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STRONG STABILIZATION AND DECAY ESTIMATE FOR A CLASS OF NON-HOMOGENEOUS SEMILINEAR DELAYED SYSTEMS OF NEUTRAL TYPE

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Abstract. In this paper, we investigate the problem of feedback stabilization for a class of non-homogeneous semilinear delayed systems of neutral type evolving on a real Hilbert space. To achieve this goal, based on an appropriate decomposition of the state space, we propose a bounded feedback control that depends only on the state projection on a suitable subspace to study the strong stabilization. Sufficient conditions are formulated in terms of observability like assumptions. The rate of strong stabilization is explicitly given. Finally, we illustrate the obtained results on some functional differential equations of neutral type.
Keywords. Decay estimate; Non-homogeneous systems; Neutral systems; Semilinear systems; Time delay, Strong stabilization.

1. Introduction

Semilinear systems arise in various fields, including nuclear, thermal, chemical, and social processes, as they balance linearity and nonlinearity, enabling more accurate modeling of complex phenomena. In many real-world applications, time delays naturally occur due to information transmission, reaction times, or signal propagation. These delays can appear in the system state, control input, or measurements, significantly affecting stability and performance, and can take various forms, including constant, time-varying, distributed, or state-dependent delays. Systems with delays in the state are known as retarded systems, whereas those with delays in both the state and its derivatives are classified as neutral systems. The literature on related retarded and neutral functional differential equations is extensive, and we refer the reader to the book [16] for more details. Due to the potential instabilities caused by time delay in system [13], researchers have been motivated to study the stabilization problem of such systems.

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Recently, there has been growing interest in the stabilization analysis of retarded infinite dimensional systems with variable delay [12, 19], and the references therein, [5, 6] for retarded systems with distributed delay, and [7, 14, 22] for retarded systems with finite time delay. On the other hand, there are also some works on the infinite dimensional neutral systems where the state and its derivatives are affected by time delays [8–11]. The authors in [10] investigated strong stabilization for a class of infinite dimensional bilinear neutral systems with finite time delay, by constructing a bounded feedback control characterized as the solution to a minimization problem, ensuring a polynomial decay estimate of the stabilized state. Whereas, the authors in [8] employed a state space decomposition approach to establish sufficient conditions for strong and exponential stabilization of the non-homogeneous case.

In this work, we focus on the problem of feedback stabilization for a class of non-homogeneous infinite dimensional semilinear neutral system with finite time delay $r > 0$, described as follows:

$$\begin{cases} \dfrac{d\mathrm{D}z_t}{dt} = A\mathrm{D}z_t + Nz(t) + v(t)(Bz(t) + b), \quad t > 0, \\ z_0 = \varphi \in \mathscr{C} := C\big([-r,0], H\big), \end{cases} \tag{1.1}$$

where $z(.)$ is the state on a real Hilbert space H endowed with the inner product $\langle .,. \rangle$ and its corresponding norm $\|.\|$. Moreover, the linear operator $A : \mathscr{D}(A) \subset H \to H$ generates a strongly continuous semigroup $S(t)$ on H. If $z \in C([-r,+\infty[, H)$ and $t \geq 0$, then $z_t \in \mathscr{C}$ is defined by $z_t(\theta) = z(t+\theta)$ for all $\theta \in [-r,0]$, where $\mathscr{C} = C([-r,0], H)$ denotes the Banach space of continuous functions defined from $[-r,0]$ into H, endowed with the supremum norm $\|\psi\|_{\mathscr{C}} = \sup_{\theta\in[-r,0]} \|\psi(\theta)\|$ and $\varphi \in \mathscr{C}$ is a given initial function. B is the control operator which is supposed to be linear and bounded from H to H. N is a nonlinear operator from H to H such that $N(0) = 0$, and $b \in H$ is a fixed vector, whereas $t \longmapsto v(t)$ is a scalar function which represents the control. The operator $\mathrm{D} : \mathscr{C} \to H$ is linear and bounded defined by $\mathrm{D}\psi = \psi(0) - K\psi(-r)$, where $K : H \to H$ is a linear and bounded operator which satisfies $\|K\|_{\mathscr{L}(H)} < 1$, so that D is stable (see, e.g, [23]). In particular, we have $\mathrm{D}z_t = z(t) - Kz(t-r)$, $t \geq 0$. Historically, [2] were the first to introduce the subject of feedback stabilization of semilinear system, described by

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = Az(t) + v(t)Bz(t), t \geq 0,$$

where B is a possibly nonlinear operator from H into H (Hilbert space). Later, the stabilization question for infinite dimensional semilinear systems without delay was considered and extensively discussed in many works by using a various types of feedback control, including switching control, quadratic control, normalized feedback control, and sliding mode control. For more details about this subject, the reader can refer to, for examples, ([4, 18, 21, 24] and the references cited therein) for the homogeneous systems, and [1, 3, 15] for the non-homogenous case. If A generates a compact semi-group, [3] gave necessary and sufficient conditions for the strong stabilization by using bounded feedback control. In [15] and [1], the authors used the decomposition of the state space to obtain sufficient conditions for the strong and exponential stabilization, respectively. However, a constraint that should be taken into consideration is that if the system contain time delays. Therefore, the novelty of this work is to propose a delayed feedback control for systems like (1.1) and provide two main sufficient conditions: the first one (3.9) secures the strong stabilization of system (1.1) with polynomial decay (3.32), while the second (3.42) can be used to obtain the strong stabilization, when (3.9) does not hold.

The rest of this paper is structured as follows: In Section 2, we propose a delayed feedback control and discuss the existence and uniqueness of the solution to the corresponding controlled neutral system. In Section 3, we provide sufficient conditions for strong and polynomial stabilization via an appropriate decomposition of the state space. The last section, Section 4, is dedicated to some illustrating examples governed by functional differential equations of neutral type.

In the sequel, the following assumption will be required:

$(\mathscr{H})$: The operator N is Lipschitz, i.e.,

$$\exists L_N > 0, \forall x, y \in H: \ \|Nx - Ny\| \leq Ł_N \|x - y\|. \tag{1.2}$$

2. Existence and Uniqueness of the Global Mild Solution

We consider the following delayed feedback control:

$$v(t) = -\rho \frac{\langle Bz(t) + b, \mathrm{D}z_t \rangle}{1 + |\langle Bz(t) + b, \mathrm{D}z_t \rangle|}, \ \rho > 0,$$

where the parameter $\rho > 0$ represents the gain control. This leads to the closed loop system:

$$\begin{cases} \dfrac{d\mathrm{D}z_t}{dt} = A\mathrm{D}z_t + F(z_t), \quad t > 0, \\ z_0 = \varphi \in \mathscr{C}, \end{cases} \tag{2.1}$$

where the function $F : \mathscr{C} \longrightarrow H$ is defined by

$$F(\psi) = -\rho \frac{\langle B\psi(0) + b, \mathrm{D}\psi \rangle}{1 + |\langle B\psi(0) + b, \mathrm{D}\psi \rangle|} \big(B\psi(0) + b\big) + N\psi(0), \forall \psi \in \mathscr{C}. \tag{2.2}$$

The following definition offers a clear understanding of the mild solution for systems like (2.1).

Definition 2.1. [23] Let $T > 0$. A function $z \in C([-r, T], H)$ is said to be a mild solution to system (2.1) if it satisfies

$$\begin{cases} \mathrm{D}z_t = S(t)\mathrm{D}\varphi + \displaystyle\int_0^t S(t-\tau)F(z_\tau)d\tau, \quad t \in [0, T], \\ z_0 = \varphi \in \mathscr{C}. \end{cases}$$

We first present the following main result concerning the existence and uniqueness of the global mild solution of closed-loop system (2.1).

Theorem 2.2. *Assume that*

(1) The operator A generates a contraction semigroup $S(t)$ on H and $B \in \mathscr{L}(H)$.
(2) The nonlinear operator N satisfies condition (1.2), and, for all $\psi \in \mathscr{C}$,

$$\langle N\psi(0), \mathrm{D}\psi \rangle \leq 0. \tag{2.3}$$

Then, system (2.1) possesses a unique global mild solution $z \in C([-r, +\infty[, H)$. Moreover,

$$\|\mathrm{D}z_t\|^2 - \|\mathrm{D}z_s\|^2 \leq -2\rho \int_s^t \frac{|\langle Bz(\tau) + b, \mathrm{D}z_\tau \rangle|^2}{1 + |\langle Bz(\tau) + b, \mathrm{D}z_\tau \rangle|} d\tau, \textit{ for all } 0 \leq s \leq t.$$

Proof. Let $R>0$ and $\psi,\phi\in\mathscr{B}(0,R)=\{\phi\in\mathscr{C}:\|\phi\|_{\mathscr{C}}\le R\}$. Take $F(\psi)=\frac{F_1(\psi)}{F_2(\psi)}+N\psi(0)$, where $F_1(\psi)=-\rho\langle B\psi(0)+b,\mathrm{D}\psi\rangle\big(B\psi(0)+b\big)$ and $F_2(\psi)=1+|\langle B\psi(0)+b,\mathrm{D}\psi\rangle|$. Note that F_1 and F_2 are locally Lipschitz such that

$$\|F_1(\psi)-F_1(\phi)\|\le L_{F_1}\|\psi-\phi\|_{\mathscr{C}},\text{ and } |F_2(\psi)-F_2(\phi)|\le L_{F_2}\|\psi-\phi\|_{\mathscr{C}}, \tag{2.4}$$

where $L_{F_1}=2\rho\left(3R^2\|B\|^2+4R\|B\|\|b\|+\|b\|^2\right)$, and $L_{F_2}=4R\|B\|+2\|b\|$. Using the assumption $(\mathscr{H})$ and the fact that $\dfrac{1}{F_2(\psi)F_2(\phi)}<1$, we see from (2.4) that

$$\|F(\psi)-F(\phi)\|\le L_F\|\psi-\phi\|_{\mathscr{C}},$$

where $L_F=2RL_{F_1}L_{F_2}+L_N$. Thus the function F defined by (2.2) is locally Lipschitz. Using similar techniques as in the proof of [8, Theorem 2.1], along with the fact that F is locally Lipschitz, we prove that system (2.1) admits a unique global mild solution $z\in C\big([-r,+\infty[,H\big)$ given by the following variation of constants formula

$$\begin{cases}\mathrm{D}z_t=S(t)\mathrm{D}\varphi+\displaystyle\int_0^t S(t-\tau)F(z_\tau)d\tau, & t\in[0,+\infty[,\\ z_0=\varphi\in\mathscr{C},\end{cases}$$

and satisfies

$$\|\mathrm{D}z_t\|^2-\|\mathrm{D}z_s\|^2\le-2\rho\int_s^t\frac{|\langle Bz(\tau)+b,\mathrm{D}z_\tau\rangle|^2}{1+|\langle Bz(\tau)+b,\mathrm{D}z_\tau\rangle|}d\tau+2\int_s^t\langle Nz(\tau),\mathrm{D}z_\tau\rangle d\tau.$$

Employing (2.3), we conclude that

$$\|\mathrm{D}z_t\|^2-\|\mathrm{D}z_s\|^2\le-2\rho\int_s^t\frac{|\langle Bz(\tau)+b,\mathrm{D}z_\tau\rangle|^2}{1+|\langle Bz(\tau)+b,\mathrm{D}z_\tau\rangle|}d\tau\le 0,$$

which implies that $\|\mathrm{D}z_t\|\le\|\mathrm{D}\varphi\|$, for all $t\ge0$. In addition, we have

$$\|z_t\|_{\mathscr{C}}\le\frac{3\|\varphi\|_{\mathscr{C}}}{1-\|K\|},\ \forall t\ge 0. \tag{2.5}$$

$\square$

3. Stabilization Results

In the sequel, we assume that the state space H can be decomposed according to

$$H=H_u\oplus H_s, \tag{3.1}$$

where H_u and H_s are two closed subspaces of H which are orthogonal to each one and invariants under $S(t)$, i.e, $S(t)H_u\subset H_u$ and $S(t)H_s\subset H_s$. Note that $S_u(t)$ is the strongly continuous semigroup induced by $S(t)$ on H_u, and $S_s(t)$ is the strongly continuous semigroup induced by $S(t)$ on H_s. Moreover, we consider the following assumptions:

$$B=B_u+B_s,\ BH_u\subset H_u\text{ and } BH_s\subset H_s, \tag{3.2}$$

and

$$K=K_u+K_s,\ KH_u\subset H_u\text{ and } KH_s\subset H_s. \tag{3.3}$$

Under assumptions (3.2) and (3.3), system (1.1) can be decomposed into the two following subsystems:

$$\begin{cases} \dfrac{d\mathrm{D}z_t^u}{dt} = A_u \mathrm{D}z_t^u + Nz^u(t) + v(t)(B_u z^u(t) + b_u), \quad t > 0, \\ z_0^u = \varphi_u \in \mathscr{C}_u := C([-r,0], H_u), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \dfrac{d\mathrm{D}z_t^s}{dt} = A_s \mathrm{D}z_t^s + Nz^s(t) + v(t)(B_s z^s(t) + b_s), \quad t > 0, \\ z_0^s = \varphi_s \in \mathscr{C}_s := C([-r,0], H_s), \end{cases} \tag{3.5}$$

where A_u and B_u are, respectively, the restrictions of A and B in H_u, A_s and B_s are, respectively, the restrictions of A and B in H_s. Moreover, $b = b_u + b_s$, with $b_u \in H_u$ and $b_s \in H_s$, $\varphi(t) = \varphi_u(t) + \varphi_s(t)$, $\forall t \in [-r,0]$. Going forward, we consider the following feedback control

$$v_u(t) = -\rho \frac{\langle B_u z^u(t) + b_u, \mathrm{D}z_t^u \rangle}{1 + |\langle B_u z^u(t) + b_u, \mathrm{D}z_t^u \rangle|}, \forall t \geq 0, \; \rho > 0. \tag{3.6}$$

Before we state the main results, let us recall the following existing results which are needed in the stabilization analysis.

Lemma 3.1. *Assume that*

(1) The operator A generates a contraction semigroup $S(t)$ on H and $B \in \mathscr{L}(H)$.
(2) The nonlinear operator N satisfies assumptions (1.2) and (2.3) .

Then, the solution of system (2.1) satisfies the inequality

$$\|z_t\|_{\mathscr{C}} \leq \alpha e^{-\beta t} \|\varphi\|_{\mathscr{C}} + \frac{4}{1 - \|K\|} \|\mathrm{D}z_{\frac{t}{2}}\|, \; t \geq 0,$$

for some constants $\alpha, \beta > 0$.

Proof. The proof of this lemma relies on the same arguments as in the proof of [10, Lemma 3.1], along with the fact that the operator D is stable. □

Lemma 3.2. [7] *Let $\theta : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ be a non-increasing function satisfying $C\theta^2(t) \leq \theta(t) - \theta(t+T)$, for all $t \geq 0$, where $C, T > 0$ are two constants. Then, $\theta(t) = \mathscr{O}\left(\frac{1}{t}\right)$ as $t \to +\infty$.*

Lemma 3.3. [17] *Let $\Phi(t)$, $\Psi(t)$, $\chi(t)$, and $\omega(t)$ be continuous functions in $[0,T]$, let $\omega(t)$ be nonnegative in $[0,T]$, and suppose that*

$$\chi(t) \leq \Psi(t) + \int_0^t [\omega(\tau)\chi(\tau) + \Phi(\tau)] \, d\tau, \; t \in [0,T].$$

Then,

$$\chi(t) \leq \int_0^t \Phi(\tau) d\tau + \sup_{t \in [0,T]} |\Psi(t)| \exp\left(\int_0^t \omega(\tau) d\tau \right), \; t \in [0,T].$$

3.1. **Strong stabilization and polynomial decay rate estimate.** The following result provides sufficient conditions for the strong stabilization of system (3.4), with an explicit polynomial decay estimate.

Theorem 3.4. *Assume that*

(1) A_u generates a contraction semigroup $S_u(t)$ on H_u.

(2) *For all* $\psi \in \mathscr{C}_u$,

$$\langle N\psi(0), \mathrm{D}\psi\rangle \leq 0. \tag{3.7}$$

(3) *There exists* $\mu > 0$ *such that*

$$|\langle B_u\psi(0)+b_u, \mathrm{D}\psi\rangle| \geq \mu\|N\psi(0)\|^2,\ \forall \psi \in \mathscr{C}_u. \tag{3.8}$$

(4) *There exist* $T > 0$ *and* $\delta > 0$ *such that*

$$\int_0^T |\langle B_u S_u(\tau)\phi + b_u, S_u(\tau)\phi\rangle| d\tau \geq \delta\|\phi\|,\ \forall \phi \in H_u. \tag{3.9}$$

(5) *The operator* K_u *satisfies*

$$\|K_u\| < \frac{\delta}{\delta + 3T\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}. \tag{3.10}$$

Then, feedback control (3.6) *strongly stabilizes system* (3.4) *with the following decay estimate*

$$\|z^u(t)\| = \mathscr{O}\left(\frac{1}{\sqrt{t}}\right),\ \text{as } t \to +\infty. \tag{3.11}$$

Proof. Using feedback control (3.6), we see that system (3.4) is reduced to

$$\begin{cases} \dfrac{d\mathrm{D}z_t^u}{dt} = A_u\mathrm{D}z_t^u + Nz^u(t) - \rho\dfrac{\langle B_u z^u(t)+b_u, \mathrm{D}z_t^u\rangle}{1+|\langle B_u z^u(t)+b_u, \mathrm{D}z_t^u\rangle|}\big(B_u z^u(t)+b_u\big), \quad t>0, \\ z_0^u = \varphi_u \in \mathscr{C}_u. \end{cases} \tag{3.12}$$

According to Theorem 2.2, system (3.12) possesses a unique global mild solution $z^u \in C([-r,+\infty[,H_u)$ given by the following variation of constants formula

$$\begin{cases} \mathrm{D}z_t^u = S_u(t)\mathrm{D}\varphi_u + \displaystyle\int_0^t S_u(t-\tau)\Big(v_u(\tau)\big(B_u z^u(\tau)+b_u\big) + Nz^u(\tau)\Big)d\tau, \quad t \geq 0, \\ z_0^u = \varphi_u \in \mathscr{C}_u, \end{cases} \tag{3.13}$$

and satisfies the following estimate

$$\|\mathrm{D}z_t^u\|^2 - \|\mathrm{D}z_s^u\|^2 \leq -2\rho\int_s^t \frac{|\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle|^2}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle|}d\tau \leq 0,\ \forall s \in [0,t], \tag{3.14}$$

which gives

$$\|\mathrm{D}z_t^u\| \leq \|\mathrm{D}\varphi_u\|,\ \forall t \geq 0. \tag{3.15}$$

Furthermore, by virtue of (2.5), we have

$$\|z_t^u\|_{\mathscr{C}_u} \leq \frac{3\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|},\ \forall t \geq 0. \tag{3.16}$$

Now, let $\xi(t) = \mathrm{D}z_t^u - S_u(t)\mathrm{D}\varphi_u$, $\forall t \geq 0$. It comes from (3.13) that

$$\begin{aligned} \xi(t) = &-\rho\int_0^t S_u(t-\tau)\frac{\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle|}\big(B_u z_u(\tau)+b_u\big)d\tau \\ &+ \int_0^t S_u(t-\tau)Nz^u(\tau)d\tau,\ \forall t \geq 0. \end{aligned} \tag{3.17}$$

Note that

$$\frac{|\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle|}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle|} \leq |\langle B_u z^u(\tau)+b_u, \mathrm{D}z_\tau^u\rangle|.$$

Since $S_u(t)$ is a contraction semigroup, it follows from (3.17) and ((3.16)) that

$$\|\xi(t)\| \leq \rho\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\int_0^t |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau + \int_0^t \|Nz^u(\tau)\| d\tau.$$

Applying Schwartz's inequality on the last term of the previous inequality, we have, for any $T>0$,

$$\begin{aligned} \|\xi(t)\| \leq \rho\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau \\ + \sqrt{T}\left(\int_0^T \|Nz^u(\tau)\|^2 d\tau\right)^{\frac{1}{2}}, \ \forall t \in [0,T]. \end{aligned} \tag{3.18}$$

Employing (3.8), we obtain from (3.18) that

$$\begin{aligned} \|\xi(t)\| \leq \rho\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau \\ + \sqrt{\frac{T}{\mu}}\left(\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau\right)^{\frac{1}{2}}, \end{aligned}$$

which yields

$$\begin{aligned} \|\xi(t)\|^2 \leq \Bigg(\rho\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau \\ + \sqrt{\frac{T}{\mu}}\left(\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau\right)^{\frac{1}{2}}\Bigg)^2. \end{aligned}$$

Since $(p+q)^2 \leq 2p^2+2q^2$ for all $(p,q)\in\mathbb{R}^2$, we obtain

$$\begin{aligned} \|\xi(t)\|^2 \leq 2\rho^2\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^2\left(\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau\right)^2 \\ + \frac{2T}{\mu}\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau. \end{aligned}$$

Applying Schwarz's inequality again, we conclude that

$$\begin{aligned} \|\xi(t)\|^2 \leq 2T\rho^2\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^2\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2 d\tau \\ + \frac{2T^{\frac{3}{2}}}{\mu}\left(\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2 d\tau\right)^{\frac{1}{2}}, \end{aligned}$$

which gives

$$\begin{aligned} \|\xi(t)\|^2 \leq & 2T\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^3\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle| d\tau \\ & + \frac{2T^{\frac{3}{2}}}{\mu}\left(\int_0^T |\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2 d\tau\right)^{\frac{1}{2}}. \end{aligned}$$

Using the Schwartz's inequality once again, we find

$$\begin{aligned}\|\xi(t)\|^2 \leq & 2T^{\frac{3}{2}}\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^3\left(\int_0^T|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|^2d\tau\right)^{\frac{1}{2}}\\ &+\frac{2T^{\frac{3}{2}}}{\mu}\left(\int_0^T|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|^2d\tau\right)^{\frac{1}{2}},\end{aligned}$$

which implies that

$$\|\xi(t)\|^2 \leq 2T^{\frac{3}{2}}\left[\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^3+\frac{1}{\mu}\right]\left(\int_0^T|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|^2d\tau\right)^{\frac{1}{2}}.$$

Thus

$$\begin{aligned}\|\xi(t)\| \leq & \sqrt{2}T^{\frac{3}{4}}\left[\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^3+\frac{1}{\mu}\right]^{\frac{1}{2}}\\ &\times\left(\int_0^T|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|^2d\tau\right)^{\frac{1}{4}},\ \forall t\in[0,T].\end{aligned} \tag{3.19}$$

Since

$$\begin{aligned}\langle B_u S_u(\tau)\mathrm{D}z^u_0+b_u,S_u(\tau)\mathrm{D}z^u_0\rangle = & -\langle B_u\xi(\tau),S_u(\tau)\mathrm{D}z^u_0\rangle-\langle B_u\mathrm{D}z^u_\tau+b_u,\xi(\tau)\rangle\\ &+\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle-\langle B_u K_u z^u(\tau-r),\mathrm{D}z^u_\tau\rangle,\ \forall\tau\geq 0,\end{aligned}$$

and $S_u(t)$ is a contraction semigroup, it follows by using (3.15) that

$$\begin{aligned}|\langle B_u S_u(\tau)\mathrm{D}z^u_0+b_u,S_u(\tau)\mathrm{D}z^u_0\rangle| \leq & \left(2\|\mathrm{D}\varphi_u\|+\|b_u\|\right)\|\xi(\tau)\|+|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|\\ &+|\langle B_u K_u z^u(\tau-r),\mathrm{D}z^u_\tau\rangle|.\end{aligned} \tag{3.20}$$

Utilizing (3.19), we deduce from (3.20) that

$$\begin{aligned}&|\langle B_u S_u(\tau)\mathrm{D}z^u_0+b_u,S_u(\tau)\mathrm{D}z^u_0\rangle|\\ &\leq \sqrt{2}T^{\frac{3}{4}}\left(2\|\mathrm{D}\varphi_u\|+\|b_u\|\right)\left[\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)^3+\frac{1}{\mu}\right]^{\frac{1}{2}}\\ &\quad\times\left(\int_0^T|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|^2d\tau\right)^{\frac{1}{4}}+|\langle B_u z^u(\tau)+b_u,\mathrm{D}z^u_\tau\rangle|+|\langle B_u K_u z^u(\tau-r),\mathrm{D}z^u_\tau\rangle|.\end{aligned} \tag{3.21}$$

Replacing z_0^u by z_t^u, we obtain by (3.21) that

$$
\begin{aligned}
&|\langle B_u S_u(\tau)\mathrm{D}z_t^u + b_u, S_u(\tau)\mathrm{D}z_t^u\rangle| \\
&\le \sqrt{2}T^{\frac{3}{4}}\left(2\|\mathrm{D}z_t^u\| + \|b_u\|\right)\left[\rho^2\|\mathrm{D}z_t^u\|\left(\frac{3\|B_u\|\|z_t^u\|_{\mathscr{C}_u}}{1-\|K_u\|} + \|b_u\|\right)^3 + \frac{1}{\mu}\right]^{\frac{1}{2}} \\
&\quad \times\left(\int_t^{t+T}|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2 d\tau\right)^{\frac{1}{4}} + |\langle B_u z^u(\tau+t) + b_u, \mathrm{D}z_{\tau+t}^u\rangle| \\
&\quad + |\langle B_u K_u z^u(t+\tau-r), \mathrm{D}z_{t+\tau}^u\rangle|.
\end{aligned}
$$

Employing (3.15) and (3.16) with the fact that the function $t \longmapsto \|\mathrm{D}z_t^u\|$ is non-increasing, we see that

$$
\begin{aligned}
&|\langle B_u S_u(\tau)\mathrm{D}z_t^u + b_u, S_u(\tau)\mathrm{D}z_t^u\rangle| \\
&\le \sqrt{2}T^{\frac{3}{4}}\left(2\|\mathrm{D}\varphi_u\| + \|b_u\|\right)\left[\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{9\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{(1-\|K_u\|)^2} + \|b_u\|\right)^3 + \frac{1}{\mu}\right]^{\frac{1}{2}} \\
&\times\left(\int_t^{t+T}|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2 d\tau\right)^{\frac{1}{4}} + |\langle B_u z^u(\tau+t) + b_u, \mathrm{D}z_{\tau+t}^u\rangle| \\
&+ \|B_u\|\|K_u\|\,\|\mathrm{D}z_t^u\|\|z^u(t+\tau-r)\|.
\end{aligned}
\tag{3.22}
$$

Integrating (3.22) with respect to τ over the interval $[0,T]$, we see that

$$
\begin{aligned}
&\int_0^T|\langle B_u S_u(\tau)\mathrm{D}z_t^u + b_u, S_u(\tau)\mathrm{D}z_t^u\rangle| d\tau \\
&\le\sqrt{T}\left(C\sqrt{T}+1\right)\left(\int_t^{t+T}|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2 d\tau\right)^{\frac{1}{4}} \\
&\quad + \|B_u\|\|K_u\|\,\|\mathrm{D}z_t^u\|\int_0^T\|z^u(t+\tau-r)\| d\tau,
\end{aligned}
\tag{3.23}
$$

where the constant $C > 0$ is given by

$$
C = \sqrt{2}T^{\frac{3}{4}}\left(2\|\mathrm{D}\varphi_u\| + \|b_u\|\right)\left[\rho^2\|\mathrm{D}\varphi_u\|\left(\frac{9\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{(1-\|K_u\|)^2} + \|b_u\|\right)^3 + \frac{1}{\mu}\right]^{\frac{1}{2}}.
$$

Using (3.16), we see from (3.23) that

$$
\begin{aligned}
&\int_0^T|\langle B_u S_u(\tau)\mathrm{D}z_t^u + b_u, S_u(\tau)\mathrm{D}z_t^u\rangle| d\tau \\
&\le\sqrt{T}\left(C\sqrt{T}+1\right)\left(\int_t^{t+T}|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2 d\tau\right)^{\frac{1}{4}} \\
&\quad + \frac{3T\|B_u\|\|K_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}\|\mathrm{D}z_t^u\|.
\end{aligned}
\tag{3.24}
$$

On the other hand, we obtain by (3.16) that

$$\frac{|\langle B_u z^u(t)+b_u, \mathrm{D}z^u_t\rangle|^2}{1+|\langle B_u z^u(t)+b_u, \mathrm{D}z^u_t\rangle|} \geq \frac{|\langle B_u z^u(t)+b_u, \mathrm{D}z^u_t\rangle|^2}{1+\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\|\mathrm{D}\varphi_u\|}, \ \forall t \geq 0, \tag{3.25}$$

which yields from (3.24) that

$$\begin{aligned} &\int_0^T |\langle B_u S_u(\tau)\mathrm{D}z^u_t + b_u, S_u(\tau)\mathrm{D}z^u_t\rangle| d\tau \\ &\leq \left[1+\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\|\mathrm{D}\varphi_u\|\right]^{\frac{1}{4}} \\ &\times \sqrt{T}\left(C\sqrt{T}+1\right)\left(\int_t^{t+T} \frac{|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|} d\tau\right)^{\frac{1}{4}} \\ &+ \frac{3T\|B_u\|\|K_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}\|\mathrm{D}z^u_t\|. \end{aligned} \tag{3.26}$$

Utilizing the condition (3.9), we deduce from (3.26) that

$$\begin{aligned} &\left(\delta - \frac{3T\|B_u\|\|K_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}\right)\|\mathrm{D}z^u_t\| \\ &\leq \left[1+\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\|\mathrm{D}\varphi_u\|\right]^{\frac{1}{4}} \\ &\times \sqrt{T}\left(C\sqrt{T}+1\right)\left(\int_t^{t+T} \frac{|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|} d\tau\right)^{\frac{1}{4}}. \end{aligned} \tag{3.27}$$

In view of (3.10), we have

$$\|\mathrm{D}z^u_t\| \leq \nu\left(\int_t^{t+T} \frac{|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|} d\tau\right)^{\frac{1}{4}}, \tag{3.28}$$

where

$$\nu = \frac{\sqrt{T}\left(C\sqrt{T}+1\right)\left[1+\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\|\mathrm{D}\varphi_u\|\right]^{\frac{1}{4}}}{\delta - \frac{3T\|B_u\|\|K_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}}.$$

Furthermore, from estimate (3.14), we have

$$\|\mathrm{D}z^u_{t+T}\|^2 - \|\mathrm{D}z^u_t\|^2 \leq -2\rho\int_t^{t+T} \frac{|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|^2}{1+|\langle B_u z^u(\tau)+b_u, \mathrm{D}z^u_\tau\rangle|} d\tau, \ \forall t \geq 0.$$

Employing (3.28), we deduce from the last inequality that

$$\|\mathrm{D}z^u_{t+T}\|^2 - \|\mathrm{D}z^u_t\|^2 \leq \frac{-2\rho}{\nu^4}\|\mathrm{D}z^u_t\|^4, \ \forall t \geq 0,$$

which implies that $\dfrac{2\rho}{\nu^4}\|\mathrm{D}z_t^u\|^4 \le \|\mathrm{D}z_t^u\|^2 - \|\mathrm{D}z_{t+T}^u\|^2$. If we set $\theta(t) = \|\mathrm{D}z_t^u\|^2, \forall t \ge 0$, we derive that $\frac{2\rho}{\nu^4}\theta^2(t) \le \theta(t) - \theta(t+T)$. Applying Lemma 3.2, it follows that

$$\|\mathrm{D}z_t^u\| \le \frac{\lambda}{\sqrt{t}}, \text{ as } t \to +\infty, \text{ where } \lambda > 0. \tag{3.29}$$

Employing Lemma 3.1 yields that

$$\|z^u(t)\| = \mathscr{O}\left(\frac{1}{\sqrt{t}}\right), \text{ as } t \to +\infty.$$

□

The next theorem examines the stabilization of full system (1.1).

Theorem 3.5. *Suppose that all the hypothesis of Theorem 3.4 hold, and assume further that*

(1) The semigroup $S_s(t)$ is exponentially stable., i.e. there exist $M, \eta > 0$ such that

$$\|S_s(t)\| \le Me^{-\eta t}, \ \forall t \ge 0. \tag{3.30}$$

(2) The Lipschitz constant L_N satisfies

$$L_N < \frac{\eta}{M_2}, \ \textit{where } M_2 > 0. \tag{3.31}$$

Then, for ρ sufficiently small, feedback (3.6) strongly stabilizes system (1.1) with the following decay estimate

$$\|z(t)\| = \mathscr{O}\left(\frac{1}{\sqrt{t}}\right), \ \textit{as } t \to +\infty. \tag{3.32}$$

Proof. By using the feedback control (3.6), system (1.1) becomes

$$\begin{cases} \dfrac{d\mathrm{D}z_t}{dt} = A\mathrm{D}z_t + G(z_t), \quad t > 0, \\ z_0 = \varphi \in \mathscr{C}, \end{cases}$$

where the function $G : \mathscr{C} \to H$ is defined by

$$G(\psi) = -\rho \frac{\langle B_u\psi_u(0) + b, \mathrm{D}\psi_u\rangle}{1 + |\langle B_u\psi_u(0) + b_u, \mathrm{D}\psi_u\rangle|}\big(B\psi(0) + b\big) + N\psi(0), \forall \psi \in \mathscr{C}.$$

It is clear that G is a locally Lipschitz function, then, the system (1.1) controlled by (3.6) possesses a unique mild solution defined on a maximal interval $[0, t_{max}[$, and given by the following variation of constants formula

$$\mathrm{D}z_t = S(t)\mathrm{D}\varphi + \int_0^t S(t-\tau)\big(v_u(\tau)\big(Bz(\tau) + b\big) + Nz(\tau)\big)d\tau.$$

Thus

$$\begin{cases} \mathrm{D}z_t^s = S_s(t)\mathrm{D}\varphi_s + \displaystyle\int_0^t S_s(t-\tau)\big(v_u(\tau)\big(B_sz^s(\tau) + b_s\big) + Nz^s(\tau)\big)d\tau, \quad 0 \le t < t_{max}, \\ z_0^s = \varphi_s \in \mathscr{C}_s. \end{cases} \tag{3.33}$$

Now, let us show that $z^s(t)$ is defined for all $t \geq -r$. Note that $\|S_s(t)\| \leq M$ and $|v_u(t)| < \rho$. It follows from (3.33) that

$$\|\mathrm{D}z_t^s\| \leq 2M\|\varphi_s\|_{\mathscr{C}_s} + M\rho \int_0^t \left(\|B_s\|\|z^s(\tau)\| + \|b_s\|\right)d\tau + M\int_0^t L_N\|z^s(\tau)\|d\tau,$$

which implies that

$$\|\mathrm{D}z_t^s\| \leq 2M\|\varphi_s\|_{\mathscr{C}_s} + M\rho\|b_s\|t + M(\rho\|B_s\| + L_N)\int_0^t \|z^s(\tau)\|d\tau.$$

Using the fact that $\mathrm{D}z_t^s = z^s(t) - K_s z^s(t-r)$, we arrive at

$$\|z^s(t)\| \leq 2M\|\varphi_s\|_{\mathscr{C}_s} + M\rho\|b_s\|t + \|K_s\|\|z^s(t-r)\| + M(\rho\|B_s\| + L_N)\int_0^t \|z^s(\tau)\|d\tau.$$

Thus

$$\|z^s(t)\| \leq 2M\|\varphi_s\|_{\mathscr{C}_s} + m_s\|K_s\| + M\rho\|b_s\|t + M(\rho\|B_s\| + L_N)\int_0^t \|z^s(\tau)\|d\tau,$$

where $m_s = \sup\limits_{-r\leq\sigma\leq t_{max}-r} \|z^s(\sigma)\|$. Applying Gronwall inequality, we find that

$$\begin{aligned}\|z^s(t)\| \leq & 2M\|\varphi_s\|_{\mathscr{C}_s} + m_s\|K_s\| + M\rho\|b_s\|t \\ & + M(\rho\|B_s\| + L_N)\int_0^t (2M\|\varphi_s\|_{\mathscr{C}_s} + m_s\|K_s\| + M\rho\|b_s\|\tau)e^{M(\rho\|B_s\|+L_N)(t-\tau)}d\tau,\end{aligned}$$

which yields that

$$\|z^s(t)\| \leq \left(2M\|\varphi_s\|_{\mathscr{C}_s} + m_s\|K_s\| + M\rho\|b_s\|t\right)e^{M(\rho\|B_s\|+L_N)t},\ 0 \leq t < t_{max}.$$

Thus $\|z^s(t)\|$ is bounded on $[0, t_{max}[$. Hence, $t_{max} = +\infty$. We deduce that the system (3.5) controlled by (3.6) admits a unique global mild solution $z^s \in C([-r, +\infty[, H_s)$. Now, employing (3.30) together with the fact that $\mathrm{D}z_t^s = z^s(t) - K_s z^s(t-r)$ and $|v_u(t)| < \rho$, it comes that

$$\begin{aligned}\|z^s(t)\| \leq & \|K_s\|\|z^s(t-r)\| + 2M\|\varphi_s\|_{\mathscr{C}_s}e^{-\eta t} + Me^{-\eta t}\left(\rho\|B_s\| + L_N\right)\int_0^t e^{\eta\tau}\|z^s(\tau)\|d\tau \\ & + M\|b_s\|e^{-\eta t}\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau,\end{aligned}$$

which implies that

$$\begin{aligned}\|z^s(t)\|e^{\eta t} \leq & \|K_s\|\|z^s(t-r)\|e^{\eta(t-r)}e^{\eta r} + 2M\|\varphi_s\|_{\mathscr{C}_s} \\ & + M\left(\rho\|B_s\| + L_N\right)\int_0^t e^{\eta\tau}\|z^s(\tau)\|d\tau + M\|b_s\|\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau.\end{aligned}$$

Let $\theta \in [0, t]$. Then

$$\begin{aligned}\|z^s(\theta)\|e^{\eta\theta} \leq & \|K_s\|\|z^s(\theta-r)\|e^{\eta(\theta-r)}e^{\eta r} + 2M\|\varphi_s\|_{\mathscr{C}_s} \\ & + M\left(\rho\|B_s\| + L_N\right)\int_0^\theta e^{\eta\tau}\|z^s(\tau)\|d\tau + M\|b_s\|\int_0^\theta e^{\eta\tau}|v_u(\tau)|d\tau,\end{aligned}$$

which gives

$$\begin{aligned}\|z^s(\theta)\|e^{\eta\theta} \leq & \|K_s\|e^{\eta r}\sup_{-r\leq\sigma\leq t}\|z^s(\sigma)\|e^{\eta\sigma} + 2M\|\varphi_s\|_{\mathscr{C}_s} \\ & + M\left(\rho\|B_s\| + L_N\right)\int_0^t e^{\eta\tau}\|z^s(\tau)\|d\tau + M\|b_s\|\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau.\end{aligned}$$

Hence,

$$\begin{aligned}\sup_{0\le\sigma\le t}\|z^s(\sigma)\|e^{\eta\sigma} \le&\|K_s\|e^{\eta r}\sup_{-r\le\sigma\le t}\|z^s(\sigma)\|e^{\eta\sigma}+2M\|\varphi_s\|_{\mathscr{C}_s}\\ &+M\big(\rho\|B_s\|+L_N\big)\int_0^t e^{\eta\tau}\|z^s(\tau)\|d\tau+M\|b_s\|\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau.\end{aligned} \tag{3.34}$$

On the other hand, it comes for $\theta\in[-r,0]$ and $M\ge 1$ that

$$\|z^s(\theta)\|e^{\eta\theta}\le\|\varphi_s\|_{\mathscr{C}_s}\le 2M\|\varphi_s\|_{\mathscr{C}_s}. \tag{3.35}$$

Combining (3.34) and (3.35), we have

$$\begin{aligned}\big(1-\|K_s\|e^{\eta r}\big)\sup_{-r\le\sigma\le t}\|z^s(\sigma)\|e^{\eta\sigma} \le&2M\|\varphi_s\|_{\mathscr{C}_s}+M\big(\rho\|B_s\|+L_N\big)\int_0^t e^{\eta\tau}\|z^s(\tau)\|d\tau\\ &+M\|b_s\|\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau,\end{aligned}$$

Taking $\eta>0$ such that $1-\|K_s\|e^{\eta r}>0$, we arrive at

$$\|z^s(t)\|e^{\eta t}\le M_1+M_2\big(\rho\|B_s\|+L_N\big)\int_0^t e^{\eta\tau}\|z^s(\tau)\|d\tau+M_3\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau,$$

where $M_1=\dfrac{2M\|\varphi_s\|_{\mathscr{C}_s}}{1-\|K_s\|e^{\eta r}}$, $M_2=\dfrac{M}{1-\|K_s\|e^{\eta r}}$ and $M_3=\dfrac{M\|b_s\|}{1-\|K_s\|e^{\eta r}}$. Let $\chi(t)=\|z^s(t)\|e^{\eta t}$. In view of 3.3, we see that

$$\chi(t)\le M_1e^{M_2(\rho\|B_s\|+L_N)t}+M_3\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau,$$

therefore

$$\|z^s(t)\|\le M_1e^{\left(M_2(\rho\|B_s\|+L_N)-\eta\right)t}+M_3e^{-\eta t}\int_0^t e^{\eta\tau}|v_u(\tau)|d\tau. \tag{3.36}$$

Moreover, feedback control (3.6) satisfies

$$|v_u(t)|\le\rho\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right)\|\mathrm{D}z_t^u\|,\ \forall t\ge 0.$$

Taking into consideration (3.29), it follows that $|v_u(t)|\le\frac{\zeta_\rho}{\sqrt{t}}$, as $t\to+\infty$, where

$$\zeta_\rho=\rho\lambda\left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|}+\|b_u\|\right).$$

Hence, we deduce from (3.36) that

$$\|z^s(t)\|\le M_1e^{\left(M_2(\rho\|B_s\|+L_N)-\eta\right)t}+M_3\zeta_\rho e^{-\eta t}\int_0^t\frac{e^{\eta\tau}}{\sqrt{\tau}}d\tau,$$

which can be written as

$$\|z^s(t)\|\le M_1e^{\left(M_2(\rho\|B_s\|+L_N)-\eta\right)t}+M_3\zeta_\rho\frac{2\Gamma(\sqrt{\eta t})}{\sqrt{\eta}}, \tag{3.37}$$

where Γ is the Dawson function defined as $\Gamma(t) = e^{-t^2}\int_0^t e^{x^2}dx$. Using the fact that $\Gamma(t) \simeq \frac{1}{2t}$, as $t \to +\infty$, and for $L_N < \frac{\eta}{M_2}$, it yields from (3.37) by taking $\rho < \frac{\eta - M_2 L_N}{M_2\|B_s\|}$ that

$$\|z^s(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right), \text{ as } t \to +\infty. \tag{3.38}$$

Combining (3.11) and (3.38), we conclude that $\|z(t)\| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$ as $t \to +\infty$. □

Remark 3.6. (1) There is no constraint on $\dim H_u$.

(2) Note that feedback (3.6) depends only on the unstable part $z^u(t)$. Therefore, it requires less information about the state, as it is based on a partial projection of the state. This restriction leads to lower control energy, as only the critical dynamics are influenced. In addition, $|v_u(t)| = \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$, as $t \to +\infty$.
(3) If $B_u = 0$, it follows from (3.27) that system (1.1) is polynomially stable without any condition on K_u.
(4) If $B_s = 0$, no restriction on the gain control ρ is needed.
(5) If condition (3.30) is satisfied for $\eta > 0$, then it is also satisfied for all $\eta' \in]0,\eta[$.
(6) To ensure the stability of system (3.5), the proof of Theorem 3.5 provides a new approach with respect to the one considered in [8] which is based on Lemma 3.1.
(7) Theorem 3.5 extends the results of [21] and [24] to non-homogeneous delayed semilinear systems.
(8) It should be noted that the obtained result generalizes that of [14] to the case of semilinear neutral systems.

Example 3.7. *Consider the following mono-dimensional semilinear heat equation with Neumann boundary conditions:*

$$\begin{cases} \frac{\partial}{\partial t}[z(x,t) - kz(x,t-r)] = \frac{\partial^2}{\partial x^2}[z(x,t) - kz(x,t-r)] + Nz(x,t) & \\ \qquad\qquad\qquad\qquad +v(t)\big(Bz(x,t) + b(x)\big), & x \in (0,1), t \geq 0, \\ \frac{\partial z(0,t)}{\partial x} = \frac{\partial z(1,t)}{\partial x} = 0, & t \in [0,+\infty[, \\ z(x,t) = \varphi(x,t), & x \in (0,1), t \in [-r,0], \end{cases} \tag{3.39}$$

where $-1 < k < 1$ and $r > 0$. System (3.39) has the form of (1.1) if $H = L^2(0,1)$ and the operator A is defined by

$$Az = \frac{\partial^2 z}{\partial x^2}, \text{ with } \mathcal{D}(A) = \left\{z \in H^2(0,1) : \tfrac{\partial z(0,t)}{\partial x} = \tfrac{\partial z(1,t)}{\partial x} = 0\right\}.$$

The spectrum of A is given by the simple eigenvalues $\lambda_n = -\pi^2(n-1)^2, \forall n \geq 1$, and its corresponding eigenfunctions is defined by

$$e_n(x) = \begin{cases} 1, & n = 1, \\ \sqrt{2}\cos((n-1)\pi x), & n \geq 2. \end{cases}$$

In this case, the unstable subspace is given by $H_u = \text{vect}\{e_1\}$ and $S_u(t) = I_{H_u}$, namely $S_u(t)$ is a semigroup of isometries. Let $Bz = \sum_{n=2}^{+\infty} \frac{1}{n^2}\langle z, e_n\rangle e_n$ and $b \in L^2(0,1)$, such that $\|b_u\|_{L^2(0,1)} \neq 0$. It follows that $BH_u = \{0\} \subset H_u$ and $BH_s \subset H_s$. According to the third point in 3.6, the fact that $B_u = 0$ implies that no restriction on $|k|$ is needed. The nonlinear operator N is defined by

$$Nz = \frac{1}{1+|\langle z, \phi\rangle|}\sum_{n=2}^{+\infty}\frac{1}{n^2}\langle z, e_n\rangle e_n,$$

where $\phi \in H$ is chosen such that assumption (3.31) is satisfied. It is clear that N is Lipschitz. Moreover, by observing that for all $\psi \in \mathscr{C}_u$, $N\psi(0) = 0$, it yields that assumptions (3.7) and (3.8) hold. Letting $T > 0$, we have

$$\int_0^T |\langle B_u S_u(\tau)\phi + b_u, S_u(\tau)\phi\rangle| d\tau = \int_0^T |\langle b_u, \phi\rangle| d\tau \geq \delta \|\phi\|_{L^2(0,1)}, \ \forall \phi \in H_u,$$

where $\delta = T\|b_u\|_{L^2(0,1)}$, and therefore, hypothesis (3.9) is satisfied. Applying Theorem 3.5, we deduce that the feedback control

$$v(t) = -\rho\frac{\int_0^1 b_u(x)\big(z^u(x,t) - kz^u(x,t-r)\big)dx}{1+\left|\int_0^1 b_u(x)\big(z^u(x,t) - kz^u(x,t-r)\big)dx\right|}, \ t \geq 0, \ \rho > 0,$$

where the parameter ρ is chosen sufficiently small, strongly stabilizes system (3.39) with the following decay estimate

$$\int_0^1 z^2(x,t)dx = \mathscr{O}\left(\frac{1}{t}\right), \ \text{as } t \to +\infty.$$

Remark 3.8. *Note that estimate (3.9) is unverified when $b_u = 0$. Indeed, if $b_u = 0$, we see from (3.9) that $\int_0^T |\langle B_u S_u(\tau)\phi, S_u(\tau)\phi\rangle| d\tau \geq \delta\|\phi\|$, for all $\phi \in H_u$. Replacing ϕ by $\eta\phi$, where $\eta > 0$ sufficiently small, we have*

$$\eta\int_0^T |\langle B_u S_u(\tau)\phi, S_u(\tau)\phi\rangle| d\tau \geq \delta\|\phi\|, \ \forall \phi \in H_u,$$

which is contradictory for $\eta \to 0^+$.

3.2. **Strong stabilization.** In this section, we establish the strong stabilization of system (1.1) under a weak observability assumption (3.42) with respect to (3.9).

Theorem 3.9. *Assume that:*

1. *A_u generates a contraction semigroup $S_u(t)$ on H_u.*
2. *For all $\psi \in \mathscr{C}_u$, we have*
$$\langle N\psi(0), \mathrm{D}\psi\rangle \leq 0. \tag{3.40}$$
3. *There exists $\mu > 0$ such that*
$$|\langle B_u\psi(0) + b_u, \mathrm{D}\psi\rangle| \geq \mu\|N\psi(0)\|^2, \ \forall \psi \in \mathscr{C}_u. \tag{3.41}$$
4. *There exist $T > 0$ and $\delta > 0$ such that*
$$\int_0^T |\langle B_u S_u(\tau)\phi + b_u, S_u(\tau)\phi\rangle| d\tau \geq \delta\|\phi\|^2, \ \forall \phi \in H_u. \tag{3.42}$$

(5) The operator K_u satisfies

$$\|K_u\| < \frac{\delta}{\delta + 4T\|B_u\|}. \tag{3.43}$$

(5) The semigroup $S_s(t)$ is exponentially stable, i.e. there exist $M, \eta > 0$ such that

$$\|S_s(t)\| \le M e^{-\eta t}, \ \forall t \ge 0.$$

(6) The Lipschitz constant L_N satisfies

$$L_N < \frac{\eta}{M_2}, \text{ where } M_2 > 0. \tag{3.44}$$

Then, the feedback (3.6) strongly stabilizes system (1.1).

Proof. Combining (3.23) and (3.25), we obtain

$$\begin{aligned}
&\int_0^T |\langle B_u S_u(\tau)\mathrm{D}z_t^u + b_u, S_u(\tau)\mathrm{D}z_t^u\rangle| d\tau \\
&\le \left[1 + \left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|} + \|b_u\|\right)\|\mathrm{D}\varphi_u\|\right]^{\frac{1}{4}} \\
&\times \sqrt{T}\left(C\sqrt{T}+1\right)\left(\int_t^{t+T} \frac{|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2}{1+|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|} d\tau\right)^{\frac{1}{4}} \\
&+ \|B_u\|\|K_u\|\,\|\mathrm{D}z_t^u\| \int_0^T \|z^u(t+\tau-r)\| d\tau.
\end{aligned} \tag{3.45}$$

Let $\tau_t \in [0,T]$ such that $\|z^u(t+\tau_t-r)\| = \sup_{0\le\tau\le T} \|z^u(t+\tau-r)\|$. According to Lema 3.1, it follows, for all $t \ge r$, that

$$\|z^u(t+\tau-r)\| \le \alpha_u e^{-\beta_u(t+\tau_t-r)}\|\varphi_u\|_{\mathscr{C}_u} + \frac{4}{1-\|K_u\|}\left\|\mathrm{D}z^u_{\frac{t+\tau_t-r}{2}}\right\|, \ \forall \tau \in [0,T], \tag{3.46}$$

for some constants $\alpha_u, \beta_u > 0$. Applying (3.42) and using (3.46), we see from (3.45) that

$$\begin{aligned}
\delta\|\mathrm{D}z_t^u\|^2 \le & C_1\left(\int_t^{t+T} \frac{|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2}{1+|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|} d\tau\right)^{\frac{1}{4}} \\
& + T\|B_u\|\|K_u\|\,\|\mathrm{D}z_t^u\|\left(\alpha_u e^{-\beta_u(t+\tau_t-r)}\|\varphi_u\|_{\mathscr{C}_u} + \frac{4}{1-\|K_u\|}\left\|\mathrm{D}z^u_{\frac{t+\tau_t-r}{2}}\right\|\right),
\end{aligned} \tag{3.47}$$

where

$$C_1 = \sqrt{T}\left(C\sqrt{T}+1\right)\left[1 + \left(\frac{3\|B_u\|\|\varphi_u\|_{\mathscr{C}_u}}{1-\|K_u\|} + \|b_u\|\right)\|\mathrm{D}\varphi_u\|\right]^{\frac{1}{4}}.$$

On the other hand, it comes from (3.14) that

$$2\rho\int_0^t \frac{|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2}{1+|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|} d\tau \le \|\mathrm{D}\varphi_u\|^2, \ \forall t \ge 0,$$

which implies that $\int_0^t \frac{|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2}{1+|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|} d\tau$ converges for all $t \ge 0$. Consequently, we conclude from the Cauchy criterion that

$$\lim_{t\to+\infty}\int_t^{T+t} \frac{|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|^2}{1+|\langle B_u z^u(\tau) + b_u, \mathrm{D}z_\tau^u\rangle|} d\tau = 0.$$

The fact that $t \longmapsto \|\mathrm{D}z_t^u\|$ is non-increasing function implies that there exists $\varsigma \in [0,+\infty[$ such that $\|\mathrm{D}z_t^u\| \underset{t\to+\infty}{\longrightarrow} \varsigma$. Hence, at the limit as $t \to +\infty$, we find from (3.47) that

$$\delta\varsigma^2 \leq \frac{4T\|B_u\|\|K_u\|}{1-\|K_u\|}\varsigma^2. \tag{3.48}$$

Employing (3.43), we have $\varsigma = 0$ which implies that $\mathrm{D}z_t^u \longrightarrow 0$, as $t \to +\infty$. Using Lemma 3.1, we deduce that

$$z^u(t) \longrightarrow 0, \text{ as } t \to +\infty. \tag{3.49}$$

Let $\varepsilon > 0$. Since $\mathrm{D}z_t^u$ strongly converges to 0 as $t \to +\infty$, there exists $t_1 \geq 0$ such that, $\forall t \geq t_1$, $|v_u(t)| \leq \rho\varepsilon$. On the other hand, it was shown in the proof of Theorem 3.5 that the system (3.5) controlled by (3.6) admits a unique global mild solution $z^s \in C([-r,+\infty[,H_s)$ given by the following variation of the constants formula:

$$\begin{cases} \mathrm{D}z_t^s = S_s(t)\mathrm{D}\varphi_s + \displaystyle\int_0^t S_s(t-\tau)\big(v_u(\tau)\big(B_s z^s(\tau)+b_s\big)+Nz^s(\tau)\big)d\tau, \quad t\geq 0, \\ z_0^s = \varphi_s \in \mathscr{C}_s. \end{cases} \tag{3.50}$$

In this case, it follows from (3.50) that

$$\begin{aligned} \|\mathrm{D}z_t^s\| \leq 2M\|z_{t_1}^s\|_{\mathscr{C}_s} e^{-\eta(t-t_1)} + \rho\varepsilon M\int_{t_1}^t e^{-\eta(t-\tau)}\Big(\|B_s\|\|z^s(\tau)\|+\|b_s\|\Big)d\tau \\ +M\int_{t_1}^t e^{-\eta(t-\tau)}L_N\|z^s(\tau)\|d\tau. \end{aligned}$$

Hence,

$$\begin{aligned} \|\mathrm{D}z_t^s\| &\leq 2M\|z_{t_1}^s\|_{\mathscr{C}_s} e^{-\eta(t-t_1)} + Me^{-\eta t}\Big(\rho\varepsilon\|B_s\|+L_N\Big)\int_{t_1}^t e^{\eta\tau}\|z^s(\tau)\|d\tau \\ &\quad +\rho\varepsilon M\|b_s\|e^{-\eta t}\int_{t_1}^t e^{\eta\tau}d\tau. \end{aligned}$$

Taking $\eta > 0$ such that $1-\|K_s\|e^{\eta r} > 0$, we demonstrate by using similar techniques as in the proof of Theorem 3.5 that

$$\|z^s(t)\|e^{\eta t} \leq M_1 + M_2\Big(\rho\varepsilon\|B_s\|+L_N\Big)\int_{t_1}^t e^{\eta\tau}\|z^s(\tau)\|d\tau + \varepsilon M_3\int_{t_1}^t e^{\eta\tau}d\tau,$$

where $M_1 = \dfrac{2M\|z_{t_1}^s\|_{\mathscr{C}_s}e^{\eta t_1}}{1-\|K_s\|e^{\eta r}}$, $M_2 = \dfrac{M}{1-\|K_s\|e^{\eta r}}$ and $M_3 = \frac{\rho M\|b_s\|}{1-\|K_s\|e^{\eta r}}$, which can be written as

$$\|z^s(t)\|e^{\eta t} \leq a(t) + M_2\Big(\rho\varepsilon\|B_s\|+L_N\Big)\int_{t_1}^t e^{\eta\tau}\|z^s(\tau)\|d\tau,$$

where $a(t) = \dfrac{\varepsilon M_3}{\eta}e^{\eta t} + w(t_1)$ with $w(t_1) = M_1 - \dfrac{\varepsilon M_3}{\eta}e^{\eta t_1}$. Let $\chi(t) = \|z^s(t)\|e^{\eta t}$. It yields by applying Gronwall's inequality that

$$\chi(t) \leq a(t) + M_2\Big(\rho\varepsilon\|B_s\|+L_N\Big)\int_{t_1}^t a(\tau)e^{M_2\left(\rho\varepsilon\|B_s\|+L_N\right)(t-\tau)}d\tau,$$

which gives

$$\chi(t) \leq a(t) + M_2\Big(\rho\varepsilon\|B_s\|+L_N\Big)e^{M_2(\rho\varepsilon\|B_s\|+L_N)t}\int_{t_1}^t a(\tau)e^{-M_2(\rho\varepsilon\|B_s\|+L_N)\tau}d\tau,$$

thus

$$\|z^s(t)\| \le a(t)e^{-\eta t} + M_2\Big(\rho\varepsilon\|B_s\| + L_N\Big)e^{\big(M_2(\rho\varepsilon\|B_s\|+L_N)-\eta\big)t}\int_{t_1}^{t} a(\tau)e^{-M_2(\rho\varepsilon\|B_s\|+L_N)\tau}d\tau. \tag{3.51}$$

Note that $a(t)e^{-\eta t} = \dfrac{\varepsilon M_3}{\eta} + w(t_1)e^{-\eta t}$, which implies that

$$\lim_{t\to+\infty} a(t)e^{-\eta t} = \frac{M_3}{\eta}\varepsilon. \tag{3.52}$$

On the other hand, we have

$$\begin{aligned}\int_{t_1}^{t} a(\tau)e^{-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\tau}d\tau = \frac{\varepsilon M_3}{\eta}\int_{t_1}^{t} e^{\Big(\eta-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)\tau}d\tau \\ + w(t_1)\int_{t_1}^{t} e^{-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\tau}d\tau,\end{aligned}$$

which leads to

$$\begin{aligned}&\int_{t_1}^{t} a(\tau)e^{-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\tau}d\tau \\ &= \frac{\varepsilon M_3}{\eta\Big(\eta - M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)}\Bigg(e^{\Big(\eta-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)t} - e^{\Big(\eta-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)t_1}\Bigg) \\ &+ \frac{w(t_1)}{M_2\left(\rho\varepsilon\|B_s\|+L_N\right)}\Bigg(e^{-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)t_1} - e^{-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)t}\Bigg).\end{aligned} \tag{3.53}$$

For $L_N < \dfrac{\eta}{M_2}$, it comes from (3.53) by choosing $\varepsilon < \dfrac{\eta - M_2 L_N}{M_2\rho\|B_s\|}$ that

$$\begin{aligned}&\lim_{t\to+\infty} e^{\Big(M_2\left(\rho\varepsilon\|B_s\|+L_N\right)-\eta\Big)t}\int_{t_1}^{t} a(\tau)e^{-M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\tau}d\tau \\ &= \frac{\varepsilon M_3}{\eta\Big(\eta - M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)}.\end{aligned} \tag{3.54}$$

Combining (3.52) and (3.54), we derive from (3.51) that

$$\lim_{t\to+\infty}\|z^s(t)\| \le \left(\frac{M_3}{\eta} + \frac{M_2M_3L_N}{\eta\Big(\eta - M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)}\right)\varepsilon + \frac{\rho M_2M_3\|B_s\|}{\eta\Big(\eta - M_2\left(\rho\varepsilon\|B_s\|+L_N\right)\Big)}\varepsilon^2,$$

which implies that

$$\lim_{t\to+\infty}\|z^s(t)\| = 0. \tag{3.55}$$

Hence, we conclude from (3.49) and (3.55) that $z(t) \longrightarrow 0$, as $t \to +\infty$. □

Remark 3.10. (1) In Theorem 3.9, no restriction on the gain control ρ is needed.

(2) If $B_u = 0$, it follows from (3.48) that system (1.1) is strongly stabilizable without any condition on K_u.

(3) The assumptions (3.9) and (3.42) does not depend on the delay r, which makes it easier to check.
(4) Analogous results to Theorem 3.9, for undelayed systems, have been established in [15].
(5) Note that Theorem 3.9 extends the result of [8] to the case of semilinear neutral systems.

The next result discusses the case where $\dim H_u < +\infty$.

Corollary 3.11. *Let A_u generate a contraction semigroup $S_u(t)$ on H_u. Moreover, assume that $\dim H_u < +\infty$ and $b_u = 0$ hold, provided that, for all $\phi \in H_u$,*

$$\langle B_u S_u(t)\phi, S_u(t)\phi\rangle = 0, \ \forall t \geq 0 \implies \phi = 0, \tag{3.56}$$

is satisfied. Then, (3.6) strongly stabilizes the system (1.1).

Proof. If $b_u = 0$, then condition (3.42) reads as follows: There exist $T > 0$ and $\delta > 0$ such that

$$\int_0^T |\langle B_u S_u(\tau)\phi, S_u(\tau)\phi\rangle| d\tau \geq \delta \|\phi\|^2, \ \forall \phi \in H_u. \tag{3.57}$$

If $\dim H_u < +\infty$, then (3.56) and (3.57) are equivalent [20, Lemma 3.1] Thus the result follows from Theorem 3.9 immediately. ☐

Example 3.12. *Consider the following uncoupled systems:*

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[y(x,t) - ky(x,t-r)] = \dfrac{\partial^2}{\partial x^2}[y(x,t) - ky(x,t-r)] & \\ \qquad\qquad + v(t)\dfrac{\partial y(x,t)}{\partial t}, & x \in (0,1), t \geq 0, \\ \dfrac{\partial}{\partial t}[w(x,t) - kw(x,t-r)] = \dfrac{\partial^2}{\partial x^2}[w(x,t) - kw(x,t-r)] - \dfrac{\alpha w(x,t)}{1 + \|w(t)\|_{L^2(0,1)}} & \\ \qquad\qquad + v(t)\big(w(x,t) + q(x)\big), & x \in (0,1), t \geq 0, \\ y(0,t) = y(1,t) = 0, & t \in [0,+\infty[, \\ w(0,t) = w(1,t) = 0, & t \in [0,+\infty[, \\ y(x,t) = \varphi_1(x,t), \ \dfrac{\partial y(x,t)}{\partial t} = \varphi_2(x,t), \ w(x,t) = \varphi_3(x,t), & x \in (0,1), t \in [-r,0], \end{cases} \tag{3.58}$$

where $-1 < k < 1$, $r > 0$, $\alpha > 0$ and $q \in L^2(0,1)$ is a fixed vector. System (3.58) has the form of (1.1) if we set $H = H_0^1(0,1) \times (L^2(0,1))^2$, with the inner product

$$\langle (x_1,x_2,x_3),(y_1,y_2,y_3)\rangle = \langle x_1,y_1\rangle_{H_0^1(0,1)} + \langle x_2,y_2\rangle_{L^2(0,1)} + \langle x_3,y_3\rangle_{L^2(0,1)}.$$

Taking

$$z(t) = \begin{pmatrix} y(.,t) \\ \dfrac{\partial y(.,t)}{\partial t} \\ w(.,t) \end{pmatrix} \ \text{and } \mathrm{D}z_t = \begin{pmatrix} y(.,t) - ky(.,t-r) \\ \dfrac{\partial y(.,t)}{\partial t} - k\dfrac{\partial y(.,t-r)}{\partial t} \\ w(.,t) - kw(.,t-r) \end{pmatrix},$$

it comes that system (3.58) can be written as:

$$\begin{cases} \dfrac{d\mathrm{D}z_t}{dt} = A\mathrm{D}z_t + Nz(t) + v(t)(Bz(t) + b), & t \geq 0, \\ z(t) = \begin{pmatrix} \varphi_1(.,t) \\ \varphi_2(.,t) \\ \varphi_3(.,t) \end{pmatrix}, & t \in [-r,0], \end{cases}$$

where the operators A, B, N and the vector b are defined as follows:

$$A = \begin{pmatrix} 0 & \mathrm{Id}_{L^2(0,1)} & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & \Delta \end{pmatrix}, \text{ with}$$

$\mathscr{D}(A) = (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1) \times (H^2(0,1) \cap H_0^1(0,1))$.

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{Id}_{L^2(0,1)} & 0 \\ 0 & 0 & \mathrm{Id}_{L^2(0,1)} \end{pmatrix}, \ N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix}, \text{ and } b = \begin{pmatrix} 0 \\ 0 \\ q \end{pmatrix},$$

where $F : L^2(0,1) \longrightarrow L^2(0,1)$ is defined by

$$Fw = -\frac{\alpha w(x,t)}{1 + \|w(t)\|_{L^2(0,1)}}, \ \forall w \in L^2(0,1),$$

where the parameter $\alpha > 0$ is chosen such that assumption (3.44) is satisfied. The state space $H = H_0^1(0,1) \times (L^2(0,1))^2$ can be decomposed into $H = H_u \oplus H_s$, with $H_u = H_0^1(0,1) \times L^2(0,1) \times \{0\}$ and $H_s = \{0\} \times L^2(0,1)$. The operator A can be decomposed into $A = A_u \oplus A_s$, where

$$A_u = \begin{pmatrix} 0 & \mathrm{Id}_{L^2(0,1)} & 0 \\ \Delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta \end{pmatrix}.$$

Furthermore, $B_u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathrm{Id}_{L^2(0,1)} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $b_u = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. *In this case, it follows for all $\psi \in \mathscr{C}_u$ where $\psi(0) = (\psi_1(0), \psi_2(0), 0) \in H_u$ that* $N\psi(0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & F \end{pmatrix} \begin{pmatrix} \psi_1(0) \\ \psi_2(0) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, *which implies that assumptions (3.40) and (3.41) hold. The set of function $(e_n)_{n \geq 1}$ expressed by $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n \geq 1$ is a complete orthonormal eigenfunctions system of $\dfrac{\partial^2}{\partial x^2}$ associated to the eigenvalues $c_n = -(n\pi)^2$, $n \geq 1$. Let* $z^u = \sum_{n=1}^{+\infty} \begin{pmatrix} a_n \\ (-c_n)^{\frac{1}{2}} p_n \\ 0 \end{pmatrix} e_n \in H_u$, *where $(a_n, p_n) \in \mathbb{R}^2$, $n \geq 1$. The operator A_u generates the following semigroup of isometries given by:*

$$S_u(t)z^u = \sum_{n=1}^{+\infty} \begin{pmatrix} a_n \cos(\sqrt{-c_n}t) + p_n \sin(\sqrt{-c_n}t) \\ -a_n\sqrt{-c_n}\sin(\sqrt{-c_n}t) + p_n\sqrt{-c_n}\cos(\sqrt{-c_n}t) \\ 0 \end{pmatrix} e_n,$$

which yields

$$\begin{aligned} \langle B_u S_u(t)z^u, S_u(t)z^u \rangle &= \sum_{n=1}^{+\infty} c_n \big(a_n \sin(n\pi t) - p_n \cos(n\pi t)\big)^2 \\ &= \sum_{n=1}^{+\infty} c_n \big(a_n^2 \sin^2(n\pi t) + p_n^2 \cos^2(n\pi t) - a_n p_n \sin(2n\pi t)\big). \end{aligned}$$

Integrating this relation from 0 *to* 2*, we obtain that*

$$\int_0^2 |\langle B_u S_u(t) z^u, S_u(t) z^u \rangle| dt = \sum_{n=1}^{+\infty} -c_n(a_n^2 + p_n^2) = \|z^u\|^2.$$

Consequently, condition (3.42) *holds. Applying Theorem* 3.9*, we conclude, for* $|k|$ *sufficiently small, that system* (3.58) *is strongly stabilsable by using the feedback control*

$$v(t) = -\rho \frac{\displaystyle\int_0^1 \frac{\partial y(x,t)}{\partial t}\left(\frac{\partial y(x,t)}{\partial t} - k\frac{\partial y(x,t-r)}{\partial t}\right)dx}{1 + \left|\displaystyle\int_0^1 \frac{\partial y(x,t)}{\partial t}\left(\frac{\partial y(x,t)}{\partial t} - k\frac{\partial y(x,t-r)}{\partial t}\right)dx\right|}, \; t \geq 0, \; \rho > 0.$$

4. Conclusion

In this work, we considered the problem of strong stabilization of a non-homogeneous semi-linear delayed system of neutral type. The system was decomposed into the stable and the unstable parts. It was shown that stabilizing such a system reduces stabilizing only the unstable part which can make the whole system stable. Sufficient conditions for both polynomial and strong stabilization were given with an explicit decay estimate in the polynomial stabilization case.

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