J. Nonlinear Funct. Anal. 2026 (2026) 21 https://doi.org/10.23952/jnfa.2026.21

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

NEW NORM-NUMERICAL RADIUS BOUNDS FOR CERTAIN OPERATOR FORMS

MOHAMMAD SABABHEH[1,2,*], HAMID REZA MORADI[3]

[1]Department of Mathematics, College of Integrative Studies, Abdullah Al Salem University, Kuwait
[2]Department of Basic Sciences, Princess Sumaya University for Technology, Amman, Jordan
[3]Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran

Abstract. The main goal of this paper is to present new bounds that work for the norm and numerical radius. The importance of this study is the simple calculations, which allow to obtain some accessible forms. Our results, which are demonstrated the superiorization over existing results, improve several existing bounds.
Keywords. Numerical radius; Operator norm; Operator matrix.
2020 Mathematics Subject Classification. 47A12, 47A30, 47A63.

1. Introduction

It is of interest to find possible relations among some norms on the C^*-algebra $\mathbb{B}(\mathbb{H})$ of all bounded linear operators on a complex Hilbert space $\mathbb{H}$. The most well-studied norms are the usual operator norm and the numerical radius, which are defined respectively for $A \in \mathbb{B}(\mathbb{H})$, as follows

$$\|A\| = \sup_{\|x\|=1} \|Ax\| \text{ and } \omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

These two norms are equivalent, as we have [6, Theorem 1.3-1]

$$\frac{1}{2}\|A\| \le \omega(A) \le \|A\|. \tag{1.1}$$

While the second inequality becomes equality when A is normal, the first becomes equality if $A^2 = O$, where O is the zero operator.

A useful observation about the numerical radius is that

$$\omega(A) = \sup_{\theta \in \mathbb{R}} \left\|\Re\left(e^{\mathrm{i}\theta} A\right)\right\|, \tag{1.2}$$

*Corresponding author.
E-mail address: mohammad.sababheh@aasu.edu.kw; sababheh@yahoo.com (M. Sababheh).
Received July 18, 2025; Accepted May 30, 2026.

where $\Re(\cdot)$ refers to the real part which is defined as $\Re(A) = \frac{A+A^*}{2}$, where A^* denotes the adjoint of A. Using (1.2), the authors in [8] showed that if $A,B \in \mathbb{B}(\mathbb{H})$, then

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) = \sup_{\theta} \frac{1}{2}\left\|A + e^{\mathrm{i}\theta}B^*\right\|, \tag{1.3}$$

where the supremum is taken over all real θ; a convention throughout this paper. The operator $\begin{bmatrix} O & A \\ B & O \end{bmatrix}$ is an operator on $\mathbb{H} \oplus \mathbb{H}$, and is referred to as an operator matrix. From (1.3), it immediately follows that

$$\frac{1}{2}\|A + B^*\| \le \omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \le \frac{\|A\| + \|B\|}{2}. \tag{1.4}$$

In [9], it was shown that if $A,B \in \mathbb{B}(\mathbb{H})$, then

$$\|A + B^*\| \le \|\, |A^*| + |B| \,\|^{\frac{1}{2}} \,\|\, |A| + |B^*| \,\|^{\frac{1}{2}}.$$

Replacing B in the above inequality by $e^{\mathrm{i}\theta}B$ and applying (1.3) imply

$$\omega\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix}\right) \le \frac{1}{2}\|\, |A^*| + |B| \,\|^{\frac{1}{2}} \,\|\, |A| + |B^*| \,\|^{\frac{1}{2}}. \tag{1.5}$$

Operator matrices, as the one above and others, have been used extensively in the literature as they provide a broader perspective and a different scope of applications; see, e.g., [3, 4, 7, 8, 11–15].

In [10], it was shown that if $A,B \ge O$ (i.e., if they are positive in the sense that $\langle Ax,x\rangle, \langle Bx,x\rangle \ge 0$, for all $x \in \mathbb{H}$), then

$$\|A - B\|^2 \le \left\|A^2 + B^2 - 2BA\right\|, \tag{1.6}$$

which play an important role. In addition, it is shown in [2, Inequality (2.21)] that if A,B are self adjoint, then

$$\|A + B\|^2 \le \|A^2 + B^2\| + \omega(AB). \tag{1.7}$$

Further, it was proved in [2, Theorem 6] that, for $A,B \in \mathbb{B}(\mathbb{H})$,

$$\|A + B^*\|^2 \le \max\{\|A^*A + BB^*\|, \|AA^* + B^*B\|\} + 2\max\{\omega(AB), \omega(BA)\}. \tag{1.8}$$

Among those basic bounds, the numerical radius is

$$\omega(A^*B) \le \frac{1}{2}\|A^*A + B^*B\|, A,B \in \mathbb{B}(\mathbb{H}). \tag{1.9}$$

This inequality follows immediately from the definition. Moreover, we have [5]

$$\omega(S^*T \pm TS) \le 2\|S\|\,\omega(T). \tag{1.10}$$

A more elaborated bound is stated in the next lemma [1, Remark 2.2].

Lemma 1.1. *Let $A,X,Y,B \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(AX + YB^*) \le \sqrt{\left\||A^*|^2 + |B^*|^2\right\| \left\||X|^2 + |Y^*|^2\right\|}.$$

In particular,

$$\omega^2(A + B^*) \le 2\left\||A^*|^2 + |B|^2\right\|.$$

In this paper, we present new, simple bounds that refine the above-mentioned results and several others. A distinguishing feature of this work is the use of simple calculations, which lead to accessible and easily applicable bounds. We compare our results with existing ones in the literature, showing that several of our bounds improve known inequalities, while others are complementary.

2. Main Results

This section contains our main results, and we start with the following significant improvement of (1.6), (1.7), and (1.8).

Theorem 2.1. *Let $A,B \in \mathbb{B}(\mathbb{H})$. Then*

$$\|A+B^*\| \leq \sqrt{\omega\left(A^*A+BB^*+2BA\right)}.$$

Proof. Using the basic identity $\|X\|^2 = \|X^*X\|$, for $X \in \mathbb{B}(\mathbb{H})$, we can write

$$\begin{aligned}\|A+B^*\|^2 &= \|(A^*+B)(A+B^*)\| \\ &= \|A^*A+BB^*+BA+A^*B^*\| \\ &= \|\Re\left(A^*A+BB^*+2BA\right)\| \\ &\leq \omega\left(A^*A+BB^*+2BA\right),\end{aligned}$$

where the last equality follows because A^*A+BB^* is self-adjoint, and the last inequality follows from the simple fact that $\|\Re X\| \leq \omega(X)$ for $X \in \mathbb{B}(\mathbb{H})$. This completes the proof. □

Remark 2.2. *If we replace B by $-B$ in Theorem 2.1, then*

$$\|A-B^*\|^2 \leq \omega(A^*A+BB^*-2BA) \leq \|A^*A+BB^*-2BA\|.$$

In particular, if $A,B \geq O$, so that $A^*=A$ and $B^*=B$, we have

$$\|A-B\|^2 \leq \omega(A^2+B^2-2BA \leq \left\|A^2+B^2-2BA\right\|,$$

which is a refinement of (1.6).

Remark 2.3. *Notice that Theorem 2.1 provides a refinement of (1.7) and (1.8).*

One can easily see that

$$\|A+B^*\|^2 \leq 2\,\|A^*A+BB^*\|. \tag{2.1}$$

Indeed, it follows from the proof of Theorem 2.1 that

$$\begin{aligned}\|A+B^*\|^2 &= \|A^*A+BB^*+2\Re\left(BA\right)\| \\ &\leq \|A^*A+BB^*\|+2\,\|\Re\left(BA\right)\| \\ &\leq \|A^*A+BB^*\|+2\omega\left(BA\right).\end{aligned}$$

Using (1.9), we deduce (2.1) immediately.

In the following result, we present a refinement of this estimate.

Corollary 2.4. *Let $A,B \in \mathbb{B}(\mathbb{H})$. Then*

$$\|A+B^*\|^2 \leq \omega\left(A^*A+BB^*+2BA\right) \leq 2\,\|A^*A+BB^*\|,$$

and

$$\|A+B^*\|^2 \leq \omega\left(AA^*+B^*B+2AB\right) \leq 2\,\|AA^*+B^*B\|.$$

Proof. Applying Theorem 2.1 and using Lemma 1.1 and (2.1), we have

$$\begin{aligned}\|A+B^*\|^2 &\le \omega\left((A^*+B)A+B(A+B^*)\right)\\ &\le \sqrt{2\left\||A+B^*|^2\right\|\left\||A|^2+|B^*|^2\right\|}\\ &= \sqrt{2}\,\|A+B^*\|\left\||A|^2+|B^*|^2\right\|^{\frac{1}{2}}\\ &\le 2\,\|A^*A+BB^*\|,\end{aligned}$$

which completes the proof of the first inequality. The second inequality follows from the first, due to $\|A+B^*\| = \|A^*+B\|$. □

Remark 2.5. *It follows from Corollary 2.4 that*

$$\|A+B^*\| \le \min\left\{\omega^{\frac{1}{2}}\left(A^*A+BB^*+2BA\right), \omega^{\frac{1}{2}}\left(AA^*+B^*B+2AB\right)\right\}.$$

Equivalently,

$$\|A^*+B\| \le \min\left\{\omega^{\frac{1}{2}}\left(A\left(A^*+B\right)+\left(A^*+B\right)^*B\right), \omega^{\frac{1}{2}}\left(\left(A^*+B\right)A+B(A^*+B)^*\right)\right\}. \tag{2.2}$$

If $T=\Re T+\mathrm{i}\Im T$ is the Cartesian decomposition of $T\in\mathbb{B}(\mathbb{H})$, letting $A=\Re T$ and $B=\mathrm{i}\Im T$ in (2.2), we get

$$\begin{aligned}\|T\| &\le \min\left\{\omega^{\frac{1}{2}}\left((\Re T)T+\mathrm{i}T^*(\Im T)\right), \omega^{\frac{1}{2}}\left(T(\Re T)+\mathrm{i}(\Im T)T^*\right)\right\}\\ &= \min\left\{\omega^{\frac{1}{2}}\left(T^*T+\mathrm{i}\Im T^2\right), \omega^{\frac{1}{2}}\left(TT^*+\mathrm{i}\Im T^2\right)\right\}.\end{aligned}$$

In particular, if A,B are normal operators, then

$$\begin{aligned}\|A+B^*\|^2 &\le \min\left\{\omega\left(A^*A+B^*B+2BA\right), \omega\left(A^*A+B^*B+2AB\right)\right\}\\ &\le \max\left\{\omega\left(A^*A+B^*B+2BA\right), \omega\left(A^*A+B^*B+2AB\right)\right\}\\ &\le 2\,\|A^*A+B^*B\|.\end{aligned}$$

Corollary 2.6. *Let $T\in\mathbb{B}(\mathbb{H})$. Then*

$$\begin{aligned}\|T\|^2 &\le \min\left\{\omega\left(T^*T+\mathrm{i}\Im T^2\right), \omega\left(TT^*+\mathrm{i}\Im T^2\right)\right\}\\ &\le \max\left\{\omega\left(T^*T+\mathrm{i}\Im T^2\right), \omega\left(TT^*+\mathrm{i}\Im T^2\right)\right\}\\ &\le \|TT^*+T^*T\|.\end{aligned}$$

Proof. If we replace A by $\Re T$ and B by $\mathrm{i}\Im T$, in Corollary 2.4, we get

$$\|T\|^2 \le \omega\left(T^*T+\mathrm{i}\Im T^2\right) \le \|TT^*+T^*T\|,$$

and

$$\|T\|^2 \le \omega\left(TT^*+\mathrm{i}\Im T^2\right) \le \|TT^*+T^*T\|.$$

We get the desired result by combining the above two inequalities immediately. □

Interestingly, Corollary 2.6 could be used to obtain the following refinement of the first inequality in (1.1).

Theorem 2.7. *Let $T \in \mathbb{B}(\mathbb{H})$. Then*

$$\begin{aligned}\frac{1}{2}\|T\| &\le \frac{1}{2}\min\left\{\omega^{\frac{1}{2}}\left(T^*T + \mathrm{i}\Im T^2\right), \omega^{\frac{1}{2}}\left(TT^* + \mathrm{i}\Im T^2\right)\right\}\\ &\le \frac{1}{2}\max\left\{\omega^{\frac{1}{2}}\left(T^*T + \mathrm{i}\Im T^2\right), \omega^{\frac{1}{2}}\left(TT^* + \mathrm{i}\Im T^2\right)\right\}\\ &\le \omega(T).\end{aligned}$$

Proof. Notice that

$$\begin{aligned}\|TT^* + T^*T\| &= 2\left\|(\Re T)^2 + (\Im T)^2\right\|\\ &\le 2\left(\left\|(\Re T)^2\right\| + \left\|(\Im T)^2\right\|\right)\\ &= 2\left(\|\Re T\|^2 + \|\Im T\|^2\right)\\ &\le 4\omega^2(T).\end{aligned}$$

Therefore, from Corollary 2.6, we obtain the desired result immediately. □

Continuing with the same theme, we have the following refinements of the basic known inequalities that if $A \in \mathbb{B}(\mathbb{H})$, then

$$\Re A \le \omega(A) \text{ and } \Im A \le \omega(A).$$

Corollary 2.8. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$\|\Re A\| \le \frac{\sqrt{2}}{2}\,\omega^{\frac{1}{2}}\left((\Re A)A + A(\Re A)\right) \le \omega(A),$$

and

$$\|\Im A\| \le \frac{\sqrt{2}}{2}\omega^{\frac{1}{2}}\left((\Im A)A + A(\Im A)\right) \le \omega(A).$$

Proof. We prove the first inequality. Notice first that if $\Re A = O$, then the result follows immediately. So, we may assume that $\Re A \ne O$. Noting that $\Re A = \frac{A+A^*}{2}$, it follows from Theorem 2.1 that

$$\|\Re A\| \le \frac{1}{2}\sqrt{\omega\left(AA^* + A^*A + 2A^2\right)},$$

which can be written in the following form

$$\begin{aligned}\|\Re A\|^2 &\le \frac{1}{4}\omega\left(A(A+A^*) + (A+A^*)A\right)\\ &= \frac{1}{2}\omega\left(A(\Re A) + (\Re A)A\right)\\ &= \frac{1}{2}\omega\left(\left(A(\Re A) + (\Re A)A\right)^*\right)\\ &= \frac{1}{2}\omega\left((\Re A)A^* + A^*(\Re A)\right).\end{aligned}$$

It follows from (1.10) that

$$\|\Re A\|^2 \le \frac{1}{2}\omega\left((\Re A)A^* + A^*(\Re A)\right) \le \|\Re A\|\,\omega(A).$$

Now, we infer the first inequality by using the fact that $\|\Re A\| \le \omega(A)$, noting the assumption $\Re A \ne O$.

If we replace A by $\mathrm{i}A^*$, we get the second inequality. $\square$

Remark 2.9. *It follows from Corollary 2.8 that*

$$\frac{1}{2}\|A\| \leq \frac{\sqrt{2}}{4}\left(\omega^{\frac{1}{2}}\left((\Im A)A + A(\Im A)\right) + \omega^{\frac{1}{2}}\left((\Re A)A + A(\Re A)\right)\right) \leq \omega(A),$$

due to

$$\begin{aligned}\|A\| &\leq \|\Re A\| + \|\Im A\| \\ &\leq \frac{\sqrt{2}}{2}\left(\omega^{\frac{1}{2}}\left((\Im A)A + A(\Im A)\right) + \omega^{\frac{1}{2}}\left((\Re A)A + A(\Re A)\right)\right) \\ &\leq 2\omega(A).\end{aligned}$$

This provides another refinement of the first inequality in (1.1).

Remark 2.10. *To see that the coefficient of $\frac{\sqrt{2}}{2}$ in the middle term in the inequality shown in Corollary 2.8 cannot be replaced with a smaller coefficient, it suffices to assume that the operator A is self-adjoint. In this case, all three values in this inequality will equal $\|A\|$.*

We have seen that (1.2) provides an equivalent identity for the numerical radius, where the supremum of a certain norm is equal to the numerical radius. We witnessed such relations in the literature, where other equivalent forms have been shown in [8, 11, 12, 14] for an operator or an operator matrix, for example. In the following, we add a new identity.

Corollary 2.11. *Let $A \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega^2(A) = \frac{1}{2}\sup_{\theta\in\mathbb{R}} \omega\left(\left(\Re\left(e^{\mathrm{i}\theta}A\right)\right)A + A\left(\Re\left(e^{\mathrm{i}\theta}A\right)\right)\right).$$

Proof. By replacing A by $e^{\mathrm{i}\theta}A$, in Corollary 2.8, we get

$$\left\|\Re\left(e^{\mathrm{i}\theta}A\right)\right\| \leq \frac{\sqrt{2}}{2}\,\omega^{\frac{1}{2}}\left(\left(\Re\left(e^{\mathrm{i}\theta}A\right)\right)A + A\left(\Re\left(e^{\mathrm{i}\theta}A\right)\right)\right) \leq \omega(A).$$

Taking the supremum over $\theta \in \mathbb{R}$ gives the desired result, due to (1.2). $\square$

The next result improves the first inequality in (1.4).

Corollary 2.12. *Let $S, T \in \mathbb{B}(\mathbb{H})$. Then*

$$\begin{aligned}\|S+T\| &\leq \max\left\{\omega^{\frac{1}{2}}\left((S+T)S^* + T(S+T)^*\right), \omega^{\frac{1}{2}}\left((S+T)^*T + S^*(S+T)\right)\right\} \\ &\leq 2\omega\left(\begin{bmatrix} O & S \\ T^* & O\end{bmatrix}\right).\end{aligned}$$

Proof. Let $A=\begin{bmatrix} O & S \\ T^* & O \end{bmatrix}$. Applying Corollary 2.8 on this A yields

$$\begin{aligned}
\|\Re A\| &= \frac{1}{2}\|S+T\| \\
&= \left\| \begin{bmatrix} O & \frac{S+T}{2} \\ \frac{(S+T)^*}{2} & O \end{bmatrix} \right\| \\
&\le \frac{\sqrt{2}}{2}\omega^{\frac{1}{2}}\left(\begin{bmatrix} \frac{(S+T)T^*+S(S+T)^*}{2} & O \\ O & \frac{(S+T)^*S+T^*(S+T)}{2} \end{bmatrix} \right) \\
&= \frac{1}{2}\omega^{\frac{1}{2}}\left(\begin{bmatrix} (S+T)T^*+S(S+T)^* & O \\ O & (S+T)^*S+T^*(S+T) \end{bmatrix} \right) \\
&= \frac{1}{2}\max\left\{ \omega^{\frac{1}{2}}\left((S+T)T^*+S(S+T)^*\right), \omega^{\frac{1}{2}}\left((S+T)^*S+T^*(S+T)\right) \right\} \\
&\le \omega\left(\begin{bmatrix} O & S \\ T^* & O \end{bmatrix} \right),
\end{aligned}$$

where we have used the facts that

$$\left\| \begin{bmatrix} O & X \\ X^* & O \end{bmatrix} \right) = \|X\|, \omega\left(\begin{bmatrix} X & O \\ O & Y \end{bmatrix} \right) = \max\{\omega(X), \omega(Y)\}$$

in the above calculations. This completes the proof. □

Remark 2.13. *It is known that if $X,Y \ge O$, then [3]*

$$\omega\left(\begin{bmatrix} O & X \\ Y & O \end{bmatrix} \right) = \frac{\|X+Y\|}{2}.$$

This together with Corollary 2.12 implies

$$\begin{aligned}
\frac{1}{2}\|S+T\| &= \frac{1}{2}\max\left\{ \omega^{\frac{1}{2}}\left((S+T)T+S(S+T)\right), \omega^{\frac{1}{2}}\left((S+T)S+T(S+T)\right) \right\} \\
&= \omega\left(\begin{bmatrix} O & S \\ T & O \end{bmatrix} \right),
\end{aligned}$$

provided that $S,T \ge O$. If $S,T \ge O$, this is equivalent to

$$\frac{1}{2}\|S+T\| = \frac{1}{2}\omega^{1/2}\left(S^2+T^2+2ST\right) = \omega\left(\begin{bmatrix} O & S \\ T & O \end{bmatrix} \right).$$

We conclude with the following excellent refinement of the second inequality in (1.4).

Theorem 2.14. *Let $A,B \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega^2\left(\begin{bmatrix} O & A \\ B & O \end{bmatrix} \right) \le \frac{1}{2}\omega\left(\begin{bmatrix} O & 2BA \\ A^*A+BB^* & O \end{bmatrix} \right).$$

Proof. If we replace A by $e^{\mathrm{i}\theta}A$, in Theorem 2.1, we obtain

$$\begin{aligned}\left\|e^{\mathrm{i}\theta}A+B^*\right\| &\le \sqrt{\omega\left(A^*A+BB^*+2e^{\mathrm{i}\theta}BA\right)}\\ &\le \sqrt{\left\|A^*A+BB^*+2e^{\mathrm{i}\theta}BA\right\|}\\ &\le \sqrt{2\omega\left(\begin{bmatrix} O & A^*A+BB^* \\ 2A^*B^* & O\end{bmatrix}\right)}\end{aligned}$$

That is,

$$\left\|e^{\mathrm{i}\theta}A+B^*\right\| \le \sqrt{2\omega\left(\begin{bmatrix} O & 2BA \\ A^*A+BB^* & O\end{bmatrix}\right)}.$$

If we take the supremum over $\theta\in\mathbb{R}$, we get

$$\begin{aligned}\sup_{\theta\in\mathbb{R}}\left\|e^{\mathrm{i}\theta}A+B^*\right\| &= 2\omega\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right)\\ &\le \sqrt{2\omega\left(\begin{bmatrix} O & 2BA \\ A^*A+BB^* & O\end{bmatrix}\right)},\end{aligned}$$

which is equivalent to the desired conclusion. $\square$

Remark 2.15. *Applying* (1.4) *implies that*

$$\begin{aligned}\frac{1}{2}\omega\left(\begin{bmatrix} O & 2BA \\ A^*A+BB^* & O\end{bmatrix}\right) &\le \frac{1}{4}\left(2\|BA\|+\|A^*A+BB^*\|\right)\\ &\le \frac{1}{4}\left(2\|BA\|+\|A^*A\|+\|BB^*\|\right)\\ &\le \frac{1}{4}\left(2\|A\|\|B\|+\|A^*A\|+\|BB^*\|\right)\\ &= \frac{1}{4}\left(2\|A\|\|B\|+\|A\|^2+\|B\|^2\right)\\ &= \frac{1}{4}(\|A\|+\|B\|)^2.\end{aligned}$$

So, Theorem 2.14 is sharper than the second inequality in (1.4).

Remark 2.16. *We give an example where Theorem 2.14 can be better than* (1.5). *Let* $A=\begin{bmatrix} -1 & 1 \\ 1 & 0\end{bmatrix}$ *and* $B=\begin{bmatrix} 1 & -1 \\ 0 & 1\end{bmatrix}$. *Numerical calculations show that*

$$\omega^2\left(\begin{bmatrix} O & A \\ B & O\end{bmatrix}\right)=2.5, \frac{1}{4}\left\||A^*|+|B|\right\|\left\||A|+|B^*|\right\|\approx 2.53262,$$

and

$$\frac{1}{2}\omega\left(\begin{bmatrix} O & 2BA \\ A^*A+BB^* & O\end{bmatrix}\right)=2.5,$$

showing that Theorem 2.14 is sharper than (1.5) *for this example. We point out that* (1.5) *can be better than Theorem 2.14 for other examples. In fact, Theorem 2.14 is sharp, as the above numerical example indicates.*

Acknowledgements
This research was done when the first author was on a sabbatical leave from Princess Sumaya University for Technology to Abdullah Al Salem University. The first author is grateful to both institutions.

REFERENCES

[1] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for products and commutators of operators, Houston J. Math. 41 (2015) 1163–1173.
[2] A. Abu-Omar, F. Kittaneh, Generalized spectral radius and norm inequalities for Hilbert space operators, Int. J. Math. 26 (2015) 1550097.
[3] A. Abu-Omar, F. Kittaneh, Numerical radius inequalities for $n \times n$ operator matrices, Linear Algebra Appl. 468 (2015) 18–26.
[4] C. Conde, F. Kittaneh, H. R. Moradi and M. Sababheh, Numerical radii of operator matrices in terms of certain complex combinations of operators, Georgian Math, J. 31 (2024) 575–586.
[5] C.-K. Fong, J. A. R. Holbrook, Unitarily invariant operator norms, Can. J. Math. 35 (1983) 274–299.
[6] K. E. Gustafson, D. K. M. Rao, Numerical Range: The Field of Values of Linear Operators and Matrices, Springer, New York, 1997.
[7] M. Hajmohamadi, R. Lashkaripour, M. Bakherad, Some generalizations of numerical radius on off-diagonal part of 2×2 operator matrices, J. Math. Inequal. 12 (2018) 447–457.
[8] O. Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices, Integral Equ. Oper. Theory 71 (2011) 129–147.
[9] R. Horn, X. Zhan, Inequalities for C-S seminorms and Lieb functions, Linear Algebra Appl. 291 (1999) 103–113.
[10] F. Kittaneh, Norm inequalities for commutators of positive operators and applications, Math Z. 258 (2008) 845–849.
[11] F. Kittaneh, H. R. Moradi, M. Sababheh, On the numerical radius of the product of Hilbert space operators, Hokkaido Math. J. 53 (2024) 1–26.
[12] F. Kittaneh, M. S. Moslehian, T. Yamazaki, Cartesian decomposition and numerical radius inequalities, Linear Algebra Its Appl. 471 (2015) 46–53.
[13] Hi. Qiao, G. Hai, E. Bai, Some refinements of numerical radius inequalities for 2×2 operator matrices, J. Math. Inequal. 16 (2022) 425–444.
[14] M. Sababheh, C. Conde, H.R. Moradi, On the numerical radius of an operator matrix, Math. Inequal. Appl. 27 (2024) 417–434.
[15] K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices II, Linear Algebra Appl. 523 (2017) 1–12.