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A VISCOSITY ALGORITHM FOR THE SPLIT COMMON FIXED POINT PROBLEM WITH MULTIPLE OUTPUT SETS

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Abstract. In the paper, we study the split common fixed point problem with multiple output sets for demicontractive mappings in Hilbert spaces. To solve this problem, we propose a multi-parameter viscosity iterative method and its error-sequenced version. Furthermore, we prove the strong convergence of the algorithms under some mild conditions. Our results generalize some existing strong convergence results. Finally, two applications are provided to support our convergence theorems.

Keywords. Demicontractive mapping; Multiple output sets; Split common fixed point problem; Viscosity.

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1. Introduction

The Split Feasibility Problem (SFP), originally introduced by Censor and Elfving [1], is a significant class of inverse problems. Due to its extensive applications in applied disciplines such as signal processing, image reconstruction, and intensity-modulated radiation therapy, it has attracted considerable research attention in recent years; see, e.g., [2–5]

As an important generalization of the SFP, Censor and Segal [6] introduced the Split Common Fixed Point Problem (SCFPP). Specifically, let H, H_1 be two real Hilbert spaces, $A : H \to H_1$ be a linear and bounded operator, and let $T_0 : H \to H$, $T_1 : H_1 \to H_1$ be two nonlinear mappings. The SCFPP is to find $x^* \in H$ such that

$$x^* \in \mathrm{Fix}(T_0) \cap A^{-1}\big(\mathrm{Fix}(T_1)\big), \tag{1.1}$$

where $A^{-1}\big(\mathrm{Fix}(T_1)\big) = \{x \in H : Ax \in \mathrm{Fix}(T_1)\}$, $\mathrm{Fix}(T_0)$ and $\mathrm{Fix}(T_1)$ denote the fixed point sets of operators T_0 and T_1, respectively. We usually assume that the SCFPP is consistent, meaning that the set of solutions is not empty, denoted by $\Omega \neq \varnothing$.

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In the past, scholars have devoted efforts to solving the SCFPP. Byrne [7] proposed the famous CQ algorithm for the SFP, and Censor and Segal [6] proposed the algorithm, which is designed for directed operators. For any initial guess $x_0 \in H$, this algorithm generates a sequence $\{x_n\}$ via the following recursive process:

$$x_{n+1} = T_0\Big(x_n + \gamma A^t(T_1 - I)Ax_n\Big),\ n \geq 0, \tag{1.2}$$

where $\gamma \in (0, 2/\lambda)$ and λ is the largest eigenvalue of the matrix A^tA (A^t denotes the matrix transpose). They proved that algorithm (1.2) converges weakly to a solution of problem (1.1).

In 2010, to extend the study to more general nonlinear mappings, Moudafi [8] proposed the following algorithm for demicontractive operators T_0 and T_1, with an initial point $x_0 \in H$:

$$x_{n+1} = (1-\alpha_n)u_n + \alpha_n T_0 u_n,\ n \geq 0, \tag{1.3}$$

where $u_n = x_n + \gamma A^*(T_1 - I)Ax_n$, $\gamma \in (0, 1-\mu/\lambda)$ with λ being the spectral radius of A^*A, and $\{\alpha_n\} \subset (0,1)$. It was shown that, for demicontractive operators, algorithm (1.3) converges weakly to a solution of problem (1.1).

In 2015, for a k_0-demicontractive mapping $T_0 : H \to H$ and a k_1-demicontractive mapping $T_1 : H_1 \to H_1$, Shehu and Cholamjiak [9] modified the algorithm (1.3) into the following form: an initial guess $x_0 \in H$,

$$x_{n+1} = (1-\beta_n)(\lambda_n y_n) + \beta_n T_0 y_n, \quad n \geq 1, \tag{1.4}$$

where $y_n = x_n + \gamma A^*(T_1 - I)Ax_n$, $\{\beta_n\}, \{\lambda_n\} \subset (0,1)$ and $\gamma \in \left(0, \frac{1-k_1}{\|A\|^2}\right)$. They proved the strong convergence of algorithm (1.4) under the following conditions:

$$\lim_{n\to\infty} \lambda_n = 1, \qquad \sum_{n=1}^{\infty}(1-\lambda_n) = \infty;$$

$$\beta_n \in \left[\varepsilon, \tfrac{\lambda_n(1-k_1)}{1-(1-k_1)(1-\lambda_n)}\right] \text{ for some } \varepsilon > 0 \text{ and } \limsup_{n\to\infty} \beta_n < 1-k_1.$$

In 2019, Wang, Fang and Kim [10] proposed a viscosity-type iterative method for the SCFPP: starting from an arbitrary initial point $x_0 \in H$, the algorithm generated a sequence $\{x_n\}$ by

$$x_{n+1} = \beta_n f(x_n) + (1-\beta_n)\Big[x_n - \tau_n\big(x_n - T_0x_n + A^*(I-T_1)Ax_n\big)\Big], \tag{1.5}$$

where $f : H \to H$ is a contraction with coefficient $\alpha \in (0, 1/\sqrt{2})$, $\{\beta_n\} \subset (0,1)$, and the stepsize τ_n is chosen as

$$\tau_n = \frac{\rho_n}{\|x_n - T_0x_n + A^*(I-T_1)Ax_n\|}, \tag{1.6}$$

with $\{\rho_n\} \subset (0, +\infty)$ and $\sum_{n=0}^{\infty} \rho_n^2 < \infty$. They proved the strong convergence of the algorithm (1.5) under stepsize rule (1.6). For further research on SCFPP algorithms, we refer to [11–15].

However, in practical applications, we often face more complex constraints. Motivated by practical applications involving multiple constraints, several extensions of the SFP and the SCFPP have been studied in some literature; see, e.g., [16–18]. One such extension is the split common fixed point problem with multiple output sets. For each $i = 1, 2, \ldots, N$, let H and H_i be real Hilbert spaces, $A_i : H \to H_i$ be a linear and bounded operator, $T_0 : H \to H$, $T_i : H_i \to H_i$

be nonlinear mappings. The split common fixed point problem with multiple output sets (for short SCFPPMOS) is to find a point $x^* \in H$ such that

$$x^* \in \mathrm{Fix}(T_0) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}\left(\mathrm{Fix}(T_i)\right) \right), \tag{1.7}$$

where $A_i^{-1}\left(\mathrm{Fix}(T_i)\right) = \{x \in H : A_i x \in \mathrm{Fix}(T_i)\}$. This model provides a unified framework for dealing with multi-constraint inverse problems.

In 2021, Reich et al. [19] introduced a new self-adaptive viscosity-type algorithm for solving the SCFPPMOS:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) u_n, \tag{1.8}$$

where $u_n = x_n - \delta_n \Theta_n^*(\Theta_n x_n - t_n)$,

$$\delta_n = \rho_n \frac{\|\Theta_n x_n - t_n\|^2}{\|\Theta_n^*(\Theta_n x_n - t_n)\|^2 + a_n},$$

$\Theta_n = I$ or A_{i_n}, $t_n = T_0 x_n$ or $T_{i_n}(A_{i_n} x_n)$ according to the maximal residual is attained in the original space or in one of the output spaces. Here, $f : H \to H$ is a strict contraction with a contraction coefficient $c \in [0,1)$, $\{\alpha_n\} \subset (0,1)$, $\{\rho_n\} \subset [c,d] \subset (0,1)$ and $\{a_n\}$ is a sequence of positive real numbers. They proved the strong convergence of the algorithm (1.8).

In 2022, Wang [20] addressed this problem by extending the scope of research to the case of demicontractive mappings. To get strong convergence, he proposed the following two algorithms.

Fixed Step-Size Algorithm:

$$x_{n+1} = \gamma_n f(x_n) + (1-\gamma_n)\, T_{0,\lambda} \left(x_n - \tau \sum_{i=1}^{N} A_i^*(I - T_i) A_i x_n \right), \tag{1.9}$$

where $f : H \to H$ is a ρ-contractive mapping with $0 < \rho < 1$, $0 < \lambda < 1 - \kappa_0$, $0 < \tau < \frac{\kappa}{\sum_{i=1}^{N} \|A_i\|^2}$, and $T_{0,\lambda} = (1-\lambda)I + \lambda T_0$. They proved the strong convergence of algorithm (1.9) under some proper conditions.

Variable Step Size Algorithm:

$$x_{n+1} = \gamma_n f(x_n) + (1-\gamma_n)\, T_{0,\lambda} \left(x_n - \tau_n \sum_{i=1}^{N} A_i^*(I - T_i) A_i x_n \right), \tag{1.10}$$

where the step size satisfies $\tau_n = 0$ if $\left\|\sum_{i=1}^{N} A_i^*(I - T_i) A_i x_n\right\| = 0$; otherwise

$$\tau_n = \frac{\min_{1 \le i \le N}(1-\kappa_i) \sum_{i=1}^{N} \|(I - T_i) A_i x_n\|^2}{2\left\|\sum_{i=1}^{N} A_i^*(I - T_i) A_i x_n\right\|^2}. \tag{1.11}$$

They established the strong convergence for the method (1.10) under stepsize condition (1.11).

In 2024, Cui and Wang [21] proposed the following two algorithms:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) \left[x_n - \tau \sum_{i=0}^{N} A_i^* \left(A_i x_n - T_i(A_i x_n) \right) \right], \tag{1.12}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n) \left[x_n - \tau_n \sum_{i=0}^{N} A_i^* \left(A_i x_n - T_i(A_i x_n) \right) \right], \tag{1.13}$$

where $\{\alpha_n\} \subseteq [0,1]$, the stepsizes τ and τ_n satisfy the following conditions, respectively:

$$0 < \tau \left(\sum_{i=0}^{N} \|A_i\|^2 \right) < \kappa, \tag{1.14}$$

$$\tau_n = \begin{cases} 0, \text{if } \left\| \sum_{i=0}^{N} A_i^*(I - T_i)A_i x_n \right\| = 0, \\ \dfrac{\kappa \sum_{i=0}^{N} \|A_i x_n - T_i(A_i x_n)\|^2}{2\left\|\sum_{i=0}^{N} A_i^*(I-T_i)A_i x_n\right\|^2}, \text{otherwise.} \end{cases} \tag{1.15}$$

They proved the strong convergence of algorithms (1.12) and (1.13) under stepsize conditions (1.14) and (1.15), respectively.

Most recently, Cui and Wang [22] proposed the following algorithm:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\Big[(1-\lambda)y_n + \lambda T_0 y_n\Big], \tag{1.16}$$

where $y_n = x_n - \tau \sum_{i=1}^{q} A_i^*(I - T_i)A_i x_n$, f is a given contraction, $0 < \lambda < 1 - \kappa_0$, $\{\alpha_n\} \subset [0,1]$ and $0 < \tau < 1/\big(\sum_{i=1}^{q} \|A_i\|^2/(1-\kappa_i)\big)$. They proved that the algorithm (1.16) converges strongly to a solution of problem (1.7).

Inspired by the above works, we propose a multi-parameter viscosity iterative algorithm and consider its extension with errors to solve problem (1.7) in the paper. Our aim is to obtain the strong convergence for the two algorithms under some mild conditions that do not require prior knowledge of operator norms. Finally, two applications are provided in the paper.

2. PRELIMINARIES

This section briefly reviews some basic concepts of Hilbert spaces and related lemmas required for the subsequent analysis in the paper.

In the paper, let H be a real Hilbert space and let C be a nonempty, convex, and closed set in space H. We denote by $\langle \cdot, \cdot \rangle$ the inner product on H and by $\|\cdot\|$ the norm induced by $\langle \cdot, \cdot \rangle$, i.e., $\|x\| = \sqrt{\langle x, x\rangle}$ for all $x \in H$. For a sequence $\{x_n\}$ and a point $x \in H$, the symbol $x_n \rightharpoonup x$ denotes that the sequence $\{x_n\}$ converges weakly to x, and the symbol $x_n \to x$ denotes that the sequence $\{x_n\}$ converges strongly to x.

Definition 2.1. *If there exists a constant $\rho \in [0,1)$ such that, for any $x, y \in H$, $\|f(x) - f(y)\| \leq \rho\|x - y\|$, then f is called a contraction mapping.*

Definition 2.2. *Let $T : H \to H$ be a nonlinear mapping, and let* $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$ *be the fixed point set of T.*

(a) If, for any $x, y \in H$, $\|Tx - Ty\| \leq \|x - y\|$, then T is called a nonexpansive mapping;

(b) If, for any $x \in H$ and $p \in Fix(T)$, which is nonempty, $\|Tx - Tp\| \leq \|x - p\|$, then T is called a quasi-nonexpansive mapping;

(c) If, for any $x, y \in H$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$, then T is called a firmly nonexpansive mapping;

(d) If, for any $x, y \in H$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(x - y) - (Tx - Ty)\|^2$, then T is called a pseudocontractive mapping;

(e) If there exists a constant $\kappa \in (0,1)$ such that, for any $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(x - y) - (Tx - Ty)\|^2,$$

then T is called a strictly pseudocontractive mapping;

(f) If there exists a constant $\kappa < 1$ such that, for any $x \in H$ and $p \in Fix(T)$,

$$\|Tx - p\|^2 \le \|x - p\|^2 + \kappa\|x - Tx\|^2,$$

then T is called a κ-demicontractive mapping.

Obviously, nonexpansive mappings are contained in quasi-nonexpansive mappings, and quasi-nonexpansive mappings are special cases of demicontractive mappings when $\kappa = 0$.

Example 2.3. *Let R be the set of real numbers, and define $T : R \to R$ by $Tx = -2x$. It is easy to check that T is continuous and $\frac{1}{3}$-demicontractive with a unique fixed point 0. However, T is not quasi-nonexpansive. Indeed, let $x_0 = 1$. Then $\|Tx_0 - 0\| = 2 > 1 = \|x_0 - 0\|$.*

Example 2.4. *[20] Define $T : H \to H$ by*

$$Tx = \begin{cases} \frac{1}{2}x, & \|x\| > 1, \\ -3x, & \|x\| \le 1. \end{cases}$$

It is easy to check that T is discontinuous and $\frac{1}{2}$-demicontractive with a unique fixed point 0. However, T is not quasi-nonexpansive. Indeed, fix any $x_0 \in H$ with $0 < \|x_0\| \le 1$. Then

$$\|Tx_0\| = 3\|x_0\| > \|x_0\|.$$

Definition 2.5. *Let C be a nonempty, convex, and closed subset of H. The metric projection from H onto C, denoted by P_C, is defined by $\|x - P_C x\| = \min_{y \in C} \|x - y\|$ for all x in H.*

Lemma 2.6. *For any $x \in H$ and $z \in C$, $z = P_C x$ if and only if*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.1}$$

Lemma 2.7. *For any $x, y \in H$ and $\lambda \in R$, the following conclusions hold:*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$;

(ii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

Lemma 2.8. *[8] For any $\lambda > 0$, let $T_\lambda = (1 - \lambda)I + \lambda T$. If the operator T is a κ-demicontractive mapping, then the following conclusions hold:*

(i) *$Fix(T)$ is a closed convex subset;*

(ii) *For every $(x, z) \in H \times Fix(T)$, we have*

$$\langle x - Tx, x - z\rangle \ge \frac{1 - \kappa}{2}\|x - Tx\|^2. \tag{2.2}$$

(iii) *For every $(x, z) \in H \times Fix(T)$, we have*

$$\|T_\lambda x - z\|^2 \le \|x - z\|^2 - \lambda(1 - \lambda - \kappa)\|(I - T)x\|^2. \tag{2.3}$$

Lemma 2.9. *[23] Let C be a closed and convex subset of a Hilbert space H and $T : C \to H$ be a nonexpansive mapping. If the sequence $\{x_n\} \subset C$ converges weakly to x^*, and $(I - T)x_n$ converges strongly to 0, then $x^* \in Fix(T)$.*

Lemma 2.10. *[24] Let $s_{n+1} \le (1 - \gamma_n)s_n + \gamma_n\delta_n + \varepsilon_n$ for all $n \ge 0$, where $\{\gamma_n\} \subset (0, 1)$, δ_n, and ε_n satisfy*

(i) $\lim_{n\to\infty} \gamma_n = 0, \quad \sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \gamma_n |\delta_n| < \infty$;

(iii) $\varepsilon_n \geq 0, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

3. Main results

In the section, we propose two algorithms to solve the split common fixed point problem with multiple output sets. To this end, we assume that problem (1.7) is consistent, namely its solution set, denoted by S, is nonempty.

In the following, we assume $\kappa_i \in (0,1)$ for each $i = 0,1,2,\ldots,N$, and define $T_{0,\lambda} = (1-\lambda)I + \lambda T_0$ with $0 < \lambda < 1-\kappa_0$. Given an arbitrary initial point $x_0 \in H$, the algorithm generates the sequence $\{x_n\}$ recursively according to

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \delta_n T_{0,\lambda}\left[x_n - \tau_n \sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\right], \tag{3.1}$$

where $\alpha_n, \beta_n, \delta_n \in (0,1)$, $\alpha_n + \beta_n + \delta_n = 1$, and $\tau_n = 0$ if $\left\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\right\| = 0$, otherwise

$$\tau_n = \frac{\min_{1\leq i\leq N}(1-\kappa_i)\sum_{i=1}^{N}\|(I-T_i)A_i x_n\|^2}{2\left\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\right\|^2}. \tag{3.2}$$

Theorem 3.1. *Let $A_i : H \to H_i$ be a linear and bounded operator. Let $T_0 : H \to H$, $T_i : H_i \to H_i$, and T_i be κ_i-demicontractive and fulfill the demiclosedness principle for each $i = 1,2,\ldots,N$, and let $f : H \to H$ be a ρ-contraction for some $\rho \in [0,1)$. Suppose that condition (3.2) holds and the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n\to\infty} \frac{\alpha_n}{\delta_n} = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to a solution z of problem (1.7), which is also the unique solution of the variational inequality:

$$z \in S,\ \langle (I-f)z, x-z\rangle \geq 0,\ \forall x \in S.$$

Alternatively, z is the unique fixed point of the contraction $P_S f$, that is, $z = P_S f(z)$.

Proof. Let $z \in S$ and $z = P_S f(z)$. For convenience, set $\kappa = \min_{1\leq i\leq N}(1-\kappa_i)$ and let $y_n = x_n - \tau_n \sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n$. If $\left\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\right\| \neq 0$, we can deduce from (2.2) that

$$\begin{aligned} 2\langle x_n - z, A_i^*(I-T_i)A_i x_n\rangle &= 2\langle A_i x_n - A_i z, (I-T_i)A_i x_n\rangle \\ &\geq (1-\kappa_i)\|(I-T_i)A_i x_n\|^2 \geq \kappa\|(I-T_i)A_i x_n\|^2, \end{aligned}$$

which immediately yields

$$\begin{aligned} \|y_n - z\|^2 &= \|x_n - z\|^2 - 2\tau_n \sum_{i=1}^{N}\langle x_n - z, A_i^*(I-T_i)A_i x_n\rangle + \tau_n^2\Big\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\Big\|^2 \\ &\leq \|x_n - z\|^2 - \kappa\tau_n \sum_{i=1}^{N}\|(I-T_i)A_i x_n\|^2 + \tau_n^2\Big\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\Big\|^2. \end{aligned}$$

From the definition of τ_n, we have

$$\|y_n - z\|^2 \le \|x_n - z\|^2 - \frac{\kappa\tau_n}{2}\sum_{i=1}^{N}\big\|(I-T_i)A_ix_n\big\|^2.$$

It is clear that the above inequality also holds true if $\big\|\sum_{i=1}^N A_i^*(I-T_i)A_ix_n\big\| = 0$. Moreover, from (2.3), we see that

$$\begin{aligned}\|T_{0,\lambda}y_n - z\|^2 &\le \|y_n - z\|^2 - \lambda(1-\lambda-\kappa_0)\|(I-T_0)y_n\|^2\\ &\le \|x_n - z\|^2 - \frac{\kappa\tau_n}{2}\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2 - \lambda(1-\lambda-\kappa_0)\|(I-T_0)y_n\|^2. \qquad (3.3)\end{aligned}$$

For convenience, we set $\rho_n = \frac{\delta_n}{1-\alpha_n}$, and $u_n = (1-\rho_n)x_n + \rho_n T_{0,\lambda}y_n$. Then we can rewrite algorithm (3.1) as $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n$. On the other hand, since $\alpha_n + \beta_n + \delta_n = 1$ and $\beta_n, \delta_n > 0$, it follows that

$$0 < \rho_n = \frac{\delta_n}{1-\alpha_n} = \frac{\delta_n}{\beta_n + \delta_n} < 1, \quad \text{for any } n \ge 0. \qquad (3.4)$$

To show the strong convergence, we divide the proof into four steps.

Step 1. Show the sequence $\{x_n\}$ is bounded.

By Lemma 2.7(ii), (3.3) and (3.4), we see that

$$\begin{aligned}\|u_n - z\|^2 &= (1-\rho_n)\|x_n - z\|^2 + \rho_n\|T_{0,\lambda}y_n - z\|^2 - \rho_n(1-\rho_n)\|x_n - T_{0,\lambda}y_n\|^2\\ &\le \|x_n - z\|^2 - \rho_n\Big(\frac{\kappa\tau_n}{2}\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2 + \lambda(1-\lambda-\kappa_0)\|(I-T_0)y_n\|^2\Big). \qquad (3.5)\end{aligned}$$

Since $0 < \lambda < 1-\kappa_0$, it follows from (3.5) that $\|u_n - z\| \le \|x_n - z\|$. Then we can deduce that

$$\begin{aligned}\|x_{n+1} - z\| &\le \alpha_n\|f(x_n) - f(z) + f(z) - z\| + (1-\alpha_n)\|u_n - z\|\\ &\le \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|u_n - z\|\\ &\le \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\|\\ &\le \max\left\{\|x_n - z\|, \frac{\|f(z)-z\|}{1-\rho}\right\}.\end{aligned}$$

By induction, we obtain

$$\|x_n - z\| \le \max\left\{\|x_0 - z\|, \frac{\|f(z)-z\|}{1-\rho}\right\}$$

for all $n \ge 0$. Hence, $\{x_n\}$ is bounded.

Step 2. Show the following inequality holds:

$$s_{n+1} \le (1-\tilde{\alpha}_n)s_n + \tilde{\alpha}_n b_n, \qquad (3.6)$$

where $\tilde{\alpha}_n = \alpha_n(1-\rho^2)$, $s_n = \|x_n - z\|^2$, and

$$\begin{aligned}b_n = &\frac{2}{1-\rho^2}\langle f(z) - z, x_{n+1} - z\rangle\\ &- \frac{(1-\alpha_n)\rho_n}{\alpha_n(1-\rho^2)}\left(\frac{\kappa\tau_n}{2}\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2 + \lambda(1-\lambda-\kappa_0)\|(I-T_0)y_n\|^2\right).\end{aligned}$$

By Lemma 2.7(i), we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\alpha_n(f(x_n)-f(z))+(1-\alpha_n)(u_n-z)+\alpha_n(f(z)-z)\|^2\\ &\le \|\alpha_n(f(x_n)-f(z))+(1-\alpha_n)(u_n-z)\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &\le \alpha_n\rho^2\|x_n-z\|^2+(1-\alpha_n)\|u_n-z\|^2+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle. \end{aligned}\tag{3.7}$$

Substituting (3.5) into the above inequality yields

$$\begin{aligned}\|x_{n+1}-z\|^2 \le{}& \alpha_n\rho^2\|x_n-z\|^2+(1-\alpha_n)\Bigg[\|x_n-z\|^2-\rho_n\Bigg(\frac{\kappa\tau_n}{2}\sum_{i=1}^N\|(I-T_i)A_ix_n\|^2\\ &+\lambda(1-\lambda-\kappa_0)\|(I-T_0)y_n\|^2\Bigg)\Bigg]+2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ ={}& \big[1-\alpha_n(1-\rho^2)\big]\|x_n-z\|^2\\ &+\alpha_n(1-\rho^2)\Bigg[-\frac{(1-\alpha_n)\rho_n}{\alpha_n(1-\rho^2)}\Bigg(\frac{\kappa\tau_n}{2}\sum_{i=1}^N\|(I-T_i)A_ix_n\|^2\\ &+\lambda(1-\lambda-\kappa_0)\|(I-T_0)y_n\|^2\Bigg)+\frac{2}{1-\rho^2}\langle f(z)-z,x_{n+1}-z\rangle\Bigg].\end{aligned}$$

Hence, the desired inequality at once follows.

Step 3. Show that $-\delta\le\limsup_{n\to\infty}b_n<+\infty$ for some $\delta>0$, which indicates that $\limsup_{n\to\infty}b_n$ is finite. Since $\{x_n\}$ is bounded, we have

$$\sup_{n\ge0}b_n\le\sup_{n\ge0}\frac{2}{1-\rho^2}\|f(z)-z\|\,\|x_{n+1}-z\|<+\infty.$$

We next prove $\limsup_{n\to\infty}b_n\ge-\delta$. To this aim, we proceed by contradiction. Assume that $\limsup_{n\to\infty}b_n<-\delta$, which implies that there exists $n_0\in N$ such that $b_n<-\delta$ for all $n\ge n_0$. It follows that

$$s_{n+1}\le(1-\tilde{\alpha}_n)s_n+\tilde{\alpha}_nb_n\le s_n-\tilde{\alpha}_n(s_n+\delta)\le s_n-(1-\rho^2)\delta\,\alpha_n.$$

By induction, we have

$$s_{n+1}\le s_{n_0}-(1-\rho^2)\delta\sum_{i=n_0}^n\alpha_i,\quad \forall n\ge n_0.$$

Thus, $\limsup_{n\to\infty}s_n<s_{n_0}-(1-\rho^2)\delta\sum_{i=n_0}^\infty\alpha_i=-\infty$. However, $\{s_n\}$ is a nonnegative real sequence, which is a contradiction. Therefore, $\limsup_{n\to\infty}b_n$ is finite.

Step 4. Prove that $\{x_n\}$ converges to $z=(P_Sf)z$. From Step 3, it can be seen that there exists a sequence $\{n_k\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}b_n=\lim_{k\to\infty}b_{n_k}=\lim_{k\to\infty}\Bigg[&\frac{2}{1-\rho^2}\langle f(z)-z,x_{n_k+1}-z\rangle\\ &-\frac{(1-\alpha_{n_k})\rho_{n_k}}{\alpha_{n_k}(1-\rho^2)}\Bigg(\frac{\kappa\tau_{n_k}}{2}\sum_{i=1}^N\|(I-T_i)A_ix_{n_k}\|^2+\lambda(1-\lambda-\kappa_0)\|(I-T_0)y_{n_k}\|^2\Bigg)\Bigg].\end{aligned}$$

Since the sequence $\{\langle f(z)-z, x_{n_k+1}-z\rangle\}$ is bounded, we may assume that, without loss of generality, there exists the limit $\lim_{k\to\infty}\langle f(z)-z, x_{n_k+1}-z\rangle$. Thus the following limit exists:

$$\lim_{k\to\infty}\frac{\delta_{n_k}}{\alpha_{n_k}}\Big(\frac{\kappa\tau_{n_k}}{2}\sum_{i=1}^{N}\|(I-T_i)A_ix_{n_k}\|^2+\lambda(1-\lambda-\kappa_0)\|(I-T_0)y_{n_k}\|^2\Big).$$

Moreover, combining with condition (ii), we have

$$\lim_{k\to\infty}\tau_{n_k}\sum_{i=1}^{N}\|(I-T_i)A_ix_{n_k}\|^2=\lim_{k\to\infty}\|(I-T_0)y_{n_k}\|^2=0. \tag{3.8}$$

According to (3.2) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\tau_n=\frac{\kappa\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2}{2\left\|\sum_{i=1}^{N}A_i^*(I-T_i)A_ix_n\right\|^2}&\geq\frac{\kappa\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2}{2\left(\sum_{i=1}^{N}\|A_i^*(I-T_i)A_ix_n\|\right)^2}\\&\geq\frac{\kappa\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2}{2\left(\sum_{i=1}^{N}\|A_i^*\|\,\|(I-T_i)A_ix_n\|\right)^2}\\&\geq\frac{\kappa}{2\sum_{i=1}^{N}\|A_i\|^2}>0,\end{aligned}$$

which together with (3.8) yields

$$\lim_{k\to\infty}\sum_{i=1}^{N}\|(I-T_i)A_ix_{n_k}\|^2=0,\ \text{i.e.}\lim_{k\to\infty}\|(I-T_i)A_ix_{n_k}\|=0, i=1,2,3... \tag{3.9}$$

On the other hand, we infer from (3.2) that

$$\|y_{n_k}-x_{n_k}\|^2=\left\|\tau_{n_k}\sum_{i=1}^{N}A_i^*(I-T_i)A_ix_{n_k}\right\|^2=\frac{\kappa\tau_{n_k}}{2}\sum_{i=1}^{N}\|(I-T_i)A_ix_{n_k}\|^2,$$

which together with (3.8) yields

$$\|y_{n_k}-x_{n_k}\|\to 0. \tag{3.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$. Combining (3.9) and Lemma 2.9, we obtain $A_ix^*\in Fix(T_i)$. Similarly, by (3.8), (3.10) and Lemma 2.9, we have $x^*\in Fix(T_0)$. Thus, $x^*\in S$. Note that

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\|&\leq\|x_{n_k+1}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\\&\leq\alpha_{n_k}\|f(x_{n_k})-y_{n_k}\|+(1-\alpha_{n_k})\|u_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|\\&=\alpha_{n_k}\|f(x_{n_k})-y_{n_k}\|+(1-\alpha_{n_k})\Big\|(1-\rho_{n_k})(x_{n_k}-y_{n_k})+\rho_{n_k}(T_{0,\lambda}y_{n_k}-y_{n_k})\Big\|\\&\quad+\|y_{n_k}-x_{n_k}\|\\&\leq\alpha_{n_k}\|f(x_{n_k})-y_{n_k}\|+(1-\alpha_{n_k})(1-\rho_{n_k})\|x_{n_k}-y_{n_k}\|\\&\quad+(1-\alpha_{n_k})\rho_{n_k}\|T_{0,\lambda}y_{n_k}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\|.\end{aligned} \tag{3.11}$$

Since $T_{0,\lambda} = (1-\lambda)I + \lambda T_0$, it follows from (3.11) that

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\le \alpha_{n_k}\|f(x_{n_k}) - y_{n_k}\| + (1-\alpha_{n_k})(1-\rho_{n_k})\|x_{n_k} - y_{n_k}\| \\ &\quad + (1-\alpha_{n_k})\rho_{n_k}\lambda\|T_0 y_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\ &\le \alpha_{n_k}\|f(x_{n_k}) - y_{n_k}\| + (1-\rho_{n_k})\|x_{n_k} - y_{n_k}\| \\ &\quad + \rho_{n_k}\lambda\|T_0 y_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\|.\end{aligned}$$

From (3.8), (3.10) and condition(i), we obtain $\|x_{n_k+1} - x_{n_k}\| \to 0$, which implies that $\{x_{n_k}\}$ and $\{x_{n_k+1}\}$ have the same weak cluster point set. Hence, $x_{n_k+1} \rightharpoonup x^*$. Then, we conclude from $z = P_S f(z)$ and (2.1) that

$$\limsup_{n\to\infty} b_n \le \lim_{k\to\infty} \frac{2}{1-\rho^2}\langle f(z) - z, x_{n_k+1} - z\rangle = \frac{2}{1-\rho^2}\langle f(z) - z, x^* - z\rangle \le 0.$$

Consequently, applying Lemma 2.10 to (3.6), we have $\|x_n - z\| \to 0$. □

Remark 3.2. *Theorem 3.1 establishes the strong convergence of the proposed multi-parameter viscosity-type iterative method. In particular, if $\beta_n = 0$, then algorithm (3.1) reduces to algorithm (1.10). Therefore, Theorem 3.1 can be regarded as an extension of the corresponding result of [20].*

In the following, we assume $\kappa_i \in (0,1)$ for each $i = 0,1,2,\ldots,N$, and define $T_{0,\lambda} = (1-\lambda)I + \lambda T_0$ with $0 < \lambda < 1-\kappa_0$. We next consider the error-sequenced version of algorithm (3.1): for any initial point $x_0 \in H$, the algorithm generates the sequence $\{x_n\}$ according to

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \delta_n T_{0,\lambda}\left[x_n - \tau_n \sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\right] + e_n, \tag{3.12}$$

where $\alpha_n, \beta_n, \delta_n \in (0,1)$, $\alpha_n + \beta_n + \delta_n = 1$, and $\tau_n = 0$ if $\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\| = 0$, otherwise

$$\tau_n = \frac{\kappa \sum_{i=1}^{N}\|(I-T_i)A_i x_n\|^2}{2\left\|\sum_{i=1}^{N} A_i^*(I-T_i)A_i x_n\right\|^2}. \tag{3.13}$$

By the arguments used in the proof of Theorem 3.1 above, we can get the following convergence theorem.

Theorem 3.3. *Let $A_i : H \to H_i$ be a linear and operator. Let $T_0 : H \to H$, $T_i : H_i \to H_i$ be κ_i-demicontractive and fulfill the demiclosedness principle for each $i = 1,2,\ldots,N$, and let $f : H \to H$ be a contraction with coefficient $\rho \in [0,1)$. Suppose that condition (3.13) holds and the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\alpha_n}{\delta_n} = 0$;
(iii) $\sum_{n=0}^{\infty}\|e_n\| < \infty$ *or* $\|e_n\|/\alpha_n \to 0$.

Then the sequence $\{x_n\}$ generated by (3.12) converges strongly to a solution z of problem (1.7), which is also the unique solution of the variational inequality:

$$z \in S, \ \langle (I-f)z, x-z\rangle \ge 0, \ \forall x \in S.$$

Alternatively, z is the unique fixed point of the contraction $P_S f$, that is, $z = P_S f(z)$.

Proof. Let z be the unique fixed point of $P_S f$. Define $\rho_n = \frac{\delta_n}{1-\alpha_n}$, $y_n = x_n - \tau_n \sum_{i=1}^N A_i^*(I-T_i)A_i x_n$, and $u_n = (1-\rho_n)x_n + \rho_n T_{0,\lambda} y_n$. Then algorithm (3.12) can be rewritten as

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)u_n + e_n.$$

From Step 1 of the proof of Theorem 3.1, it is obvious that

$$\|x_{n+1} - z\| \le [1-\alpha_n(1-\rho)]\|x_n - z\| + \alpha_n\|f(z) - z\| + \|e_n\|. \tag{3.14}$$

The rest of the proof will be divided into two cases.

Case 1: $\sum_{n=0}^\infty \|e_n\| < \infty$.
Set

$$M_0 = \frac{\|f(z)-z\|}{1-\rho}.$$

Then (3.14) becomes

$$\begin{aligned}\|x_{n+1} - z\| &\le [1-\alpha_n(1-\rho)]\|x_n - z\| + \alpha_n(1-\rho)M_0 + \|e_n\|\\ &\le \max\{\|x_n - z\|, M_0\} + \|e_n\|.\end{aligned}$$

By induction, one has

$$\|x_n - z\| \le \max\{\|x_0 - z\|, M_0\} + \sum_{k=0}^{n-1}\|e_k\|, \qquad \forall n \ge 1.$$

Since $\sum_{n=0}^\infty \|e_n\| < \infty$, we conclude that $\{x_n\}$ is bounded.

Next, we show

$$s_{n+1} \le (1-\tilde{\alpha}_n)s_n + \tilde{\alpha}_n b_n + \eta_n, \tag{3.15}$$

where s_n, $\tilde{\alpha}_n$, and b_n are defined as in Theorem 3.1, $\eta_n = M\|e_n\|$. Actually, similarly to (3.7) and Lemma 2.7(i), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\le \|\alpha_n(f(x_n) - f(z)) + (1-\alpha_n)(u_n - z)\|^2\\ &\quad + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle + 2\langle e_n, x_{n+1} - z\rangle\\ &\le [1-\alpha_n(1-\rho^2)]\|x_n - z\|^2 + \alpha_n(1-\rho^2)b_n + 2\langle e_n, x_{n+1} - z\rangle.\end{aligned} \tag{3.16}$$

Since $\{x_n\}$ is bounded, there exists $M > 0$ such that $\|x_{n+1} - z\| \le \frac{M}{2}$. Hence,

$$2\langle e_n, x_{n+1} - z\rangle \le 2\|e_n\|\,\|x_{n+1} - z\| \le M\|e_n\|.$$

Therefore, inequality (3.15) holds. From the definition of η_n and $\sum_{n=0}^\infty \|e_n\| < \infty$, we have

$$\eta_n \ge 0, \quad \sum_{n=0}^\infty \eta_n < \infty. \tag{3.17}$$

Moreover, we show that $\limsup_{n\to\infty} b_n$ is finite. Since $\{x_n\}$ is bounded, it follows from the definition of b_n that

$$\sup_{n\ge 0} b_n \le \sup_{n\ge 0} \frac{2}{1-\rho^2}\|f(z) - z\|\|x_{n+1} - z\| < +\infty.$$

It remains to show that $\limsup_{n\to\infty} b_n > -\infty$. On the contrary, suppose that $\limsup_{n\to\infty} b_n = -\infty$. Then there exists $n_0 \in N$ such that $b_n < -1$ for all $n \ge n_0$. By (3.15), we have

$$s_{n+1} \le (1-\tilde{\alpha}_n)s_n + \tilde{\alpha}_n b_n + \eta_n \le (1-\tilde{\alpha}_n)s_n - \tilde{\alpha}_n + \eta_n \le s_n - (1-\rho^2)\alpha_n + \eta_n.$$

By induction, we obtain

$$s_{n+1} \le s_{n_0} - (1-\rho^2)\sum_{i=n_0}^{n}\alpha_i + \sum_{i=n_0}^{n}\eta_i.$$

Thus $\limsup_{n\to\infty} s_n < s_{n_0} - (1-\rho^2)\sum_{i=n_0}^{\infty}\alpha_i + \sum_{i=n_0}^{\infty}\eta_i = -\infty$. However, $\{s_n\}$ is a nonnegative real sequence, which is a contradiction. Therefore, $\limsup_{n\to\infty} b_n$ is finite. b_n is defined in the same way as in Theorem 3.1. Moreover, the following estimates in Theorem 3.1 depend only on the definition of b_n. Therefore, we obtain

$$\lim_{k\to\infty}\|(I-T_0)y_{n_k}\| = \lim_{k\to\infty}\|(I-T_i)A_i x_{n_k}\| = 0, i = 1,2,3..., \tag{3.18}$$

and

$$\|y_{n_k} - x_{n_k}\| \to 0. \tag{3.19}$$

By (3.18), (3.19) and Lemma 2.9, it follows immediately that any weak cluster point of $\{x_{n_k}\}$ belongs to S. Note that

$$\begin{aligned}
\|x_{n_k+1} - x_{n_k}\| &\le \|x_{n_k+1} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\
&\le \alpha_{n_k}\|f(x_{n_k}) - y_{n_k}\| + (1-\alpha_{n_k})\|u_{n_k} - y_{n_k}\| + \|e_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\
&= \alpha_{n_k}\|f(x_{n_k}) - y_{n_k}\| + (1-\alpha_{n_k})\big\|(1-\rho_{n_k})(x_{n_k} - y_{n_k}) + \rho_{n_k}(T_{0,\lambda}y_{n_k} - y_{n_k})\big\| \\
&\quad + \|e_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\
&\le \alpha_{n_k}\|f(x_{n_k}) - y_{n_k}\| + (1-\alpha_{n_k})(1-\rho_{n_k})\|x_{n_k} - y_{n_k}\| \\
&\quad + (1-\alpha_{n_k})\rho_{n_k}\|T_{0,\lambda}y_{n_k} - y_{n_k}\| + \|e_{n_k}\| + \|y_{n_k} - x_{n_k}\| \\
&\le \alpha_{n_k}\|f(x_{n_k}) - y_{n_k}\| + (1-\rho_{n_k})\|x_{n_k} - y_{n_k}\| \\
&\quad + \rho_{n_k}\lambda\|T_0 y_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| + \|e_{n_k}\|.
\end{aligned} \tag{3.20}$$

Since $\sum_{n=0}^{\infty}\|e_n\| < +\infty$, we have $\|e_n\| \to 0$. Therefore, from (3.18), (3.19) and condition (i), we obtain $\|x_{n_k+1} - x_{n_k}\| \to 0$, which implies that $\{x_{n_k}\}$ and $\{x_{n_k+1}\}$ have the same weak cluster point set. Without loss of generality, we may suppose that $\{x_{n_k+1}\}$ converges weakly to $\bar{x} \in S$. Then, we conclude from $z = P_S f(z)$ and (2.1) that

$$\limsup_{n\to\infty} b_n \le \lim_{k\to\infty}\frac{2}{1-\rho^2}\langle f(z) - z, x_{n_k+1} - z\rangle = \frac{2}{1-\rho^2}\langle f(z) - z, \bar{x} - z\rangle \le 0.$$

Consequently, applying Lemma 2.10 to (3.15), we have $\|x_n - z\| \to 0$.

Case 2: $\|e_n\|/\alpha_n \to 0$.

By the assumption $\|e_n\|/\alpha_n \to 0$, there exists n_0 such that, for all $n \ge n_0$, $\|e_n\| \le \alpha_n$. Hence, for $n \ge n_0$, inequality (3.14) can be rewritten as

$$\|x_{n+1} - z\| \le [1 - \alpha_n(1-\rho)]\|x_n - z\| + \alpha_n\big(\|f(z) - z\| + 1\big).$$

Let

$$M_0' = \frac{\|f(z) - z\| + 1}{1-\rho}.$$

Then it follows that

$$\begin{aligned}
\|x_{n+1} - z\| &\le [1 - \alpha_n(1-\rho)]\|x_n - z\| + \alpha_n(1-\rho)M_0' \\
&\le \max\{\|x_n - z\|, M_0'\}.
\end{aligned}$$

By induction, we obtain $\|x_n - z\| \le \max\{\|x_{n_0} - z\|, M_0'\}$ for all $n \ge n_0$. Thus $\{x_n\}$ is bounded.

Next, we show the following inequality holds:

$$s_{n+1} \le (1-\tilde{\alpha}_n)s_n + \tilde{\alpha}_n(b_n + \varepsilon_n), \tag{3.21}$$

where s_n, $\tilde{\alpha}_n$, and b_n are defined as in Theorem 3.1, $\varepsilon_n = \frac{M'\|e_n\|}{\tilde{\alpha}_n}$. Since $\{x_n\}$ is bounded, there exists a constant $M' > 0$ such that $\|x_{n+1} - z\| \le \frac{M'}{2}$. Hence,

$$2\langle e_n, x_{n+1} - z\rangle \le 2\|e_n\|\|x_{n+1} - z\| \le M'\|e_n\|.$$

Combining (3.16), we have $s_{n+1} \le (1-\tilde{\alpha}_n)s_n + \tilde{\alpha}_n b_n + M'\|e_n\|$. Then, the desired inequality (3.21) at once follows.

We next show that $\limsup_{n\to\infty} b_n$ is finite. Since $\{x_n\}$ is bounded, it follows from the definition of b_n that

$$\sup_{n\ge 0} b_n \le \sup_{n\ge 0} \frac{2}{1-\rho^2}\|f(z) - z\|\|x_{n+1} - z\| < +\infty.$$

Since $\|e_n\|/\alpha_n \to 0$, we have $\varepsilon_n \to 0$. If $\limsup_{n\to\infty} b_n = -\infty$, then there exists $n_1 \in N$ such that, for all $n \ge n_1$, $b_n + \varepsilon_n \le -1$. By (3.21), we have

$$s_{n+1} \le (1-\tilde{\alpha}_n)s_n + \tilde{\alpha}_n(b_n + \varepsilon_n) \le (1-\tilde{\alpha}_n)s_n - \tilde{\alpha}_n \le s_n - \tilde{\alpha}_n.$$

By induction,

$$s_{n+1} \le s_{n_1} - \sum_{i=n_1}^{n} \tilde{\alpha}_i = s_{n_1} - (1-\rho^2)\sum_{i=n_1}^{n} \alpha_i.$$

Since $\sum_{n=0}^{\infty} \alpha_n = +\infty$, we obtain $s_{n+1} \to -\infty$, which contradicts the fact that $\{s_n\}$ is nonnegative. Therefore, $\limsup_{n\to\infty} b_n$ is finite. Since b_n has the same form as before, (3.18) and (3.19) still hold. Hence, by Lemma 2.9, any weak cluster point of $\{x_{n_k}\}$ belongs to S. Using the definitions of x_{n+1} and u_n, we can derive (3.20) in the same way. Since $\|e_n\|/\alpha_n \to 0$ and $\alpha_n \to 0$, we have $\|e_n\| = \alpha_n \frac{\|e_n\|}{\alpha_n} \to 0$. Combining (3.18), (3.19), and condition (i), we obtain $\|x_{n_k+1} - x_{n_k}\| \to 0$, which implies that $\{x_{n_k}\}$ and $\{x_{n_k+1}\}$ have the same weak cluster point set. Without loss of generality, we may suppose that $\{x_{n_k+1}\}$ converges weakly to $\bar{x} \in S$. Then, we infer from $z = P_S f(z)$ and (2.1) that

$$\limsup_{n\to\infty} b_n \le \lim_{k\to\infty} \frac{2}{1-\rho^2}\langle f(z) - z, x_{n_k+1} - z\rangle = \frac{2}{1-\rho^2}\langle f(z) - z, \bar{x} - z\rangle \le 0.$$

On the other hand, we can deduce from $\varepsilon_n \to 0$ that

$$\limsup_{n\to\infty}(b_n + \varepsilon_n) \le \limsup_{n\to\infty} b_n + \lim_{n\to\infty} \varepsilon_n \le 0. \tag{3.22}$$

Finally, in view of (3.21), (3.22), and Lemma 2.10, we have $\|x_n - z\| \to 0$. $\square$

Remark 3.4. *Theorem 3.3 shows that the proposed algorithm still converges strongly even when errors are incorporated into the iterative process.*

4. Applications

In this section, we give two applications in Hilbert spaces.

4.1. **Split feasibility problems with multiple output sets.** Let C be a nonempty, convex and closed set in H, and let Q_i be a nonempty, convex, and closed set in H_i for each $i = 1,2,\dots,N$. Let $A_i : H \to H_i$ be a linear and bounded mapping. Consider the split feasibility problem with multiple output sets [16], which requires finding a point $x^* \in H$ such that

$$x^* \in C \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(Q_i) \right). \tag{4.1}$$

Denote the solution set of problem (4.1) by S and assume that $S \neq \varnothing$. Since

$$\mathrm{Fix}(P_C) = C, \qquad \mathrm{Fix}(P_{Q_i}) = Q_i, \qquad i = 1,2,\dots,N,$$

it follows that

$$S = \mathrm{Fix}(P_C) \cap \Big(\bigcap_{i=1}^{N} A_i^{-1}(\mathrm{Fix}(P_{Q_i})) \Big).$$

This shows that the split feasibility problem with multiple output sets is a special case of the SCFPPMOS studied in this paper. Moreover, since metric projections P_C and P_{Q_i} in Hilbert spaces are firmly nonexpansive, they satisfy the demiclosedness principle. Therefore, by Theorem 3.1, we obtain the following result.

Theorem 4.1. *Let C be a nonempty, convex and closed set in H, and let Q_i be a nonempty, convex and closed set in H_i for each $i = 1,2,\dots,N$. Let $A_i : H \to H_i$ be a linear and bounded operator, and let $f : H \to H$ be a ρ-contraction for some $\rho \in [0,1)$. For any initial point $x_0 \in H$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \delta_n P_C \left(x_n - \tau_n \sum_{i=1}^{N} A_i^*(I - P_{Q_i}) A_i x_n \right), \tag{4.2}$$

where $\alpha_n, \beta_n, \delta_n \in (0,1)$, $\alpha_n + \beta_n + \delta_n = 1$, and $\tau_n = 0$ if $\left\| \sum_{i=1}^{N} A_i^*(I - P_{Q_i}) A_i x_n \right\| = 0$, otherwise

$$\tau_n = \frac{\sum_{i=1}^{N} \|(I - P_{Q_i}) A_i x_n\|^2}{\left\| \sum_{i=1}^{N} A_i^*(I - P_{Q_i}) A_i x_n \right\|^2}.$$

Suppose there hold the conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{\alpha_n}{\delta_n} = 0$.

Then the sequence $\{x_n\}$ generated by (4.2) converges strongly to a solution z of problem (4.1), which is also the unique solution of the variational inequality:

$$z \in S, \ \langle (I - f)z, x - z \rangle \geq 0, \ \forall x \in S.$$

Alternatively, z is the unique fixed point of the contraction $P_S f$, that is $z = P_S f(z)$.

4.2. **Split variational inequality problems with multiple output sets.** Let C, Q_i be nonempty, convex and closed subsets of real Hilbert spaces H and H_i, $i = 1,2,\dots,N$, respectively. Let $A_i : H \to H_i$ be a linear and bounded mapping for each $i = 1,2,\dots,N$. Consider the split variational inequality problem with multiple output sets [25], which requires finding a point $x^* \in H$ such that

$$x^* \in \mathrm{VI}(C, F_0) \cap \left(\bigcap_{i=1}^{N} A_i^{-1}\big(\mathrm{VI}(Q_i, F_i)\big) \right), \tag{4.3}$$

where

$$\mathrm{VI}(C,F_0)=\{x\in C:\langle F_0x,y-x\rangle\geq 0,\ \forall y\in C\},$$

and

$$\mathrm{VI}(Q_i,F_i)=\{u\in Q_i:\langle F_iu,v-u\rangle\geq 0,\ \forall v\in Q_i\},\qquad i=1,2,\ldots,N.$$

Denote the solution set of problem (4.3) by S and assume that $S\neq\varnothing$. Assume that $F_0:H\to H$ is η_0-inverse strongly monotone, and for each $i=1,2,\ldots,N$, $F_i:H_i\to H_i$ is η_i-inverse strongly monotone, where $\eta_0>0$ and $\eta_i>0$. That is,

$$\langle F_0x-F_0y,x-y\rangle\geq\eta_0\|F_0x-F_0y\|^2,\qquad\forall x,y\in H,$$

and

$$\langle F_iu-F_iv,u-v\rangle\geq\eta_i\|F_iu-F_iv\|^2,\qquad\forall u,v\in H_i,\quad i=1,2,\ldots,N.$$

Let

$$\mu_0\in(0,2\eta_0),\qquad\mu_i\in(0,2\eta_i),\quad i=1,2,\ldots,N,$$

and define

$$T_0=P_C(I-\mu_0F_0),\qquad T_i=P_{Q_i}(I-\mu_iF_i),\quad i=1,2,\ldots,N.$$

Note that

$$\mathrm{Fix}(T_0)=\mathrm{VI}(C,F_0),\qquad\mathrm{Fix}(T_i)=\mathrm{VI}(Q_i,F_i),\quad i=1,2,\ldots,N.$$

Hence,

$$S=\mathrm{Fix}(T_0)\cap\left(\bigcap_{i=1}^{N}A_i^{-1}\big(\mathrm{Fix}(T_i)\big)\right).$$

This shows that the split variational inequality problem with multiple output sets is a special case of the SCFPPMOS studied in this paper. Moreover, since P_C and P_{Q_i} are firmly nonexpansive, and $I-\mu_0F_0$ and $I-\mu_iF_i$ are nonexpansive for $\mu_0\in(0,2\eta_0)$ and $\mu_i\in(0,2\eta_i)$, it follows that T_i is nonexpansive for each $i=0,1,2,\ldots,N$. Therefore, T_i is 0-demicontractive and satisfies the demiclosedness principle for each $i=0,1,2,\ldots,N$. Hence, by Theorem 3.1, we obtain the following result.

Theorem 4.2. *Let C, Q_i be nonempty, convex and closed subsets of real Hilbert spaces H and H_i, $i=1,2,\ldots,N$, respectively. Let $A_i:H\to H_i$ be a linear and bounded mapping for each $i=1,2,\ldots,N$. Assume that $F_0:H\to H$ is η_0-inverse strongly monotone, and for each $i=1,2,\ldots,N$, $F_i:H_i\to H_i$ is η_i-inverse strongly monotone, where $\eta_0>0$ and $\eta_i>0$. Let $f:H\to H$ be a ρ-contraction for some $\rho\in[0,1)$. Let*

$$T_0=P_C(I-\mu_0F_0),\quad T_i=P_{Q_i}(I-\mu_iF_i),\quad i=1,2,\ldots,N.$$

For any initial point $x_0\in H$, define a sequence $\{x_n\}$ by

$$x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\delta_nT_{0,\lambda}\left(x_n-\tau_n\sum_{i=1}^{N}A_i^*(I-T_i)A_ix_n\right),\tag{4.4}$$

where $\alpha_n,\beta_n,\delta_n\in(0,1)$, $\alpha_n+\beta_n+\delta_n=1$, $T_{0,\lambda}=(1-\lambda)I+\lambda T_0$ with $0<\lambda<1$, and $\tau_n=0$ if $\left\|\sum_{i=1}^{N}A_i^*(I-T_i)A_ix_n\right\|=0$, otherwise

$$\tau_n=\frac{\sum_{i=1}^{N}\|(I-T_i)A_ix_n\|^2}{2\left\|\sum_{i=1}^{N}A_i^*(I-T_i)A_ix_n\right\|^2}.$$

Suppose that there hold the conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\lim_{n\to\infty}\frac{\alpha_n}{\delta_n} = 0$;
(iii) $0 < \mu_0 < 2\eta_0, \quad 0 < \mu_i < 2\eta_i$.

Then the sequence $\{x_n\}$ generated by (4.4) converges strongly to a solution z of problem (4.3), which is also the unique solution of the variational inequality:

$$z \in S,\ \langle (I-f)z, x-z\rangle \geq 0,\ \forall x \in S.$$

Alternatively, z is the unique fixed point of the contraction $P_S f$, that is, $z = P_S f(z)$.

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