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A TWO-STEP INERTIAL ALGORITHM FOR SOLVING THE VARIATIONAL INEQUALITY OVER THE SOLUTION SET OF A SPLIT COMMON FIXED POINT PROBLEM

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Abstract. We investigate the variational inequality problem over the solution set of a split common fixed point problem with multiple output sets for demicontractive mappings in real Hilbert spaces. In order to solve this problem, we propose a new two-step inertial iterative algorithm with a self-adaptive step size rule. Under mild conditions on the control parameters, a strong convergence result for the proposed algorithm is established. A numerical example is provided to illustrate the effectiveness of our proposed algorithm.

Keywords. Demicontractive mapping; Multiple output sets; Split common fixed point problem; Two-step inertial method; Variational inequality.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{H}_i$ $(i = 1,2,\ldots,N)$ be real Hilbert spaces endowed with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $F : \mathscr{H} \to \mathscr{H}$ be an operator. The *variational inequality problem*, denoted by $\mathrm{VIP}(F,\Omega)$, is to find a point u^* in Ω such that

$$\langle Fu^*, u-u^*\rangle \geq 0, \quad \forall u \in \Omega, \tag{1.1}$$

where Ω is a nonempty, convex, and closed subset of $\mathscr{H}$. This problem was originally introduced by Hartman and Stampacchia [8] and has been widely applied in nonlinear analysis, economics, operations research, and engineering sciences. By the classical projection technique, problem (1.1) can be re-written equivalently to a fixed point equation $u^* = P_\Omega(I-\lambda F)u^*$, where P_Ω denotes the metric projection onto Ω, I is the identity operator, and λ is some positive real number.

The *split feasibility problem* (SFP), originally introduced by Censor and Elfving [6] in finite-dimensional Euclidean spaces, has attracted attention recently due to its wide applications in

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signal processing, medical image reconstruction, and intensity-modulated radiation therapy (IMRT) treatment planning [1, 4, 5]. Specifically, given nonempty, closed, and convex subsets $C \subseteq \mathscr{H}_1$ and $Q \subseteq \mathscr{H}_2$ of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, and a bounded and linear operator $A\colon \mathscr{H}_1 \to \mathscr{H}_2$, the SFP is to find a point $x^\dagger \in C$ such that $Ax^\dagger \in Q$.

In 2020, Reich et al. [18] introduced and studied the *split feasibility problem with multiple output sets* (SFPMOS): for each $j = 1,2,\dots,N$, let $C \subseteq \mathscr{H}$ and $Q_j \subseteq \mathscr{H}_j$ be nonempty, convex, and closed subsets, and let $A_j : \mathscr{H} \to \mathscr{H}_j$ be a linear and bounded operator. The SFPMOS seeks a point $x^\dagger$ in $\mathscr{H}$ such that

$$x^\dagger \in C \cap \left(\bigcap_{j=1}^{N} A_j^{-1}(Q_j) \right),$$

where $A_j^{-1}(Q_j) = \{x \in \mathscr{H} : A_j x \in Q_j\}$.

Reich et al. [18] proposed the iterative algorithm

$$x_{k+1} = P_C\left(x_k - \tau \sum_{i=1}^{N} A_i^*(I - P_{Q_i})A_i x_k \right)$$

and established its weak convergence under the condition $0 < \tau < 2/\big(\max_{1\le i\le N} \|A_i\|^2\big)$. They also proposed another algorithm

$$y_{k+1} = \gamma_k f(y_k) + (1-\gamma_k)P_C\left(y_k - \tau \sum_{i=1}^{N} A_i^*(I - P_{Q_i})A_i y_k \right)$$

to achieve strong convergence, where f is a contraction and $\{\gamma_k\} \subseteq (0,1)$. However, the step size τ in both algorithms depends on $\max_{1\le i\le N} \|A_i\|^2$, which is generally difficult to compute in practice.

In 2022, Wang [24] extended the split feasibility framework to the setting of *demicontractive mappings*. For each $i = 0,1,\dots,N$, let T_i be a κ_i-demicontractive mapping. The *split common fixed point problem* (SCFP) with multiple output sets is to find x^* in $\mathscr{H}$ such that

$$x^* \in \mathrm{Fix}(T_0) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(\mathrm{Fix}(T_i)) \right). \tag{1.2}$$

They used the self-adaptive step size

$$\tau_k = \frac{\min_{1\le i\le N}(1-\kappa_i) \displaystyle\sum_{i=1}^{N} \|(I-T_i)A_i x_k\|^2}{\left\| \displaystyle\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_k \right\|^2}$$

to propose the following iterative algorithm

$$x_{k+1} = T_{0,\lambda}\left[x_k - \tau_k \sum_{i=1}^{N} A_i^*(I-T_i)A_i x_k \right],$$

where $T_{0,\lambda} = (1-\lambda)I + \lambda T_0$ with $\lambda \in (0, 1-\kappa_0)$, and proved that the sequence $\{x_k\}$ converges weakly to a solution of problem (1.2). The advantage of this self-adaptive step size is that it does not require prior computation of the operator norms. However, the convergence result is only weak, and no acceleration technique was employed.

Recently, Thi Thu Thuy et al. [22] proposed the following iterative algorithm with a self-adaptive step size $\tilde{\gamma}_k$ for solving the variational inequality problem:

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}), \\ y^k = \beta_k w^k + (1-\beta_k) P_C(I - \lambda \mathscr{A}) w^k, \\ z^k = y^k - \tilde{\gamma}_k \sum_{i=1}^{N} A_i^* \big[I - P_{Q_i}(I - \lambda B_i)\big] A_i y^k, \\ x^{k+1} = z^k - \alpha_k F z^k, \end{cases} \tag{1.3}$$

where $F\colon \mathscr{H} \to \mathscr{H}$ is a strongly monotone and Lipschitz continuous operator, and they established a strong convergence result. In (1.3), the term $\theta_k(x^k - x^{k-1})$ is called the *inertial term*, which is a key factor for improving the computational efficiency of the algorithm. Inertial acceleration, originally introduced by Polyak [17] in 1964, is an effective technique for improving the convergence speed of iterative algorithms and is motivated by the continuous analogue of the heavy ball method, namely the second-order dissipative dynamical system

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f\big(x(t)\big) = 0,$$

known as the *heavy ball system*, which models a particle moving under a potential field f with viscous friction coefficient $\gamma > 0$. The inertial term $\ddot{x}(t)$ captures the tendency of the particle to continue in the direction of its previous motion. Discretizing this continuous system yields an iterative scheme in which each new iterate incorporates an extrapolation step $x_k + \theta_k(x_k - x_{k-1})$, mimicking the memory effect of the continuous model. Since then, inertial techniques have been widely incorporated into various iterative schemes (see, e.g., [2, 3, 12, 13, 19–21] and the references therein).

Recently, Thuy and Nghia [23] and Thuy and Tung [22] proposed algorithms employing a single inertial parameter to accelerate the solution of the *variational inequality problem*. However, it was observed in [10] that the two-step inertial extrapolation can offer more flexibility and better performance compared with the single-step inertial extrapolation.

Consider the following form:

$$w_k = x_k + \theta_k(x_k - x_{k-1}) + \delta_k(x_{k-1} - x_{k-2}),$$

where θ_k and δ_k are inertial parameters. The introduction of two inertial parameters enhances stability and robustness, and provides superior acceleration compared to the classical one-step inertial technique (see, e.g., [11] and the reference therein).

In this paper, motivated by the works mentioned above, we first propose a new two-step inertial iterative algorithm for solving the variational inequality problem over the solution set of the split common fixed point problem with multiple output sets for demicontractive mappings. Then, we prove that the sequence generated by the proposed algorithm converges strongly to the unique solution of the *variational inequality problem*. Finally, we present an example including its numerical experiments to illustrate the effectiveness and computational advantage of the proposed algorithm.

2. Preliminaries

In this section, we collect some definitions and auxiliary results that are needed in the convergence analysis.

Let $\mathscr{H}$ denote a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C denote a nonempty, convex, and closed subset of $\mathscr{H}$. We write $x_k \rightharpoonup x$ ($x_k \to x$) to indicate that the sequence $\{x_k\}$ converges weakly (strongly) to x. The set of all fixed points of a mapping $T : C \to \mathscr{H}$ is denoted by $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$.

Definition 2.1. *Let C be a nonempty, convex, and closed subset of a real Hilbert space $\mathscr{H}$. A mapping $T : C \to C$ is said to be*
(1) k-Lipschitz continuous *if there exists a constant $k \geq 0$ such that*

$$\|Tx - Ty\| \leq k\|x - y\|,\ \forall x, y \in C.$$

If $k \in (0,1)$, then T is called a contraction*. If $k = 1$, then T is said to be* nonexpansive*.*
(2) firmly nonexpansive *if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2,\ \forall x, y \in C.$$

(3) k-strictly pseudocontractive *if there exists a constant $k \in (0,1)$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2,\ \forall x, y \in C.$$

(4) monotone *if*

$$\langle Tx - Ty, x - y\rangle \geq 0,\ \forall x, y \in C.$$

(5) η-strongly monotone *if there exists a constant $\eta > 0$ such that*

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2,\ \forall x, y \in C.$$

(6) κ-demicontractive *if $Fix(T) \neq \emptyset$ and there exists a constant $\kappa \in [0,1)$ such that*

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \kappa\|x - Tx\|^2,\ \forall x \in C, p \in \mathrm{Fix}(T).$$

Remark 2.2. The class of demicontractive mappings was introduced by Hicks and Kubicek [9] and Măruşter [15]. It is a broad class that encompasses the class of nonexpansive mappings, the class of quasi-nonexpansive mappings, and the class of strictly pseudocontractive mappings with nonempty fixed point sets as special cases.

We next recall several fundamental identities and lemmas.

Lemma 2.3. *Let $\mathscr{H}$ be a real Hilbert space. Then, for all $x, y \in \mathscr{H}$ and $\lambda \in (0,1)$, the following equalities hold:*
(1) $2\langle x, y\rangle = \|x + y\|^2 - \|x\|^2 - \|y\|^2 = \|x\|^2 + \|y\|^2 - \|x - y\|^2$.
(2) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2$.

Lemma 2.4 ([7])**.** *Let C be a nonempty, convex, and closed subset of a real Hilbert space $\mathscr{H}$, and let $T : C \to \mathscr{H}$ be a nonexpansive mapping. Then $I - T$ is demiclosed at the origin; that is, whenever $\{x_k\}$ is a sequence in C such that $x_k \rightharpoonup u^*$ and $(I - T)x_k \to 0$, it follows that $u^* \in \mathrm{Fix}(T)$.*

Lemma 2.5 ([16])**.** *Let $\mathscr{H}$ be a real Hilbert space, and let $T : \mathscr{H} \to \mathscr{H}$ be a κ-demicontractive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For any $\lambda > 0$, let $T_\lambda = (1-\lambda)I + \lambda T$. Then*
(1) $\mathrm{Fix}(T) = \mathrm{Fix}(T_\lambda)$ is a closed convex subset of $\mathscr{H}$;
(2) for each $x \in \mathscr{H}$ and $p \in \mathrm{Fix}(T)$, $\langle x - Tx, x - p\rangle \geq \frac{1-\kappa}{2}\|x - Tx\|^2$;
(3) for each $x \in \mathscr{H}$ and $p \in \mathrm{Fix}(T)$, $\|T_\lambda x - p\|^2 \leq \|x - p\|^2 - \lambda(1 - \lambda - \kappa)\|(I - T)x\|^2$.

Lemma 2.6 ([14])**.** *Let $\{s_k\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{s_{k_j}\}$ satisfying $s_{k_j} < s_{k_j+1}$ for all $j \in \mathbb{N}$. Define the integer sequence $\{v(k)\}_{k \geq k_0}$ by*

$v(k) = \max\{m \le k : s_m < s_{m+1}\}$, where $k_0 \in \mathbb{N}$ is chosen such that the set $\{m \le k_0 : s_m < s_{m+1}\}$ is nonempty. Then the following hold:
(1) $v(k_0) \le v(k_0+1) \le \cdots$ and $v(k) \to \infty$ as $k \to \infty$.
(2) $s_{v(k)} \le s_{v(k)+1}$ and $s_k \le s_{v(k)+1}$ for all $k \ge k_0$.

Lemma 2.7 ([26]). *Let $\{s_k\}$ be a sequence of nonnegative real numbers, $\{a_k\} \subset (0,1)$ with $\sum_{k=0}^{\infty} a_k = \infty$, and $\{b_k\}$ be a sequence of real numbers with $\limsup_{k\to\infty} b_k \le 0$. If $s_{k+1} \le (1-a_k)s_k + a_k b_k$, for all $k \ge 0$, then $\lim_{k\to\infty} s_k = 0$.*

3. The Two-step Inertial Algorithm and Its Convergence

In this section, we present a new two-step inertial iterative algorithm with a self-adaptive step size for solving the strongly monotone variational inequality problem over the solution set of the split common fixed point problem for demicontractive mappings in real Hilbert spaces. Under mild conditions on the control parameters, we establish the strong convergence of the proposed method.

Throughout this section, we impose the following assumptions:
(A1) $\mathscr{H}$ and $\mathscr{H}_i$ $(i = 1,2,\dots,N)$ are real Hilbert spaces, and $A_i\colon \mathscr{H} \to \mathscr{H}_i$ is a linear and bounded operator with adjoint A_i^* for each $i = 1,2,\dots,N$;
(A2) $F : \mathscr{H} \to \mathscr{H}$ is an η-strongly monotone and ℓ-Lipschitz continuous operator;
(A3) For each $i = 0,1,2,\dots,N$, $T_i\colon \mathscr{H}_i \to \mathscr{H}_i$ is a κ_i-demicontractive mapping with $\kappa_i \in [0,1)$ such that $I - T_i$ is demiclosed at the origin, where $\mathscr{H}_0 := \mathscr{H}$;
(A4) $\tilde{\Omega} = \mathrm{Fix}(T_0) \cap \bigcap_{i=1}^{N} A_i^{-1}\big(\mathrm{Fix}(T_i)\big) \ne \emptyset$.
And the control parameters satisfy the following conditions:
(C1) $\{\alpha_k\} \subset (0,1)$, $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$;
(C2) $\liminf_{k\to\infty} \rho_k(\tau - \rho_k) > 0$, where $\tau = \min_{1\le i\le N}(1-\kappa_i)$ and $\{\rho_k\} \subset (0,\tau)$;
(C3) $\liminf_{k\to\infty}(\beta_k - \kappa_0)(1-\beta_k) > 0$, $\{\beta_k\} \subset (\kappa_0, 1)$;
(C4) $\{\eta_k\} \subset (0,\infty)$, $\lim_{k\to\infty} \eta_k/\alpha_k = 0$.

We now state the proposed algorithm and the main convergence theorem.

Theorem 3.1. *Let $\mathscr{H}$, $\mathscr{H}_i$, F, A_i, T_i, $\tilde{\Omega}$ $(i = 1,2,\dots,N)$ satisfy assumptions* (A1)–(A4), *and $\{\alpha_k\}$, $\{\beta_k\}$, $\{\eta_k\}$ and $\{\rho_k\}$ satisfy conditions* (C1)–(C4). *For arbitrary $x^0, x^1, x^2 \in \mathscr{H}$, let the sequence $\{x^k\}$ be generated by*

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}) + \delta_k(x^{k-1} - x^{k-2}), \\ y^k = \beta_k w^k + (1-\beta_k)T_0 w^k, \\ z^k = y^k - \tilde{\gamma}_k \displaystyle\sum_{i=1}^{N} A_i^*(I - T_i)A_i y^k, \\ x^{k+1} = z^k - \alpha_k F z^k, \end{cases} \tag{3.1}$$

where $\{\theta_k\}$ and $\{\delta_k\}$ are defined respectively by

$$\theta_k = \begin{cases} \min\left\{\dfrac{\eta_k}{\|x^k - x^{k-1}\|}, \bar{\theta}\right\}, & \text{if } x^k \ne x^{k-1}, \\ \bar{\theta}, & \text{otherwise}, \end{cases} \tag{3.2}$$

and

$$\delta_k = \begin{cases} \max\left\{\dfrac{-\eta_k}{\|x^{k-1}-x^{k-2}\|}, -\bar{\delta}\right\}, & \textit{if } x^{k-1} \neq x^{k-2}, \\ -\bar{\delta}, & \textit{otherwise}, \end{cases} \tag{3.3}$$

with $\bar{\theta}, \bar{\delta} > 0$ *given constants and the self-adaptive step size* $\tilde{\gamma}_k$ *is defined by*

$$\tilde{\gamma}_k = \begin{cases} \rho_k \dfrac{\sum_{i=1}^{N} \|(I-T_i)A_i y^k\|^2}{\left\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i y^k\right\|^2}, & \textit{if } \sum_{i=1}^{N} A_i^*(I-T_i)A_i y^k \neq 0, \\ 0, & \textit{otherwise}. \end{cases} \tag{3.4}$$

Then the sequence $\{x^k\}$ *generated by* (3.1) *converges strongly to the point* $u^* \in \tilde{\Omega}$*, which is the unique solution of the variational inequality problem*

$$\langle Fu^*, u-u^*\rangle \geq 0, \quad \forall\, u \in \tilde{\Omega}.$$

Remark 3.2. From the definitions of $\{\theta_k\}$ and $\{\delta_k\}$ in (3.2)–(3.3), it is clear that $\theta_k \geq 0$ and $\delta_k \leq 0$ for all k. Moreover,

$$\lim_{k\to\infty} \frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = \lim_{k\to\infty} \frac{|\delta_k|}{\alpha_k}\|x^{k-1}-x^{k-2}\| = 0.$$

Indeed, by the definition of θ_k, we have $\theta_k\|x^k - x^{k-1}\| \leq \eta_k$, and hence

$$\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq \frac{\eta_k}{\alpha_k} \to 0 \quad (k\to\infty).$$

Similarly, by the definition of δ_k, we have $|\delta_k|\|x^{k-1}-x^{k-2}\| \leq \eta_k$, so that

$$\frac{|\delta_k|}{\alpha_k}\|x^{k-1}-x^{k-2}\| \leq \frac{\eta_k}{\alpha_k} \to 0 \quad (k\to\infty).$$

Furthermore, there exist two constants $M_1, M_2 > 0$ such that

$$\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq M_1, \quad \frac{|\delta_k|}{\alpha_k}\|x^{k-1}-x^{k-2}\| \leq M_2, \quad \forall\, k \geq 2.$$

Before proving Theorem 3.1, we establish two key estimates.

Lemma 3.3. *Let* $\{w^k\}$*,* $\{y^k\}$*, and* $\{z^k\}$ *be the sequences generated by* (3.1)*, and let* $u \in \tilde{\Omega}$*. Then for all* $k \geq 2$*,*

$$\|y^k - u\|^2 \leq \|w^k - u\|^2 - (\beta_k - \kappa_0)(1-\beta_k)\|(I-T_0)w^k\|^2 \tag{3.5}$$

and

$$\|z^k - u\|^2 \leq \|y^k - u\|^2 - (\tau - \rho_k)\tilde{\gamma}_k \sum_{i=1}^{N} \|(I-T_i)A_i y^k\|^2. \tag{3.6}$$

Proof. Let $u \in \tilde{\Omega}$. Then $u \in \mathrm{Fix}(T_0)$ and $A_i u \in \mathrm{Fix}(T_i)$ for all $i = 1,2,\ldots,N$. Using (3.1) and Lemma 2.3(2), we have

$$\begin{aligned} \|y^k - u\|^2 &= \|\beta_k(w^k - u) + (1-\beta_k)(T_0 w^k - u)\|^2 \\ &= \beta_k\|w^k - u\|^2 + (1-\beta_k)\|T_0 w^k - u\|^2 - \beta_k(1-\beta_k)\|w^k - T_0 w^k\|^2. \end{aligned}$$

Since T_0 is κ_0-demicontractive and $u \in \mathrm{Fix}(T_0)$, we have

$$\|T_0 w^k - u\|^2 \le \|w^k - u\|^2 + \kappa_0 \|w^k - T_0 w^k\|^2.$$

Therefore,

$$\begin{aligned}\|y^k - u\|^2 &\le \beta_k \|w^k - u\|^2 + (1-\beta_k)\left[\|w^k - u\|^2 + \kappa_0 \|w^k - T_0 w^k\|^2\right] \\ &\quad - \beta_k(1-\beta_k)\|w^k - T_0 w^k\|^2 \\ &= \|w^k - u\|^2 - (\beta_k - \kappa_0)(1-\beta_k)\|(I - T_0)w^k\|^2.\end{aligned}$$

Thus (3.5) holds. By (3.1) and the property of the adjoint operator, we have

$$\|z^k - u\|^2 = \|y^k - u\|^2 + \tilde{\gamma}_k^2 \left\| \sum_{i=1}^N A_i^*(I - T_i)A_i y^k \right\|^2 - 2\tilde{\gamma}_k \sum_{i=1}^N \langle A_i y^k - A_i u, (I - T_i)A_i y^k \rangle.$$

By Lemma 2.5 (2), for each $i = 1, 2, \ldots, N$, since T_i is κ_i-demicontractive and $A_i u \in \mathrm{Fix}(T_i)$, we deduce

$$\langle A_i y^k - A_i u, (I - T_i)A_i y^k \rangle \ge \frac{1-\kappa_i}{2}\|(I - T_i)A_i y^k\|^2.$$

Thus,

$$\|z^k - u\|^2 \le \|y^k - u\|^2 + \tilde{\gamma}_k^2 \left\| \sum_{i=1}^N A_i^*(I - T_i)A_i y^k \right\|^2 - \tau \tilde{\gamma}_k \sum_{i=1}^N \|(I - T_i)A_i y^k\|^2, \tag{3.7}$$

where $\tau = \min_{1 \le i \le N}(1 - \kappa_i)$. We consider two cases.

Case 1. If $\sum_{i=1}^N A_i^*(I - T_i)A_i y^k = 0$, then $\tilde{\gamma}_k = 0$ and $z^k = y^k$, so (3.6) holds trivially.

Case 2. If $\sum_{i=1}^N A_i^*(I - T_i)A_i y^k \neq 0$, substituting the definition of $\tilde{\gamma}_k$ from (3.4) into (3.7), we obtain

$$\|z^k - u\|^2 \le \|y^k - u\|^2 - (\tau - \rho_k)\tilde{\gamma}_k \sum_{i=1}^N \|(I - T_i)A_i y^k\|^2.$$

This completes the proof. □

We are now in a position to prove the main result.

Proof of Theorem 3.1. Since F is η-strongly monotone and ℓ-Lipschitz continuous and $\tilde{\Omega} \neq \emptyset$, by [25, Theorem 3.1, Step 1], we see that problem $\mathrm{VIP}(F, \tilde{\Omega})$ has a unique solution u^*.

Step 1. *We prove that sequence $\{x^k\}$ is bounded.*

For any $u \in \tilde{\Omega}$, by the triangle inequality, we have

$$\|x^{k+1} - u\| \le \|x^{k+1} - u + \alpha_k F u\| + \alpha_k \|F u\|, \quad k \ge 1. \tag{3.8}$$

Choose $\varepsilon \in \left(0, \frac{2\eta}{\ell^2}\right)$. Since $\lim_{k\to\infty} \alpha_k = 0$, there exists $k_0 \in \mathbb{N}$ such that $\alpha_k < \varepsilon$ for all $k \ge k_0$. By (3.1), for all $k \ge k_0$, we have

$$\begin{aligned}\|x^{k+1} - u + \alpha_k F u\| &= \left\| \left(1 - \frac{\alpha_k}{\varepsilon}\right)(z^k - u) + \frac{\alpha_k}{\varepsilon}\left[z^k - u - \varepsilon(F z^k - F u)\right] \right\| \\ &\le \left(1 - \frac{\alpha_k}{\varepsilon}\right)\|z^k - u\| + \frac{\alpha_k}{\varepsilon}\|z^k - u - \varepsilon(F z^k - F u)\|. \end{aligned} \tag{3.9}$$

Using the η-strong monotonicity and ℓ-Lipschitz continuity of F, we find

$$\begin{aligned}\|z^k-u-\varepsilon(Fz^k-Fu)\|^2 &= \|z^k-u\|^2+\varepsilon^2\|Fz^k-Fu\|^2-2\varepsilon\langle z^k-u,\,Fz^k-Fu\rangle\\ &\le \left[1-\varepsilon(2\eta-\varepsilon\ell^2)\right]\|z^k-u\|^2. \end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10), we obtain

$$\|x^{k+1}-u+\alpha_k Fu\| \le \left(1-\frac{\alpha_k\rho}{\varepsilon}\right)\|z^k-u\|, \quad \forall\, k\ge k_0, \tag{3.11}$$

where $\rho := 1-\sqrt{1-\varepsilon(2\eta-\varepsilon\ell^2)} \in (0,1)$. By (3.5), (3.6) and the conditions (C2)–(C3), there exists $k_1\ge k_0$ such that, for all $k\ge k_1$,

$$\|z^k-u\| \le \|y^k-u\| \le \|w^k-u\|. \tag{3.12}$$

By (3.1) and the triangle inequality, we obtain

$$\|w^k-u\| \le \|x^k-u\| + \alpha_k\left(\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\| + \frac{|\delta_k|}{\alpha_k}\|x^{k-1}-x^{k-2}\|\right).$$

By Remark 3.2, setting $M_3 := M_1+M_2$, we arrive at

$$\|w^k-u\| \le \|x^k-u\| + \alpha_k M_3. \tag{3.13}$$

Combining (3.8), (3.11), (3.12), and (3.13), for $k\ge k_1$, we have

$$\begin{aligned}\|x^{k+1}-u\| &\le \left(1-\frac{\alpha_k\rho}{\varepsilon}\right)(\|x^k-u\|+\alpha_k M_3)+\alpha_k\|Fu\|\\ &\le \left(1-\frac{\alpha_k\rho}{\varepsilon}\right)\|x^k-u\| + \frac{\alpha_k\rho}{\varepsilon}\cdot\frac{\varepsilon(M_3+\|Fu\|)}{\rho}\\ &\le \max\left\{\|x^k-u\|,\ \frac{\varepsilon(M_3+\|Fu\|)}{\rho}\right\}.\end{aligned}$$

By induction, for every $k\ge k_1$, we find that

$$\|x^k-u\| \le \max\left\{\|x^{k_1}-u\|,\ \frac{\varepsilon(M_3+\|Fu\|)}{\rho}\right\}.$$

Therefore, $\{x^k\}$ is bounded, and consequently $\{w^k\}$, $\{y^k\}$, $\{z^k\}$ and $\{Fz^k\}$ are all bounded.

Step 2. *We prove that $\{x^k\}$ converges strongly to u^*.*

From (3.11), using $\|a+b\|^2\le\|a\|^2+2\langle b,a+b\rangle$, we obtain

$$\begin{aligned}\|x^{k+1}-u\|^2 &\le \|x^{k+1}-u+\alpha_k Fu\|^2+2\alpha_k\langle Fu,\,u-x^{k+1}\rangle\\ &\le \left(1-\frac{\alpha_k\rho}{\varepsilon}\right)^2\|z^k-u\|^2+2\alpha_k\langle Fu,\,u-x^{k+1}\rangle\\ &\le \left(1-\frac{\alpha_k\rho}{\varepsilon}\right)\|z^k-u\|^2+2\alpha_k\langle Fu,\,u-x^{k+1}\rangle. \end{aligned}\tag{3.14}$$

By (3.1), we have

$$\begin{aligned}\|w^k-u\|^2 &= \|x^k-u+\theta_k(x^k-x^{k-1})+\delta_k(x^{k-1}-x^{k-2})\|^2\\ &\le \|x^k-u\|^2+2\theta_k\|x^k-u\|\,\|x^k-x^{k-1}\|+2|\delta_k|\|x^k-u\|\,\|x^{k-1}-x^{k-2}\|\\ &\quad+\theta_k^2\|x^k-x^{k-1}\|^2+2\theta_k|\delta_k|\|x^k-x^{k-1}\|\,\|x^{k-1}-x^{k-2}\|+\delta_k^2\|x^{k-1}-x^{k-2}\|^2. \end{aligned}\tag{3.15}$$

Using (3.12), (3.14), and (3.15), for all $k \geq k_1$, we conclude

$$\begin{aligned} \|x^{k+1}-u\|^2 \leq \Big(1-\frac{\alpha_k\rho}{\varepsilon}\Big)\|x^k-u\|^2 + \frac{\alpha_k\rho}{\varepsilon}\cdot\frac{\varepsilon}{\rho}\Big[\frac{2\theta_k}{\alpha_k}\|x^k-u\|\,\|x^k-x^{k-1}\| \\ + \frac{2|\delta_k|}{\alpha_k}\|x^k-u\|\,\|x^{k-1}-x^{k-2}\| + \frac{\theta_k^2}{\alpha_k}\|x^k-x^{k-1}\|^2 \\ + \frac{2\theta_k|\delta_k|}{\alpha_k}\|x^k-x^{k-1}\|\,\|x^{k-1}-x^{k-2}\| + \frac{\delta_k^2}{\alpha_k}\|x^{k-1}-x^{k-2}\|^2 \\ + 2\langle Fu, u-x^{k+1}\rangle\Big]. \end{aligned} \tag{3.16}$$

From (3.13), we also obtain

$$\|w^k-u^*\|^2 \leq \|x^k-u^*\|^2 + \alpha_k M_4, \quad k \geq 1, \tag{3.17}$$

where $M_4 := \sup_{k\geq 1}\{\alpha_k M_3^2 + 2M_3\|x^k-u^*\|\} < \infty$. Using Lemma 3.3, (3.14), (3.17) and the conditions (C2)–(C3), for all $k \geq k_1$, we have

$$\begin{aligned} 0 &\leq (\beta_k-\kappa_0)(1-\beta_k)\|(I-T_0)w^k\|^2 + (\tau-\rho_k)\tilde{\gamma}_k\sum_{i=1}^N\|(I-T_i)A_iy^k\|^2 \\ &\leq \big(\|x^k-u^*\|^2-\|x^{k+1}-u^*\|^2\big) - \frac{\alpha_k\rho}{\varepsilon}\|z^k-u^*\|^2 + 2\alpha_k\langle Fu^*, u^*-x^{k+1}\rangle + \alpha_k M_4. \end{aligned} \tag{3.18}$$

We now consider two cases.

Case 1. *Suppose there exists an integer $k^* \geq k_1$ such that $\{\|x^k-u^*\|^2\}_{k\geq k^*}$ is monotonically non-increasing.*

In this case, $\lim_{k\to\infty}\|x^k-u^*\|^2$ exists. Since $\{x^k\}$ and $\{y^k\}$ are bounded, from conditions (C1)–(C3) and (3.18), we deduce

$$\lim_{k\to\infty}\|(I-T_0)w^k\| = 0 \tag{3.19}$$

and

$$\lim_{k\to\infty}(\tau-\rho_k)\tilde{\gamma}_k\sum_{i=1}^N\|(I-T_i)A_iy^k\|^2 = 0. \tag{3.20}$$

Furthermore, we have

$$(\tau-\rho_k)\rho_k\frac{\Big(\sum_{i=1}^N\|(I-T_i)A_iy^k\|^2\Big)^2}{\Big\|\sum_{i=1}^N A_i^*(I-T_i)A_iy^k\Big\|^2} \to 0 \quad (k\to\infty),$$

which together with condition (C2) implies that

$$\lim_{k\to\infty}\frac{\sum_{i=1}^N\|(I-T_i)A_iy^k\|^2}{\Big\|\sum_{i=1}^N A_i^*(I-T_i)A_iy^k\Big\|} = 0. \tag{3.21}$$

Let $M>0$ be such that $\max_{1\leq i\leq N}\|A_i^*\| \leq M$. By the definition of $\tilde{\gamma}_k$ in (3.4), we have

$$\tilde{\gamma}_k = \rho_k\frac{\sum_{i=1}^N\|(I-T_i)A_iy^k\|^2}{\Big\|\sum_{i=1}^N A_i^*(I-T_i)A_iy^k\Big\|^2} \geq \frac{\rho_k}{M^2N}.$$

In view of condition (C2) and (3.20), we find

$$\lim_{k\to\infty}\sum_{i=1}^{N}\|(I-T_i)A_iy^k\|^2=0. \tag{3.22}$$

By (3.1), $\{\beta_k\}\subset(0,1)$, (3.19) and (3.21), we have

$$\|y^k-w^k\|=(1-\beta_k)\|(I-T_0)w^k\|\to 0 \quad (k\to\infty), \tag{3.23}$$

$$\|z^k-y^k\|\le\tilde{\gamma}_k\left\|\sum_{i=1}^{N}A_i^*(I-T_i)A_iy^k\right\|=\rho_k\frac{\sum_{i=1}^{N}\|(I-T_i)A_iy^k\|^2}{\left\|\sum_{i=1}^{N}A_i^*(I-T_i)A_iy^k\right\|}\to 0 \quad (k\to\infty). \tag{3.24}$$

By (3.1) and Remark 3.2, we see that

$$\begin{aligned}\|w^k-x^k\|&\le\theta_k\|x^k-x^{k-1}\|+|\delta_k|\|x^{k-1}-x^{k-2}\|\\&=\alpha_k\left(\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\frac{|\delta_k|}{\alpha_k}\|x^{k-1}-x^{k-2}\|\right)\to 0 \quad (k\to\infty).\end{aligned} \tag{3.25}$$

Combining (3.23), (3.24), and (3.25), we obtain

$$\|z^k-x^k\|\le\|z^k-y^k\|+\|y^k-w^k\|+\|w^k-x^k\|\to 0 \quad (k\to\infty).$$

We now show that $\limsup_{k\to\infty}\langle Fu^*,u^*-x^{k+1}\rangle\le 0$. Since $\{z^k\}$ is bounded, there exists a subsequence $\{z^{k_l}\}$ such that $z^{k_l}\rightharpoonup\hat{u}$ and

$$\limsup_{k\to\infty}\langle Fu^*,u^*-z^k\rangle=\lim_{l\to\infty}\langle Fu^*,u^*-z^{k_l}\rangle=\langle Fu^*,u^*-\hat{u}\rangle. \tag{3.26}$$

We claim that $\hat{u}\in\tilde{\Omega}$. From $z^{k_l}\rightharpoonup\hat{u}$, (3.23) and (3.24), it follows that $y^{k_l}\rightharpoonup\hat{u}$ and $w^{k_l}\rightharpoonup\hat{u}$. Since $I-T_0$ is demiclosed at the origin, (3.19) yields $\hat{u}\in\mathrm{Fix}(T_0)$. Moreover, since each A_i is a bounded linear operator, $A_iy^{k_l}\rightharpoonup A_i\hat{u}$. Since $I-T_i$ is demiclosed at the origin, (3.22) gives $A_i\hat{u}\in\mathrm{Fix}(T_i)$ for all $i=1,2,\dots,N$. Therefore, $\hat{u}\in\tilde{\Omega}$. Since u^* is the unique solution of VIP$(F,\tilde{\Omega})$, we have $\langle Fu^*,\hat{u}-u^*\rangle\ge 0$, which together with (3.26) implies that $\limsup_{k\to\infty}\langle Fu^*,u^*-z^k\rangle\le 0$. In view of the boundedness of $\{Fz^k\}$ and $\lim_{k\to\infty}\alpha_k=0$, we have

$$\begin{aligned}\limsup_{k\to\infty}\langle Fu^*,u^*-x^{k+1}\rangle&=\limsup_{k\to\infty}\left[\langle Fu^*,u^*-z^k\rangle+\alpha_k\langle Fu^*,Fz^k\rangle\right]\\&=\limsup_{k\to\infty}\langle Fu^*,u^*-z^k\rangle\le 0.\end{aligned} \tag{3.27}$$

Setting $u=u^*$ in (3.16), we can write

$$\|x^{k+1}-u^*\|^2\le\left(1-\frac{\alpha_k\rho}{\varepsilon}\right)\|x^k-u^*\|^2+\frac{\alpha_k\rho}{\varepsilon}\cdot b_k, \quad k\ge k_1, \tag{3.28}$$

where

$$\begin{aligned}b_k=\frac{\varepsilon}{\rho}\Bigg[&\frac{2\theta_k}{\alpha_k}\|x^k-u^*\|\,\|x^k-x^{k-1}\|+\frac{2|\delta_k|}{\alpha_k}\|x^k-u^*\|\,\|x^{k-1}-x^{k-2}\|\\&+\frac{\theta_k^2}{\alpha_k}\|x^k-x^{k-1}\|^2+\frac{2\theta_k|\delta_k|}{\alpha_k}\|x^k-x^{k-1}\|\,\|x^{k-1}-x^{k-2}\|\\&+\frac{\delta_k^2}{\alpha_k}\|x^{k-1}-x^{k-2}\|^2+2\langle Fu^*,u^*-x^{k+1}\rangle\Bigg].\end{aligned}$$

By (3.27) and Remark 3.2, we obtain

$$\limsup_{k\to\infty} b_k \leq 0. \tag{3.29}$$

Since $0 < \alpha_k < \varepsilon$ for $k \geq k_0$, we have $\frac{\alpha_k \rho}{\varepsilon} \in (0,1)$ for $k \geq k_0$. By using (3.28), condition (C1), (3.29) and Lemma 2.7, we obtain $\lim_{k\to\infty} \|x^k - u^*\|^2 = 0$, i.e., $x^k \to u^* \quad (k \to \infty)$.

Case 2. *Assume that there exists a subsequence $\{k_{m_l}\} \subset \{k\}$ such that $\|x^{k_{m_l}} - u^*\| < \|x^{k_{m_l}+1} - u^*\|$ for all $m_l \geq 0$.*

By Lemma 2.6, there exists a nondecreasing sequence $\{\nu(k)\}_{k\geq\tilde{k}}$ with $\nu(k) \to \infty$ as $k \to \infty$ such that

$$\|x^{\nu(k)} - u^*\| \leq \|x^{\nu(k)+1} - u^*\| \tag{3.30}$$

and

$$\|x^k - u^*\| \leq \|x^{\nu(k)+1} - u^*\|, \quad \forall\, k \geq \tilde{k}. \tag{3.31}$$

Choose $k_2 \geq \max\{k_1, \tilde{k}\}$ such that $\nu(k) \geq k_1$ for all $k \geq k_2$. From (3.16) , for all $k \geq k_2$, we have

$$\|x^{\nu(k)+1} - u^*\|^2 \leq \left(1 - \frac{\alpha_{\nu(k)}\rho}{\varepsilon}\right) \|x^{\nu(k)} - u^*\|^2 + \alpha_{\nu(k)}\, c_{\nu(k)}, \tag{3.32}$$

where

$$\begin{aligned} c_{\nu(k)} = {} & \frac{2\theta_{\nu(k)}}{\alpha_{\nu(k)}} \|x^{\nu(k)} - u^*\|\, \|x^{\nu(k)} - x^{\nu(k)-1}\| + \frac{2|\delta_{\nu(k)}|}{\alpha_{\nu(k)}} \|x^{\nu(k)} - u^*\|\, \|x^{\nu(k)-1} - x^{\nu(k)-2}\| \\ & + \frac{\theta^2_{\nu(k)}}{\alpha_{\nu(k)}} \|x^{\nu(k)} - x^{\nu(k)-1}\|^2 + \frac{2\theta_{\nu(k)}|\delta_{\nu(k)}|}{\alpha_{\nu(k)}} \|x^{\nu(k)} - x^{\nu(k)-1}\|\, \|x^{\nu(k)-1} - x^{\nu(k)-2}\| \\ & + \frac{\delta^2_{\nu(k)}}{\alpha_{\nu(k)}} \|x^{\nu(k)-1} - x^{\nu(k)-2}\|^2 + 2\langle Fu^*, u^* - x^{\nu(k)+1}\rangle. \end{aligned}$$

From (3.30) and (3.32), we obtain

$$0 \leq \|x^{\nu(k)+1} - u^*\|^2 - \|x^{\nu(k)} - u^*\|^2 \leq -\frac{\alpha_{\nu(k)}\rho}{\varepsilon} \|x^{\nu(k)} - u^*\|^2 + \alpha_{\nu(k)}\, c_{\nu(k)},$$

which implies $\frac{\rho}{\varepsilon}\|x^{\nu(k)} - u^*\|^2 \leq c_{\nu(k)}$. From (3.18) and (3.30), we can repeat the similar arguments as in Case 1 to obtain

$$\lim_{k\to\infty} \|(I - T_0)w^{\nu(k)}\| = 0, \quad \lim_{k\to\infty} \sum_{i=1}^{N} \|(I - T_i)A_i y^{\nu(k)}\|^2 = 0,$$

and consequently $\limsup_{k\to\infty} \langle Fu^*, u^* - z^{\nu(k)}\rangle \leq 0$. By the boundedness of $\{Fz^k\}$ and $\lim_{k\to\infty} \alpha_k = 0$, we have $\limsup_{k\to\infty} c_{\nu(k)} \leq 0$. Using (3.30)–(3.32), we have

$$\|x^{\nu(k)+1} - u^*\|^2 \leq \left(1 - \frac{\alpha_{\nu(k)}\rho}{\varepsilon}\right) \|x^{\nu(k)+1} - u^*\|^2 + \alpha_{\nu(k)}\, c_{\nu(k)},$$

which gives $\|x^{\nu(k)+1} - u^*\|^2 \leq \frac{\varepsilon}{\rho} c_{\nu(k)}$. For all $k \geq k_2$, it follows from (3.31) that

$$\|x^k - u^*\|^2 \leq \|x^{\nu(k)+1} - u^*\|^2 \leq \frac{\varepsilon}{\rho} c_{\nu(k)}. \tag{3.33}$$

Letting $k \to \infty$, we arrive at $\limsup_{k\to\infty} \|x^k - u^*\|^2 \leq 0$, which implies $x^k \to u^*$. This completes the proof. $\square$

Remark 3.4. Theorem 3.1 extends and develops Theorem 3.1 in [22] from the following aspects:
(a) the single-step inertial extrapolation is extended to the two-step inertial extrapolation;
(b) nonexpansive mappings are extended to demicontractive mappings.

From Theorem 3.1 we have the following corollary.

Corollary 3.5. *Let $\mathscr{H}$, $\mathscr{H}_i$, F, A_i, T_i $(i = 1,2,\ldots,N)$ satisfy assumptions* (A1)–(A2)*, and the sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{\eta_k\}$, $\{\delta_k\}$, $\{\rho_k\}$ satisfy conditions (C1)–(C4) with $\kappa_i = 0$ for all $i = 0,1,\ldots,N$. For arbitrary $x^0, x^1, x^2 \in \mathscr{H}$, let the sequence $\{x^k\}$ be generated by*

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}) + \delta_k(x^{k-1} - x^{k-2}), \\ y^k = \beta_k w^k + (1-\beta_k) P_C(w^k), \\ z^k = y^k - \tilde{\gamma}_k \sum_{i=1}^{N} A_i^*(I - P_{Q_i}) A_i y^k, \\ x^{k+1} = z^k - \alpha_k F z^k, \end{cases} \tag{3.34}$$

where θ_k, δ_k and $\tilde{\gamma}_k$ are defined by (3.2), (3.3) *and* (3.4)*, respectively. Suppose that $\hat{\Omega} = C \cap \big(\bigcap_{i=1}^N A_i^{-1}(Q_i)\big) \neq \emptyset$. Then the sequence $\{x^k\}$ defined by* (3.34) *converges strongly to $u^* \in \hat{\Omega}$, which is the unique solution to the variational inequality, $\langle Fu^*, u - u^*\rangle \geq 0$, for all u in $\hat{\Omega}$.*

Proof. Set $T_0 = P_C$ and $T_i = P_{Q_i}$ for $i = 1,2,\ldots,N$. Since the metric projection onto a convex and closed set is nonexpansive, each T_i $(i = 0,1,\ldots,N)$ is nonexpansive. By Remark 2.2, each T_i $(i = 0,1,\ldots,N)$ is 0-demicontractive. It follows from Lemma 2.4 that $I - T_i$ $(i = 0,1,\ldots,N)$ is demiclosed at the origin. Thus, assumption (A3) is satisfied.

Since $\mathrm{Fix}(P_C) = C$ and $\mathrm{Fix}(P_{Q_i}) = Q_i$ for each $i = 1,2,\ldots,N$, we have

$$\hat{\Omega} = \mathrm{Fix}(T_0) \cap \Big(\bigcap_{i=1}^{N} A_i^{-1}\big(\mathrm{Fix}(T_i)\big)\Big) = C \cap \Big(\bigcap_{i=1}^{N} A_i^{-1}(Q_i)\Big) \neq \emptyset,$$

which implies that assumption (A4) is satisfied. Thus, by Theorem 3.1, the conclusion holds. □

4. Numerical Example

In this section, we present a numerical example to demonstrate the efficiency of our algorithm (3.34) in comparison with Algorithm 3.1 (with $\mathscr{A} = 0$, $B_j = 0$) in Thuy and Tung [22].

Example 4.1. *Let $\mathscr{H} = \mathbb{R}^2$, $\mathscr{H}_i = \mathbb{R}^{2+i}$ for $i = 1,2,\ldots,5$, and $N = 5$. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. Define the constraint set $C = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 + 2x_2 \leq 1\}$ and the closed balls $Q_i = \{y \in \mathbb{R}^{2+i} : \|y\| \leq i\}$ for $i = 1,2,\ldots,5$. For each $i = 1,2,\ldots,5$, define the bounded linear operator $A_i\colon \mathbb{R}^2 \to \mathbb{R}^{2+i}$ by the matrix*

$$A_i = \begin{pmatrix} 0 & 1 \\ -2 & 0 \\ 1 & 0 \\ 2 & 0 \\ \vdots & \\ i & 0 \end{pmatrix}_{(2+i)\times 2}.$$

Then the sequence $\{x^k\}$ generated by the algorithm (3.34) converges strongly to $u^* = (0,0)^\top \in \hat{\Omega}$, where $\hat{\Omega} = C \cap \left(\bigcap_{i=1}^5 A_i^{-1}(Q_i)\right)$.

Solution. It is easy to see that F is 2-strongly monotone. Moreover, $\|F\| = \sqrt{5}$, so F is $\sqrt{5}$-Lipschitz continuous. Hence the assumption (A2) is satisfied. Set $T_0 = P_C$ and $T_i = P_{Q_i}$ for $i = 1,2,\dots,5$. Then, each T_i $(i = 0,1,\dots,5)$ is 0-demicontractive with $\kappa_i = 0$ and $I - T_i$ is demiclosed at the origin. Thus, assumption (A3) is satisfied. For any $i \in \{1,2,\dots,5\}$, the projection formulas are given by

$$P_C(x) = x - \max\left\{0,\ \frac{x_1 + 2x_2 - 1}{5}\right\}\begin{pmatrix}1\\2\end{pmatrix}, \qquad P_{Q_i}(y) = \begin{cases} y, & \text{if } \|y\| \le i,\\ \dfrac{i \cdot y}{\|y\|}, & \text{if } \|y\| > i. \end{cases}$$

It is straightforward to verify that $u^* = (0,0)^\top \in C \cap \left(\bigcap_{i=1}^5 A_i^{-1}(Q_i)\right) = \hat{\Omega} \neq \emptyset$, and the unique solution of $\mathrm{VIP}(F,\hat{\Omega})$ is $u^* = (0,0)^\top$. Hence, assumption (A4) is satisfied. Take $\alpha_k = 0.3/(k+1)$, $\eta_k = 100/(k+1)^{1.5}$, $\beta_k = 0.7$, $\rho_k = 0.95$, $\bar{\theta} = 0.15$, and $\bar{\delta} = 0.35$. It can easily be checked that all the conditions (C1)–(C4) in Corollary 3.5 are satisfied. Algorithm (3.34) is reduced to the following:

$$\begin{cases} w^k = x^k + \theta_k(x^k - x^{k-1}) + \delta_k(x^{k-1} - x^{k-2}),\\ y^k = 0.7\, w^k + 0.3\, P_C(w^k),\\ z^k = y^k - \tilde{\gamma}_k \displaystyle\sum_{i=1}^{5} A_i^\top (I - P_{Q_i}) A_i y^k,\\ x^{k+1} = z^k - \dfrac{0.3}{k+1} F z^k, \end{cases} \tag{4.1}$$

where

$$\theta_k = \begin{cases} \min\left\{\dfrac{100/(k+1)^{1.5}}{\|x^k - x^{k-1}\|},\ 0.15\right\}, & \text{if } x^k \neq x^{k-1},\\ 0.15, & \text{otherwise}, \end{cases} \tag{4.2}$$

$$\delta_k = \begin{cases} \max\left\{\dfrac{-100/(k+1)^{1.5}}{\|x^{k-1} - x^{k-2}\|},\ -0.35\right\}, & \text{if } x^{k-1} \neq x^{k-2},\\ -0.35, & \text{otherwise}, \end{cases} \tag{4.3}$$

and

$$\tilde{\gamma}_k = \begin{cases} \dfrac{0.95 \displaystyle\sum_{i=1}^{5} \|(I - P_{Q_i}) A_i y^k\|^2}{\left\| \displaystyle\sum_{i=1}^{5} A_i^\top (I - P_{Q_i}) A_i y^k \right\|^2}, & \text{if } \displaystyle\sum_{i=1}^{5} A_i^\top (I - P_{Q_i}) A_i y^k \neq 0,\\ 0, & \text{otherwise}. \end{cases} \tag{4.4}$$

Hence, the sequence $\{x^k\}$ generated by (4.1) converges strongly to $u^* = (0,0)^\top \in \hat{\Omega}$.

The Algorithm 3.1 (with $\mathscr{A} = 0$, $B_j = 0$) in Thuy and Tung [22] uses the same parameters but with $\delta_k \equiv 0$ (single-step inertial). We choose two different initial values to compare our algorithm with Algorithm 3.1 in [22]:

- Case (a): $x^0 = (5,8)^\top$, $x^1 = (4,7)^\top$, $x^2 = (3,6)^\top$;

- Case (b): $x^0 = (3,-5)^\top$, $x^1 = (2,-4)^\top$, $x^2 = (1,-3)^\top$.

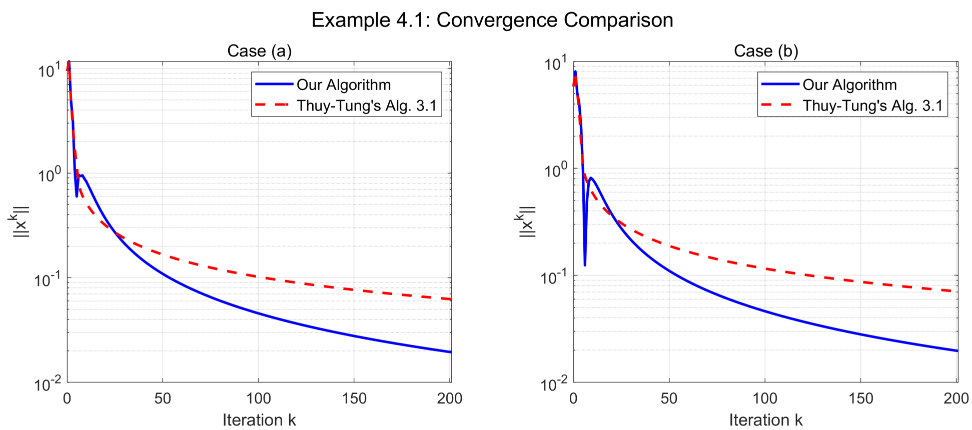

FIGURE 1. Numerical results for Example 4.1: Case (a) (left) and Case (b) (right).

TABLE 1. Computational results for Example 4.1, Case (a).

k	Our Algorithm	Thuy–Tung's Alg. 3.1
0	9.4340e+00	9.4340e+00
1	1.1662e+01	1.1662e+01
3	3.1325e+00	2.7682e+00
5	5.9748e−01	1.1709e+00
10	8.3214e−01	5.1710e−01
15	5.3857e−01	3.8858e−01
20	3.6743e−01	3.1728e−01
25	2.7118e−01	2.7111e−01
30	2.1213e−01	2.3841e−01
40	1.4515e−01	1.9464e−01
50	1.0886e−01	1.6630e−01
60	8.6356e−02	1.4623e−01
70	7.1135e−02	1.3116e−01
80	6.0206e−02	1.1937e−01
90	5.2008e−02	1.0985e−01
100	4.5650e−02	1.0198e−01
120	3.6463e−02	8.9669e−02
140	3.0180e−02	8.0427e−02
160	2.5632e−02	7.3195e−02
180	2.2200e−02	6.7357e−02
200	1.9527e−02	6.2530e−02

From Figure 1 and Tables 1–2, it is obvious that our algorithm converges faster than the Algorithm 3.1 in [22].

TABLE 2. Computational results for Example 4.1, Case (b).

k	Our Algorithm	Thuy–Tung's Alg. 3.1
0	5.8310e+00	5.8310e+00
1	8.0623e+00	8.0623e+00
3	4.0390e+00	3.4790e+00
5	9.1925e−01	1.0453e+00
10	7.8849e−01	5.8713e−01
15	5.4606e−01	4.4123e−01
20	3.7349e−01	3.6027e−01
25	2.7506e−01	3.0784e−01
30	2.1497e−01	2.7071e−01
40	1.4705e−01	2.2101e−01
50	1.1029e−01	1.8883e−01
60	8.7489e−02	1.6604e−01
70	7.2068e−02	1.4893e−01
80	6.0996e−02	1.3554e−01
90	5.2691e−02	1.2473e−01
100	4.6248e−02	1.1580e−01
120	3.6942e−02	1.0182e−01
140	3.0576e−02	9.1324e−02
160	2.5968e−02	8.3111e−02
180	2.2492e−02	7.6483e−02
200	1.9783e−02	7.1002e−02

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