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HU STABILITY OF i-$\mathscr{F}$-HOM-DER ON BANACH ALGEBRAS: SOLUTIONS AND METHODS

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Abstract. In this paper, we define a new class of system of additive-quadratic functional equations on Banach algebras, which we call the system of i-mappings, where $i \in \{1,2\}$. We introduce the notion of a i-$\mathscr{F}$-homomorphism-derivations (abbreviated i-$\mathscr{F}$-hom-ders), with $i \in \{1,2\}$: for $i = 1$, $\mathscr{F}$ is a linear homomorphism, and for $i = 2$, f is a quadratic homomorphism on Banach algebras. Finally, using a fixed-point method, we investigate Hyers–Ulam stability for the system of additive-quadratic functional equations and i-$\mathscr{F}$-hom-ders, employing Gavruta-type, Rassias-type and JMRassias-type control functions on Banach algebras.

Keywords. Additive-quadratic functional equations; Banach algebras; Fixed-point method; Hyers–Ulam stability; Homomorphism.

1. Introduction

In 1940, Ulam [26] presented a talk at the University of Wisconsin, in which he raised several unresolved mathematical questions, one of which concerned the robustness of homomorphisms. The question proposed by Ulam sparked widespread interest in stability research and the study of functional equations ($\mathbb{FE}$s). In 1941, Hyers [8] gave an answer to Ulam's question for additive mappings on Banach spaces, giving rise to the concept, now called HU stability. In 1978, Rassias [23] extended Hyers' result by replacing the original control term with $\theta(\|u\|^p + \|v\|^p)$ for constants $p < 1$ and $\theta > 0$, and established the generalized HU stability. In 1982, Rassias [22] proposed an alternative control term of the form $\theta(\|u\|^p \cdot \|v\|^p)$ with $\theta > 0$ and $p < 1$, which introduces a different family of admissible controls. In 1994, Gǎvruta [7] replaced these particular bounds by a general control function $\varphi(u,v)$ and analyzed HU stability in this flexible setting. The topic has attracted considerable attention, and researchers have investigated

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stability issues for derivations and homomorphisms, explored problems involving the Fibonacci sequence, and addressed various space questions by using diverse control functions and both FP and direct methods. In 2018, Elqorachi and Rassias [5] investigated HU stability of a class of generalized trigonometric $\mathbb{FE}$s on a semigroup S equipped with an involutive morphism σ and a multiplicative function μ satisfying $\mu(u\sigma(u)) = 1$ for all $u \in S$. They also applied their results to stability theorems on amenable monoids. In 2022, Sharma and Chandok [25] introduced a new class of Picard operators for certain contraction mappings in $\mathscr{F}$-metric spaces and derived existence and uniqueness results. They applied these results to HU stability of FP problems, the Cauchy functional equation, and integral equations. They also discussed the well-posedness. In 2024, Keshavarz and Heydari [12] introduced a novel notion of derivations, termed the generalized Lie bracket of derivations, and established HU stability results for them. In 2025, Sarfraz et al. [24] studied Hyers–Ulam stability for norm-additive functional equations on groups by using (δ, ε)-isometries for surjective (or δ-surjective) maps $\xi : G \to X$, where G is a (possibly noncommutative) group and X a (real) Banach space. For more results, we refer to, for example, [9, 10, 15] and the references therein .

Driven by the importance of differential equations (D$\mathbb{E}$s), research on HU stability for D$\mathbb{E}$s began with Obloza [18] in 1993. In 2024, Alfwzan et al. [1] conducted a qualitative analysis of a fractional equations ($\mathbb{FE}$s) hybrid pantograph system with a p-Laplacian operator and established existence and uniqueness criteria for solutions under Caputo-type boundary conditions by using Banach's and Arzelà–Ascoli theorems. They also investigated HU stability and illustrated the results via examples. In 2025, Alzabut et al. [3] examined discrete fractional calculus and developed a discrete fractional model with Riemann–Liouville two-point boundary conditions to study various types of HU stability. El Ghazouani et al. [4] studied boundary-value problems for Caputo-type FD$\mathbb{E}$s of order $1 < q < 2$, proved the existence of mild solutions via measures of noncompactness and Darbo's FP theorem under weak conditions, and investigated generalized HU stability. The results on differentiable mappings are due to Alsina and Ger [2]. Applications to D$\mathbb{E}$s and operators were also developed. In 2024, Keshavarz et al. [14] introduced the new concept of D$\mathbb{E}$s and operators on weighted Hardy spaces, studied their boundedness, and proved HU stability results. The $\mathbb{FE}$ $\mathscr{F}(u+v) = \mathscr{F}(u) + \mathscr{F}(v)$ is now known as the additive equation. Any function satisfying the equation is called an additive mapping. Building on this concept and Jensen's equation, Kefayati et al. [11] introduced the orthogonally Jensen ρ-$\mathbb{FE}$ in 2025. The $\mathbb{FE}$ $\mathscr{Q}(u+v) + \mathscr{Q}(u-v) = 2\mathscr{Q}(u) + 2\mathscr{Q}(v)$ is known as the quadratic equation. In 2012, Eshaghi Gordji et al. [6] analyzed Ulam stability and superstability for the m-$\mathbb{FE}$s, where $m = 1, 2, 3, 4$, and determined the general solution of the m-$\mathbb{FE}$s. In 2025, Park et al. [21] proposed a system of quadratic $\mathbb{FE}$s.

The organization of this paper is as follows. In Section 2, using the notions of additive and quadratic $\mathbb{FE}$s, we introduce the new concept of the system of additive and quadratic $\mathbb{FE}$s. We define the new i-$\mathscr{F}$-hom-der, where $i \in \{1, 2\}$, on Banach algebra. We also investigate some examples of the defined concepts. In Section 3, we first solve the system of additive–quadratic $\mathbb{FE}s$. For $i = 1$, the mappings are additive, i.e., they satisfy (2.1) and for $i = 2$, the mappings are quadratic. Then, by using FP theorem, we prove HU stability of system of additive-quadratic $\mathbb{FE}s$ on BAs. In Section 4, we show that i-$\mathscr{F}$-hom-der can be stable between BAs with two control functions of Găvruta-type, Rassias-type and JMRassias-type, where $i = 1, 2$ such that $i = 1$, these functions are $\mathbb{C}$-linear $\mathscr{F}$-hom-ders and for $i = 2$, they are quadratic $\mathscr{F}$-hom-ders.

2. The Basic Notions

Let $\mathscr{X}$ and $\mathscr{Y}$ be normed spaces (NS) and $\mathfrak{A}$ be a BA. In the following definition, the system of additive–quadratic $\mathbb{FE}$s are constructed by combining the properties of additive $\mathbb{FE}$s with those of quadratic $\mathbb{FE}$s, yielding a framework that simultaneously captures linear and second-degree relations within a single set of $\mathbb{FE}$s.

Definition 2.1. *The mappings $\mathscr{F}, \mathscr{G} : \mathscr{X} \to \mathscr{Y}$ are called a system of additive–quadratic $\mathbb{FE}$s if they satisfy*

$$\begin{cases} \mathscr{F}(2u+v) + \mathscr{F}(2u-v) = 2^{i-2}[\mathscr{G}(u+v) + \mathscr{G}(u-v) + 6\mathscr{G}(u)], \\ \mathscr{G}(2u+v) + \mathscr{G}(2u-v) = 2^{i-2}[\mathscr{F}(u+v) + \mathscr{F}(u-v) + 6\mathscr{F}(u)]. \end{cases} \tag{2.1}$$

For all $u, v \in \mathscr{X}$ and $i \in \{1,2\}$: when $i = 1$, $\mathscr{F}$ and $\mathscr{G}$ satisfy (2.1) and are additive; when $i = 2$, they satisfy (2.1) and are quadratic.

The monomial $\mathfrak{J}(t_1) = wt_1^i$ ($w \neq 0, \pm 1$ is a fixed integers, $t_1 \in \mathbb{R}$ and $i = 1,2$) is a solution of the additive and quadratic $\mathbb{FE}$s. In 2019, Park et al. [20] introduced an additive s-functional inequality. Using FP and direct methods, they proved HU stability results for that inequality and for hom-derivations in complex BA. In 2025, Keshavarz et al. [13] introduced the ternary hom-multiplier on a ternary BA and proved its HU type stability via the FP method. In 2023, Paokanta et al. [19] solved the system of additive $\mathbb{FE}$s, proved the HU stability of this system in complex BA, and established the HU stability of f-hom-derivations in BA.

In the following definition, using the concept of $\mathbb{C}$-linear $\mathscr{F}$-hom-ders and quadratic $\mathscr{F}$-hom-ders, we introduce the new concept i-$\mathscr{F}$-hom-der on BAs, where $i = 1,2$ such that $i = 1$, a function is $\mathbb{C}$-linear $\mathscr{F}$-hom-ders and for $i = 2$, it is quadratic $\mathscr{F}$-hom-ders.

Definition 2.2. *Let $\mathscr{F} : \mathfrak{A} \to \mathfrak{A}$ be an i-homomorphism.*
(1) $\mathbb{C}$-linear mapping $\mathscr{G} : \mathfrak{A} \to \mathfrak{A}$ is called an 1-$\mathscr{F}$-hom-der if it satisfies the following condition:

$$\mathscr{G}_1(u)\mathscr{G}_1(v) = \mathscr{F}_1(u)\mathscr{G}_1(v) + \mathscr{G}_1(u)\mathscr{F}_1(v)$$

(2) quadratic mapping $\mathscr{G}_i : \mathfrak{A} \to \mathfrak{A}$ is called a 2-$\mathscr{F}$-hom-der if it satisfies the following condition:

$$\mathscr{G}_2(u)\mathscr{G}_2(v) = \mathscr{F}_2(u)\mathscr{G}_2(v) + \mathscr{G}_2(u)\mathscr{F}_2(v)$$

for all $u, v \in \mathfrak{A}$ and $i \in \{1,2\}$.

We now examine several examples of the concept of an i-$\mathscr{F}$-hom-der.

Example 2.3. *Let M_n denote the algebra consisting of all $n \times n$ complex matrices. Define $\mathscr{F}$ from M_n to M_n by $\mathscr{F}(u) = u^i$ and $\mathscr{G} : M_n \to M_n$ by $\mathscr{G}(u) = 2u^i$, where $i \in \{1,2\}$. Then $\mathscr{F}$ is an i-homomorphism (of the specified type) and $\mathscr{G}$ is an i-$\mathscr{F}$-hom-der.*

Example 2.4. *Let $C_0(X)$ be the complex BA of complex-valued continuous functions on a locally compact X (Hausdorff space). Define the mappings $\mathscr{F}$ and $\mathscr{G}$ from $C_0(X)$ to $C_0(X)$ by $\mathscr{F}(M) = M^i$ and $\mathscr{G}(M) = 2M^i$, with $i \in \{1,2\}$. Then $\mathscr{F}$ is an i-homomorphism and $\mathscr{G}$ is an i-$\mathscr{F}$-hom-der.*

Using the following theorem, we can prove the HU stability of the newly defined concepts on BA.

Theorem 2.5. *[16]. Let $(\mathfrak{A},d)$ be a complete generalized metric space, and assume $\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ is strictly contractive with Lipschitz constant $0<\upsilon<1$. Then, for each $u\in\mathfrak{A}$, exactly one of the following alternatives occurs: Either $d(\mathscr{G}^n u,\mathscr{G}^{n+1}u)$ are infinite for all integers $n\geq 0$; or there exists $n_0\in\mathbb{N}$ such that, $d\left(\mathscr{G}^n(u),\mathscr{G}^{n+1}(u)\right)<\infty$ for every $n>n_0$. If the second alternative holds, then*

a) *$(\mathscr{G}^n u)$ is convergent to a FP u^* of $\mathscr{G}$;*
b) *u^* is the unique FP of $\mathscr{G}$ in $\mathfrak{A}^*=\{v\in\mathfrak{A}: d(\mathscr{G}^{n_0}u,v)<\infty\}$;*
c) *ff $u^*\in\mathfrak{A}^*$, then $d(v,u^*)\leq\frac{1}{1-\upsilon}d(v,\mathscr{G}v)$.*

3. The System of i-Mappings: Solving and Stability

In this section, one always supposes that $\mathfrak{A}$ be a BA. We investigate the mappings $\mathscr{F}$ and $\mathscr{G}$ that satisfy (2.1). For $i=1$, they are additive (i.e., it satisfies (2.1)) and for $i=2$, they are quadratic.

Lemma 3.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be NS, and let $\mathscr{F}$ and $\mathscr{G}$ be mappings from $\mathscr{X}$ to $\mathscr{Y}$ satisfying equation (2.1). Then*

i) *for $i=1$, they are are additive;*
ii) *for $i=2$, they are quadratic.*

Proof. Firstly, let $\mathscr{F}$ and $\mathscr{G}$ from $\mathscr{X}$ to $\mathscr{Y}$ satisfy (2.1). Letting $u=v=0$ in (2.1), we see that $\mathscr{F}(0)=\mathscr{G}(0)=0$. In the following, we solve the case that $i=1$. If we set 0 instead of v, we obtain

$$\begin{cases}\mathscr{F}(2u)=2\mathscr{G}(u)\\ \mathscr{G}(2u)=2\mathscr{F}(u)\end{cases}\tag{3.1}$$

for all $u\in\mathscr{X}$. By using (3.1) and substituting $u=0$ into (2.1), we have $\mathscr{F}(-u)=-\mathscr{F}(u)$. The same holds for $\mathscr{G}(v)$.

In the following, we solve the case $i=2$. If we set 0 instead of v, then

$$\begin{cases}\mathscr{F}(2u)=4\mathscr{G}(u)\\ \mathscr{G}(2u)=4\mathscr{F}(u)\end{cases}\tag{3.2}$$

for all $u\in\mathscr{X}$. By using (3.2) and replacing v with $v+2v$ in (2.1), we then obtain

$$\begin{cases}\mathscr{F}(3u+2v)+\mathscr{F}(u-2v)=\mathscr{G}(2u+2v)+\mathscr{G}(-2v)+6\mathscr{G}(u),\\ \mathscr{G}(3u+2v)+\mathscr{G}(u-2v)=\mathscr{F}(2u+2v)+\mathscr{F}(-2v)+6\mathscr{F}(u),\end{cases}$$

Next, putting $u=u+v$ and $v=2u$ yields

$$\begin{cases}\mathscr{F}(5u+3v)+\mathscr{F}(-u+v)=\mathscr{G}(4u+2v)+\mathscr{G}(-2u)+6\mathscr{G}(u+v),\\ \mathscr{G}(5u+3v)+\mathscr{G}(-u+v)=\mathscr{F}(4u+2v)+\mathscr{F}(-2s)+6\mathscr{F}(u+v),\end{cases}\tag{3.3}$$

Substituting $v=u$ into (3.3), we obtain

$$\begin{cases}\mathscr{F}(8u)=\mathscr{G}(6u)+\mathscr{G}(-2u)+6\mathscr{G}(2u),\\ \mathscr{G}(8u)=\mathscr{G}(6u)+\mathscr{G}(-2u)+6\mathscr{G}(2u),\end{cases}\tag{3.4}$$

Finally, setting $u = \frac{u}{2}$ and using (3.2) in (3.4), we obtain $\mathscr{F}(-v) = \mathscr{F}(v)$. The same holds for $\mathscr{G}(v)$. This shows that $\mathscr{F}$ and $\mathscr{G}$ are additive when $i = 1$ and quadratic when $i = 2$, which completes the proof. □

Before stating the main theorem, we recall the control function of Găvruta-type for system of additive–quadratic $\mathbb{FE}$s. Letting $\gamma : \mathfrak{A}^2 \to [0,\infty)$ be a function, we assume that

$$\gamma\left(\frac{u}{2}, \frac{v}{2}\right) \leq \frac{\tau}{2^i}\gamma(u,v). \tag{3.5}$$

where the constant $0 < \tau < 1$ and $i \in \{1,2\}$. If we set $u = v = 0$ in (3.5), then $\gamma(0,0) = 0$. Using (3.5), we obtain:

$$\lim_{j\to\infty} 2^{ij}\gamma\left(\frac{u}{2^j}, \frac{v}{2^j}\right) = 0. \tag{3.6}$$

Building on the preceding concepts, the next theorem studies the HU stability of system of additive–quadratic $\mathbb{FE}s$ on BAs, with a Găvruta-type control function, using the FP method.

Theorem 3.2. *Let the mapping $\gamma : \mathfrak{A}^2 \to [0,\infty)$ satisfy (3.5), and let $\mathscr{F}, \mathscr{G} : \mathfrak{A} \to \mathfrak{A}$ be two mappings such that*

$$\begin{cases} \|\mathscr{F}(2u+v) + \mathscr{F}(2u-v) - 2^{i-2}[\mathscr{G}(u+v) + \mathscr{G}(u-v) + 6\mathscr{G}(u)]\| \leq \gamma(u,v), \\ \\ \|\mathscr{G}(2u+v) + \mathscr{G}(2u-v) - 2^{i-2}[\mathscr{F}(u+v) + \mathscr{F}(u-v) + 6\mathscr{F}(u)]\| \leq \gamma(u,v). \end{cases} \tag{3.7}$$

Then, there exist unique $i-$mappings $\mathscr{T}, \mathscr{J} : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|\mathscr{T}(u) - \mathscr{F}(u)\| \leq \tfrac{\tau}{2^{i+1}(1-\tau)}\gamma(u,0) \quad \text{and} \quad \|\mathscr{J}(u) - \mathscr{G}(u)\| \leq \tfrac{\tau}{2^{i+1}(1-\tau)}\gamma(u,0), \tag{3.8}$$

Proof. Firstly, putting 0 instead of u and v in (3.7), we obtain

$$\begin{cases} \|\mathscr{F}(0) + \mathscr{F}(0) - 2^{i-2}[\mathscr{G}(0) + \mathscr{G}(0) + 6\mathscr{G}(0)]\| \leq \gamma(0,0), \\ \\ \|\mathscr{G}(0) + \mathscr{G}(0) - 2^{i-2}[\mathscr{F}(0) + \mathscr{F}(0) + 6\mathscr{F}(0)]\| \leq \gamma(0,0), \end{cases}$$

so $\mathscr{F}(0) = \mathscr{G}(0) = 0$. In the following, putting $v = 0$ into (3.7), we obtain

$$\begin{cases} \|2\mathscr{F}(2u) - 2^{i+1}\mathscr{G}(u)\| \leq \gamma(u,0), \\ \\ \|2\mathscr{G}(2u) - 2^{i+1}\mathscr{F}(u)\| \leq \gamma(u,0), \end{cases}$$

that is,

$$\begin{cases} \left\|\mathscr{F}(u) - 2^i\mathscr{F}(\tfrac{u}{2})\right\| \leq \tfrac{1}{2}\gamma\left(\tfrac{u}{2},0\right), \\ \\ \left\|\mathscr{G}(u) - 2^i\mathscr{G}(\tfrac{u}{2})\right\| \leq \tfrac{1}{2}\gamma\left(\tfrac{u}{2},0\right) \end{cases} \tag{3.9}$$

Let $\Psi := \{\mathscr{F} : \mathfrak{A} \to \mathfrak{A} \mid \mathscr{F}(0) = 0\}$. Define a metric d on Ψ by, for $\mathscr{F}, \mathscr{G} \in \Psi$,

$$d(\mathscr{F},\mathscr{G}) = \inf\left\{\kappa \in \mathbb{R}_+ : \ \|\mathscr{F}(u) - \mathscr{G}(u)\| \leq \kappa\gamma(u,0) \ \ \forall u \in \mathfrak{A}\right\}.$$

It is straightforward to verify that (Ψ, d) forms a generalized metric space (see [17] for further details). We then consider i-mappings $\Phi : \Psi \to \Psi$ that satisfy

$$\Phi\mathscr{F}(u) = 2^i\mathscr{F}\left(\frac{u}{2}\right), \quad \forall\, u \in \mathfrak{A}.$$

Let $\mathscr{F},\mathscr{G} \in (\Psi,d)$ be given with $d(\mathscr{F},\mathscr{G}) = \kappa$. Then $\|\mathscr{F}(u)-\mathscr{G}(u)\| \leq \varepsilon\omega(u,0)$. It follows that

$$\|\Phi\mathscr{F}(u)-\Phi\mathscr{G}(u)\| = \left\|2^i\mathscr{F}\left(\frac{u}{2}\right)-2^i\mathscr{G}\left(\frac{u}{2}\right)\right\| \leq 2^i\varepsilon\gamma\left(\frac{u}{2},0\right) \leq \tau\varepsilon\gamma(u,0)$$

for all $u \in \mathfrak{A}$ and $d(\Phi\mathscr{F},\Phi\mathscr{G}) \leq \tau\varepsilon$. This means that

$$d(\Phi\mathscr{F},\Phi\mathscr{G})) \leq \tau d(\mathscr{F},\mathscr{G}), \quad \forall\, \mathscr{F},\mathscr{G} \in \Psi.$$

It follows from (3.9) that

$$d(\mathscr{F},\Phi\mathscr{F}) \leq \tfrac{\tau}{2^{i+1}(1-\tau)}\gamma_i(u,0) \text{ and } d(\mathscr{G},\Phi\mathscr{G}) \leq \tfrac{\tau}{2^{i+1}(1-\tau)}\gamma_i(u,0).$$

Applying the FP alternative produces a unique FP of Φ. Then there exist mappings $\mathscr{F},\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ such that

$$\mathscr{F}(u) = 2^i\mathscr{F}\left(\frac{u}{2}\right) \qquad \text{and} \qquad \mathscr{G}(u) = 2^i\mathscr{G}\left(\frac{u}{2}\right)$$

for all $u \in \mathfrak{A}$ and $i \in \{1,2\}$. There exist positive real numbers α and δ such that

$$\|\mathscr{F}(u)-\mathscr{T}(u)\| \leq \alpha\gamma(u,0) \quad \text{and} \quad \|\mathscr{G}(u)-\mathscr{J}(u)\| \leq \delta\gamma(u,0).$$

$d(\Phi^j\mathscr{F},\mathscr{T}) \to 0$ and $d(\Phi^j\mathscr{G},\mathscr{J}) \to 0$ as $j\to\infty$. This implies

$$\lim_{j\to\infty} 2^{ij}\mathscr{F}\left(\frac{u}{2^j}\right) = \mathscr{T}(u) \quad \text{and} \quad \lim_{j\to\infty} 2^{ij}\mathscr{G}\left(\frac{u}{2^j}\right) = \mathscr{J}(u), \tag{3.10}$$

where $i \in \{1,2\}$ and for all $u \in \mathfrak{A}$. $d(\mathscr{F},\mathscr{T}) \leq \frac{1}{1-\tau}d(\mathscr{F},\Psi\mathscr{F})$ and $d(\mathscr{G},\mathscr{J}) \leq \frac{1}{1-\tau}d(\mathscr{G},\Psi\mathscr{G})$. Thus

$$\|\mathscr{F}(u)-\mathscr{T}(u)\| \leq \tfrac{1}{2^{i+1}(\tau^{-1}-1)}\gamma_i(u,0) \quad \text{and} \quad \|\mathscr{G}(u)-\mathscr{J}(u)\| \leq \tfrac{1}{2^{i+1}(\tau^{-1}-1)}\gamma_i(u,0),$$

From (3.6), (3.7) and (3.10), we deduce that

$$\begin{aligned}
&\left\|\mathscr{T}(2u+v)+\mathscr{T}(2u-v)-2^{i-2}[\mathscr{J}(u+v)+\mathscr{J}(u-v)+6\mathscr{J}(u)]\right\| \\
&= \lim_{J\to\infty} 2^{ij}\left(\left\|\mathscr{F}(\frac{2u+v}{2^j})+\mathscr{F}(\frac{2u-v}{2^j})-2^{i-2}\left[\mathscr{G}(\frac{u+v}{2^j})+\mathscr{G}(\frac{u-v}{2^j})+6\mathscr{G}(\frac{u}{2^j})\right]\right\|\right) \\
&= \lim_{j\to\infty} 2^{ij}\gamma\left(\frac{u}{2^j},\frac{v}{2^j}\right) = 0
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left\|\mathscr{J}(2u+v)+\mathscr{J}(2u-v)-2^{i-2}[\mathscr{T}(u+v)+\mathscr{T}(u-v)+6\mathscr{T}(u)]\right\| \\
&= \lim_{J\to\infty} 2^{ij}\left(\left\|\mathscr{G}(\frac{2u+v}{2^j})+\mathscr{G}(\frac{2u-v}{2^j})-2^{i-2}\left[\mathscr{F}(\frac{u+v}{2^j})+\mathscr{F}(\frac{u-v}{2^j})+6\mathscr{F}(\frac{u}{2^j})\right]\right\|\right) \\
&= \lim_{j\to\infty} 2^{ij}\gamma\left(\frac{u}{2^j},\frac{v}{2^j}\right) = 0
\end{aligned}$$

for all $u,v \in \mathfrak{A}$ and $i \in \{1,2\}$. Thus we obtain

$$\begin{cases}
\mathscr{T}(2u+v)+\mathscr{T}(2u-v) = 2^{i-2}[\mathscr{J}(u+v)+\mathscr{J}(u-v)+6\mathscr{J}(u)], \\[2ex]
\mathscr{J}(2u+v)+\mathscr{J}(2u-v) = 2^{i-2}[\mathscr{T}(u+v)+\mathscr{T}(u-v)+6\mathscr{T}(u)],
\end{cases}$$

for all $u,v\in\mathfrak{A}$ and $i\in\{1,2\}$. According to Lemma 3.1, the mappings $\mathscr{T}$ and $\mathscr{J}$ from $\mathfrak{A}$ to $\mathfrak{A}$ are $i-$mappings. This shows that $\mathscr{F}$ and $\mathscr{G}$ are additive when $i=1$ and quadratic when $i=2$, which completes the proof. □

In the next section, we examine whether Theorem 3.2 yields HU stability for system of additive–quadratic $\mathbb{FE}$s under a Rassias-type control function. Specifically, we replace $\gamma(u,v)$ with $\mu(\|u\|^s+\|v\|^s)$, where $s>i$ and μ is a nonnegative real constant.

Corollary 3.3. *If the mappings $\mathscr{F},\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ satisfy*

$$\begin{cases} \|\mathscr{F}(2u+v)+\mathscr{F}(2u-v)-2^{i-2}[\mathscr{G}(u+v)+\mathscr{G}(u-v)+6\mathscr{G}(u)]\|\leq\mu(\|u\|^s+\|v\|^s),\\ \\ \|\mathscr{G}(2u+v)+\mathscr{G}(2u-v)-2^{i-2}[\mathscr{F}(u+v)+\mathscr{F}(u-v)+6\mathscr{F}(u)]\|\leq\mu(\|u\|^s+\|v\|^s)\end{cases} \tag{3.11}$$

then there exist mappings $\mathscr{T},\mathscr{J}:\mathfrak{A}\to\mathfrak{A}$, and one of them is the unique i-mapping such that

$$\|\mathscr{F}(u)-\mathscr{T}(u)\|\leq\frac{\mu}{2(2^s-2^i)}\|u\|^s,\ \|\mathscr{G}(u)-\mathscr{J}(u)\|\leq\frac{\mu}{2(2^s-2^i)}\|u\|^s.$$

Proof. Put $\tau=2^{i-s}$. Using the proof of Theorem 3.2, we obtain the proof immediately. □

In the following, we investigate stability of system of additive-quadratic $\mathbb{FE}s$ with JMRassias-type control function on BAs.

Remark 3.4. *If the mappings $\mathscr{F},\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ satisfy*

$$\begin{cases} \|\mathscr{F}(2u+v)+\mathscr{F}(2u-v)-2^{i-2}[\mathscr{G}(u+v)+\mathscr{G}(u-v)+6\mathscr{G}(u)]\|\leq\mu(\|u\|^s.\|v\|^s),\\ \\ \|\mathscr{G}(2u+v)+\mathscr{G}(2u-v)-2^{i-2}[\mathscr{F}(u+v)+\mathscr{F}(u-v)+6\mathscr{F}(u)]\|\leq\mu(\|u\|^s.\|v\|^s),\end{cases} \tag{3.12}$$

then there exist mappings $\mathscr{T},\mathscr{J}:\mathfrak{A}\to\mathfrak{A}$ which are the unique i-mappings satisfying:

$$\|\mathscr{F}(u)-\mathscr{T}(u)\|\leq\frac{\mu}{2(2^s-2^i)}\|u\|^s,\ \|\mathscr{G}(u)-\mathscr{J}(u)\|\leq\frac{\mu}{2(2^s-2^i)}\|u\|^s$$

where $0\leq s<\frac{1}{2}$, for all $u\in\mathfrak{A}$ and $i\in\{1,2\}$.

4. i-$\mathscr{F}$-HOM-DER: HU STABILITY

In this section, one supposes that $\mathfrak{A}$ always is a BA. In the main results, applying Theorem 3.2, we study the stability of the i-$\mathscr{F}$-hom-der with a Găvruta-type control function between BA.

Theorem 4.1. *Let $\sigma:\mathfrak{A}^2\to[0,\infty)$, for some $\tau<1$, satisfy*

$$\sigma\left(\frac{u}{2},\frac{v}{2}\right)\leq\frac{\tau}{2^{2i}}\sigma(u,v), \tag{4.1}$$

where $i=1,2$. Let mappings $\mathscr{F}_i,\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ satisfy (3.7) and

$$\|\mathscr{F}_i(uv)-\mathscr{F}_i(u)\mathscr{F}_i(v)\|\leq\sigma(u,v), \tag{4.2}$$

$$\|\mathscr{G}_i(u)\mathscr{G}_i(v)-\mathscr{F}_i(u)\mathscr{G}_i(v)+\mathscr{G}_i(u)\mathscr{F}_i(v)\|\leq\sigma(u,v) \tag{4.3}$$

Then there exist unique mappings $\mathscr{H}_i,\mathscr{K}_i:\mathfrak{A}\to\mathfrak{A}$ where $i=1,2$ satisfying (3.8) and $\mathscr{H}_i$ are $i-$homomorphism and $\mathscr{K}_i$ is a i-$\mathscr{F}$-hom-der.

Proof. First, by setting $u=v=0$ in (4.1), we see $\sigma(0,0)=0$. Moreover, from (4.1), we deduce, for each $j\in\mathbb{N}$, that

$$\lim_{j\to\infty} 2^{2ij}\sigma\left(\frac{u}{2^j},\frac{v}{2^j}\right)=0.$$

By repeating the argument used in the proof of Theorem 3.2, we defin the mappings $\mathscr{H}_i,\mathscr{K}_i:\mathfrak{A}\to\mathfrak{A}$ for $i=1,2$ as follows

$$\mathscr{H}_i(u)=\lim_{j\to\infty} 2^{ij}\mathscr{F}_i\left(\frac{u}{2^j}\right),\ \text{and}\ \mathscr{K}_i(u)=\lim_{j\to\infty} 2^{ij}\mathscr{G}_i\left(\frac{u}{2^j}\right) \tag{4.4}$$

Using these definitions together with (4.2), (4.3) and (4.4) yields the desired subsequent estimates

$$\begin{aligned}&\|\mathscr{H}_i(uv)-\mathscr{H}_i(u)\mathscr{H}_i(v)\|\\ &=\lim_{j\to\infty} 2^{2ij}\left\|\mathscr{F}_i\left(\frac{u}{2^j}\frac{v}{2^j}\right)-\mathscr{F}_i\left(\frac{u}{2^j}\right)\mathscr{F}_i\left(\frac{v}{2^j}\right)\right\|\\ &\le \lim_{j\to\infty} 2^{2ij}\sigma\left(\frac{u}{2^j},\frac{v}{2^j}\right)=0\end{aligned}$$

Thus $\mathscr{H}_i(uv)=\mathscr{H}_i(u)\mathscr{H}_i(v)$. For $\mathscr{K}_i$, we have

$$\begin{aligned}&\|\mathscr{K}_i(u)\mathscr{K}_i(v)-\mathscr{H}_i(u)\mathscr{K}_i(v)-\mathscr{H}_i(u)\mathscr{K}_i(v)\|\\ &=\lim_{j\to\infty} 2^{2ij}\left\|\mathscr{G}_i\left(\frac{u}{2^j}\right)\mathscr{G}_i\left(\frac{v}{2^j}\right)-\mathscr{F}_i\left(\frac{u}{2^j}\right)\mathscr{G}_i\left(\frac{v}{2^j}\right)-\mathscr{G}_i\left(\frac{u}{2^j}\right)\mathscr{F}_i\left(\frac{v}{2^j}\right)\right\|\\ &\le \lim_{j\to\infty} 2^{2ij}\sigma\left(\frac{u}{2^j},\frac{v}{2^j}\right)=0.\end{aligned}$$

So $\mathscr{K}_i(u)\mathscr{K}_i(v)=\mathscr{H}_i(u)\mathscr{K}_i(v)+\mathscr{H}_i(u)\mathscr{K}_i(v)$. Thus the mapping $\mathscr{K}_i$ is a i-$\mathscr{F}$-hom-der on $\mathfrak{A}$. □

We investigate HU stability with a Rassias-type control function for the i-$\mathscr{F}$-hom-der by using the FP theorem. By appealing to Theorem 4.1, it is sufficient to substitute the Rassias-type control function where $s>i$ and $\mu\ge 0$.

Corollary 4.2. *Let the mappings $\mathscr{F}_i,\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ for $i=1,2$ satisfy equation (3.11),*

$$\|\mathscr{F}_i(uv)-\mathscr{F}_i(u)\mathscr{F}_i(v)\|\le\mu(\|u\|^s+\|v\|^s)$$

and

$$\left\|\mathscr{G}_i(u)\mathscr{G}_i(v)-\mathscr{F}_i(u)\mathscr{G}_i(v)+\mathscr{G}_i(u)\mathscr{F}_i(v)\right\|\le\mu(\|u\|^s+\|v\|^s)$$

for all $u,v\in\mathfrak{A}$. Then there exist unique mappings $\mathscr{H}_i,\mathscr{K}_i:\mathfrak{A}\to\mathfrak{A}$, where $i=1,2$ satisfying (3.8) *and $\mathscr{H}_i$ are i$-$homomorphism and $\mathscr{K}_i$ is a i-$\mathscr{F}$-hom-der.*

In the following remark, we investigate stability of i-$\mathscr{F}$-hom-der with JMRassias-type control function on BAs.

Remark 4.3. *If a mapping $\mathscr{F},\mathscr{G}:\mathfrak{A}\to\mathfrak{A}$ satisfies* (3.12),

$$\|\mathscr{F}_i(uv)-\mathscr{F}_i(u)\mathscr{F}_i(v)\|\le\mu(\|u\|^s.\|v\|^s)$$

and

$$\left\|\mathscr{G}_i(u)\mathscr{G}_i(v)-\mathscr{F}_i(u)\mathscr{G}_i(v)+\mathscr{G}_i(u)\mathscr{F}_i(v)\right\|\le\mu(\|u\|^s.\|v\|^s)$$

where $i=1,2$ and for all $u,v\in\mathfrak{A}$, then there exist unique mappings $\mathscr{H}_i,\mathscr{K}_i:\mathfrak{A}\to\mathfrak{A}$, where $i=1,2$, satisfying (3.8) *and $\mathscr{H}_i$ are i$-$homomorphisms and $\mathscr{K}_i$ is a i-$\mathscr{F}$-hom-der.*

5. Conclusions

We introduced a class of additive–quadratic $\mathbb{FE}s$ on BAs and the notion of i-$\mathscr{F}$-hom-der: for $i = 1$, $\mathscr{F}$ is a linear homomorphism; for $i = 2$, $\mathscr{F}$ is a quadratic homomorphism. We solved some examples about the defined concepts. Based on the FP method with Găvruta-type, Rassias-type and JMRassias-type control functions, we proved HU stability for these equation system and their i-$\mathscr{F}$-hom-der.

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