J. Nonlinear Funct. Anal. 2026 (2026) 20 https://doi.org/10.23952/jnfa.2026.20

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

C^*-ALGEBRA-VALUED $\mathscr{S}$-METRIC SPACES FOR QUARTET MAPPINGS WITH AN APPLICATION

GUNASEELAN MANI[1], BENITHA WISES SAMUEL[2], RAJAGOPALAN RAMASWAMY[3,*], A.S. HAMARSHEH [3], OLA ASHOUR ABDELNABY[3], OZGUR EGE[4]

[1]Research Centre of Excellence, Rajalakshmi Engineering College, Chennai 602105, India
[2]Department of Mathematics, Rajalakshmi Institute of Technology, Chennai 600124, India
[3]Department of Mathematics, College of Science and Humanities, Prince Sattam Bin Adbulaziz University, Alkharj 11942, Saudi Arabia
[4]Department of Mathematics, Faculty of Science, Ege University, 35100 Bornova, Turkey

Abstract. In this paper, we establish common fixed point theorems under generalized contractive conditions in a C^*-algebra-valued $\mathscr{S}$-metric framework for quartet distinct mappings. We include an example to demonstrate the applicability of the results obtained in this paper. Additionally, we provide an application to prove the existence and uniqueness of solutions for a system of Volterra-type integral equations.
Keywords. C^*-algebra-valued metric space; Compatible mappings; Common fixed point; Volterra-type integral equations.
2020 Mathematics Subject Classification. 47H10, 54E35, 26A33.

1. Introduction

A C^*-algebra-valued $\mathscr{S}$-metric space is a generalization of the classical metric space concept, where the metric is replaced by a map that takes values in a C^*-algebra, a type of algebra that arises in functional analysis. This framework is used to study geometric properties in spaces where distances are not just real numbers, but elements of a more complex algebraic structure. The study of multiple mappings in a metric space involves examining how these mappings interact with each other and with the underlying space. This interaction can include analyzing fixed points, the behavior of sequences under these mappings, and the conditions under which the mappings share common properties such as compatibility.

In 1922, Banach [1] introduced Banach fixed point theorem to demonstrate the existence of solutions for integral equations and nonlinear operator equations. In 1966, Jungck [2] introduced the concept of a common fixed point (CFP) for commuting mappings in metric spaces.

*Corresponding author.
E-mail address: r.gopalan@psau.edu.sa (R. Ramaswamy).
Received February 23, 2026; Accepted July 17, 2026.

The concept of the compatibility was introduced by Jungck [3]. In 2007, A CFP of quartet maps in complete metric spaces was proved by Sedghi et al. [4]. Later, many authors followed this research by introducing new CFP results for various mappings; see, e.g., [5–7]. In 2012, Sedghi et al. [8] introduced the notion of $\mathscr{S}$-metric spaces. Banach fixed point theorem in $\mathscr{S}$-metric spaces was generalized by Sedghi and Shobe [9]. A common fixed point theorem for expansive mappings in S-metric space was proved by Chouhan [10]. Sedghi et al. [11] proved new fixed point theorem in $\mathscr{S}$-metric spaces. Sedghi et al. [12] presented a common fixed point theorem for multivalued maps on complete S-metric spaces. Hieu et al. [13] proved a fixed point theorem for a class of functions on S-metric spaces. Ege et al. [14] introduced modular S-metric spaces and investigated their properties. Sedghi et al. [15] proved some common fixed point results for quartet mappings satisfying generalized contractive condition in S-metric space. In 2014, the notion C^*-algebra-valued metric space(C^*AVMS) was introduced by Ma et al. [16] and they investigated some nonlinear operators with contractive conditions. Fixed points on C^*AVMS was generalized and proved by Batul and Kamran [**?**]. Later, authors further generalized and presented applications to find solutions to differential and integral equations by using various contractive conditions in the setting of metric and metric like spaces including C^*AVMS; see, e.g., [17–23]. In 2018, Ege et al. [24] proved some fixed point theorems on C^*-algebra-valued $\mathscr{S}$-metric space(C^*AVSMS). Samuel et al. [25] established a coupled fixed point theorem on orthogonal $\mathscr{S}$-metric spaces. Further, the existence and uniqueness of solutions to fractional integral equations have been extensively investigated; see, e.g., [**?**, **?**, 26–30].

In this paper, we prove some common fixed point theorems for quartet mappings on C^*AVSMS under contractive conditions. We furnish suitable examples to illustrate the validity of the hypotheses of our results. Moreover, we present an application to show the existence and uniqueness of a solution for system of volterra type integral equations. The rest of the paper is organised as follows: In Section-2, we present some preliminary definitions, example, and lemmas. In Section-3, we present our main results and establish fixed point results and provide examples to supplement the derived results. In Setion-4, the derived result is applied to find solutions of integral equations. Finally, in Section 5, we conclude this paper with some problems for further study.

2. Preliminaries

Let us begin this section with some basic definitions, examples, and lemmas, which are used in the sequel. Throughout this paper, $\mathbb{Q}$ denotes an unital C^*-algebra. An involution on $\mathbb{Q}$ is a conjugate linear map $\mathrm{z} \mapsto \mathrm{z}^*$ on $\mathbb{Q}$ such that $\mathrm{z}^{**} = \mathrm{z}$ and $(\mathrm{z}\nu)^* = \nu^*\mathrm{z}^*$ for all $\mathrm{z}, \nu \in \mathbb{Q}$. The pair $(\mathbb{Q}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{Q}$ together with a complete sub multiplicative norm such that $\|\mathrm{z}^*\| = \|\mathrm{z}\|$ for all $\mathrm{z} \in \mathbb{Q}$. A C^*-algebra is a Banach $*$-algebra s.t $\|\mathrm{z}^*\mathrm{z}\| = \|\mathrm{z}\|^2$. Consider $\mathbb{Q}_\omega = \{\mathrm{z} \in \mathbb{Q} : \vartheta = \vartheta^*\}$. An positive element $\vartheta \in \mathbb{Q}$, defined by $\vartheta \geq 0_{\mathbb{Q}}$, if $\vartheta = \vartheta^*$ and $\mathrm{z}(\vartheta) \subseteq [0, +\infty)$, where $0_{\mathbb{Q}}$ is the zero element in $\mathbb{Q}$ and $\mathrm{z}(\vartheta)$ is the specturm of ϑ. There is a natural partial ordering on $\mathbb{Q}_\omega$ given by $\vartheta \leq \rho$ iff $\rho - \vartheta \geq 0_{\mathbb{Q}}$. From now on, by $\mathbb{Q}_+$, we denote the set $\{\vartheta \in \mathbb{Q} : \vartheta \geq 0_{\mathbb{Q}}\}$ and $|\vartheta| = (\vartheta^*\vartheta)^{\frac{1}{2}}$.

Definition 2.1. *[16] Let $\mathbb{Z}$ be a non-empty set. Let $\Delta : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ be a mapping such that*

(i) $0_{\mathbb{Q}} \leq \Delta(\vartheta, \rho)\ \forall \vartheta, \rho \in \mathbb{Z}$ *and* $\Delta(\vartheta, \rho) = 0_{\mathbb{Q}} \Leftrightarrow \vartheta = \rho$*;*
(ii) $\Delta(\vartheta, \rho) = \Delta(\rho, \vartheta)\ \forall \vartheta, \rho \in \mathbb{Z}$*;*

(iii) $\Delta(\vartheta,\rho)\leq\Delta(\vartheta,\xi)+\Delta(\xi,\rho)\ \forall\vartheta,\rho,\xi\in\mathbb{Z}$.

Then Δ *is called a* C^**-algebra-valued metric on* $\mathbb{Z}$ *and* $(\mathbb{Z},\mathbb{Q},\Delta)$ *is called a* C^**AVMS.*

Example 2.2. *[16] Consider* $\mathbb{Z}=\mathbb{R}$ *and* $\mathbb{Q}=\mathscr{M}_2(\mathbb{R})$ *and define* $\Delta(\vartheta,\rho)=diag(|\vartheta-\rho|,\alpha|\vartheta-\rho|)$, *where* $\vartheta,\rho\in\mathbb{R}$ *and* $\alpha\geq 0$ *is a constant,* Δ *is a* C^**-algebra-valued metric and* $(\mathbb{Z},\mathscr{M}_2(\mathbb{R}),\Delta)$ *is a complete* C^**AVMS by the completeness of* $\mathbb{R}$.

Definition 2.3. *[?] Let* $(\mathbb{Z},\mathbb{Q},d)$ *be a* C^**AVMS. A mapping* $\mathscr{F}:\mathbb{Z}\to\mathbb{Z}$ *is said to be a* C^**-algebra-valued contractive function on* $\mathbb{Z}$ *if* $\exists$ *an* $\aleph\in\mathbb{Q}$ *with* $\|\aleph\|<1$ *such that,* $\vartheta,\rho\in\aleph$, $\Delta(\mathscr{F}\vartheta,\mathscr{F}\rho)=\aleph^*\Delta(\vartheta,\rho)\aleph$.

Definition 2.4. *[13] Let* $\mathbb{Z}$ *be a non-empty set. A mapping* $\mathscr{S}:\mathbb{Z}^3\to[0,\infty)$ *is said to be an* $\mathscr{S}$*-metric on* $\mathbb{Z}$ *if, for each* $\vartheta,\rho,\xi,\mathfrak{r}\in\mathbb{Z}$,

(i) $\mathscr{S}(\vartheta,\rho,\xi)\geq 0$;

(ii) $\mathscr{S}(\vartheta,\rho,\xi)=0\quad iff\quad\vartheta=\rho=\xi$;

(iii) $\mathscr{S}(\vartheta,\rho,\xi)\leq\mathscr{S}(\vartheta,\vartheta,\mathfrak{r})+\mathscr{S}(\rho,\rho,\mathfrak{r})+\mathscr{S}(\xi,\xi,\mathfrak{r})$.

Then $(\mathbb{Z},\mathscr{S})$ *is an* $\mathscr{S}$*-metric space.*

Example 2.5. *[17] Let* $\mathbb{Z}$ *be a non-void set. Let* Δ_1 *and* Δ_2 *be two ordinary metrics on* $\mathbb{Z}$*. Then* $\mathscr{S}(\vartheta,\rho,\xi)=\Delta_1(\vartheta,\xi)+\Delta_2(\rho,\xi)$ *is an* $\mathscr{S}$*-metric on* $\mathbb{Z}$.

Definition 2.6. *[13] Let* $(\mathbb{Z},\mathscr{S})$ *be an* $\mathscr{S}$*-metric space.*

(i) A seq $\{\vartheta_\ell\}$ *in* $\mathbb{Z}$ *converges to* ϑ *if* $\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\vartheta)\to 0$ *as* $\ell\to\infty$*, i.e.,* $\forall\,\varepsilon>0\ \exists\ \ell_0\in\mathbb{N}$ *s.t* $\ell\geq\ell_0$, $\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\vartheta)<\varepsilon$.

(ii) A seq $\{\vartheta_\ell\}$ *in* $\mathbb{Z}$ *is said to be cauchy seq if, for all* $\varepsilon>0$*, there exists* $\ell_0\in\mathbb{N}$ *such that* $\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\vartheta_\mu)<\varepsilon$ *for each* $\ell,\mu\geq\ell_0$.

(iii) The $\mathscr{S}$*-metric space* $(\mathbb{Z},\mathscr{S})$ *is said to be complete if every cauchy sequence is convergent.*

Lemma 2.7. *[13] Let* $(\mathbb{Z},\mathscr{S})$ *be an* $\mathscr{S}$*-metric space. If there exist sequences* $\{\vartheta_\ell\}$ *and* $\{\rho_\ell\}$ *such that* $\lim_{\ell\to\infty}\vartheta_\ell=\vartheta$ *and* $\lim_{\ell\to\infty}\rho_\ell=\rho$*, then* $\lim_{\ell\to\infty}\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\rho_\ell)=\mathscr{S}(\vartheta,\vartheta,\rho)$.

Definition 2.8. *[15] Let* $(\mathbb{Z},\mathscr{S})$ *be an* $\mathscr{S}$*-metric space. A pair* $\{\phi,\chi\}$ *is said to be compatible iff* $\lim_{\ell\to\infty}\mathscr{S}(\phi\chi\vartheta_\ell,\phi\chi\vartheta_\ell,\chi\phi\vartheta_\ell)=0$*, whenever* $\{\vartheta_\ell\}$ *is a seq in* $\mathbb{Z}$ *such that* $\lim_{\ell\to\infty}\phi\vartheta_\ell=\lim_{\ell\to\infty}\chi\vartheta_\ell=\theta$ *for some* $\theta\in\mathbb{Z}$.

Definition 2.9. *[24] Let* $\mathbb{Z}$ *be a non-void set. Assume that a function* $\mathscr{S}:\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\to\mathbb{Q}$ *satisfies the following conditions for each* $\vartheta,\rho,\xi,\mathfrak{r}\in\mathbb{Z}$

(i) $\mathscr{S}(\vartheta,\rho,\xi)\geq 0_{\mathbb{Q}}$;

(ii) $\mathscr{S}(\vartheta,\rho,\xi)=0_{\mathbb{Q}}\quad iff\quad\vartheta=\rho=\xi$;

(iii) $\mathscr{S}(\vartheta,\rho,\xi)\leq\mathscr{S}(\vartheta,\vartheta,\mathfrak{r})+\mathscr{S}(\rho,\rho,\mathfrak{r})+\mathscr{S}(\xi,\xi,\mathfrak{r})$.

Then $(\mathbb{Z},\mathbb{Q},\mathscr{S})$ *is a* C^**-algebra-valued* $\mathscr{S}$*-metric space.*

Lemma 2.10. *[24]. In a* C^**AVSMS,* $\mathscr{S}(\vartheta,\vartheta,\rho)=\mathscr{S}(\rho,\rho,\vartheta)$.

Definition 2.11. *[24] Let* $(\mathbb{Z},\mathbb{Q},\mathscr{S})$ *be a* C^**AVSMS.*

(i) A sequence $\{\vartheta_\ell\}$ *in* $\mathbb{Z}$ *converges to* $\vartheta\in\mathbb{Z}$ *w.r.t* $\mathbb{Q}$ *iff* $\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\vartheta)\to 0_{\mathbb{Q}}$ *as* $\ell\to\infty$.

(ii) A sequence $\{\vartheta_\ell\}$ *in* $\mathbb{Z}$ *is cauchy w.r.t* $\mathbb{Q}$ *if, for each* $\varepsilon>0$*, there exists* $\mathscr{P}\in\mathbb{N}$ *s.t.* $\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\vartheta_\mu)<\varepsilon$ *for each* $\ell,\mu\geq\mathscr{P}$*, that is,* $\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\vartheta_\mu)\to 0_{\mathbb{Q}}$ *as* $\ell,\mu\to\infty$.

(iii) $(\mathbb{Z},\mathbb{Q},\mathscr{S})$ *is a complete C^*AVSMS if every cauchy sequence is convergent.*

In 2018, Ege et al. [24] proved the following theorem.

Theorem 2.12. *Let $(\mathbb{Z},\mathbb{Q},\mathscr{S})$ be a complete C^*AVSMS and $\Psi:\mathbb{Z}\to\mathbb{Z}$ be a C^*-algebra-valued contractive function. Then Ψ has a unique fixed point $\vartheta_0\in\mathbb{Z}$.*

Inspired by the work above, we establish common fixed point theorems for quartet mappings in C^*AVSMS, supported by a relevant example and applications.

3. Main Results

Definition 3.1. *Let $(\mathbb{Z},\mathbb{Q},\mathscr{S})$ be a complete C^*AVSMS. A pair $\{\phi,\chi\}$ is said to be compatible iff $\lim_{\ell\to\infty}\mathscr{S}(\phi\chi\vartheta_\ell,\phi\chi\vartheta_\ell,\chi\phi\vartheta_\ell)=0_{\mathbb{Q}}$, whenever $\{\vartheta_\ell\}$ is a sequence in $\mathbb{Z}$ such that $\lim_{\ell\to\infty}\phi\vartheta_\ell=\lim_{\ell\to\infty}\chi\vartheta_\ell=\theta$, where $\theta\in\mathbb{Z}$.*

Lemma 3.2. *Let $(\mathbb{Z},\mathbb{Q},\mathscr{S})$ be a complete C^*AVSMS. If there exists two sequences $\{\vartheta_\ell\}$ and $\{\rho_\ell\}$ such that $\lim_{\ell\to\infty}\mathscr{S}(\vartheta_\ell,\vartheta_\ell,\rho_\ell)=0_{\mathbb{Q}}$ whenever $\{\vartheta_\ell\}$ is a sequence in $\mathbb{Z}$ such that $\lim_{\ell\to\infty}\vartheta_\ell=\theta$, where $\theta\in\mathbb{Z}$. Then $\lim_{\ell\to\infty}\rho_\ell=\theta$.*

Proof. By using the triangle inequality in C^*AVSMS, one has

$$\begin{aligned}\mathscr{S}(\rho_\ell,\rho_\ell,\theta)&\le\mathscr{S}(\rho_\ell,\rho_\ell,\vartheta_\ell)+\mathscr{S}(\rho_\ell,\rho_\ell,\vartheta_\ell)+\mathscr{S}(\theta,\theta,\vartheta_\ell)\\&\le 2\mathscr{S}(\rho_\ell,\rho_\ell,\vartheta_\ell)+\mathscr{S}(\theta,\theta,\vartheta_\ell)\end{aligned}$$

Taking upper limit as $\ell\to\infty$, one sees that

$$\limsup_{\ell\to\infty}\mathscr{S}(\rho_\ell,\rho_\ell,\theta)\le 2\limsup_{\ell\to\infty}\mathscr{S}(\rho_\ell,\rho_\ell,\vartheta_\ell)+\limsup_{\ell\to\infty}\mathscr{S}(\theta,\theta,\vartheta_\ell)=3\times 0_{\mathbb{Q}}$$

Hence, $\lim_{\ell\to\infty}\rho_\ell=\theta$. □

Let $\mathbb{Q}$ be a unital C^*-algebra. We define the supremum set for subsets of the positive cone $\mathbb{Q}^+$, with respect to the natural partial order induced on $\mathbb{Q}^+$. Let $\mathscr{H}\subseteq\mathbb{Q}^+$ be a non-void subset. A positive element $\rho\in\mathbb{Q}^+$ is said to be an upper bound for $\mathscr{H}$ if $\forall\vartheta\in\mathscr{H}$ if $\vartheta\le\rho$. That is, ρ is an upper bound of $\mathscr{H}\Leftrightarrow\forall\vartheta\in\mathscr{H}$, $\rho-\vartheta\in\mathbb{Q}^+$.

Theorem 3.3. *Consider $(\mathbb{Z},\mathbb{Q}^+,\mathscr{S})$ as a complete C^*AVSMS. Assume that ϕ,χ,Ω, and Ψ are self maps, with $\phi(\mathbb{Z})\subseteq\Psi(\mathbb{Z})$, $\chi(\mathbb{Z})\subseteq\Omega(\mathbb{Z})$ and that the pairs $\{\phi,\Omega\}$ and $\{\chi,\Psi\}$ are compatible. If*

$$\begin{aligned}\mathscr{S}(\phi\vartheta,\phi\rho,\chi\xi)\le\aleph^*\sup_{\vartheta,\rho,\xi\in\mathbb{Q}^+}&\{\mathscr{S}(\Omega\vartheta,\Omega\rho,\Psi\xi),\mathscr{S}(\phi\vartheta,\phi\vartheta,\Omega\vartheta),\\&\mathscr{S}(\chi\xi,\chi\xi,\Psi\xi),\mathscr{S}(\phi\rho,\phi\rho,\chi\xi)\}\aleph,\end{aligned}\tag{3.1}$$

$\forall\ \vartheta,\rho,\xi\in\mathbb{Z}$, with $\|\aleph\|<1$. Then ϕ,χ,Ω, and Ψ have a unique common fixed point(UCFP) in $\mathbb{Z}$ provided that Ω and Ψ are continuous.

Proof. Let $\vartheta_0\in\mathbb{Z}$. Since $\phi(\mathbb{Z})\subseteq\Psi(\mathbb{Z})$, then there exists $\vartheta_1\in\mathbb{Z}$ such that $\phi\vartheta_0=\Psi\vartheta_1$, and also as $\chi\vartheta_1\in\Omega(\mathbb{Z})$, we take $\vartheta_2\in\mathbb{Z}$ s.t $\chi\vartheta_1=\Omega\vartheta_2$. In general, taking $\vartheta_{2\ell+1}\in\mathbb{Z}$ such that

$\phi\vartheta_{2\ell} = \Psi\vartheta_{2\ell+1}$ and $\vartheta_{2\ell+2} \in \mathbb{Z}$ such that $\chi\vartheta_{2\ell+1} = \Omega\vartheta_{2\ell+2}$, one sees a sequence $\{\rho_\ell\}$ in $\mathbb{Z}$ such that $\rho_{2\ell} = \phi\vartheta_{2\ell} = \Psi\vartheta_{2\ell+1}$, $\rho_{2\ell+1} = \chi\vartheta_{2\ell+1} = \Omega\vartheta_{2\ell+2}$, $\ell \geq 0$,

$$\begin{aligned}\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq &\aleph^* \sup\{\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\Psi\vartheta_{2\ell+1}),\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Omega\vartheta_{2\ell}),\\ &\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1}),\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\}\aleph\\ =&\aleph^* \sup\{\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}),\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}),\\ &\mathscr{S}(\rho_{2\ell+1},\rho_{2\ell+1},\rho_{2\ell}),\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1})\}\aleph\\ =&\aleph^* \sup\{\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}),\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1})\}\aleph.\end{aligned}$$

If $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) > \mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell})$, then $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) < \aleph^*\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1})\aleph$, which is a contradiction. Hence, $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq \mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell})$,

$$\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq \aleph^*\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell})\aleph. \tag{3.2}$$

Similarly, one has

$$\begin{aligned}\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \leq &\aleph^* \sup\{\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\Psi\vartheta_{2\ell-1}),\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Omega\vartheta_{2\ell}),\\ &\mathscr{S}(\chi\vartheta_{2\ell-1},\chi\vartheta_{2\ell-1},\Psi\vartheta_{2\ell-1}),\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\chi\vartheta_{2\ell-1})\}\aleph\\ =&\aleph^* \sup\{\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}),\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}),\\ &\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}),\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1})\}\aleph\\ =&\aleph^* \sup\{\mathscr{S}(\rho_{2\ell-2},\rho_{2\ell-2},\rho_{2\ell-1}),\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1})\}\aleph.\end{aligned}$$

If $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) > \mathscr{S}(\rho_{2\ell-2},\rho_{2\ell-2},\rho_{2\ell-1})$, then

$$\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) < \aleph^*\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1})\aleph,$$

which is a contradiction. Hence, $\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \leq \mathscr{S}(\rho_{2\ell-2},\rho_{2\ell-2},\rho_{2\ell-1})$,

$$\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \leq \aleph^*\mathscr{S}(\rho_{2\ell-2},\rho_{2\ell-2},\rho_{2\ell-1})\aleph. \tag{3.3}$$

From equation (3.2) and equation (3.3), one has $\mathscr{S}(\rho_\ell,\rho_\ell,\rho_{\ell-1}) \leq \aleph^*\mathscr{S}(\rho_{\ell-1},\rho_{\ell-1},\rho_{\ell-2})\aleph$, $\ell \geq 2$, where $\|\aleph\| < 1$. Hence, for $\ell \geq 2$,

$$\mathscr{S}(\rho_\ell,\rho_\ell,\rho_{\ell-1}) \leq \ldots \leq (\aleph^*)^{\ell-1}\mathscr{S}(\rho_1,\rho_1,\rho_0)(\aleph)^{\ell-1}. \tag{3.4}$$

Using the triangle inequality in C^*AVSMS, for $\ell > \mu$, one has

$$\begin{aligned}\mathscr{S}(\rho_\ell,\rho_\ell,\rho_\mu) \leq &2\aleph^*\mathscr{S}(\rho_\mu,\rho_\mu,\rho_{\mu+1})\aleph + 2\aleph^*\mathscr{S}(\rho_{\mu+1},\rho_{\mu+1},\rho_{\mu+2})\aleph\\ &+ \ldots + \aleph^*\mathscr{S}(\rho_{\ell-1},\rho_{\ell-1},\rho_\ell)\aleph\\ <&2\aleph^*\mathscr{S}(\rho_\mu,\rho_\mu,\rho_{\mu+1})\aleph + 2\aleph^*\mathscr{S}(\rho_{\mu+1},\rho_{\mu+1},\rho_{\mu+2})\aleph\\ &+ \ldots + 2\aleph^*\mathscr{S}(\rho_{\ell-1},\rho_{\ell-1},\rho_\ell)\aleph.\end{aligned}$$

Hence, from equation (3.4), as $\|\aleph\| < 1$,

$$\begin{aligned}
\mathscr{S}(\rho_\ell,\rho_\ell,\rho_\mu) \leq & 2\sum_{i=\mu}^{\ell-2}(\aleph^*)^i\mathscr{S}(\rho_1,\rho_1,\rho_0)\aleph^i + (\aleph^*)^{\ell-1}\mathscr{S}(\rho_1,\rho_1,\rho_0)\aleph^{\ell-1}\\
= & 2\sum_{i=\mu}^{\ell-1}(\mathscr{D}^{\frac{1}{2}}\aleph^i)^*(\mathscr{D}^{\frac{1}{2}}\aleph^i) + (\mathscr{D}^{\frac{1}{2}}\aleph^{\ell-1})^*(\mathscr{D}^{\frac{1}{2}}\aleph^{\ell-1})\\
\leq & \|2\sum_{i=\mu}^{\ell-1}|\mathscr{D}^{\frac{1}{2}}\aleph^i|^2 + |\mathscr{D}^{\frac{1}{2}}\aleph^{\ell-1}|^2\|\mathscr{I}\\
\leq & 2\sum_{i=\mu}^{\ell-1}\|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph^i\|^2\mathscr{I} + \|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph\|^{2\ell-2}\mathscr{I}\\
\leq & 2\|\mathscr{D}^{\frac{1}{2}}\|^2\sum_{i=\mu}^{\ell-1}\|\aleph\|^{2i}\mathscr{I} + \|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph\|^{2\ell-2}\mathscr{I}\\
\leq & 2\|\mathscr{D}^{\frac{1}{2}}\|^2\frac{\|\aleph\|^{2\mu}}{1-\|\aleph\|}\mathscr{I} + \|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph\|^{2\ell-2}\mathscr{I} \to 0_{\mathbb{Q}}
\end{aligned}$$

as $\mu,\ell\to\infty$, where $\mathscr{D} = \mathscr{S}(\rho_1,\rho_1,\rho_0)$. So $\{\rho_\ell\}$ is cauchy w.r.t. $\mathbb{Q}^+$. Since $(\mathbb{Z},\mathbb{Q}^+,\mathscr{S})$ is a complete C^*AVSMS, there exists $\rho\in\mathbb{Z}$ such that

$$\lim_{\ell\to\infty}\phi\vartheta_{2\ell} = \lim_{\ell\to\infty}\Psi\vartheta_{2\ell+1} = \lim_{\ell\to\infty}\chi\vartheta_{2\ell+1} = \lim_{\ell\to\infty}\Omega\vartheta_{2\ell+2} = \rho.$$

Since Ω is continuous, one has $\lim_{\ell\to\infty}\Omega^2\vartheta_{2\ell+2} = \Omega\rho$ and $\lim_{\ell\to\infty}\Omega\phi\vartheta_{2\ell} = \Omega\rho$. Since ϕ and Ω are compatible, one has $\lim_{\ell\to\infty}\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\Omega\phi\vartheta_{2\ell}) = 0_{\mathbb{Q}^+}$. By Lemma (3.2), $\lim_{\ell\to\infty}\phi\Omega\vartheta_{2\ell} = \Omega\rho$. Putting $\vartheta = \rho = \Omega\vartheta_{2\ell}$ and $\xi = \vartheta_{2\ell+1}$ in inequality (3.1), one has

$$\begin{aligned}
&\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\\
&\leq \aleph^*\sup\{\mathscr{S}(\Omega^2\vartheta_{2\ell},\Omega^2\vartheta_{2\ell},\Psi\vartheta_{2\ell+1}),\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\Omega^2\vartheta_{2\ell})\\
&\quad\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1}),\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\}\aleph.
\end{aligned}\tag{3.5}$$

Taking upper limit as $\ell\to\infty$ in equation (3.5), one sees that

$$\begin{aligned}
\mathscr{S}(\Omega\rho,\Omega\rho,\rho) \leq & \aleph^*\sup\{\lim_{\ell\to\infty}\mathscr{S}(\Omega^2\vartheta_{2\ell},\Omega^2\vartheta_{2\ell},\Psi\vartheta_{2\ell+1}),\lim_{\ell\to\infty}\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\Omega^2\vartheta_{2\ell}),\\
&\lim_{\ell\to\infty}\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1}),\lim_{\ell\to\infty}\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\}\aleph\\
\leq & \aleph^*\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph.
\end{aligned}$$

Consequently, $\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\leq\aleph^*\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph$, as $\|\aleph\|<1$, which implies that $\Omega\rho=\rho$. Since Ψ is continuous, one has $\lim_{\ell\to\infty}\Psi^2\vartheta_{2\ell+1} = \Psi\rho$ and $\lim_{\ell\to\infty}\Psi\chi\vartheta_{2\ell+1} = \Psi\rho$. Since χ and Ψ are compatible, one sees that $\lim_{\ell\to\infty}\mathscr{S}(\chi\Psi\vartheta_{2\ell+1},\chi\Psi\vartheta_{2\ell+1},\Psi\chi\vartheta_{2\ell+1}) = 0_{\mathbb{Q}^+}$. By Lemma (3.2), $\lim_{\ell\to\infty}\chi\Psi\vartheta_{2\ell+1} = \Psi\rho$. Putting $\vartheta=\rho=\vartheta_{2\ell}$ and $\xi = \Psi\vartheta_{2\ell+1}$ in inequality (3.1), we have

$$\begin{aligned}
\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\chi\Psi\vartheta_{2\ell+1}) \leq & \aleph^*\sup\{\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\Psi^2\vartheta_{2\ell+1}),\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Omega\vartheta_{2\ell}),\\
&\mathscr{S}(\chi\Psi\vartheta_{2\ell+1},\chi\Psi\vartheta_{2\ell+1},\Psi^2\vartheta_{2\ell+1}),\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\chi\Psi\vartheta_{2\ell+1})\}\aleph.
\end{aligned}\tag{3.6}$$

Taking upper limit as $\ell \to \infty$ in inequality (3.6), we obtain

$$\mathscr{S}(\rho,\rho,\Psi\rho) \preceq \aleph^* \sup\{\mathscr{S}(\rho,\rho,\Psi\rho), 0_{\mathbb{Q}^+}, 0_{\mathbb{Q}^+}, \mathscr{S}(\rho,\rho,\Psi\rho)\}\aleph,$$

that is, $\Psi\rho = \rho$. Also, we can apply equation (3.1) to obtain

$$\begin{aligned}\mathscr{S}(\phi\rho,\phi\rho,\chi\vartheta_{2\ell+1}) \preceq \aleph^* \sup\{&\mathscr{S}(\Omega\rho,\Omega\rho,\Psi\vartheta_{2\ell+1}), \mathscr{S}(\phi\rho,\phi\rho,\Omega\rho),\\ &\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1}), \mathscr{S}(\phi\rho,\phi\rho,\chi\vartheta_{2\ell+1})\}\aleph.\end{aligned} \tag{3.7}$$

Taking upper limit as $\ell \to \infty$ in equation (3.7), as $\Omega\rho = \Psi\rho = \rho$, we have

$$\begin{aligned}\mathscr{S}(\phi\rho,\phi\rho,\rho) &\preceq \aleph^* \sup\{\mathscr{S}(\Omega\rho,\Omega\rho,\rho), \mathscr{S}(\phi\rho,\phi\rho,\rho), \mathscr{S}(\rho,\rho,\rho), \mathscr{S}(\phi\rho,\phi\rho,\rho)\}\aleph\\ &= \aleph^* \mathscr{S}(\phi\rho,\phi\rho,\rho)\aleph.\end{aligned}$$

Since $\|\aleph\| < 1$, $\mathscr{S}(\phi\rho,\phi\rho,\rho) = 0_{\mathbb{Q}^+}$ and $\phi\rho = \rho$. By using equation (3.1), as $\Omega\rho = \Psi\rho = \phi\rho = \rho$, we obtain

$$\begin{aligned}&\mathscr{S}(\rho,\rho,\chi\rho)\\ &\preceq \aleph^* \sup\{\mathscr{S}(\Omega\rho,\Omega\rho,\Psi\rho), \mathscr{S}(\phi\rho,\phi\rho,\Omega\rho), \mathscr{S}(\chi\rho,\chi\rho,\Psi\rho), \mathscr{S}(\phi\rho,\phi\rho,\chi\rho)\}\aleph\\ &= \aleph^* \mathscr{S}(\rho,\rho,\chi\rho)\aleph,\end{aligned}$$

which implies that $\mathscr{S}(\rho,\rho,\chi\rho) = 0_{\mathbb{Q}^+}$ and $\chi\rho = \rho$. Thus $\Omega\rho = \Psi\rho = \phi\rho = \chi\rho = \rho$. If there exists another CFP $\rho^\star \in \mathbb{Z}$ of all ϕ, χ, Ω and Ψ, then

$$\begin{aligned}0_{\mathbb{Q}^+} \preceq \mathscr{S}(\rho^\star,\rho^\star,\rho) \preceq \aleph^* \sup\{&\mathscr{S}(\Omega\rho^\star,\Omega\rho^\star,\Psi\rho), \mathscr{S}(\phi\rho^\star,\phi\rho^\star,\Omega\rho^\star)\\ &\mathscr{S}(\chi\rho,\chi\rho,\Psi\rho), \mathscr{S}(\phi\rho^\star,\phi\rho^\star,\chi\rho)\}\aleph\\ \preceq& \aleph^* \mathscr{S}(\rho^\star,\rho^\star,\rho)\aleph.\end{aligned}$$

Since $\|\aleph\| < 1$, $\mathscr{S}(\rho^\star,\rho^\star,\rho) = 0_{\mathbb{Q}^+}$ and $\rho^\star = \rho$. Hence, ρ is a UCFP of ϕ, χ, Ω and Ψ. □

Example 3.4. *Let $\mathbb{Z} = [0,1]$ be endowed with C*AVSMS and $\mathscr{S}(\vartheta,\rho,\xi) = |\vartheta - \xi| + |\rho - \xi|$, for all $\vartheta, \rho, \xi \in \mathbb{Z}$. Define ϕ, χ, Ω and Ψ on $\mathbb{Z}$ by $\phi\vartheta = (\frac{\vartheta}{4})^8$, $\chi\vartheta = (\frac{\vartheta}{4})^4$, $\Omega\vartheta = (\frac{\vartheta}{4})^2$, and $\Psi\vartheta = \frac{\vartheta}{4}$. Since $\phi\vartheta \subseteq \Psi\vartheta$ and $\chi\vartheta \subseteq \Omega\vartheta$, $\{\phi, \Omega\}$ and $\{\chi, \Psi\}$ are compatible mappings. For each $\vartheta, \rho, \xi \in \mathbb{Z}$, we have*

$$\begin{aligned}&\mathscr{S}(\phi\vartheta,\phi\rho,\chi\xi)\\ =&|\phi\vartheta - \chi\xi| + |\phi\rho - \chi\xi|\\ =&|(\frac{\vartheta}{4})^4 - (\frac{\xi}{4})^2||(\frac{\vartheta}{4})^4 + (\frac{\xi}{4})^2| + |(\frac{\rho}{4})^4 - (\frac{\xi}{4})^2||(\frac{\rho}{4})^4 + (\frac{\xi}{4})^2|\\ \le&\frac{17}{256}|(\frac{\vartheta}{4})^2 - \frac{\xi}{4}||(\frac{\vartheta}{4})^2 + \frac{\xi}{4}| + \frac{17}{256}|(\frac{\rho}{4})^2 - \frac{\xi}{4}||(\frac{\rho}{4})^2 + \frac{\xi}{4}|\\ \le&\frac{85}{4,096}|(\frac{\vartheta}{4})^2 - \frac{\xi}{4}| + \frac{85}{4,096}|(\frac{\rho}{4})^2 - \frac{\xi}{4}|\\ \le&\frac{85}{4,096}(|\Omega\vartheta - \Psi\xi| + |\Omega\rho - \Psi\xi|)\\ =&\frac{\sqrt{85}}{64}\mathscr{S}(\Omega\vartheta,\Omega\rho,\Psi\xi)\frac{\sqrt{85}}{64}\\ \preceq&\aleph^* \sup_{\vartheta,\rho,\xi\in[0,1]}\{\mathscr{S}(\Omega\vartheta,\Omega\rho,\Psi\xi), \mathscr{S}(\phi\vartheta,\phi\vartheta,\Omega\vartheta), \mathscr{S}(\chi\xi,\chi\xi,\Psi\xi), \mathscr{S}(\phi\rho,\phi\rho,\chi\xi)\}\aleph,\end{aligned}$$

where $\|\aleph\| = \frac{\sqrt{85}}{64} < 1$. *Hence* ϕ, χ, Ω *and* Ψ *satisfies the criteria of Theorem 3.3.*

Example 3.5. *Consider* $\mathbb{Z} = [0,1]$ & $\mathbb{Q}^+ = \mathbb{M}_2(\mathbb{R})$ *with* $\|\aleph\| = \sup\{\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4\}$, *where* $\mathbb{Z}_i' s$ *are the entries of* $\aleph$. *Then* $(\mathbb{Z}, \mathbb{Q}^+, \mathscr{S})$ *is a* C^*AVSMS, *where*

$$\mathscr{S}(\vartheta, \rho, \xi) = \begin{bmatrix} |\vartheta - \xi| + |\rho - \xi| & 0 \\ 0 & |\vartheta - \xi| + |\rho - \xi| \end{bmatrix}$$

and partial ordering on $\mathbb{Q}^+$ *is given by*

$$\begin{bmatrix} \mathbb{Z}_1 & \mathbb{Z}_2 \\ \mathbb{Z}_3 & \mathbb{Z}_4 \end{bmatrix} \geq \begin{bmatrix} \nu_1 & \nu_2 \\ \nu_3 & \nu_4 \end{bmatrix} \Leftrightarrow \mathbb{Z}_i \geq \nu_i, \quad for \quad i = 1,2,3,4.$$

Define ϕ, χ, Ω *and* Ψ *on* $\mathbb{Z}$ *by* $\phi\vartheta = (\frac{\vartheta}{2})^8$, $\chi\vartheta = (\frac{\vartheta}{2})^4$, $\Omega\vartheta = (\frac{\vartheta}{2})^2$, *and* $\Psi\vartheta = \frac{\vartheta}{2}$. *Since* $\phi\vartheta \subseteq \Psi\vartheta$ *and* $\chi\vartheta \subseteq \Omega\vartheta$, $\{\phi, \Omega\}$ *and* $\{\chi, \Psi\}$ *are compatible mappings. For each* $\vartheta, \rho, \xi \in \mathbb{Z}$, *we have*

$$\begin{aligned}
&\mathscr{S}(\phi\vartheta, \phi\rho, \chi\xi) \\
&= \begin{bmatrix} |(\frac{\vartheta}{2})^8 - (\frac{\xi}{2})^4| + |(\frac{\rho}{2})^8 - (\frac{\xi}{2})^4| & 0 \\ 0 & |(\frac{\vartheta}{2})^8 - (\frac{\xi}{2})^4| + |(\frac{\rho}{2})^8 - (\frac{\xi}{2})^4| \end{bmatrix} \\
&\leq \begin{bmatrix} \frac{15}{64}|(\frac{\vartheta}{2})^2 - (\frac{\xi}{2})| + \frac{15}{64}|(\frac{\rho}{2})^2 - (\frac{\xi}{2})| & 0 \\ 0 & \frac{15}{64}|(\frac{\vartheta}{2})^2 - (\frac{\xi}{2})| + \frac{15}{64}|(\frac{\rho}{2})^2 - (\frac{\xi}{2})| \end{bmatrix} \\
&\leq \begin{bmatrix} \frac{15}{64}|\Omega\vartheta - \Psi\xi| + \frac{15}{64}|\Omega\rho - \Psi\xi| & 0 \\ 0 & \frac{15}{64}|\Omega\vartheta - \Psi\xi| + \frac{15}{64}|\Omega\rho - \Psi\xi| \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sqrt{15}}{8} & 0 \\ 0 & \frac{\sqrt{15}}{8} \end{bmatrix} \begin{bmatrix} |\Omega\vartheta - \Psi\xi| + |\Omega\rho - \Psi\xi| & 0 \\ 0 & |\Omega\vartheta - \Psi\xi| + |\Omega\rho - \Psi\xi| \end{bmatrix} \\
&\quad \begin{bmatrix} \frac{\sqrt{15}}{8} & 0 \\ 0 & \frac{\sqrt{15}}{8} \end{bmatrix} \\
&= \aleph^* \mathscr{S}(\Omega\vartheta, \Omega\rho, \Psi\xi) \aleph \\
&\leq \aleph^* \sup_{\vartheta, \rho, \xi \in [0,1]} \{\mathscr{S}(\Omega\vartheta, \Omega\rho, \Psi\xi), \mathscr{S}(\phi\vartheta, \phi\vartheta, \Omega\vartheta), \mathscr{S}(\chi\xi, \chi\xi, \Psi\xi), \mathscr{S}(\phi\rho, \phi\rho, \chi\xi)\} \aleph,
\end{aligned}$$

where $\aleph = \begin{bmatrix} \frac{\sqrt{15}}{8} & 0 \\ 0 & \frac{\sqrt{15}}{8} \end{bmatrix}$ *and* $\|\aleph\| = \frac{\sqrt{15}}{8} < 1$. *Hence* ϕ, χ, Ω *and* Ψ *satisfies the criteria of Theorem 3.3.*

Theorem 3.6. *Let* $\{\phi, \Omega\}$ *and* $\{\chi, \Psi\}$ *be compatible self functions on a complete* C^*AVSMS $(\mathbb{Z}, \mathbb{Q}, \mathscr{S})$ *and, for all* $\vartheta, \rho, \xi \in \mathbb{Z}$,

$$\begin{aligned}
\mathscr{S}(\phi\vartheta, \phi\rho, \chi\xi) \leq\, & \aleph_{\mathfrak{a}}^* \mathscr{S}(\Omega\vartheta, \Omega\rho, \Psi\xi) \aleph_{\mathfrak{a}} + \aleph_{\mathfrak{b}}^* \mathscr{S}(\phi\vartheta, \phi\vartheta, \Psi\xi) \aleph_{\mathfrak{b}} + \aleph_{\mathfrak{c}}^* \mathscr{S}(\Omega\vartheta, \Omega\rho, \chi\xi) \aleph_{\mathfrak{c}} \\
& + \aleph_{\mathsf{d}}^* \mathscr{S}(\phi\rho, \phi\rho, \Psi\xi) \aleph_{\mathsf{d}} + \aleph_{\mathfrak{e}}^* \mathscr{S}(\chi\xi, \chi\xi, \Psi\xi) \aleph_{\mathfrak{e}}, \qquad (3.8)
\end{aligned}$$

where $\aleph_{\mathfrak{a}}, \aleph_{\mathfrak{b}}, \aleph_{\mathfrak{c}}, \aleph_{\mathsf{d}}, \aleph_{\mathfrak{e}} \in \mathbb{Q}$ *with* $(\|\aleph_{\mathfrak{a}}\|^2 + 3\|\aleph_{\mathfrak{b}}\|^2 + 3\|\aleph_{\mathfrak{c}}\|^2 + 3\|\aleph_{\mathsf{d}}\|^2 + \|\aleph_{\mathfrak{e}}\|^2) < 1$. *If* $\phi(\mathbb{Z}) \subseteq \Psi(\mathbb{Z})$ *and* $\chi(\mathbb{Z}) \subseteq \Omega(\mathbb{Z})$ *and* Ω *and* Ψ *are continuous, then quartet functions* ϕ, χ, Ω *and* Ψ *have a UCFP.*

Proof. Let $\vartheta_0 \in \mathbb{Z}$. Since $\phi(\mathbb{Z}) \subseteq \Psi(\mathbb{Z})$, we let $\vartheta_1 \in \mathbb{Z}$ be such that $\Psi\vartheta_1 = \phi\vartheta_0$, and $\chi\vartheta_1 \in \Omega(\vartheta)$. Let $\vartheta_2 \in \mathbb{Z}$ be such that $\Omega\vartheta_2 = \chi\vartheta_1$. In general, $\vartheta_{2\ell+1} \in \mathbb{Z}$ is chosen such that $\Psi\vartheta_{2\ell+1} = \phi\vartheta_{2\ell}$ and $\vartheta_{2\ell+2} \in \mathbb{Z}$ with $\Omega\vartheta_{2\ell+2} = \chi\vartheta_{2\ell+1}$; $\ell = 0, 1, 2, \ldots$ Denote

$$\rho_{2\ell} = \Psi\vartheta_{2\ell+1} = \phi\vartheta_{2\ell},\ \rho_{2\ell+1} = \Omega\vartheta_{2\ell+2} = \chi\vartheta_{2\ell+1}, \quad \ell \geq 0,$$

$$\begin{aligned}\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq & \aleph_{\mathfrak{a}}^*\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{a}} + \aleph_{\mathfrak{b}}^*\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{b}}, \\ & + \aleph_{\mathfrak{c}}^*\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\aleph_{\mathfrak{c}} + \aleph_{\mathfrak{d}}^*\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{d}} \\ & + \aleph_{\mathfrak{e}}^*\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{e}} \\ \leq & \aleph_{\mathfrak{a}}^*\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell})\aleph_{\mathfrak{a}} + \aleph_{\mathfrak{c}}^*[2\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \\ & + \mathscr{S}(\rho_{2\ell+1},\rho_{2\ell+1},\rho_{2\ell})]\aleph_{\mathfrak{c}} + \aleph_{\mathfrak{e}}^*\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1})\aleph_{\mathfrak{e}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq & \|\aleph_{\mathfrak{a}}\|^2\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) + 2\|\aleph_{\mathfrak{c}}\|^2\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \\ & + (\|\aleph_{\mathfrak{c}}\|^2 + \|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}).\end{aligned} \tag{3.9}$$

Now, we prove $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq \mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell})$, for each $\ell \in \mathbb{N}$. If $\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) < \mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1})$, then inequality (3.9) yields

$$\begin{aligned}\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) < & \|\aleph_{\mathfrak{a}}\|^2\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) + 2\|\aleph_{\mathfrak{c}}\|^2\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \\ & + (\|\aleph_{\mathfrak{c}}\|^2 + \|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \\ = & (\|\aleph_{\mathfrak{a}}\|^2 + 3\|\aleph_{\mathfrak{c}}\|^2 + \|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \\ < & \mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}),\end{aligned}$$

which is a contradiction. Thus $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq \mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell})$, for all $\ell \in \mathbb{N}$. From inequality (3.9), we obtain

$$\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell+1}) \leq (\|\aleph_{\mathfrak{a}}\|^2 + 3\|\aleph_{\mathfrak{c}}\|^2 + \|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}). \tag{3.10}$$

Also, we have

$$\begin{aligned}& \mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \\ = & \mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\chi\vartheta_{2\ell-1}) \\ \leq & \aleph_{\mathfrak{a}}^*\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\Psi\vartheta_{2\ell-1})\aleph_{\mathfrak{a}} + \aleph_{\mathfrak{b}}^*\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Psi\vartheta_{2\ell-1})\aleph_{\mathfrak{b}} \\ & + \aleph_{\mathfrak{c}}^*\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\chi\vartheta_{2\ell-1})\aleph_{\mathfrak{c}} + \aleph_{\mathfrak{d}}^*\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Psi\vartheta_{2\ell-1})\aleph_{\mathfrak{d}} \\ & + \aleph_{\mathfrak{e}}^*\mathscr{S}(\chi\vartheta_{2\ell-1},\chi\vartheta_{2\ell-1},\Psi\vartheta_{2\ell-1})\aleph_{\mathfrak{e}} \\ \leq & \|\aleph_{\mathfrak{a}}\|^2\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}) + (2\|\aleph_{\mathfrak{b}}\|^2 + 2\|\aleph_{\mathfrak{d}}\|^2)\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell}) \\ & + (\|\aleph_{\mathfrak{b}}\|^2 + \|\aleph_{\mathfrak{d}}\|^2)\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}) \\ & + \|\aleph_{\mathfrak{e}}\|^2\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}).\end{aligned}$$

Hence,

$$\begin{aligned}& \mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) \\ \leq & \|\aleph_{\mathfrak{a}}\|^2\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}) + (2\|\aleph_{\mathfrak{b}}\|^2 + 2\|\aleph_{\mathfrak{d}}\|^2)\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) \\ & + (\|\aleph_{\mathfrak{b}}\|^2 + \|\aleph_{\mathfrak{d}}\|^2 + \|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}).\end{aligned} \tag{3.11}$$

Similarly, if $\mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2}) < \mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1})$, then inequality (3.11) yields

$$\begin{aligned}\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) \leq &(\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1})\\ &<\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}),\end{aligned}$$

which is a contradiction. Thus $\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) \leq \mathscr{S}(\rho_{2\ell-1},\rho_{2\ell-1},\rho_{2\ell-2})$, $\forall\ \ell \in \mathbb{N}$. From inequality (3.11), we obtain

$$\mathscr{S}(\rho_{2\ell},\rho_{2\ell},\rho_{2\ell-1}) \leq (\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho_{2\ell},\rho_{2\ell-1},\rho_{2\ell-2}). \tag{3.12}$$

Now, from equation (3.10) and (3.12), we have

$$\mathscr{S}(\rho_{\ell},\rho_{\ell},\rho_{\ell-1}) \leq \|\aleph\|\mathscr{S}(\rho_{\ell-1},\rho_{\ell-1},\rho_{\ell-2}), \quad \ell \geq 2,$$

where $\|\aleph\|= min\{\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+\|\aleph_{\mathfrak{e}}\|^2, \|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2\}$. Since $\|\aleph\|< 1$, for $\ell \geq 2$, one has

$$\mathscr{S}(\rho_{\ell},\rho_{\ell},\rho_{\ell-1}) \leq \dots \leq (\aleph^*)^{\ell-1}\mathscr{S}(\rho_1,\rho_1,\rho_0)\aleph. \tag{3.13}$$

By using the triangle inequality in C^*AVSMS, for $\ell > \mu$, one has

$$\mathscr{S}(\rho_{\ell},\rho_{\ell},\rho_{\mu}) \leq 2\mathscr{S}(\rho_{\mu},\rho_{\mu},\rho_{\mu+1})+2\mathscr{S}(\rho_{\mu+1},\rho_{\mu+1},\rho_{\mu+2})+\dots+\mathscr{S}(\rho_{\ell-1},\rho_{\ell-1},\rho_{\ell}).$$

Hence, from equation (3.13) and as $\|\aleph\|< 1$, we have

$$\begin{aligned}\mathscr{S}(\rho_{\ell},\rho_{\ell},\rho_{\mu}) \leq &2\sum_{i=\mu}^{\ell-2}(\aleph^*)^i\mathscr{S}(\rho_1,\rho_1,\rho_0)\aleph^i+(\aleph^*)^{\ell-1}\mathscr{S}(\rho_1,\rho_1,\rho_0)\aleph^{\ell-1}\\ &=2\sum_{i=\mu}^{\ell-1}(\aleph^*)^i\mathscr{D}^{\frac{1}{2}}\mathscr{D}^{\frac{1}{2}}\aleph^i+(\aleph^*)^{\ell-1}\mathscr{D}^{\frac{1}{2}}\mathscr{D}^{\frac{1}{2}}\aleph^{\ell-1}\\ &\leq\|2\sum_{i=\mu}^{\ell-1}|\mathscr{D}^{\frac{1}{2}}\aleph^i|^2+|\mathscr{D}^{\frac{1}{2}}\aleph^{\ell-1}|^2\|\mathscr{I}\\ &\leq 2\sum_{i=\mu}^{\ell-1}\|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph^i\|^2\mathscr{I}+\|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph\|^{2\ell-2}\mathscr{I}\\ &\leq 2\|\mathscr{D}^{\frac{1}{2}}\|^2\sum_{i=\mu}^{\ell-1}\|\aleph\|^{2i}\mathscr{I}+\|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph\|^{2\ell-2}\mathscr{I}\\ &\leq 2\|\mathscr{D}^{\frac{1}{2}}\|^2\frac{\|\aleph\|^{2\mu}}{1-\|\aleph\|}\mathscr{I}+\|\mathscr{D}^{\frac{1}{2}}\|^2\|\aleph\|^{2\ell-2}\mathscr{I}\to 0_{\mathbb{Q}}\end{aligned}$$

as $\mu,\ell \to \infty$, where $\mathscr{D}=\mathscr{S}(\rho_1,\rho_1,\rho_0)$. Therefore, $\{\rho_\ell\}$ is Cauchy. Let $\rho \in \mathbb{Z}$ be such that

$$\lim_{\ell\to\infty}\phi\vartheta_{2\ell}=\lim_{\ell\to\infty}\Psi\vartheta_{2\ell+1}=\lim_{\ell\to\infty}\chi\vartheta_{2\ell+1}=\lim_{\ell\to\infty}\Omega\vartheta_{2\ell+2}=\rho.$$

Since Ω is continuous, $\lim_{\ell\to\infty}\Omega^2\vartheta_{2\ell+2}=\Omega\rho, \lim_{\ell\to\infty}\Omega\phi\vartheta_{2\ell}=\Omega\rho$. Since ϕ and Ω are compatible, $\lim_{\ell\to\infty}\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\Omega\phi\vartheta_{2\ell})=0_{\mathbb{Q}}$. By Lemma (3.2), $\lim_{\ell\to\infty}\phi\Omega\vartheta_{2\ell}=\Omega\rho$.

From equation (3.8), one has

$$\begin{aligned}&\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\\ &\leq \aleph_{\mathfrak{a}}^{*}\mathscr{S}(\Omega^2\vartheta_{2\ell},\Omega^2\vartheta_{2\ell},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{b}}\\ &+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\Omega^2\vartheta_{2\ell},\Omega^2\vartheta_{2\ell},\chi\vartheta_{2\ell+1})\aleph_{\mathfrak{c}}+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\phi\Omega\vartheta_{2\ell},\phi\Omega\vartheta_{2\ell},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{d}}\\ &+\aleph_{\mathfrak{e}}^{*}\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{e}}.\end{aligned}$$

Taking upper limit as $\ell\to\infty$, one has

$$\begin{aligned}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\leq&\aleph_{\mathfrak{a}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{b}}+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{c}}\\ &+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{d}}+\aleph_{\mathfrak{e}}^{*}\mathscr{S}(\rho,\rho,\rho)\aleph_{\mathfrak{e}}\\ \leq&\aleph_{\mathfrak{a}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{b}}+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{c}}\\ &+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{d}}\\ \leq&(\|\aleph_{\mathfrak{a}}\|^2+\|\aleph_{\mathfrak{b}}\|^2+\|\aleph_{\mathfrak{c}}\|^2+\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\Omega\rho,\Omega\rho,\rho).\end{aligned}$$

Therefore, $\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\leq(\|\aleph_{\mathfrak{a}}\|^2+\|\aleph_{\mathfrak{b}}\|^2+\|\aleph_{\mathfrak{c}}\|^2+\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\Omega\rho,\Omega\rho,\rho)$ as

$$(\|\aleph_{\mathfrak{a}}\|^2+\|\aleph_{\mathfrak{b}}\|^2+\|\aleph_{\mathfrak{c}}\|^2+\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)<1.$$

Note that $\Omega\rho=\rho$. Since Ψ is continuous, one has $\lim_{\ell\to\infty}\Psi^2\vartheta_{2\ell+1}=\Psi\rho$ and $\lim_{\ell\to\infty}\Psi\chi\vartheta_{2\ell+1}=\Psi\rho$. Since χ and Ψ are compatible, $\lim_{\ell\to\infty}\mathscr{S}(\chi\Psi\vartheta_{2\ell+1},\chi\Psi\vartheta_{2\ell+1},\Psi\chi\vartheta_{2\ell+1})=0_{\mathbb{Q}}$.
Lemma 3.2 yields $\lim_{\ell\to\infty}\chi\Psi\vartheta_{2\ell+1}=\Psi\rho$. From equation (3.8), one has

$$\begin{aligned}\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\chi\Psi\vartheta_{2\ell+1})\leq&\aleph_{\mathfrak{a}}^{*}\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\Psi^2\vartheta_{2\ell+1})\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Psi^2\vartheta_{2\ell+1})\aleph_{\mathfrak{b}}\\ &+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\Omega\vartheta_{2\ell},\Omega\vartheta_{2\ell},\chi\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{c}}+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\phi\vartheta_{2\ell},\phi\vartheta_{2\ell},\Psi^2\vartheta_{2\ell+1})\aleph_{\mathfrak{d}}\\ &+\aleph_{\mathfrak{e}}^{*}\mathscr{S}(\chi\Psi\vartheta_{2\ell+1},\chi\Psi\vartheta_{2\ell+1},\Psi^2\vartheta_{2\ell+1})\aleph_{\mathfrak{e}}.\end{aligned}$$

Taking the upper limit as $\ell\to\infty$, one has

$$\begin{aligned}\mathscr{S}(\rho,\rho,\Psi\rho)\leq&\aleph_{\mathfrak{a}}^{*}\mathscr{S}(\rho,\rho,\Psi\rho)\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\rho)\aleph_{\mathfrak{b}}+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\rho,\rho,\Psi\rho)\aleph_{\mathfrak{c}}\\ &+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\rho,\rho,\Psi\rho)\aleph_{\mathfrak{d}}+\aleph_{\mathfrak{e}}^{*}\mathscr{S}(\Psi\rho,\Psi\rho,\Psi\rho)\aleph_{\mathfrak{e}}\\ \leq&(\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho,\rho,\Psi\rho),\end{aligned}$$

that is,

$$\mathscr{S}(\rho,\rho,\Psi\rho)\leq(\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho,\rho,\Psi\rho).$$

Therefore, $(\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)<1$. Note that $\Psi\rho=\rho$. From equation (3.8), one has

$$\begin{aligned}&\mathscr{S}(\phi\rho,\phi\rho,\chi\vartheta_{2\ell+1})\\ &\leq\aleph_{\mathfrak{a}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\phi\rho,\phi\rho,\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{b}}+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\Omega\rho,\Omega\rho,\chi\vartheta_{2\ell+1})\aleph_{\mathfrak{c}}\\ &+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\phi\rho,\phi\rho,\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{d}}+\aleph_{\mathfrak{e}}^{*}\mathscr{S}(\chi\vartheta_{2\ell+1},\chi\vartheta_{2\ell+1},\Psi\vartheta_{2\ell+1})\aleph_{\mathfrak{e}}.\end{aligned}$$

By taking the upper limit as $\ell\to\infty$, as $\Omega\rho=\rho$, $\Psi\rho=\rho$, we get

$$\begin{aligned}\mathscr{S}(\phi\rho,\phi\rho,\rho)\leq&\aleph_{\mathfrak{a}}^{*}\mathscr{S}(\rho,\rho,\rho)\aleph_{\mathfrak{a}}+\aleph_{\mathfrak{b}}^{*}\mathscr{S}(\phi\rho,\phi\rho,\rho)\aleph_{\mathfrak{b}}+\aleph_{\mathfrak{c}}^{*}\mathscr{S}(\rho,\rho,\rho)\aleph_{\mathfrak{c}}\\ &+\aleph_{\mathfrak{d}}^{*}\mathscr{S}(\phi\rho,\phi\rho,\rho)\aleph_{\mathfrak{d}}+\aleph_{\mathfrak{e}}^{*}\mathscr{S}(\rho,\rho,\rho)\aleph_{\mathfrak{e}}\\ \leq&(\|\aleph_{\mathfrak{b}}\|^2+\|\aleph_{\mathfrak{d}}\|^2)\mathscr{S}(\phi\rho,\phi\rho,\rho).\end{aligned}$$

Therefore,

$$\mathscr{S}(\phi\rho,\phi\rho,\rho) \leq (\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\phi\rho,\phi\rho,\rho),$$

as $(\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2) < 1$. Note that $\phi\rho = \rho$. From equation (3.8), we have $\mathscr{S}(\phi\rho,\phi\rho,\chi\rho) = 0_{\mathbb{Q}}$. Hence, $\phi\rho = \chi\rho$. Thus $\phi\rho = \chi\rho = \Omega\rho = \Psi\rho = \rho$. If there exists another common fixed point $\rho^\star$ in $\mathbb{Z}$ of quartet maps ϕ,χ,Ω and Ψ, then

$$\begin{aligned}
0_{\mathbb{Q}} \leq & \mathscr{S}(\rho^\star,\rho^\star,\rho)\\
\leq & \aleph_{\mathfrak{a}}^*\mathscr{S}(\Omega\rho^\star,\Omega\rho^\star,\Psi\rho)\aleph_{\mathfrak{a}} + \aleph_{\mathfrak{b}}^*\mathscr{S}(\phi\rho^\star,\phi\rho^\star,\Psi\rho)\aleph_{\mathfrak{b}} + \aleph_{\mathfrak{c}}^*\mathscr{S}(\Omega\rho^\star,\Omega\rho^\star,\chi\rho)\aleph_{\mathfrak{c}}\\
& + \aleph_{\mathfrak{d}}^*\mathscr{S}(\phi\rho^\star,\phi\rho^\star,\Psi\rho)\aleph_{\mathfrak{d}} + \aleph_{\mathfrak{e}}^*\mathscr{S}(\chi\rho^\star,\chi\rho^\star,\Psi\rho)\aleph_{\mathfrak{e}}\\
= & (\|\aleph_{\mathfrak{a}}\|^2+\|\aleph_{\mathfrak{b}}\|^2+\|\aleph_{\mathfrak{c}}\|^2+\|\aleph_{\mathfrak{d}}\|^2)\mathscr{S}(\rho^\star,\vartheta,\rho)\\
\leq & (\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho^\star,\rho^\star,\rho).
\end{aligned}$$

This implies

$$\mathscr{S}(\rho^\star,\rho^\star,\rho) \leq (\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2)\mathscr{S}(\rho^\star,\rho^\star,\rho).$$

Since $(\|\aleph_{\mathfrak{a}}\|^2+3\|\aleph_{\mathfrak{b}}\|^2+3\|\aleph_{\mathfrak{c}}\|^2+3\|\aleph_{\mathfrak{d}}\|^2+\|\aleph_{\mathfrak{e}}\|^2) < 1$, $\mathscr{S}(\rho^\star,\rho^\star,\rho) = 0_{\mathbb{Q}}$, i.e., $\rho^\star = \rho$. Hence, ρ is a UCFP of quartet maps ϕ,χ,Ω and Ψ. □

4. APPLICATIONS

Consider the system of integral equation

$$\vartheta(\theta) = \mathscr{A}(\theta) + \int_{\mathscr{F}} \mathscr{J}(\theta,\mathfrak{h},\Omega(\vartheta(\mathfrak{h})))d\mathfrak{h}, \quad \theta \in \mathscr{F},$$
$$\vartheta(\theta) = \mathscr{A}(\theta) + \int_{\mathscr{F}} \mathscr{K}(\theta,\mathfrak{h},\Psi(\vartheta(\mathfrak{h})))d\mathfrak{h}, \quad \theta \in \mathscr{F},$$

where $\mathscr{F}$ is a Lebesgue measurable set (L.M.S) and $m(\mathscr{F}) < \infty$. Let $\mathbb{Z} = \mathscr{L}^\infty(\mathscr{F})$ denote the class of essentially bounded measurable functions on $\mathscr{F}$, where $\mathscr{F}$ is a L.M.S such that $m(\mathscr{F}) < \infty$.

Theorem 4.1. *Suppose that*

(1) $\mathscr{J},\mathscr{K} : \mathscr{F}\times\mathscr{F}\times\mathbb{R} \to \mathbb{R}$ *and* $\mathscr{A} \in \mathscr{L}^\infty(\mathscr{F})$*;*
(2) there exist a continuous function $\mathscr{G} : \mathscr{F}\times\mathscr{F} \to \mathbb{R}$ *and* $\mathfrak{y} \in (0,1)$ *such that*

$$|\mathscr{J}(\theta,\mathfrak{h},\Omega(\vartheta(\theta))) - \mathscr{K}(\theta,\mathfrak{h},\Psi(\rho(\theta)))| \leq \mathfrak{y}\mathscr{G}(\theta,\mathfrak{h})|\Omega(\vartheta(\theta)) - \Psi(\rho(\theta))|$$
$$for\ \theta,\mathfrak{h} \in \mathscr{F}\ and\ \vartheta,\rho \in \mathbb{R};$$

(3) $\sup_{\theta\in\mathscr{F}} \int_{\mathscr{F}} |\mathscr{G}(\theta,\mathfrak{h})|d\mathfrak{h} \leq 1$.

Then the system of volterra type integral equation has a unique common solution ρ *in* $\mathscr{L}^\infty(\mathscr{F})$*.*

Proof. Let $\mathbb{Z} = \mathscr{L}^\infty(\mathscr{F})$ and $\mathscr{M} = \mathscr{L}^3(\mathscr{F})$, where $\mathscr{F}$ is a L.M.S and $\mathscr{L}(\mathscr{M})$ is the set of bounded linear operators on Hilbert space $\mathscr{M}$. Clearly $\mathscr{L}(\mathscr{M})$ is a C^*-algebra with the usual operator norm. Define $\mathscr{S} : \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z} \to \mathscr{L}(\mathscr{M})$ by $\mathscr{S}(\phi,\phi,\chi) = \pi_{|\phi-\chi|}$ for all $\phi,\chi \in \mathbb{Z}$, where $\pi_\omega : \mathscr{M} \to \mathscr{M}$ is the multiplication operator defined by $\pi_\omega(\varphi) = \omega.\varphi$ for all $\omega,\varphi \in \mathscr{M}$.

Then $\mathscr{S}$ is a C^*AVSMS and $(\mathbb{Z}, \mathscr{L}(\mathscr{M}), \mathscr{S})$ is a complete C^*AVSMS w.r.t. $\mathscr{L}(\mathscr{M})$. Let $\phi, \chi : \mathscr{L}^\infty(\mathscr{F}) \to \mathscr{L}^\infty(\mathscr{F})$ be

$$\phi\vartheta(\theta) = \mathscr{A}(\theta) + \int_{\mathscr{F}} \mathscr{J}(\theta, \mathfrak{h}, \Omega(\vartheta(\theta)))d\mathfrak{h},\ \chi\rho(\theta) = \mathscr{A}(\theta) + \int_{\mathscr{F}} \mathscr{K}(\theta, \mathfrak{h}, \Psi(\rho(\theta)))d\mathfrak{h}.$$

Setting $\aleph = \mathfrak{y}$, one sees that $\aleph \in \mathscr{L}(\mathscr{M})$ and $\|\aleph\| = \mathfrak{y} < 1$. For any $\omega \in \mathscr{M}$,

$$\begin{aligned}
&\|\mathscr{S}(\phi\vartheta, \phi\vartheta, \chi\rho)\| \\
&= \sup_{\|\omega\|=1} \int_{\mathscr{F}} \left[\left| \int_{\mathscr{F}} \mathscr{J}(\theta, \mathfrak{h}, \Omega(\vartheta(\theta))) - \mathscr{K}(\theta, \mathfrak{h}, \Psi(\rho(\theta)))d\mathfrak{h} \right| \right] \omega(\theta)\overline{\omega(\theta)}d\theta \\
&\le \sup_{\|\omega\|=1} \int_{\mathscr{F}} \left[\int_{\mathscr{F}} \left| \mathscr{J}(\theta, \mathfrak{h}, \Omega(\vartheta(\theta))) - \mathscr{K}(\theta, \mathfrak{h}, \Psi(\rho(\theta))) \right| d\mathfrak{h} \right] |\omega(\theta)|^2 d\theta \\
&\le \sup_{\|\omega\|=1} \int_{\mathscr{F}} \left[\int_{\mathscr{F}} \left| \mathfrak{y}\mathscr{G}(\theta, \mathfrak{h}) |\Omega(\vartheta(\theta)) - \Psi(\rho(\theta))| \right| d\mathfrak{h} \right] |\omega(\theta)|^2 d\theta \\
&\le \mathfrak{y} \sup_{\|\omega\|=1} \int_{\mathscr{F}} \left[\int_{\mathscr{F}} \left| \mathscr{G}(\theta, \mathfrak{h}) \right| d\mathfrak{h} \right] |\omega(\theta)|^2 d\theta \|\Omega\vartheta - \Psi\rho\|_\infty \\
&\le \mathfrak{y} \|\Omega\vartheta - \Psi\rho\|_\infty \\
&\le \|\aleph\| \|\mathscr{S}(\Omega\vartheta, \Omega\vartheta, \Psi\rho)\|.
\end{aligned}$$

Since $\|\aleph\| < 1$, by using Theorem (3.3), one sees that the system of volterra type integral equation has a unique common solution ρ in $\mathscr{L}^\infty(\mathscr{F})$. □

5. Conclusion

In this paper, we proved common fixed point theorems for quartet mappings on C^*AVSMS. An example was given to demonstrate the applications of the main results. Finally, an application of the contractive mapping theorem on complete C^*AVSMS for quartet mappings, and the existence and uniqueness of solutions for a system of Volterra-type integral equations were established. It is interesting to extend and generalize the results based on different contractive conditions and establish fixed new point results, and further investigate applications to fractional differential and integral equations.

Acknowledgements

The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/34316).

References

[1] S. Banach, Sur les operations dans les ensembles abstraits el leur application aux equations integrals, Fund. Math. 3 (1922) 133-181.

[2] G. Jungck, Commuting mappings and common fixed points, Amer. Math. Monthly 83 (1974) 261-263.

[3] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986) 771-779.

[4] S. Sedghi, N. Shobe, Common fixed point theorems for four mappings in complete metric spaces, Bull. Iranian Math. Soc. 33 (2007) 37–47.

[5] A. Latif, A.H. Alotaibi, M. Noorwali, Fixed point results via multivalued contractive type mappings involving a generalized distance on metric type spaces, J. Nonlinear Var. Anal. 8 (2024) 787-798.

[6] M. Abbas, M. Ali Khan, S. Radenović, Common coupled fixed point theorems in cone metric spaces for ω-compatible mappings, Appl. Math. Comput. 217 (2010) 195–202.

[7] H. Aydi, B. Samet, C. Vetro, Coupled fixed point results in cone metric spaces for compatible mappings, Fixed Point Theory Appl. 2011 (2011) 27.

[8] S. Sedghi, N. Shobe, A common unique random fixed point theorems in $\mathscr{S}$-metric spaces, J. Prime Res. Math. 7 (2011) 25–34

[9] S. Sedghi, N. Shobe A. Aliouche, A generalization of fixed point theorem in $\mathscr{S}$-metric spaces, Mat. Vesnik 64 (2012) 258-266.

[10] P. Chouhan, N. Malviya, A Common unique fixed point theorem for expansive type mappings in $\mathscr{S}$-metric spaces, Int. Math. Forum 8 (2013) 1287-1293.

[11] S. Sedghi, N.V. Dung, Fixed point theorems on $\mathscr{S}$-metric spaces, Mat. Vesnik 66 (2014) 113-124.

[12] S. Sedghi, I. Altun, N. Shobe, M.A. Salahshour, Some properties of $\mathscr{S}$-metric spaces and fixed point results, Kyungpook Math. J. 54 (2014) 113-122.

[13] N.T. Hieu, T.T.L. Nguyen, N.V. Dung, A generalization of Ciric quasi-contractions for maps on $\mathscr{S}$-metric spaces, Thai J. Math. 13 (2015) 369-380.

[14] M.E. Ege, C. Alaca, Some properties of modular $\mathscr{S}$-metric spaces and its fixed point results, J. Comput. Anal. Appl. 20 (2016) 24-33.

[15] S. Sedghi, N. Shobkolaei, M. Shahraki, T. Došenović, Common fixed point of four maps in S-metric spaces, Math. Sci. 12 (2018) 137–143.

[16] Z. Ma, L. Jiang, H. Sun, C^*-algebras-valued metric spaces and related fixed point theorems, Fixed Point Theory Appl. 2014 (2014) 206.

[17] Q. Xin, T. Cao, Common coupled fixed point theorems in C^*-algebra-valued metric spaces, Adv. Appl. Math. 10 (2021) 218-229.

[18] R. Ramaswamy, G. Mani, A.J. Gnanaprakasam, O.A.A. Abdelnaby, V. Stojiljković, S. Radojevic, S. Radenović, Fixed points on covariant and contravariant maps with an application, Mathematics 10 (2022) 4385.

[19] M. Gunaseelan, G. Arul Joseph, P.B. Khumalo, Y.U. Gaba, Common coupled fixed point theorems on C^*-algebra-valued partial metric spaces, J. Funct. Space 2022 (2022), 3336095.

[20] G. Mani, R. Ramaswamy, A.J. Gnanaprakasam, O.A.A. Abdelnaby, S. Radojevic, S. Radenovićl, Solution of integral equation with neutrosophic rectangular triple controlled metric spaces, Symmetry 14 (2022) 2074.

[21] R. Ramaswamy, G. Mani, A.J. Gnanaprakasam, O.A.A. Abdelnaby, S. Radenović, An application of Urysohn integral equation via complex partial metric space, Mathematics 10 (2022) 2019.

[22] G. Mani, R. Ramaswamy, A.J. Gnanaprakasam, V. Stojiljković, Z.M. Fadail, S. Radenović, Application of fixed point results in the setting of F-contraction and simulation function in the setting of bipolar metric space, AIMS Math. 8 (2023) 3269-3285.

[23] J.U. Maheswari, A. Anbarasan, M. Gunaseelan, V. Parvaneh, S. Hadi Bonab, Solving an integral equation via C^*-algebra-valued partial b-metrics, Fixed Point Theory Algorithms Sci. Eng. 2022 (2022) 18.

[24] M.E. Ege, C. Alaca, C^*-algebra-valued $\mathscr{S}$-metric spaces, Communications Faculty of Sciences University Ankara Series A1 Mathematics and Statistics, 66 (2018) 165-177.

[25] B.W. Samuel, G. Mani, Coupled fixed point theorem on orthogonal S-metric spaces and application to coupled system of nonlinear fractional Langevin equations, Adv. Fixed Point Theory, 14 (2018) 1-25.

[26] T.M.T. Sabri, , M.M. Mohammed, A.S. Mohammed, E.S. Mohammad, M. Vivas-Cortez, K. Imed, On coupled snap system with integral boundary conditions in the G-Caputo sense, AIMS Math. 8 (2023) 12576–12605.

[27] A.S. Rafeeq, T.M.T. Sabri, O.M. Mohammed, K. Imed, M. Vivas-Cortez, On Caputo-Hadamard fractional pantograph problem of two different orders with Dirichlet boundary conditions, Alexandria Eng. J. 86 (2024) 386–398.

[28] D. Baleanu, P.O. Mohammed, M. Vivas-Cortez, Y. Rangel-Oliveros, Some modifications in conformable fractional integral inequalities, Adv. Differ. Equ. 2020 (2020) 1-25.

[29] M. Vivas-Cortez, A.A. Muhammad, K. Artion, B. Huseyin, Generalizations of fractional Hermite-Hadamard-Mercer like inequalities for convex functions, AIMS Math. 6 (2021) 9397–9421.

[30] M. Vivas-Cortez, P. Kórus, V.J.E. Napoles, Some generalized Hermite–Hadamard–Fejér inequality for convex functions, Adv. Differ. Equ. 2021 (2021) 199.