting into (3.4) and applying (3.9), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 \le\ & \alpha_n(\alpha_n+\beta_n)\|u - z\|^2 - \beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n - Sy_n\|^2\\ & - \alpha_n\beta_n\|u - TM_n\|^2 + 2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1} - z\| + \|x_n - z\|^2\\ & - \lambda(2a-\lambda)\|Ah_n - Ah^*\|^2.\end{aligned}$$

Note that

$$\begin{aligned}\lambda(2a-\lambda)\|Ah_n - Ah^*\|^2 \le\ & \alpha_n(\alpha_n+\beta_n)\|u - z\|^2 - \beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n - Sy_n\|^2\\ & - \alpha_n\beta_n\|u - TM_n\|^2 + 2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1} - z\|\\ & + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_n - x_{n+1}\|.\end{aligned}$$

From (3.2) and condition (i), it follows that

$$\lim_{n\to\infty}\|Ah_n - Ah^*\| = 0. \tag{3.11}$$

From (3.4), (3.5), and (3.10), we obtain the following estimation:

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq{}& \alpha_n(\alpha_n+\beta_n)\|u-z\|^2-\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\ &-\alpha_n\beta_n\|u-TM_n\|^2+2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1}-z\|\\ &+\|P_C(I-\lambda A)P_C(I-\mu B)k_n-P_C(I-\lambda A)P_C(I-\mu B)k^*\|^2.\end{aligned}$$

Applying the nonexpansiveness of the projections and the mappings, we derive

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq{}& \alpha_n(\alpha_n+\beta_n)\|u-z\|^2-\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\ &-\alpha_n\beta_n\|u-TM_n\|^2+2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1}-z\|\\ &+\|(I-\mu B)k_n-(I-\mu B)k^*\|^2.\end{aligned}$$

Then,

$$\begin{aligned}\|x_{n+1}-z\|^2 ={}& \alpha_n(\alpha_n+\beta_n)\|u-z\|^2-\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\ &-\alpha_n\beta_n\|u-TM_n\|^2+2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1}-z\|\\ &+\|k_n-k^*-\mu(Bk_n-Bk^*)\|^2.\end{aligned}$$

Expanding the last term and using the strong monotonicity of B gives

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq{}& \alpha_n(\alpha_n+\beta_n)\|u-z\|^2-\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\ &-\alpha_n\beta_n\|u-TM_n\|^2+2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1}-z\|+\|k_n-k^*\|^2\\ &-2\mu\langle k_n-k^*,Bk_n-Bk^*\rangle+\mu^2\|Bk_n-Bk^*\|^2.\end{aligned}$$

Since B is inverse strongly monotone with constant $b>0$, we have

$$2\mu\langle k_n-k^*,Bk_n-Bk^*\rangle \geq 2\mu b\|Bk_n-Bk^*\|^2,$$

which implies

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq{}& \alpha_n(\alpha_n+\beta_n)\|u-z\|^2-\beta_n(\alpha_n+\beta_n)c_n(1-c_n)\|y_n-Sy_n\|^2\\ &-\alpha_n\beta_n\|u-TM_n\|^2+2\|(1-\alpha_n-\beta_n)z\|\|x_{n+1}-z\|+\|x_n-z\|^2\\ &-\mu(2b-\mu)\|Bk_n-Bk^*\|^2.\end{aligned}$$

Hence, we conclude that

$$\lim_{n\to\infty}\|Bk_n-Bk^*\|=0. \tag{3.12}$$

To estimate $\|y_n-z\|$, we begin with the definition $y_n=P_C(I-\lambda A)h_n$ and $z=P_C(I-\lambda A)h^*$. By the nonexpansiveness of the metric projection, we obtain

$$\begin{aligned}\|y_n-z\|^2 \leq{}& \frac{1}{2}\left[\|(h_n-\lambda Ah_n)-(h^*-\lambda Ah^*)\|^2+\|y_n-z\|^2\right.\\ &\left.-\|(h_n-\lambda Ah_n)-(h^*-\lambda Ah^*)-(y_n-z)\|^2\right]\\ \leq{}& \frac{1}{2}\left[\|h_n-h^*\|^2+\|y_n-z\|^2-\|(h_n-y_n)-(h^*-z)-\lambda(Ah_n-Ah^*)\|^2\right]\\ ={}& \frac{1}{2}\left[\|h_n-h^*\|^2+\|y_n-z\|^2-\|(h_n-y_n)-(h^*-z)\|^2\right.\\ &\left.-\lambda^2\|Ah_n-Ah^*\|^2+2\lambda\langle(h_n-y_n)-(h^*-z),Ah_n-Ah^*\rangle\right].\end{aligned}$$

From $\|h_n - h^*\| \le \|x_n - z\|$, we obtain

$$\|y_n - z\|^2 \le \|x_n - z\|^2 - \|(h_n - y_n) - (h^* - z)\|^2 + 2\lambda\|(h_n - y_n) - (h^* - z)\|\|Ah_n - Ah^*\|. \quad (3.13)$$

Substituting (3.13) into the inequality of $\|x_{n+1} - z\|^2$ from the iterative step, we see that

$$\begin{aligned}\|x_{n+1} - z\|^2 \le\ & \alpha_n(\alpha_n + \beta_n)\|u - z\|^2 - \beta_n(\alpha_n + \beta_n)c_n(1 - c_n)\|y_n - Sy_n\|^2 \\ & - \alpha_n\beta_n\|u - TM_n\|^2 + 2\|(1 - \alpha_n - \beta_n)z\|\|x_{n+1} - z\| + \|x_n - z\|^2 \\ & - \|(h_n - y_n) - (h^* - z)\|^2 + 2\lambda\|(h_n - y_n) - (h^* - z)\|\|Ah_n - Ah^*\|.\end{aligned}$$

That is,

$$\begin{aligned}\|(h_n - y_n) - (h^* - z)\|^2 \le\ & \alpha_n(\alpha_n + \beta_n)\|u - z\|^2 - \beta_n(\alpha_n + \beta_n)c_n(1 - c_n)\|y_n - Sy_n\|^2 \\ & - \alpha_n\beta_n\|u - TM_n\|^2 + 2\|(1 - \alpha_n - \beta_n)z\|\|x_{n+1} - z\| \\ & + (\|x_n - z\| - \|x_{n+1} - z\|)\|x_n - x_{n+1}\| \\ & + 2\lambda\|(h_n - y_n) - (h^* - z)\|\|Ah_n - Ah^*\|.\end{aligned}$$

From (3.2), (3.11), and condition (i), we conclude that

$$\lim_{n\to\infty}\|(h_n - y_n) - (h^* - z)\| = 0. \quad (3.14)$$

From the definitions of k_n and k^*, we compute

$$\|(z - k^*) - (x_n - k_n)\|^2 = \eta^2\|D^*(I - G_Q)Dx_n\|^2 \le \eta^2 L\|(I - G_Q)Dx_n\|^2.$$

Since D^* is bounded, (3.6) yields

$$\lim_{n\to\infty}\|(z - k^*) - (x_n - k_n)\| = 0. \quad (3.15)$$

Note that $\|(x_n - h_n) + (h^* - z)\| = \|(h_n - y_n) - (h^* - z) + (x_n - y_n)\|$. Using $h_n = P_C(I - \mu B)k_n$, $h^* = P_C(I - \mu B)k^*$, $y_n = P_C(I - \lambda A)h_n$, and $z = P_C(I - \lambda A)h^*$, we see that

$$\begin{aligned}&\|(x_n - h_n) + (h^* - z)\|^2 \\ &= \|(I - \mu B)(k_n - k^*) - (h_n - h^*) + \mu(Bk_n - Bk^*) + \eta D^*(I - G_Q)Dx_n\|^2 \\ &\le \|(I - \mu B)(k_n - k^*) - (h_n - h^*) + \mu(Bk_n - Bk^*)\|^2 \\ &\quad + 2\eta\langle(I - G_Q)Dx_n, D[(x_n - h_n) + (h^* - z)]\rangle \\ &\le \|(I - \mu B)(k_n - k^*) - (h_n - h^*) + \mu(Bk_n - Bk^*)\|^2 \\ &\quad + 2\eta\|(I - G_Q)Dx_n\|\|D[(x_n - h_n) + (h^* - z)]\| \\ &\le \|(I - \mu B)(k_n - k^*)\|^2 - \|h_n - h^*\|^2 + 2\mu\|Bk_n - Bk^*\| \cdot \|(I - \mu B)(k_n - k^*) - (h_n - h^*) \\ &\quad + \mu(Bk_n - Bk^*)\| + 2\eta\|(I - G_Q)Dx_n\| \cdot \|D[(x_n - h_n) + (h^* - z)]\| \\ &\le \|(I - \mu B)(k_n - k^*)\|^2 - \|Ty_n - Tz\|^2 + 2\mu\|Bk_n - Bk^*\| \cdot \|(I - \mu B)(k_n - k^*) - (h_n - h^*) \\ &\quad + \mu(Bk_n - Bk^*)\| + 2\eta\|(I - G_Q)Dx_n\| \cdot \|D[(x_n - h_n) + (h^* - z)]\| \\ &\le \|(I - \mu B)(k_n - k^*) - (Ty_n - z)\| \cdot (\|(I - \mu B)(k_n - k^*)\| \\ &\quad + \|Ty_n - z\|) + 2\mu\|Bk_n - Bk^*\| \cdot \|(I - \mu B)(k_n - k^*) - (h_n - h^*) \\ &\quad + \mu(Bk_n - Bk^*)\| + 2\eta\|(I - G_Q)Dx_n\| \cdot \|D[(x_n - h_n) + (h^* - z)]\|\end{aligned}$$

$$
\begin{aligned}
&= \|k_n - \mu Bk_n - k^* + \mu Bk^* - x_n + x_n - Ty_n + z\| \cdot (\|(I-\mu B)(k_n - k^*)\| + \|Ty_n - z\|) \\
&\quad + 2\mu\|Bk_n - Bk^*\| \cdot \|(I-\mu B)(k_n - k^*) - (h_n - h^*) + \mu(Bk_n - Bk^*)\| \\
&\quad + 2\eta\|(I-G_Q)Dx_n\| \cdot \|D[(x_n - h_n) + (h^* - z)]\| \\
&= \|(x_n - Ty_n) + (z - k^*) - (x_n - k_n) - \mu(Bk_n - Bk^*)\| \cdot (\|(I-\mu B)(k_n - k^*)\| + \|Ty_n - z\|) \\
&\quad + 2\mu\|Bk_n - Bk^*\|\|(I-\mu B)k_n - (I-\mu B)k^* - (h_n - h^*) + \mu(Bk_n - Bk^*)\| \\
&\quad + 2\eta\|(I-G_Q)Dx_n\| \cdot \|D[(x_n - h_n) + (h^* - z)]\| \\
&\le (\|x_n - Ty_n\| + \|(z - k^*) - (x_n - k_n)\| + \mu\|Bk_n - Bk^*\|) \cdot (\|(I-\mu B)(k_n - k^*)\| + \|Ty_n - z\|) \\
&\quad + 2\mu\|Bk_n - Bk^*\| \cdot \|(I-\mu B)(k_n - k^*) - (h_n - h^*) + \mu(Bk_n - Bk^*)\| \\
&\quad + 2\eta\|(I-G_Q)Dx_n\| \cdot \|D[(x_n - h_n) + (h^* - z)]\|.
\end{aligned}
$$

From (3.6), (3.8), (3.12) and (3.15), each term on the right-hand side tends to zero as $n \to \infty$. Therefore,$\lim_{n\to\infty} \|(x_n - h_n) + (h^* - z)\| = 0$. In view of (3.14), we obtain

$$\|x_n - y_n\| = \|(x_n - h_n + h^* - z) + (h_n - y_n - (h^* - z))\| \le \|x_n - h_n + h^* - z\| + \|h_n - y_n - (h^* - z)\|.$$

Since both $\|x_n - h_n + h^* - z\| \to 0$ and $\|h_n - y_n - (h^* - z)\| \to 0$ as $n \to \infty$, we deduce that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.16}$$

From (3.2), we observe that $\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0$ as $n \to \infty$, which implies $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. Using the definition of x_{n+1} and the relation

$$x_{n+1} - y_n = \alpha_n u + \beta_n TM_n - y_n = \alpha_n(u - y_n) - (1 - \alpha_n - \beta_n)y_n + \beta_n(TM_n - y_n),$$

together with assumptions (i) and (iv), we deduce $\lim_{n\to\infty} \|TM_n - y_n\| = 0$. Combining (3.7) and $\|TM_n - M_n\| \le \|TM_n - y_n\| + \|y_n - M_n\|$ yields

$$\lim_{n\to\infty} \|TM_n - M_n\| = 0, \tag{3.17}$$

which obtains $\|x_n - M_n\| \le \|x_n - TM_n\| + \|TM_n - M_n\| \to 0$. It follows that

$$\lim_{n\to\infty} \|x_n - M_n\| = 0. \tag{3.18}$$

Next, we show that $\|y_n - M_n\| \to 0$ as $n \to \infty$. Note that $\|y_n - M_n\| \le \|y_n - x_n\| + \|x_n - M_n\|$. From (3.16) and (3.18), we conclude $\lim_{n\to\infty} \|y_n - M_n\| = 0$. Therefore, $\{x_n\}$, $\{y_n\}$, and $\{M_n\}$ are asymptotically equivalent, i.e.,

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|y_n - M_n\| = \lim_{n\to\infty} \|x_n - M_n\| = 0.$$

This convergence plays a key role in establishing the strong convergence of the iterative process to a common solution of the problem under consideration.

Step 5. Prove that $\limsup_{n\to\infty} \langle u - x_0, x_n - x_0 \rangle \le 0$, where $x_0 = P_{\mathfrak{J}} u$.

Let $\{x_{n_j}\}$ be a subsequence such that

$$\limsup_{n\to\infty} \langle u - x_0, x_n - x_0 \rangle = \lim_{j\to\infty} \langle u - x_0, x_{n_j} - x_0 \rangle. \tag{3.19}$$

Since $\{x_n\}$ is bounded, we may assume $x_{n_j} \rightharpoonup q$ for some $q \in C$. It follows from (3.19) that

$$\limsup_{n\to\infty} \langle u - x_0, x_n - x_0 \rangle = \langle u - x_0, q - x_0 \rangle. \tag{3.20}$$

From (3.18) and the weak convergence $x_{n_j} \rightharpoonup q$, we also have $M_{n_j} \rightharpoonup q$. Using (3.17) and Lemma 2.4, it follows that $q \in F(T)$. Assume that $q \notin F(G)$. Then, by Opial's condition and (3.16), we obtain

$$\begin{aligned}\liminf_{j\to\infty}\|x_{n_j}-q\| &< \liminf_{j\to\infty}\|x_{n_j}-G(q)\| \\ &= \liminf_{j\to\infty}\|x_{n_j}-y_{n_j}+y_{n_j}-G_C(q-\eta D^*(I-G_Q)Dq)\| \\ &\le \liminf_{j\to\infty}\|x_{n_j}-y_{n_j}\|+\|G_C(x_{n_j}-\eta D^*(I-G_Q)Dx_{n_j}) \\ &\quad -G_C(q-\eta D^*(I-G_Q)Dq)\| \\ &\le \liminf_{j\to\infty}\|x_{n_j}-q\|,\end{aligned}$$

which is a contradiction. Thus, we conclude $q \in F(G)$. Now, we assume that $q \neq Sq$, and define the nonexpansive mapping $\bar{T} := cI + (1-c)S$. Since $\bar{T}$ is nonexpansive and $q \neq Sq \Leftrightarrow q \neq \bar{T}q$, the Opial's condition yields

$$\begin{aligned}\limsup_{j\to\infty}\|y_{n_j}-q\| &< \limsup_{j\to\infty}\|y_{n_j}-\bar{T}q\| \\ &\le \limsup_{j\to\infty}\left(\|y_{n_j}-M_{n_j}\|+\|M_{n_j}-\bar{T}q\|\right) \\ &\le \limsup_{j\to\infty}\left(\|M_{n_j}-\bar{T}y_{n_j}\|+\|\bar{T}y_{n_j}-\bar{T}q\|\right) \\ &\le \limsup_{j\to\infty}\left(\|c_{n_j}y_{n_j}+(1-c_{n_j})Sy_{n_j}-\bar{T}y_{n_j}\|+\|y_{n_j}-q\|\right) \\ &= \limsup_{j\to\infty}\|y_{n_j}-q\|,\end{aligned}$$

a contradiction. Hence, $q \in F(S)$. Thus $q \in \mathfrak{I}$. Finally, we obtain from (3.20) that

$$\limsup_{n\to\infty}\langle u-x_0, x_n-x_0\rangle \le 0. \tag{3.21}$$

Step 6. Prove that $\lim_{n\to\infty}\|x_n-x_0\| = 0$, where $x_0 = P_{\mathfrak{I}}u$.

From the nonexpansiveness of T, we obtain

$$\begin{aligned}\|TM_n-x_0\|^2 &\le \|c_n(y_n-x_0)+(1-c_n)(Sy_n-x_0)\|^2 \\ &= c_n\|y_n-x_0\|^2+(1-c_n)\|Sy_n-x_0\|^2-c_n(1-c_n)\|y_n-Sy_n\|^2 \\ &\le \|y_n-x_0\|^2 \le \|x_n-x_0\|^2.\end{aligned}$$

Using the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1}-x_0\|^2 &= \|\alpha_n(u-x_0)+\beta_n(TM_n-x_0)-(1-\alpha_n-\beta_n)x_0\|^2 \\ &\le \|\beta_n(TM_n-x_0)-(1-\alpha_n-\beta_n)x_0\|^2+2\alpha_n\langle u-x_0, x_{n+1}-x_0\rangle \\ &\le \beta_n^2\|TM_n-x_0\|^2+2(1-\alpha_n-\beta_n)\|x_0\|\cdot\|\beta_n(TM_n-x_0)-(1-\alpha_n-\beta_n)x_0\| \\ &\quad +2\alpha_n\langle u-x_0, x_{n+1}-x_0\rangle \\ &\le (1-\alpha_n)\|x_n-x_0\|^2+2(1-\alpha_n-\beta_n)\|x_0\|\cdot\|\beta_n(TM_n-x_0)-(1-\alpha_n-\beta_n)x_0\| \\ &\quad +2\alpha_n\langle u-x_0, x_{n+1}-x_0\rangle.\end{aligned}$$

By condition (i), Lemma 2.3, and inequality (3.21), we conclude that $\{x_n\}$ converges strongly to $x_0 = P_{\mathfrak{I}}u$.

Finally, applying Lemma 2.5, we deduce that $(x_0, y_0) \in \Omega^{A,B}_{\bar{A},\bar{B}}$, where $y_0 = P_C(I-\mu B)x_0$, $\bar{x}_0 = Dx_0$, and $\bar{y}_0 = Dy_0$. □

4. Applications

In this section, we demonstrate the applicability of our main result, Theorem 3.1, by addressing three specific problems: the split feasibility problem (SFP), the split variational inequality problem (SVIP), and a constrained minimization problem. These examples illustrate the versatility and generality of the proposed iterative scheme.

Let H_1 and H_2 be real Hilbert spaces, and let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed, and convex subsets. Consider a bounded and linear operator $D : H_1 \to H_2$ with adjoint D^*. We assume that $\mathfrak{I} = \Gamma \cap F(T) \cap F(S) \neq \emptyset$, where Γ is the solution set corresponding to each specific application.

4.1. Split feasibility problem.

We begin with the classical split feasibility problem (SFP), which seeks a point $x \in C$ such that $Dx \in Q$. This problem arises in various applied fields such as signal processing and image reconstruction. Mathematically, the SFP can be formulated as the fixed point equation $x = P_C(x - \eta D^*(I - P_Q)Dx)$, for some $\eta \in (0, \frac{2}{\lambda})$, where λ is the spectral radius of D^*D (see Xu [19]).

We now show that this problem can be addressed as a special case of our main result, Theorem 3.1, by eliminating the variational inequality components and focusing solely on the feasibility structure.

Theorem 4.1. *Let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed, and convex subsets. Let $T, S : C \to C$ be nonexpansive mappings, and let $D : H_1 \to H_2$ be a bounded and linear operator with adjoint D^*. Assume that $\mathfrak{I} = \{x \in C : Dx \in Q,\ x \in F(T) \cap F(S)\}$ is nonempty. Let $u, x_1 \in C$, and define the sequence $\{x_n\}$ by*

$$\begin{cases} y_n = P_C(x_n - \eta D^*(I - P_Q)Dx_n), \\ x_{n+1} = \alpha_n u + \beta_n T(c_n y_n + (1 - c_n)Sy_n), \end{cases}$$

where $\eta \in (0, \frac{1}{L})$ and L is the spectral radius of D^*D. Suppose the parameters satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$,
(iv) $\alpha_n + \beta_n \leq 1$ *for all n.*

Then the sequence $\{x_n\}$ defined above converges strongly to $x_0 = P_{\mathfrak{I}}u$, which is a solution to the split feasibility problem.

Proof. The result is an immediate consequence of Theorem 3.1 by setting $A \equiv B \equiv \bar{A} \equiv \bar{B} \equiv 0$, which eliminates the variational inequality components from the algorithm. In this case, the resolvent-type operators reduce to identity mappings, i.e., $G_C = P_C$ and $G_Q = P_Q$. Additionally, by assuming that the domain and image components of the solution coincide, i.e., $x^* = y^*$ and $\bar{x}^* = \bar{y}^*$, the iteration is reduced to the form stated in the theorem. Hence, the convergence result follows directly. □

Remark 4.2. *If we further take $T = I$ in Theorem 4.1, then the iteration is reduced to*

$$\begin{cases} y_n = P_C(x_n - \eta D^*(I - P_Q)Dx_n), \\ x_{n+1} = \alpha_n u + \beta_n(c_n y_n + (1 - c_n)Sy_n). \end{cases} \tag{4.1}$$

This formulation corresponds to a convex combination between y_n and Sy_n.

It follows from Theorem 4.1 that the sequence $\{x_n\}$ generated by (4.1) converges strongly to $x_0 = P_{\mathfrak{J}}u$, the projection of u onto the solution set of the SFP. Furthermore, if $S = I$, the iteration is reduced to the classical CQ algorithm proposed by Byrne [2].

4.2. **Split variational inequality problem.** The split variational inequality problem (SVIP) is to find a point $\hat{x} \in C$ such that, for all $x \in C$, $\langle f_1(\hat{x}), x - \hat{x}\rangle \geq 0$, and simultaneously, its image $\hat{y} := D\hat{x} \in Q$ satisfies $\langle f_2(\hat{y}), y - \hat{y}\rangle \geq 0$ for all $y \in Q$, where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear operators, and $D : H_1 \to H_2$ is a bounded and linear operator. The set of all solutions to the SVIP is defined by

$$\Phi := \{\hat{x} \in C : \hat{x} \in VI(C, f_1),\ D\hat{x} \in VI(Q, f_2)\}.$$

Clearly, the SVIP generalizes the split feasibility problem, which corresponds to the special case $f_1 \equiv f_2 \equiv 0$.

Theorem 4.3. *Let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed, and convex subsets of real Hilbert spaces. Suppose that $A : C \to H_1$ and $\bar{A} : Q \to H_2$ are a- and $\bar{a}$-inverse strongly monotone operators, respectively. Let $D : H_1 \to H_2$ be a bounded and linear operator with adjoint D^*. Define the resolvent-type operators by $G_C := P_C(I - \lambda A)$ and $G_Q := P_Q(I - \gamma\bar{A})$, and define the composite operator $G : C \to C$ by $G(x) := G_C\left(x - \eta D^*(I - G_Q)Dx\right)$. Let $T, S : C \to C$ be nonexpansive mappings and assume that $\mathfrak{J} := \Phi \cap F(T) \cap F(S)$ is nonempty. Given $u, x_1 \in C$, generate a sequence $\{x_n\}$ via the iterative scheme:*

$$\begin{cases} y_n = G_C\left(x_n - \eta D^*(I - G_Q)Dx_n\right), \\ x_{n+1} = \alpha_n u + \beta_n T\left(c_n y_n + (1 - c_n)Sy_n\right), \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{c_n\}$ lie in $[0,1]$. The parameters are chosen such that $\lambda \in (0, 2d)$ and $\gamma \in (0, 2\bar{d})$, where d and $\bar{d}$ are the inverse-strong monotonicity constants of A and $\bar{A}$, respectively. In addition, $\eta \in \left(0, \frac{1}{L}\right)$, where L denotes the spectral radius of the operator D^*D. Assume the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$,
(iv) $\alpha_n + \beta_n \leq 1$ *for all n.*

Then the sequence $\{x_n\}$ generated above converges strongly to $x_0 := P_{\mathfrak{J}}u$.

Proof. The result follows from Theorem 3.1 by setting $B \equiv \bar{B} \equiv 0$ and identifying $x^* = y^*$ and $\bar{x}^* = \bar{y}^*$. Under these substitutions, the operator G and the iteration scheme coincide with those in the present theorem. All the assumptions are satisfied, and thus the strong convergence conclusion holds. □

4.3. **Constrained minimization problem.** Let C be a nonempty, closed, and convex subset of a real Hilbert space H_1. We consider the constrained minimization problem

$$\min_{x\in C} f(x), \tag{4.2}$$

where $f : H_1 \to \mathbb{R}$ is a convex and continuously differentiable function. The set of minimizers of f over C is denoted by $\Gamma_f := \{x \in C : f(x) = \min_{z\in C} f(z)\}$.

Proposition 4.4 (Ceng et al. [4])**.** *Let $C \subset H_1$, $Q \subset H_2$ be nonempty, closed, and convex sets in real Hilbert spaces, and let $A : H_1 \to H_2$ be a bounded and linear operator with adjoint A^*. Define the proximity function $f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2$, for all $x \in H_1$, where P_Q is the metric projection onto Q. Then the following statements are equivalent:*

(1) $x^* \in C$ is a minimizer of f, i.e., $f(x^*) = \min_{x\in C} f(x)$;
(2) x^* is a fixed point of the mapping $P_C(I - \lambda\nabla f)$ for some $\lambda \in \big(0, 1/L\big)$, where L is the spectral radius of A^*A;
(3) x^* solves the variational inequality, $\langle \nabla f(x^*), x - x^*\rangle \geq 0$ for all $x \in C$, where $\nabla f(x) = A^*(I - P_Q)Ax$.

Remark 4.5. *Proposition 4.4 is stated in the split framework involving A and Q. However, in the general constrained minimization problem (4.2) with a convex differentiable function f, the same equivalence holds by taking $A = \nabla f$. That is, the optimality condition of (4.2) can be reformulated as the variational inequality* $\mathrm{VI}(C, \nabla f)$*:*

$$\langle \nabla f(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C.$$

Remark 4.6. *If ∇f is L-Lipschitz continuous, then it is $\frac{1}{L}$-inverse strongly monotone (i.e., cocoercive). This property ensures that the mapping $P_C(I - \eta\nabla f)$ is nonexpansive whenever* $\eta \in (0, 1/L)$.

Theorem 4.7. *Let $C \subset H_1$ be a nonempty closed convex subset of a real Hilbert space, and let f be defined as in Proposition 4.4. Let $T, S : C \to C$ be nonexpansive mappings and $\Im := \Gamma_f \cap F(T) \cap F(S) \neq \emptyset$. For $u, x_1 \in C$, generate a sequence $\{x_n\}$ by*

$$y_n = P_C(x_n - \eta\nabla f(x_n)), \tag{4.3}$$

$$x_{n+1} = \alpha_n u + \beta_n T(c_n y_n + (1 - c_n)Sy_n), \tag{4.4}$$

where $\{\alpha_n\}, \{\beta_n\}, \{c_n\} \subset [0, 1]$, and $\eta \in (0, 1/L)$. Suppose the following hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,
(iii) $\sum_{n=1}^{\infty} (1 - \alpha_n - \beta_n) < \infty$,
(iv) $\alpha_n + \beta_n \leq 1$ *for all n.*

Then $\{x_n\}$ converges strongly to $x_0 := P_{\Im} u$.

Proof. By Remark 4.6, ∇f is $\frac{1}{L}$-inverse strongly monotone. Hence, $P_C(I - \eta\nabla f)$ is nonexpansive with the fixed point set $F(P_C(I - \eta\nabla f)) = \Gamma_f$ by Proposition 4.4. Therefore, scheme (4.3)–(4.4) falls within the framework of Theorem 3.1 when $\bar{A} = B = \bar{B} = 0$, $D = I$, and $A = \nabla f$. Consequently, all the required assumptions are satisfied, and x_n converges strongly to $P_{\Im} u$. □

Remark 4.8. *If $T \equiv I$, then the iteration is reduced to*

$$y_n = P_C(x_n - \eta \nabla f(x_n)),$$
$$x_{n+1} = \alpha_n u + \beta_n (c_n y_n + (1-c_n) S y_n).$$

This formulation corresponds to a generalization of projection-based methods for constrained convex minimization and includes as special cases various known algorithms such as those presented in [11].

5. Numerical Results

Let $\mathbb{R}$ be the set of real numbers. Let $H_1 = H_2 = \mathbb{R}^3$. Define the sets

$$C = [-100,100] \times [-100,100] \times [-100,100]$$

and

$$Q = [-100,100] \times [-100,100] \times [-100,100].$$

Define the operators A, B, $\bar{A}$, and $\bar{B}$ as follows:

$$A(x) = \left(\frac{x_1 - 3}{3}, \frac{2x_2 - 6}{5}, \frac{x_3 - 3}{4} \right), \ B(x) = \left(\frac{x_1 - 3}{3}, \frac{x_2 - 3}{3}, \frac{x_3 - 3}{5} \right),$$

and

$$\bar{A}(x) = \left(\frac{x_1}{5}, \frac{x_2}{2}, \frac{2x_3}{3} \right), \ \bar{B}(x) = \left(\frac{x_1}{3}, \frac{x_2}{5}, \frac{x_3}{7} \right).$$

The linear operator $D : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$D(x_1,x_2,x_3) = (2x_1 - x_2 - x_3, x_1 - 2x_2 + x_3, x_1 + x_2 - 2x_3).$$

The adjoint operator D^* is defined by

$$D^*(x_1,x_2,x_3) = (2x_1 + x_2 + x_3, -x_1 - 2x_2 + x_3, -x_1 + x_2 - 2x_3).$$

The spectral radius of D^*D is 9. The iteration process uses the parameters: $\eta = 0.008$. The sequence α_n, β_n, and c_n are defined by

$$\alpha_n = \frac{0.85}{(n+10)^{1.5}}, \quad \beta_n = 1 - \alpha_n - \frac{\varepsilon}{n+1}, \text{ where } \varepsilon = 0.001, \quad c_n = 1 - \frac{1}{n+10}.$$

The iterative process is governed by the following equations. First, compute

$$y_n = G_C(x_n - \eta D^*(I - G_Q)Dx_n)$$

where G_C and G_Q are projection mappings defined as:

$$G_C(x) = P_C(I - 0.5A)P_C(I - B)x, \ G_Q(x) = P_Q(I - \bar{A})P_Q(I - 0.5\bar{B})x.$$

Then, update x_n by $x_{n+1} = \alpha_n u + \beta_n T(c_n y_n + (1-c_n)S(y_n))$, where T is a mapping from C to C defined by

$$T(x) = \left(\frac{x_1 + 6}{3}, \frac{2x_2 + 6}{4}, \frac{3x_3 + 6}{5} \right), \ S(x) = \left(\frac{x_1 + 3}{2}, \frac{x_2 + 9}{4}, \frac{2x_3 + 3}{3} \right).$$

The initial values are:

$$x_1 = (30,30,30), \ u = (10,10,10).$$

By the definition of A, B, $\bar{A}$, $\bar{B}$, D, T, and S, we have $\{3\} \in F(G) \cap F(T) \cap F(S)$, where $3 = \{3,3,3\}$. From Theorem 3.1, we can conclude that $x_n = \{x_n^1, x_n^2, x_n^3\}$ converges strongly to 3.

The values of x_n and y_n for each iteration from 1 to 50 are presented in Table 1. The sequences

Iteration	x_n^1	x_n^2	x_n^3	y_n^1	y_n^2	y_n^3
1	30	30	30	18	17.4	21.9
2	7.82073325	9.71101414	13.8962558	5.71362708	6.60945707	10.5430197
3	3.99097364	4.79890743	7.45079052	3.56999087	3.97960587	6.06454892
4	3.30552175	3.57925084	4.88483623	3.17835313	3.31919819	4.29505361
5	3.16937212	3.26155395	3.85905735	3.09778477	3.14427969	3.59041249
6	3.13296106	3.16943760	3.44316858	3.07551296	3.09257359	3.30524791
7	3.11659723	3.13606505	3.26946594	3.06559131	3.07363909	3.18619340
8	3.10547319	3.11927642	3.19269611	3.05906863	3.06419548	3.13352897
9	3.09650678	3.10799390	3.15532912	3.05393942	3.05796480	3.10783840
10	3.08886857	3.09909321	3.13445247	3.04962654	3.05312033	3.09344010
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	3.04694738	3.05223297	3.06692493	3.02619588	3.02796731	3.04655979
⋮	⋮	⋮	⋮	⋮	⋮	⋮
50	3.01611180	3.01791872	3.02273775	3.00898896	3.00959245	3.01582161

TABLE 1. Values of x_n and y_n from Iteration 1 to 50

$\{x_n\}$ and $\{y_n\}$ converge to the fixed point $(3,3,3)$. The convergence rate decreases over the iterations, as illustrated in Figures 1 and 2.

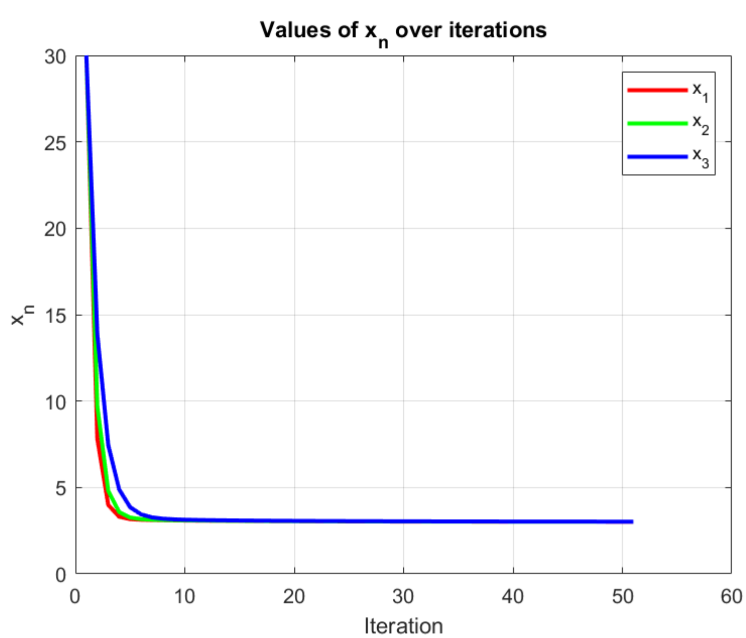

FIGURE 1. Convergence of x values over iterations

From Theorem 3.1, if we choose $C_n = 1$ and $\beta_n = 1 - \alpha_n$, then the theorem in [15] becomes a special case of the Theorem 3.1.

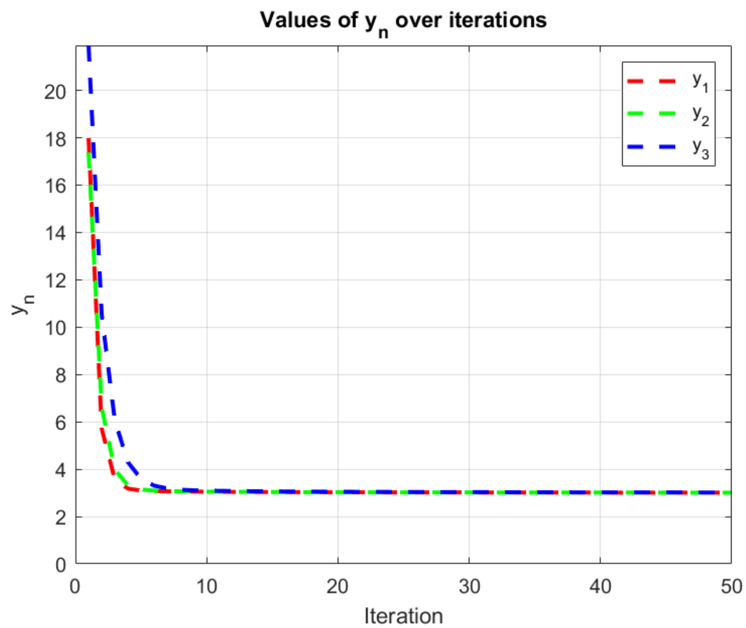

FIGURE 2. Convergence of y values over iterations

6. CONCLUSION

This paper presents a new iterative algorithm for finding a common solution to a split general system of variational inequalities (SGSV) and fixed point problems of two nonexpansive mappings in real Hilbert spaces. The proposed scheme is constructed via a hybrid projection involving resolvent-type operators and viscosity approximation. A rigorous convergence analysis was provided, showing that under standard assumptions, the sequence generated by the method converges strongly to the projection of a given point onto the solution set. In addition to the theoretical development, we demonstrated the practical effectiveness of the proposed algorithm through a numerical example in $\mathbb{R}^3$. The example illustrates fast convergence toward the exact solution, thereby supporting the theoretical convergence results. Furthermore, we sown that the algorithm is applicable to the split feasibility problem (SFP), the split variational inequality problem (SVIP), and a constrained convex minimization problem. These applications highlight the generality and adaptability of our method. The results of this work contribute to the growing literature on algorithms for solving coupled variational inequality problems and provide a unified framework that encompasses and improves several existing methods. Future work may consider extensions of this algorithm to Banach spaces, the incorporation of stochastic elements, or real-world applications such as network equilibrium, image recovery, and machine learning.

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