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OPTIMAL A PRIORI $L^\infty(J;L^2(\Omega))$-NORM ERROR ESTIMATES OF A FINITE VOLUME ELEMENT METHOD FOR PSEUDO-PARABOLIC OPTIMAL CONTROL PROBLEMS

HUIYING HOU, WEIBO GUO, ZIAN WANG, TIANLIANG HOU*

School of Mathematics and Statistics, Beihua University, Jilin 132013, China

Abstract. This paper presents a priori error analysis of finite volume element (FVE) method for linear pseudo-parabolic optimal control problems subject to integral control constraints. A novel FVE scheme is constructed based on the discretize-then-optimize approach, in which the state and co-state variables are approximated by continuous piecewise linear finite elements, while the control variable is discretized by using piecewise constant functions. Optimal a priori error estimates in the $L^\infty(J;L^2(\Omega))$-norm for the all variables are proved.

Keywords. Error estimate; Finite volume; Optimal control; Pseudo-parabolic equation.

1. Introduction

Finite element approximation plays an important role in the numerical solutions of optimal control problems (OCPs). Extensive theoretical and numerical studies were carried out on the finite element approximation of various types of OCPs. It is not even possible to give a very brief overview here. For a priori error estimates, superconvergence and a posteriori error estimates of finite element approximations for OCPs, we refer to [1–3, 5, 10, 11, 16–19]. In particular, Chen et al. [2, 3, 5] provided significant contributions to the investigation of a priori error estimates and superconvergence properties of Raviart-Thomas mixed finite element methods within this framework. Chen and Dai [4] examined superconvergence properties for semilinear elliptic OCPs discretized by the linear finite element method.

The FVE method is an effective numerical method for solving partial differential equations. A key advantage of this method is that it can maintain the local physical conservation law. Now it has been used to calculate various OCPs. Luo et al. [13, 14] used the FVE method to solve the distributed OCPs controlled by elliptic and hyperbolic equations respectively, and obtained optimal a priori error estimates of all the variables involved. Ge and Sun [8] proposed a hybrid

*Corresponding author.
E-mail address: 270854140@qq.com (T. Hou).
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approximation scheme for the OCP governed by elliptic equations with random coefficients. The finite volume element is used for spatial discretization, and the sparse grid random configuration method based on Smolyak approximation is used for probability space approximation. Lin et al. [15] developed the Fourier FVE method to deal with distributed control and Dirichlet boundary control problems. Fourier discretization was used in the azimuth direction of polar coordinates, and FVE approximation was used in the radial direction. Kumar et al. [9] proposed a series of discrete schemes for OCPs of immiscible displacement equation modeling in porous media under the framework of optimize-then-discretize, combined with mixed and discontinuous finite volume element methods. We note that all the above studies on the FVE method for OCPs adopt the method of optimize-then-discretize to construct the discreteization schemes.

The main purpose of this paper is to construct the FVE approximation of the following pseudo-parabolic OCP by the discretize-then-optimize method:

$$\min_{q\in Q_{ad}} \frac{1}{2}\Big(\|y-y_d\|^2_{L^2(J;L^2(\Omega))}+\|q\|^2_{L^2(J;L^2(\Omega))}\Big), \tag{1.1}$$

$$y_t-\operatorname{div}\big(A(x)\nabla y_t+A(x)\nabla y\big)+c(x)y=f+q,\ x\in\Omega,\ t\in J, \tag{1.2}$$

$$y(x,t)\big|_{\partial\Omega}=0,\ t\in J, \tag{1.3}$$

$$y(x,0)=y_0(x),\ x\in\Omega, \tag{1.4}$$

where $\Omega\subset\mathbb{R}^2$ is a convex polygonal domain, $J=(0,T]$ and

$$Q^{ad}=\left\{q\in L^2(J;L^2(\Omega)):\int_0^T\int_\Omega q\,dxdt\geq 0\right\}.$$

Assume that $y_d\in L^2(J;H^2(\Omega))$, $y_0\in H^2(\Omega)$, $0\leq c(x)\in W^{1,\infty}(\Omega)$. Let the matrix-valued function $A(x)=(a_{ij}(x))$ be symmetric, satisfying $a_{ij}(x)\in W^{1,\infty}(\Omega)$ and

$$c_*|\zeta|^2\leq\sum_{i,j=1}^{2}a_{ij}(x)\zeta_i\zeta_j\leq c^*|\zeta|^2,\ \forall\,\zeta\in\mathbb{R}^2,\ x\in\bar{\Omega},$$

where $c^*>c_*>0$.

The paper proceeds as follows. In Section 2, we present a novel FVE approximation for the OCP (1.1)-(1.4) and derive its equivalent optimality conditions. Section 3 discusses the $L^\infty(L^2)$-norm errors with optimal convergence order for all variables. Finally, in the last section, we summarize the results obtained and discusses potential directions for future research.

In this paper, we denote $W^{m,k}(\Omega)$ as the standard Sobolev space, with its norm $\|\cdot\|_{m,k}$ defined by $\|v\|^k_{m,k}=\sum_{|\alpha|\leq m}\|D^\alpha v\|^k_{L^k(\Omega)}$ and the semi-norm $|\cdot|_{m,k}$ defined by $|v|^k_{m,k}=\sum_{|\alpha|=m}\|D^\alpha v\|^k_{L^k(\Omega)}$. We further define $W^{m,k}_0(\Omega)=\{v\in W^{m,k}(\Omega):v|_{\partial\Omega}=0\}$. For $k=2$, we write $H^m(\Omega)=W^{m,2}(\Omega)$, $H^m_0(\Omega)=W^{m,2}_0(\Omega)$, and simplify the notations as $\|\cdot\|_m=\|\cdot\|_{m,2}$ and $\|\cdot\|=\|\cdot\|_{0,2}$. $L^s(J;W^{m,k}(\Omega))$ denotes the Banach space consisting of all L^s-integrable from J into $W^{m,k}(\Omega)$, with the corresponding norm defined by $\|v\|_{L^s(J;W^{m,k}(\Omega))}=\left(\int_0^T\|v\|^s_{W^{m,k}(\Omega)}\,dt\right)^{1/s}$, where $s\in[1,+\infty)$, for $s=+\infty$, the norm is defined in the standard manner. For simplicity, in the following, we use $\|v\|_{L^s(W^{m,k})}$ to represent $\|v\|_{L^s(J;W^{m,k}(\Omega))}$. Spaces such as $H^1(W^{m,k})$ and $C^k(W^{m,k})$ can be defined analogously.Throughout this paper, C denotes a generic positive constant independent of the mesh size h, the spatial discretization parameter.

2. FVE Approximation

This section constructs FVE scheme for problem (1.1)-(1.4) and presents several necessary lemmas. Let $(L^2(\Omega))^2 := \{v = (v_1, v_2) | v_i \in L^2(\Omega), i = 1, 2\}$.

The weak form of (1.1)-(1.4) is to find $(y, q) \in L^2(H^1) \times Q^{ad}$ such that

$$\min_{q\in Q^{ad}} \frac{1}{2}\left(\|y-y_d\|^2_{L^2(L^2)} + \|q\|^2_{L^2(L^2)}\right), \tag{2.1}$$

$$(y_t, v) + (A\nabla y_t, \nabla v) + (A\nabla y, \nabla v) + (cy, v) = (f+q, v),\ \forall\, v \in H^1_0(\Omega), \tag{2.2}$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$.

From [12], it is known that the OCP (2.1)-(2.2) has a unique solution $(y, q) \in L^2(H^1) \times Q^{ad}$. Moreover, for any $v \in H^1_0(\Omega)$, $x \in \Omega$, $\dot{q} \in Q^{ad}$, there exists a co-state $p \in L^2(H^1)$ such that (y, p, q) satisfy

$$\big(y_t, v\big) + \big(A\nabla y_t, \nabla v\big) + \big(A\nabla y, \nabla v\big) + \big(cy, v\big) = \big(f+q, v\big), \tag{2.3}$$

$$y(x,0) = y_0(x), \tag{2.4}$$

$$-\big(p_t, v\big) - \big(A\nabla p_t, \nabla v\big) + \big(A\nabla p, \nabla v\big) + \big(cp, v\big) = \big(y-y_d, v\big), \tag{2.5}$$

$$p(x,T) = 0, \tag{2.6}$$

$$\int_0^T (q+p, \dot{q}-q)\, dt \geq 0. \tag{2.7}$$

From (2.7), we obtain

$$q = \max\{0, \bar{p}\} - p, \tag{2.8}$$

where $\bar{p} = \frac{\int_0^T \int_\Omega p dxdt}{\int_0^T \int_\Omega 1 dxdt}$.

For a polygonal domain Ω, we consider a quasi-uniform and regular triangulation T_h consisting of closed triangular elements K satisfying $\bar{\Omega} = \bigcup_{K\in T_h} K$. Let $\mathscr{N}_h$ denote the set of all nodes (vertices) of T_h, and let the set of interior nodes be defined as $\mathscr{N}_h^0 = \mathscr{N}_h \cap \Omega$.

Next, based on T_h, we construct the dual mesh T_h^*. For each element $K \in T_h$ with vertices x_i, x_j, x_k, we select a point Q inside K, and select a point x_{ij} on each edge $\overline{x_i x_j}$ of K. We link Q to each x_{ij} by straight-line segments r_{ij}. For a given vertex x_i, we define V_i as the polygon formed by the union of all segments r_{ij} for which x_i is a vertex of K. The region V_i is called the control volume centered at x_i. We have $\bar{\Omega} = \bigcup_{x_i\in\mathscr{N}_h} V_i$, and the dual mesh T_h^* is defined as the set of all such control volumes.

If there exists a positive constant $C > 0$ satisfying the following conditions, then the control volume mesh T_h^* is quasi-uniform and regular.

$$C^{-1}h^2 \leq meas(V_i) \leq Ch^2,\ \forall\, V_i \in T_h^*,$$

where h denotes the maximum diameter of all elements $K \in T_h$.

Multiple ways may be utilized to construct a regular dual mesh T_h^*, which depends on the choice of an interior point Q in each element $K \in T_h$ and the selection of points x_{ij} on the edges of K. In this study, we employ a commonly adopted construction where Q is taken as the barycenter of each triangular element $K \in T_h$, and the points x_{ij} are chosen to be the midpoints of the edges of K. This control volume construction is applicable to any triangulation T_h and enables relatively simple computational implementations. Furthermore if T_h is locally regular, then the corresponding dual mesh T_h^* is also locally regular. Let S_h stand for the standard piecewise linear finite element space defined on the triangulation T_h:

$$S_h = \{v \in C(\Omega) : v|_K \text{ is linear}, \ \forall\, K \in T_h; v|_{\partial\Omega} = 0\},$$

and its dual volume element space S_h^* on T_h^*,

$$S_h^* = \left\{v \in L^2(\Omega) : v|_{V_i} \text{ is constant}, \ \forall\, V_i \in T_h^*; v|_{V_i} = 0, \text{ if } x_i \in \partial\Omega\right\}.$$

Thus, we have $S_h = \text{span}\{\phi_i(x) : x_i \in \mathscr{N}_h^0\}$ and $S_h^* = \text{span}\{\phi_i^*(x) : x_i \in \mathscr{N}_h^0\}$, where $\phi_i(x)$ is the standard nodal basis function for node x_i, and $\phi_i^*(x)$ is the characteristic function of the control volume V_i.

The space of piecewise constant functions is defined as follows

$$W_h := \{w_h \in L^2(\Omega) : w_h|_K \text{ is constant}, \forall\, K \in T_h\}.$$

We introduce the linear interpolation operator $I_h : H_0^1(\Omega) \cap H^2(\Omega) \to S_h$, $\forall\ v \in H_0^1(\Omega) \cap H^2(\Omega)$, by $I_h v = \sum_{x_i \in \mathscr{N}_h} v(x_i)\phi_i(x)$. The approximation properties are as follows

$$\|\psi - I_h\psi\| + h\|\psi - I_h\psi\|_1 \le Ch^2\|\psi\|_2, \ \forall\ \psi \in H^2(\Omega). \tag{2.9}$$

We introduce the interpolation operator $I_h^* : S_h \to S_h^*$ by

$$I_h^* v_h = \sum_{x_i \in \mathscr{N}_h} v_h(x_i)\phi_i^*(x).$$

We introduce the $L^2(\Omega)$-projection [6] $P_h : L^2(\Omega) \to W_h$, which satisfies, for any $\psi \in L^2(\Omega)$,

$$(P_h\psi - \psi, w_h) = 0, \ \ \forall\ w_h \in W_h, \tag{2.10}$$

$$\|\psi - P_h\psi\|_{0,l} \le Ch\|\psi\|_{1,l}, \ \ 2 \le l \le \infty, \ \forall\ \psi \in W^{1,l}(\Omega). \tag{2.11}$$

We define the elliptic projection $R_h : H_0^1(\Omega) \to S_h$ [7], which satisfies: for any $\psi \in H_0^1(\Omega)$

$$(A\nabla(\psi - R_h\psi), \nabla v_h) = 0, \ \forall\ v_h \in S_h, \tag{2.12}$$

$$\|\psi - R_h\psi\| + h\|\psi - R_h\psi\|_1 \le Ch^2\|\psi\|_2, \ \forall\ \psi \in H_0^1(\Omega) \cap H^2(\Omega), \tag{2.13}$$

$$\|(\psi - R_h\psi)_t\| + h\|(\psi - R_h\psi)_t\|_1 \le Ch^2\|\psi_t\|_2, \ \forall\ \psi_t \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.14}$$

On S_h and S_h^*, we define several discrete inner products and norms

$$(v_h, p_h)_{0,h} = \sum_{x_i \in N_h} \text{meas}(V_i) v_{hi} p_{hi} = (I_h^* v_h, I_h^* p_h), \ |v_h|_{0,h}^2 = (v_h, v_h)_{0,h}, \ |||v_h|||_0^2 = (v_h, I_h^* v_h),$$

$$|v_h|_{1,h}^2 = \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \text{meas}(V_i)((v_{hi} - v_{hj})/d_{ij})^2, \ \|v_h\|_{1,h}^2 = |v_h|_{0,h}^2 + |v_h|_{1,h}^2,$$

where $d_{ij} = |x_i - x_j|$. Let $a_h(\cdot,\cdot)$ denote

$$a_h(y_h, I_h^* v_h) := -\sum_{x_i \in N_h} \int_{\partial V_i} \left(\overline{A}\nabla y_h\right) \cdot \mathbf{n} I_h^* v_h ds = -\sum_{x_i \in N_h} v_h(x_i) \int_{\partial V_i} \left(\overline{A}\nabla y_h\right) \cdot \mathbf{n} ds,$$

where **n** is the outer-normal vector. By defining a new approximation for the target functional, we propose the following FVE scheme: Find discrete solution $(y_h, q_h) \in L^2(S_h) \times Q_h^{ad}$ such that

$$\min_{q_h \in Q_h^{ad}} \left\{ \frac{1}{2} \int_0^T \left(|||y_h - I_h y_d|||_0^2 + \|q_h\|^2 \right) dt \right\}, \tag{2.15}$$

$$(y_{ht}, I_h^* v_h) + (\overline{A}\nabla y_{ht}, \nabla v_h) + (\overline{A}\nabla y_h, \nabla v_h) + (\bar{c} y_h, I_h^* v_h) = (f + q_h, I_h^* v_h), \ \forall\, v_h \in S_h, \tag{2.16}$$

where $Q_h^{ad} = L^2(W_h) \cap Q^{ad}$, $y_h(x,0) = R_h y_0(x)$, $(\overline{A}\nabla y_h, \nabla v_h) = a_h(y_h, I_h^* v_h)$ and $\overline{A}\big|_K = A_K$, $\overline{c}|_K = c_K$, $A_K = \frac{1}{\text{meas}(K)} \int_K A(x) dx$, $c_K = \frac{1}{\text{meas}(K)} \int_K c(x) dx$, $\forall\, K \in T_h$.

The solution $(y_h, q_h) \in L^2(S_h) \times Q_h^{ad}$ to discrete problem (2.15)-(2.16) is unique. When (y_h, q_h) solves problem (2.15)-(2.16), there exists a discrete co-state $p_h \in L^2(S_h)$ such that (y_h, p_h, q_h) satisfy the discrete optimality conditions:

$$\left(y_{ht}, I_h^* v_h\right) + \left(\overline{A}\nabla y_{ht}, \nabla v_h\right) + \left(\overline{A}\nabla y_h, \nabla v_h\right) + \left(\bar{c} y_h, I_h^* v_h\right) = \left(f + q_h, I_h^* v_h\right), \tag{2.17}$$

$$y_h(x,0) = R_h y_0(x), \tag{2.18}$$

$$-\left(p_{ht}, I_h^* v_h\right) - \left(\overline{A}\nabla p_{ht}, \nabla v_h\right) + \left(\overline{A}\nabla p_h, \nabla v_h\right) + \left(\bar{c} p_h, I_h^* v_h\right) = \left(y_h - I_h y_d, I_h^* v_h\right), \tag{2.19}$$

$$p_h(x,T) = 0, \tag{2.20}$$

$$\int_0^T \left(q_h + I_h^* p_h, \dot{q}_h - q_h\right) dt \geq 0, \tag{2.21}$$

where $v_h \in S_h$, $x \in \Omega$, $\dot{q}_h \in Q_h^{ad}$. By (2.21), we obtain

$$q_h = \max\left\{0, \overline{I_h^* p_h}\right\} - P_h\left(I_h^* p_h\right), \tag{2.22}$$

where $\overline{I_h^* p_h} = \frac{\int_0^T \int_\Omega I_h^* p_h dx dt}{\int_0^T \int_\Omega 1 dx dt}$. For any $\ddot{q} \in L^2(L^2)$, we define the discrete intermediate variables $(y_h(\ddot{q}), p_h(\ddot{q})) \in L^2(S_h) \times L^2(S_h)$ such that

$$\begin{aligned} &\left(y_{ht}(\ddot{q}), I_h^* v_h\right) + \left(\overline{A}\nabla y_{ht}(\ddot{q}), \nabla v_h\right) + \left(\overline{A}\nabla y_h(\ddot{q}), \nabla v_h\right) + \left(\bar{c} y_h(\ddot{q}), I_h^* v_h\right) \\ &\qquad\qquad\qquad = \left(f + \ddot{q}, I_h^* v_h\right), \ \forall\, v_h \in S_h, \end{aligned} \tag{2.23}$$

$$y_h(\ddot{q})(x,0) = R_h y_0(x), \ \forall\, x \in \Omega, \tag{2.24}$$

$$\begin{aligned} &-\left(p_{ht}(\ddot{q}), I_h^* v_h\right) - \left(\overline{A}\nabla p_{ht}(\ddot{q}), \nabla v_h\right) + \left(\overline{A}\nabla p_h(\ddot{q}), \nabla v_h\right) + \left(\bar{c} p_h(\ddot{q}), I_h^* v_h\right) \\ &\qquad\qquad\qquad = \left(y_h(\ddot{q}) - I_h y_d, I_h^* v_h\right), \ \forall\, v_h \in S_h, \end{aligned} \tag{2.25}$$

$$p_h(\ddot{q})(x,T) = 0, \ \forall\, x \in \Omega. \tag{2.26}$$

Thus the exact solution (y, p) can be further written as $(y(q), p(q))$. Similarly, the discrete solution (y_h, p_h) can be written as $(y_h(q_h), p_h(q_h))$.

According to [7], we have the following results.

Lemma 2.1. *There exists a positive constant C such that $\|v_h - I_h^* v_h\| \leq Ch\|v_h\|_{1,h}$, for all $v_h \in S_h$.*

Lemma 2.2. *For any $v_h \in S_h$, there exist positive constants $C_1 - C_5$ such that*

$$|v_h|_{0,h} = \|I_h^* v_h\| \leq C_1 \|v_h\|, \tag{2.27}$$

$$C_2 \|v_h\| \leq |||v_h|||_0 \leq C_3 \|v_h\|, \tag{2.28}$$

$$C_4 \|v_h\|_1 \leq \|v_h\|_{1,h} \leq C_5 \|v_h\|_1. \tag{2.29}$$

Lemma 2.3. *It holds that* $\int_K v_h dx = \int_K I_h^* v_h dx$, *for all* $K \in T_h$ *and* $v_h \in S_h$.

Lemma 2.4. *It holds that* $(w_h, I_h^* v_h) = (v_h, I_h^* w_h)$, *for all* $w_h, v_h \in S_h$.

3. Optimal a Priori Error Estiamtes

In this section, we first derive some superclose results between the projection of the exact solution and the discrete solution, and subsequently establish the $L^\infty(L^2)$-norm error with optimal convergence order for the state, co-state and control variables.

Lemma 3.1. *Let* y *and* p *satisfy the continuous optimality conditions* (2.3)-(2.7), *while* $(y_h(q), p_h(q))$ *denotes the discrete solution to* (2.23)-(2.26) *with the choice* $\ddot{q} = q$. *If* $y, p \in L^\infty(H^2)$, $y_t, p_t \in L^2(H^2)$, $f, q \in L^2(H^1)$, *then*

$$\|y - y_h(q)\|_{L^\infty(L^2)} + \|p - p_h(q)\|_{L^\infty(L^2)} \le Ch^2. \tag{3.1}$$

Proof. Let

$$y - y_h(q) = y - R_h y + R_h y - y_h(q) =: r_y + \xi_y$$

and

$$p - p_h(q) = p - R_h p + R_h p - p_h(q) =: r_p + \xi_p.$$

Subtract (2.23) from (2.3) and (2.25) from (2.5), respectively. With the aid of (2.12), we derive the following two error equations:

$$\begin{aligned}
&(\xi_{yt}, I_h^* v_h) + (\overline{A}\nabla \xi_{yt}, \nabla v_h) + (\overline{A}\nabla \xi_y, \nabla v_h) + (\overline{c}\xi_y, I_h^* v_h)\\
=&-(r_{yt}, I_h^* v_h) - (\overline{c} r_y, v_h) - ((c-\overline{c})y, v_h)\\
&+(f + q - y_t - \overline{c} R_h y, v_h - I_h^* v_h), \ \forall\, v_h \in S_h,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&-(\xi_{p_t}, I_h^* v_h) - (\overline{A}\nabla \xi_{p_t}, \nabla v_h) + (\overline{A}\nabla \xi_p, \nabla v_h) + (\overline{c}\xi_p, I_h^* v_h)\\
=&-(r_{pt}, I_h^* v_h) - (\overline{c} r_p, v_h) - ((c-\overline{c})p, v_h) + (y - y_h(q), I_h^* v_h)\\
&+(I_h y_d - y_d, I_h^* v_h) + (y - y_d + p_t - \overline{c} R_h p, v_h - I_h^* v_h), \ \forall\, v_h \in S_h.
\end{aligned} \tag{3.3}$$

Letting $v_h = \xi_y$ in (3.2), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|||\xi_y|||_0^2 + ||\overline{A}^{\frac{1}{2}}\nabla \xi_y||^2 \right) + \|\overline{A}^{\frac{1}{2}}\nabla \xi_y\|^2 + (\overline{c}\xi_y, I_h^* \xi y)\\
=&-(r_{yt}, I_h^* \xi_y) - (\overline{c} r_y, \xi_y) - ((c-\overline{c})y, \xi_y)\\
&+(f + q - y_t - \overline{c} R_h y, \xi_y - I_h^* \xi_y).
\end{aligned} \tag{3.4}$$

Integrating (3.4) over $[0,t]$ and combining $\xi_y(0) = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}|||\xi_y|||_0^2 + \frac{1}{2}\|\overline{A}^{\frac{1}{2}}\nabla \xi_y\|^2 + \int_0^t \|\overline{A}^{\frac{1}{2}}\nabla \xi_y\|^2 ds + \int_0^t (\overline{c}\xi_y, I_h^* \xi_y) ds\\
=&-\int_0^t (r_{yt}, I_h^* \xi_y) ds - \int_0^t (\overline{c} r_y, \xi_y) ds - \int_0^t ((c-\overline{c})y, \xi_y) ds\\
&+\int_0^t (f + q - y_t - \overline{c} R_h y, \xi_y - I_h^* \xi_y) ds.
\end{aligned} \tag{3.5}$$

By Cauchy-Schwarz inequality and (2.13), we derive

$$|(r_{yt}, I_h^* \xi_h)| \le \|r_{yt}\| \cdot \|I_h^* \xi_y\| \le Ch^2 \|y_t\|_2 \|\xi_y\| \tag{3.6}$$

and

$$|(\bar{c}r_y,\xi_y)|\le C\|r_y\|\cdot\|\xi_y\|\le Ch^2\|y\|_2\cdot\|\xi_y\|. \tag{3.7}$$

By (2.10)-(2.11), Lemma 2.1, (2.14), (2.27), (2.29), and Poincare's inequality, we obtain

$$\begin{aligned}|((c-\bar{c})y,\,\xi_y)|&\le C\|c-\bar{c}\|\cdot\|y\xi_y-P_h(y\xi_y)\|\\&\le Ch^2\|c\|_1\|y\xi_y\|_1\\&\le Ch^2\|c\|_1\|y\|_{1,\infty}\|\xi_y\|_1\\&\le Ch^2\|y\|\cdot\|\nabla\xi_y\|,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}|(f+q-y_t-\bar{c}R_hy,\xi_y-I_h^*\xi_y)|&=|(f+q-y_t-\bar{c}R_hy-P_h(f+q-y_t-\bar{c}R_hy),\xi_y-I_h^*\xi_y)|\\&\le\|f+q-y_t-\bar{c}R_hy-P_h(f+q-y_t-\bar{c}R_hy)\|\cdot\|\xi_y-I_h^*\xi_y\|\\&\le Ch\|f+q-y_t-\bar{c}R_hy\|_1\|\xi_y-I_h^*\xi_y\|\\&\le Ch^2(\|f+q-y_t\|_1+\|\bar{c}R_hy\|_1)\|\xi_y\|_{1,h}\\&\le Ch^2(\|f+q-y_t\|_1+\|y\|_2)\|\xi_y\|_1\\&\le Ch^2(\|f+q-y_t\|_1+\|y\|_2)\|\nabla\xi_y\|.\end{aligned} \tag{3.9}$$

Using (3.5)-(3.9) and Young's inequality, we see that

$$\begin{aligned}&\frac{1}{2}|||\xi_y|||_0^2+\frac{1}{2}\|\overline{A}^{\frac{1}{2}}\nabla\xi_y\|^2+\int_0^t\|\overline{A}^{\frac{1}{2}}\nabla\xi_y\|^2ds+\int_0^t(\bar{c}\xi_y,I_h^*\xi_y)ds\\&\le Ch^4\int_0^t\left(\|y_t\|_2^2+\|y\|_2^2+\|f+q\|_1^2\right)ds+C\int_0^t\|\xi_y\|^2ds+C\int_0^t\|\nabla\xi_y\|^2ds.\end{aligned} \tag{3.10}$$

For (3.10), we use Gronwall's lemma and the assumptions on A and c to further derive

$$\|\xi_y\|^2+\|\nabla\xi_y\|^2\le Ch^4\int_0^t\left(\|y_t\|_2^2+\|y\|_2^2+\|f+q\|_1^2\right)ds. \tag{3.11}$$

By (2.13) and (3.11), we find that

$$\|y-y_h(q)\|_{L^\infty(L^2)}+\|\nabla(R_hy-y_h(q))\|_{L^\infty(L^2)}\le Ch^2. \tag{3.12}$$

Setting $v_h=\xi_p$ in (3.3), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|||\xi_p|||_0^2+\|\overline{A}^{\frac{1}{2}}\nabla\xi_p\|^2)+\|\overline{A}^{\frac{1}{2}}\nabla\xi_p\|^2+(\bar{c}\xi_p,I_h^*\xi_p)\\&=-(r_{pt},I_h^*\xi_p)-(\bar{c}r_p,\xi_p)-((c-\bar{c})p,\xi_p)\\&\quad+(y-y_h(q),I_h^*\xi_p)+(I_hy_d-y_d,I_h^*\xi_p)+(y-y_d+p_t-\bar{c}R_hp,\xi_p-I_h^*\xi_p).\end{aligned} \tag{3.13}$$

Integrating (3.13) over $[t,T]$ and using the condition $\xi_p(T)=0$, we deduce

$$\begin{aligned}&\frac{1}{2}|||\xi_p|||_0^2+\frac{1}{2}\|\overline{A}^{\frac{1}{2}}\nabla\xi_p\|^2+\int_t^T\|\overline{A}^{\frac{1}{2}}\nabla\xi_p\|^2ds+\int_t^T(\bar{c}\xi_p,I_h^*\xi_p)ds\\&=-\int_t^T(r_{pt},I_h^*\xi_p)ds-\int_t^T(\bar{c}r_p,\xi_p)ds-\int_t^T((c-\bar{c})p,\xi_p)ds+\int_t^T(y-y_h(q),I_h^*\xi_p)ds\\&\quad+\int_t^T(I_hy_d-y_d,I_h^*\xi_p)ds+\int_t^T(y-y_d+p_t-\bar{c}R_hp,\xi_p-I_h^*\xi_p)ds.\end{aligned} \tag{3.14}$$

Similar to (3.6)-(3.9), we find that

$$|(r_{pt}, I_h^*\xi_p)| \leq Ch^2\|p_t\|_2\|\xi_p\|, \tag{3.15}$$

$$|(\bar{c}r_p, \xi_p)| \leq Ch^2\|p\|_2\|\xi_p\|, \tag{3.16}$$

$$|((c-\bar{c})p,\, \xi_p)| \leq Ch^2\|p\|\cdot\|\nabla\xi_p\|, \tag{3.17}$$

and

$$|(y-y_d+p_t-\bar{c}R_hp, \xi_p - I_h^*\xi_p)| \leq Ch^2\left(\|y\|_1+\|y_d\|_1+\|p_t\|_1+\|p\|_2\right)\|\nabla\xi_p\|. \tag{3.18}$$

It follows from lemma 2.2 and (2.13) that

$$\begin{aligned}|(y-y_h(q), I_h^*\xi_p)| \leq & \|r_y\|\cdot\|I_h^*\xi_p\| + \|\xi_y\|\cdot\|I_h^*\xi_p\| \\ \leq & C\|y-R_hy\|\cdot\|\xi_p\| + C\|\xi_y\|\cdot\|\xi_p\| \\ \leq & Ch^2\|y\|_2\|\xi_p\| + C\|\xi_y\|\cdot\|\xi_p\|.\end{aligned} \tag{3.19}$$

According to (2.9) and (2.27), we deduce

$$|(I_hy_d - y_d, I_h^*\xi_p)| \leq \|I_hy_d-y_d\|\cdot\|I_h^*\xi_p\| \leq Ch^2\|y_d\|_2\|\xi_p\|. \tag{3.20}$$

Substituting (3.15)-(3.20) into (3.14) and applying Young's inequality, we obtain

$$\begin{aligned}&\frac{1}{2}|||\xi_p|||_0^2 + \frac{1}{2}\|\bar{A}^{\frac{1}{2}}\nabla\xi_p\|^2 + \int_t^T \|\bar{A}^{\frac{1}{2}}\nabla\xi_p\|^2ds + \int_t^T (\bar{c}\xi_p, I_h^*\xi_p)ds \\ \leq & Ch^4\int_t^T (\|y\|_1^2+\|p_t\|_2^2+\|p\|_2^2+\|y_d\|_2^2)ds + C\int_t^T \|\xi_y\|^2ds \\ &+ C\int_t^T \|\xi_p\|^2ds + C\int_t^T \|\nabla\xi_p\|^2ds.\end{aligned} \tag{3.21}$$

Applying Gronwall's lemma to (3.21), and using (3.11), the assumptions on A and c, we deduce that

$$\|\xi_p\| + \|\nabla\xi_p\| \leq Ch^2. \tag{3.22}$$

By (2.13), (3.12), (3.22), and the triangle inequality, we complete the proof. □

Lemma 3.2. *Let $q \in L^2(H^1)$, and let $\ddot{q} = q$ and $\ddot{q} = P_hq$ in (2.23)-(2.26), respectively. Then*

$$\|y_h(q)-y_h(P_hq)\|_{L^\infty(L^2)} + \|\nabla(y_h(q)-y_h(P_hq))\|_{L^\infty(L^2)} \leq Ch^2,$$

$$\|p_h(q)-p_h(P_hq)\|_{L^\infty(L^2)} + \|\nabla(p_h(q)-p_h(P_hq))\|_{L^\infty(L^2)} \leq Ch^2.$$

Proof. By (2.23)-(2.26), we find

$$\begin{aligned}&\big(y_{ht}(q)-y_{ht}(P_hq), I_h^*v_h\big) + \big(\bar{A}\nabla\big(y_{ht}(q)-y_{ht}(P_hq)\big), \nabla v_h\big) \\ &+ \big(\bar{A}\nabla\big(y_h(q)-y_h(P_hq)\big), \nabla v_h\big) + \big(\bar{c}\big(y_h(q)-y_h(P_hq)\big), I_h^*v_h\big) \\ =&\big(q-P_hq, I_h^*v_h\big),\ \forall\, v_h \in S_h,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&-\big(p_{ht}(q)-p_{ht}(P_hq), I_h^*v_h\big) - \big(\bar{A}\nabla\big(p_{ht}(q)-p_{ht}(P_hq)\big), \nabla v_h\big) \\ &+ \big(\bar{A}\nabla\big(p_h(q)-p_h(P_hq)\big), \nabla v_h\big) + \big(\bar{c}\big(p_h(q)-p_h(P_hq)\big), I_h^*v_h\big) \\ =&\big(y_h(q)-y_h(P_hq), I_h^*v_h\big),\ \forall\, v_h \in S_h.\end{aligned} \tag{3.24}$$

For the purpose of stability analysis, we set $v_h = y_h(q) - y_h(P_h q)$ in (3.23) to get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|||y_h(q)-y_h(P_hq)|||_0^2+\frac{1}{2}\frac{d}{dt}\|\overline{A}^{\frac{1}{2}}\nabla(y_h(q)-y_h(P_hq))\|^2\\&+\|\overline{A}^{\frac{1}{2}}\nabla(y_h(q)-y_h(P_hq))\|^2+\|\overline{c}^{\frac{1}{2}}(y_h(q)-y_h(P_hq))\|^2\\=&(q-P_hq,I_h^*(y_h(q)-y_h(P_hq))).\end{aligned}\tag{3.25}$$

Integrating (3.25) over $[0,t]$ and using the condition $y_h(q)(x,0) - y_h(P_h q)(x,0) = 0$, we obtain

$$\begin{aligned}&\frac{1}{2}|||y_h(q)-y_h(P_hq)|||_0^2+\frac{1}{2}\|\overline{A}^{\frac{1}{2}}\nabla(y_h(q)-y_h(P_hq))\|^2\\&+\int_0^t\|\overline{A}^{\frac{1}{2}}\nabla(y_h(q)-y_h(P_hq))\|^2ds+\int_0^t\|\overline{c}^{\frac{1}{2}}(y_h(q)-y_h(P_hq))\|^2ds\\=&\int_0^t(q-P_hq,I_h^*(y_h(q)-y_h(P_hq)))ds.\end{aligned}\tag{3.26}$$

From (2.10)-(2.11), Lemma 2.1, Cauchy inequality, and Young's inequality, we deduce

$$\begin{aligned}(q-P_hq,I_h^*(y_h(q)-y_h(P_hq)))=&(q-P_hq,I_h^*(y_h(q)-y_h(P_hq))-(y_h(q)-y_h(P_hq)))\\&+(q-P_hq,(y_h(q)-y_h(P_hq))-P_h(y_h(q)-y_h(P_hq)))\\\le&Ch^2\|q\|_1\|y_h(q)-y_h(P_hq)\|_1\\\le&Ch^4\|q\|_1^2+C\|y_h(q)-y_h(P_hq)\|_1^2.\end{aligned}\tag{3.27}$$

By Gronwall's lemma, (3.26)-(3.27), (2.28), and the assumptions on A and c, we readily obtain

$$\|y_h(q)-y_h(P_hq)\|^2+\|\nabla(y_h(q)-y_h(P_hq))\|^2\le Ch^4\int_0^t\|q\|_1^2ds.\tag{3.28}$$

Thus

$$\|y_h(q)-y_h(P_hq)\|_{L^\infty(L^2)}+\|\nabla(y_h(q)-y_h(P_hq))\|_{L^\infty(L^2)}\le Ch^2.\tag{3.29}$$

Next, setting $v_h = p_h(q) - p_h(P_h q)$ in (3.24), we have

$$\begin{aligned}&-\frac{1}{2}\frac{d}{dt}|||p_h(q)-p_h(P_hq)|||_0^2-\frac{1}{2}\frac{d}{dt}\|\overline{A}^{\frac{1}{2}}\nabla(p_h(q)-p_h(P_hq))\|^2\\&+\|\overline{A}^{\frac{1}{2}}\nabla(p_h(q)-p_h(P_hq))\|^2+\|\overline{c}^{\frac{1}{2}}(p_h(q)-p_h(P_hq))\|^2\\=&(y_h(q)-y_h(P_hq),I_h^*(p_h(q)-p_h(P_hq))).\end{aligned}\tag{3.30}$$

Integrating (3.30) over $[t,T]$, and using $p_h(q)(x,T) - p_h(P_h q)(x,T) = 0$ as well as

$$(y_h(q)-y_h(P_hq),I_h^*(p_h(q)-p_h(P_hq)))\le C\|y_h(q)-y_h(P_hq)\|\cdot\|p_h(q)-p_h(P_hq)\|,$$

we deduce that

$$\begin{aligned}&\frac{1}{2}|||p_h(q)-p_h(P_hq)|||_0^2+\frac{1}{2}\|\overline{A}^{\frac{1}{2}}\nabla(p_h(q)-p_h(P_hq))\|^2\\&+\int_t^T\|\overline{A}^{\frac{1}{2}}\nabla(p_h(q)-p_h(P_hq))\|^2ds+\int_t^T\|\overline{c}^{\frac{1}{2}}(p_h(q)-p_h(P_hq))\|^2ds\\\le&C\int_t^T\|y_h(q)-y_h(P_hq)\|\cdot\|p_h(q)-p_h(P_hq)\|ds.\end{aligned}$$

Similar to (3.28)-(3.29), we obtain

$$\|p_h(q)-p_h(P_hq)\|_{L^\infty(L^2)}+\|\nabla(p_h(q)-p_h(P_hq))\|_{L^\infty(L^2)}\le C\|y_h(q)-y_h(P_hq)\|_{L^2(L^2)}. \quad (3.31)$$

Thus, we conclude Lemma 3.2 by (3.29) and (3.31). □

Lemma 3.3. *For the discrete solutions (y_h,p_h) and $(y_h(P_hq),p_h(P_hq))$, we have*

$$\|y_h(P_hq)-y_h\|_{L^\infty(L^2)}+\left\|\nabla\left(y_h(P_hq)-y_h\right)\right\|_{L^\infty(L^2)}\le C\,\|P_hq-q_h\|_{L^2(L^2)},$$
$$\|p_h(P_hq)-p_h\|_{L^\infty(L^2)}+\left\|\nabla\left(p_h(P_hq)-p_h\right)\right\|_{L^\infty(L^2)}\le C\,\|P_hq-q_h\|_{L^2(L^2)}.$$

Proof. By (2.17)-(2.21) and (2.23)-(2.26), we derive

$$\begin{aligned}&\left(y_{ht}(P_hq)-y_{ht},I_h^*v_h\right)+\left(\overline{A}\nabla\left(y_{ht}(P_hq)-y_{ht}\right),\nabla v_h\right)\\&+\left(\overline{A}\nabla\left(y_h(P_hq)-y_h\right),\nabla v_h\right)+\left(\overline{c}\left(y_h(P_hq)-y_h\right),I_h^*v_h\right)\\=&\left(P_hq-q_h,I_h^*v_h\right),\ \forall\ v_h\in S_h,\end{aligned} \quad (3.32)$$

$$\begin{aligned}&-\left(p_{ht}(P_hq)-p_{ht},I_h^*v_h\right)-\left(\overline{A}\nabla\left(p_{ht}(P_hq)-p_{ht}\right),\nabla v_h\right)\\&+\left(\overline{A}\nabla\left(p_h(P_hq)-p_h\right),\nabla v_h\right)+\left(\overline{c}\left(p_h(P_hq)-p_h\right),I_h^*v_h\right)\\=&\left(y_h(P_hq)-y_h,I_h^*v_h\right),\ \forall\ v_h\in S_h.\end{aligned} \quad (3.33)$$

From Lemma 2.3, we obtain $(P_hq-q_h,I_h^*v_h)=(P_hq-q_h,v_h)$, for all $v_h\in S_h$. Thus we readily obtain the expected results by the stability analysis as in Lemma 3.2. □

Theorem 3.4. *Let $y,p\in L^\infty(H^2)$, $y_t,p_t\in L^2(H^2)$, and $f,q\in L^2(H^1)$. Then $\|P_hq-q_h\|_{L^2(L^2)}\le Ch^2$.*

Proof. Choosing $\dot{q}=q_h$ and $\dot{q}_h=P_hq$ in inequalities (2.7) and (2.21), we have

$$\int_0^T(q+p,q_h-q)\,dt\ge 0,\ \int_0^T(q_h+I_h^*p_h,P_hq-q_h)dt\ge 0. \quad (3.34)$$

It follows from (3.34) and (2.10) that

$$\begin{aligned}&\|P_hq-q_h\|_{L^2(L^2)}^2\\=&\int_0^T(P_hq-q_h,P_hq-q)dt+\int_0^T(P_hq-q_h,q-q_h)dt\\=&\int_0^T(P_hq-q_h,q+p)dt-\int_0^T(P_hq-q_h,q_h+I_h^*p_h)dt+\int_0^T(P_hq-q_h,I_h^*p_h-p)dt\\\le&\int_0^T(P_hq-q_h,q+p)dt+\int_0^T(P_hq-q_h,I_h^*p_h-p)dt\\=&\int_0^T(q+p,P_hq-q_h)dt+\int_0^T(P_hq-q_h,I_h^*p_h-p)dt\\=&\int_0^T(q+p,P_hq-q)dt+\int_0^T(q+p,q-q_h)dt+\int_0^T(P_hq-q_h,I_h^*p_h-p)dt\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^T (q+p, P_hq-q)dt + \int_0^T (I_h^* p_h - p, P_hq-q_h)dt \\
&= \int_0^T (I_h^* p_h - I_h^* p_h(P_hq), P_hq-q_h)dt + \int_0^T (I_h^* p_h(P_hq) - I_h^* p_h(q), P_hq-q_h)dt \\
&\quad + \int_0^T (I_h^* p_h(q) - p, P_hq-q_h)dt + \int_0^T (q+p, P_hq-q)dt.
\end{aligned} \tag{3.35}
$$

By integrating (3.32)-(3.33) over $[0,T]$, and setting $v_h = p_h(P_hq) - p_h$ and $v_h = y_h(P_hq) - y_h$, we readily verify that

$$
\begin{aligned}
&\int_0^T \big(I_h^* p_h - I_h^* p_h(P_hq), P_hq-q_h\big)dt = -\int_0^T \big(P_hq-q_h, I_h^*\big(p_h(P_hq)-p_h\big)\big)dt \\
&= -\int_0^T \big(y_{ht}(P_hq) - y_{ht}, I_h^*\big(p_h(P_hq)-p_h\big)\big)dt - \int_0^T \big(\overline{A}\nabla\big(y_{ht}(P_hq)-y_{ht}\big), \nabla\big(p_h(P_hq)-p_h\big)\big)dt \\
&\quad - \int_0^T \big(\overline{A}\nabla\big(y_h(P_hq)-y_h\big), \nabla\big(p_h(P_hq)-p_h\big)\big)dt - \int_0^T \big(\overline{c}\big(y_h(P_hq)-y_h\big), I_h^*\big(p_h(P_hq)-p_h\big)\big)dt \\
&= -\int_0^T \frac{d}{dt}\big(y_h(P_hq)-y_h, I_h^*\big(p_h(P_hq)-p_h\big)\big)dt + \int_0^T \big(y_h(P_hq)-y_h, I_h^*\big(p_{ht}(P_hq)-p_{ht}\big)\big)dt \\
&\quad - \int_0^T \frac{d}{dt}\big(\overline{A}\nabla\big(y_h(P_hq)-y_h\big), \nabla\big(p_h(P_hq)-p_h\big)\big)dt \\
&\quad + \int_0^T \big(\overline{A}\nabla\big(y_h(P_hq)-y_h\big), \nabla\big(p_{ht}(P_hq)-p_{ht}\big)\big)dt \\
&\quad - \int_0^T \big(\overline{A}\nabla\big(y_h(P_hq)-y_h\big), \nabla\big(p_h(P_hq)-p_h\big)\big)dt - \int_0^T \big(\overline{c}\big(y_h(P_hq)-y_h\big), I_h^*\big(p_h(P_hq)-p_h\big)\big)dt \\
&= -\big(y_h(P_hq)-y_h, I_h^*\big(p_h(P_hq)-p_h\big)\big)\Big|_0^T - \big(\overline{A}\nabla\big(y_h(P_hq)-y_h\big), \nabla\big(p_h(P_hq)-p_h\big)\big)\Big|_0^T \\
&\quad + \int_0^T \big(p_{ht}(P_hq)-p_{ht}, I_h^*\big(y_h(P_hq)-y_h\big)\big)dt + \int_0^T \big(\overline{A}\nabla\big(p_{ht}(P_hq)-p_{ht}\big), \nabla\big(y_h(P_hq)-y_h\big)\big)dt \\
&\quad - \int_0^T \big(\overline{A}\nabla\big(p_h(P_hq)-p_h\big), \nabla\big(y_h(P_hq)-y_h\big)\big)dt - \int_0^T \big(\overline{c}\big(p_h(P_hq)-p_h\big), I_h^*\big(y_h(P_hq)-y_h\big)\big)dt \\
&= -\int_0^T \big(y_h(P_hq)-y_h, I_h^*\big(y_h(P_hq)-y_h\big)\big)dt \\
&\leq -C_2 \int_0^T \|y_h(P_hq)-y_h\|^2 dt \leq 0,
\end{aligned} \tag{3.36}
$$

where we also used $y_h(P_hq)(0) = y_h(0)$, $p_h(P_hq)(T) = p_h(T) = 0$, Lemma 2.4, and (2.28). By Lemma 2.3, Lemma 3.2, Cauchy inequality, and (3.1), we have

$$
\begin{aligned}
\int_0^T (I_h^* p_h(P_hq) - I_h^* p_h(q), P_hq-q_h)\,dt &= \int_0^T (p_h(P_hq) - p_h(q), P_hq-q_h)\,dt \\
&\leq \int_0^T \|p_h(q) - p_h(P_hq)\| \cdot \|P_hq-q_h\|\,dt \\
&\leq Ch^2 \int_0^T \|P_hq-q_h\|\,dt
\end{aligned} \tag{3.37}
$$

and

$$\begin{aligned}\int_0^T (I_h^* p_h(q)-p, P_h q-q_h)\,dt &= \int_0^T (p_h(q)-p, P_h q-q_h)\,dt \\ &\leq \int_0^T \|p-p_h(q)\|\cdot\|P_h q-q_h\| dt \\ &\leq Ch^2\int_0^T \|P_h q-q_h\|\,dt.\end{aligned} \tag{3.38}$$

By (2.8) and (2.10), we easily have

$$\int_0^T (q+p, P_h q-q)\,dt = \int_0^T (\max\{0,\overline{p}\}, P_h q-q)\,dt = 0. \tag{3.39}$$

Substituting (3.36)-(3.39) into (3.35), one sees that $\|P_h q-q_h\|_{L^2(L^2)} \leq Ch^2$. Thus the proof is complete. □

Next, we can further obtain the following optimal $L^\infty(L^2)$-norm error estimates.

Theorem 3.5. *If $y,p\in L^\infty(H^2)$, $y_t,p_t\in L^2(H^2)$, and $f,q\in L^2(H^1)$, then*

$$\|q-q_h\|_{L^\infty(L^2)} \leq Ch, \tag{3.40}$$

$$\|y-y_h\|_{L^\infty(L^2)}+\|p-p_h\|_{L^\infty(L^2)} \leq Ch^2. \tag{3.41}$$

Proof. By (2.8) and (2.22), we deduce that

$$\begin{aligned}|q-q_h| &= |\max\{0,\bar{p}\}-p-\max\{0,\overline{I_h^* p_h}\}+P_h(I_h^* p_h)| \\ &\leq |\bar{p}-\overline{I_h^* p_h}|+|p-P_h(I_h^* p_h)| \\ &\leq C\|p-I_h^* p_h\|_{L^2(L^2)}+|p-p_h|+|p_h-P_h p_h|+|P_h(p_h-I_h^* p_h)|.\end{aligned}$$

Thus

$$\begin{aligned}\|q-q_h\| \leq& C\|p-I_h^* p_h\|_{L^2(L^2)}+C\|p-p_h\|+C\|p_h-P_h p_h\|+C\|P_h(p_h-I_h^* p_h)\| \\ \leq& C\|p-p_h\|_{L^2(L^2)}+C\|p_h-I_h^* p_h\|_{L^2(L^2)}+C\|p-p_h\| \\ &+C\|p_h-P_h p_h\|+C\|p_h-I_h^* p_h\|,\end{aligned} \tag{3.42}$$

where we also used $\|P_h(p_h-I_h^* p_h)\| \leq C\|p_h-I_h^* p_h\|$. Based on (3.42), (2.11), Lemma 2.1, and (2.29), we deduce that

$$\begin{aligned}\|q-q_h\|_{L^\infty(L^2)} \leq& C\|p-p_h\|_{L^2(L^2)}+C\|p_h-I_h^* p_h\|_{L^2(L^2)}+C\|p-p_h\|_{L^\infty(L^2)} \\ &+C\|p_h-P_h p_h\|_{L^\infty(L^2)}+C\|p_h-I_h^* p_h\|_{L^\infty(L^2)} \\ \leq& C\|p-p_h\|_{L^\infty(L^2)}+Ch\|p_h\|_{L^\infty(H^1)}.\end{aligned} \tag{3.43}$$

Applying the stability analysis as in Lemma 3.2, we obtain

$$\begin{aligned}
\|p_h\|_{L^\infty(H^1)} &\le C\big(\|y_h\|_{L^2(L^2)} + \|I_h y_d\|_{L^2(L^2)}\big)\\
&\le C\big(\|R_h y_0\| + \|\nabla R_h y_0\| + \|f\|_{L^2(L^2)} + \|q_h\|_{L^2(L^2)} + \|I_h y_d\|_{L^2(L^2)}\big)\\
&\le C\big(\|R_h y_0 - y_0\| + \|y_0\| + \|\nabla (R_h y_0 - y_0)\| + \|\nabla y_0\|\\
&\quad + \|f\|_{L^2(L^2)} + \|q_h\|_{L^2(L^2)} + \|y_d - I_h y_d\|_{L^2(L^2)} + \|y_d\|_{L^2(L^2)}\big)\\
&\le C\big(h^2\|y_0\|_2 + \|y_0\| + h\|y_0\|_2 + \|\nabla y_0\| + \|f\|_{L^2(L^2)}\\
&\quad + \|q_h\|_{L^2(L^2)} + \|y_d\|_{L^2(L^2)} + h^2\|y_d\|_{L^2(H^2)}\big) \qquad (3.44)\\
&\le C\big(\|y_0\|_2 + \|f\|_{L^2(L^2)} + \|y_d\|_{L^2(H^2)} + \|q\|_{L^2(L^2)} + \|q - q_h\|_{L^2(L^2)}\big)\\
&\le C\big(\|y_0\|_2 + \|f\|_{L^2(L^2)} + \|y_d\|_{L^2(H^2)} + \|q\|_{L^2(L^2)}\big)\\
&\quad + C\big(\|q - P_h q\|_{L^2(L^2)} + \|P_h q - q_h\|_{L^2(L^2)}\big)\\
&\le C\big(\|y_0\|_2 + \|f\|_{L^2(L^2)} + \|y_d\|_{L^2(H^2)} + \|q\|_{L^2(L^2)}\big) + Ch\|q\|_{L^2(H^1)} + Ch^2\\
&\le C,
\end{aligned}$$

where we also used (2.9), (2.10), (2.13), and Theorem 3.4. For a sufficiently small h, we can prove the following result by (3.43) and (3.44)

$$\|q - q_h\|_{L^\infty(L^2)} \le Ch + C\|p - p_h\|_{L^\infty(L^2)}. \qquad (3.45)$$

By (3.1), Lemmas 3.2–3.3, and Theorem 3.4, we obtain the desired estimate (3.41). Then, (3.40) is derived from (3.41) and (3.45). Theorem 3.5 is thus proved. □

4. CONCLUSIONS

In this paper, we designed and analyzed a novel FVE scheme for linear pseudo-parabolic OCPs subject to integral constraint based on the discretize-then-optimize approach. After defining a new approximation for the target functional, we successfully obtained the discrete optimality conditions by using the method of discretize-then-optimize. Optimal a priori error estimates in $L^\infty(L^2)$-norm for the state variable, the co-state variable and the control variable were discussed. In the future, our aim is to consider a posteriori error estimates and design adaptive finite volume algorithms for OPCs of this kind.

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