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ON THE EXISTENCE OF SOLUTIONS FOR NEUTRAL DIFFERENTIAL INCLUSIONS IN BANACH ALGEBRAS

NAJIB KADDACHI

Department of Mathematics, Faculty of Sciences of Sfax, University of Sfax, Sfax, Tunisia

Abstract. This paper establishes an existence result for first-order neutral differential inclusions in Banach algebras based on the concept of noncompactness measures and the vector versions of fixed point theorems in Banach algebras. The paper concludes with a numerical example to illustrate the main findings.

Keywords. Banach algebra; Differential inclusion; Fixed point theorem; Multi-valued mappings; Noncompactness measures.
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1. Introduction

Let ε be a positive number, and define the closed and bounded intervals $J_\varepsilon = [0,\varepsilon]$ and $J_{-\varepsilon} = [-\varepsilon, 0]$. Consider a function $\varphi \in \mathscr{C}_{J_{-\varepsilon}} = \mathscr{C}([-\varepsilon,0])$, which represents the space of all continuous real-valued functions on $J_{-\varepsilon}$. We examine the following first-order two-dimensional neutral functional differential inclusion (NFDI):

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)-k(t,x(t),x_t)}{f(t,y(t),y_t)}\right) \in G\left(t, y_t, \displaystyle\int_0^t h(t,s,y_s)ds\right) \ , \ t \in J_\varepsilon \\ y(t) = \dfrac{1}{1+b(t)|x(\theta(t))|} - p\left(t, \dfrac{1}{1+b(t)|x(\theta(t))|}\right) + p(t,y(t)) \ , \ t \in J_\varepsilon \end{cases} \tag{1.1}$$

along with the homogeneous initial condition $(x(t),y(t)) = (\varphi(t),\varphi(t))\,,\, t \in J_{-\varepsilon}$, where f and k are continuous functions defined on $J_\varepsilon \times \mathbb{R} \times C_{-\varepsilon}$ mapping to $\mathbb{R}\backslash\{0\}$, and $G : J_\varepsilon \times C_{-\varepsilon} \times \mathbb{R} \longrightarrow \mathscr{P}_p(\mathbb{R})$ is a multi-valued function that is L^1-Carathéodory. The functions $x = x(t)$ and $y = y(t)$ are unknown and satisfy the following conditions

$i)$ The function $t \longrightarrow \dfrac{x(t)-k(t,x(t),x_t)}{f(t,y(t),y_t)}$ is differentiable, and

E-mail address: najibkadachi@fsegs.usf.tn
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$ii)$ There exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in G\left(t, x_t, \int_0^t h(t,s,x_s)ds\right)$ $t \in J$ such that

$$\left(\frac{x(t)-k(t,x(t),x_t)}{f(t,y(t),y_t)}\right)' = v(t),\ t \in J.$$

For any function $x \in C(J,\mathbb{R})$, we define x_t as the element of $C_{-\varepsilon}$ given by $x_t(\theta) = x(t+\theta)$, $-\varepsilon \leq \theta \leq 0$ and $t \in J_\varepsilon$. The function x_t is referred to as a history function, and the space $C_{-\varepsilon}$ is termed the history algebra of the dynamic systems under consideration. The presence of the derivative in the denominator on the left side renders (1.1) neutral, as the rate of change $\frac{\partial}{\partial t}x$ for hereditary systems is influenced by the values of x and x' over a preceding time interval. Problem (1.1) introduces a novel aspect to the theory of functional integro-differential equations with various special cases. For instance, if we set $x(t) = y(t)$, $p(t,y) = y$, $k(t,x(t),x_t) = g(t,x_t)$ and $f(t,x(t),x_t) = 1$, then problem (1.1) is simplified to the following impulsive neutral functional integro-differential inclusions

$$\begin{cases} \dfrac{d}{dt}(x(t)-g(t,x_t)) \in G\left(t, y_t, \displaystyle\int_0^t h(t,s,x_s)ds\right),\ t \in J_\varepsilon \\ x(t) = \varphi(t),\ t \in J_{-\varepsilon}. \end{cases} \tag{1.2}$$

This formulation has been the subject of numerous studies. In [5], Benchohra, Gatsori, and Ntouyas established several existence results for (1.2) by using Schaefer's fixed point theorem in conjunction with a selection theorem by Bressan and Colombo [6]. In [8], Dhage and Ntouyas investigated the scenario where the single-valued $g(t,\cdot)$ and the multi-valued $G(t,\cdot)$ are defined on functions from $[-r,0]$ into $\mathbb{R}^n$, which are continuous everywhere except for a finite number of points. They derived existence results for first and second-order impulsive neutral functional differential inclusions under mixed Lipschitz and Carathéodory conditions. In [11], Djebali, Gorniewicz, and Ouahab explored system (1.2) in both convex and non-convex cases of the multi-valued G, to uncover new existence results. In the single-valued case, for

$$G\left(t, x_t, \int_0^t h(t,s,x_s)ds\right) = \left\{g\left(t, x(t), \int_0^t h(t,s,x(s))ds\right)\right\},$$

if we take $x(t) = y(t)$, $p(t,y) = y$, $k(t,x(t),x_t) = 0$, and $f(t,x(t),x_t) = 1$ in (1.1), it reduces to the following functional differential equation with a delay

$$\begin{cases} \dfrac{d}{dt}(x(t)) = g\left(t, x(t), \displaystyle\int_0^t h(t,s,x(s))ds\right),\ t \in J_\varepsilon \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $g : J_\varepsilon \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear operator. Equation (1.3) was recently examined by Martin [17] who provided existence theorems under mixed contraction and compactness conditions. Integro-differential equations and inclusions are pivotal in modeling various practical phenomena across multiple scientific fields, including physics, chemistry, aerodynamics, polymer rheology, economics, control theory, and biophysics; see, e.g., [13, 18]. Neutral integro-differential inclusion systems are significant in physical, chemical, and engineering contexts, leading to extensive research under diverse recently; see, e.g., [1, 3, 4, 8, 11, 15] and the references cited therein.

Based on the studies mentioned above, this paper aims to explore the existence of solutions for two-dimensional neutral functional integro-differential inclusion (1.1), specifically when the multi-valued map G exhibits convex and compact values. This paper employs a new fixed point theorem based on prior theorems established in [16,17]. The findings of this study are expected to make a substantial contribution to the field of functional integro-differential inclusions.

2. Existence Theory and Auxiliary Results

We define a norm $\|\cdot\|$ and a multiplication operation "$\cdot$" in a Banach algebra $C(J,\mathbb{R})$, which consists of all continuous real-valued functions on the interval J. The norm is given by $\|x\| = \sup_{t\in J}|x(t)|$ and the multiplication of two functions is defined as $(x\cdot y)(t) = x(t)y(t)$; $t \in J$ and $x,y \in C(J,\mathbb{R})$. Let $L^1(J,\mathbb{R})$ denote the space of Lebesgue integrable functions on J with the norm defined by $\|u\|_{L^1} = \int_0^\delta u(t)dt$. We denote the power set of $\mathbb{R}$ by $\mathscr{P}(\mathbb{R})$ and the class of all nonempty subsets of $\mathbb{R}$ with property p by $\mathscr{P}_p(\mathbb{R})$. The classes $\mathscr{P}_{cl}(\mathbb{R}), \mathscr{P}_{bd}(\mathbb{R})$, $\mathscr{P}_{cp}(\mathbb{R})$ and $\mathscr{P}_{cv}(\mathbb{R})$ represent the sets of all closed, bounded, compact, and convex subsets of $\mathbb{R}$, respectively. Similarly, $\mathscr{P}_{cl,bd}(\mathbb{R})$ and $\mathscr{P}_{cp,cv}(\mathbb{R})$ denote the classes of closed-bounded and compact-convex subsets of $\mathbb{R}$. For any multi-valued mapping $Q : J\times\mathbb{R} \longrightarrow \mathscr{P}_{cp}(\mathbb{R})$, the integral of Q is defined as

$$\int_0^t Q(s,x(s))ds = \left\{\int_0^t v(s)ds \; ; \; v \in S_Q^1(x)\right\},$$

where $S_Q^1(x) = \{v \in L^1(J,\mathbb{R}) \; ; \; v(t) \in Q(t,x(t)), \forall t \in J\}, x \in C(J,\mathbb{R})$. A point $x \in X$ is considered a fixed point of Q if $x \in Qx$. For convenience, we define $Q(S) = \cup_{x\in S}Qx$ and $Q^+(S) = \{x \in X \; ; \; Q(x) \subset S\}$, for all subsets S of X. We say that Q is upper semi-continuous if, for every open subset V of X, set $Q^+(V)$ is open in X. For $x \in X$ and $S_1, S_2 \in \mathscr{P}_{cl}(X)$, we denote by $D(x,S_1) = \inf\{\|x-y\| \; ; \; y \in S_1\}$. We define a function $d_H : \mathscr{P}_{cl}(X)\times\mathscr{P}_{cl}(X) \longrightarrow \mathbb{R}_+$ by using

$$d_H(S_1,S_2) = \max\{\sup_{x\in S_1} D(x,S_2), \sup_{y\in S_2} D(S_1,y)\}.$$

The function d_H is known as the Hausdorff metric on X. The Hausdorff metric was utilized to establish fixed point and coincidence point results on metric spaces recently. System (1.1) can be reformulated as a fixed point problem

$$\begin{cases} x(t) \in k(t,x(t),x_t) + f(t,y(t),y_t)\cdot\left(\varphi(0) + \displaystyle\int_0^t G\left(s,y_s,\int_0^t h(s,\rho,y_\rho)d\rho\right)ds\right), \;\; t \in J_\varepsilon \\ y(t) = \dfrac{1}{1+b(t)|x(\theta(t))|} - p\left(t, \dfrac{1}{1+b(t)|x(\theta(t))|}\right) + p(t,y(t)), \;\; t \in J_\varepsilon \\ (x(t),y(t)) = (\varphi(t),\varphi(t)) \;\; , \; t \in J_{-\varepsilon}. \end{cases} \tag{2.1}$$

Alternatively, this can also be expressed in matrix form

$$\begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix} \in \begin{pmatrix} k(t,\cdot,\cdot) & f(t,\cdot,\cdot)\cdot Q(t,\cdot,\cdot) \\ \\ \dfrac{1}{1+b(t)|id_{\theta(t)}(\cdot)|} - p\left(t, \dfrac{1}{1+b(t)|id_{\theta(t)}(\cdot)|}\right) & p(t,\cdot) \end{pmatrix} \begin{pmatrix} x(t) \\ \\ y(t) \end{pmatrix},$$

where the operators are defined as follows

$$\begin{cases} k(t,\cdot,\cdot)(x(t)) = k(t,x(t),x_t) \\ id_{\theta(t)}(\cdot)(x(t)) = x(\theta(t)), \;\; \text{and} \\ Q(t,\cdot,\cdot)(x(t)) = \varphi(0) + \displaystyle\int_0^t G\left(s,x_s,\int_0^s h(s,\rho,x_\rho)d\rho\right)ds. \end{cases}$$

It should be noted that problem (2.1) can be reformulated into a fixed point problem involving a 2×2 block operator matrix

$$\begin{pmatrix} A & B\cdot B' \\ C & D \end{pmatrix}, \tag{2.2}$$

where the entries represent generally nonlinear multi-valued operators defined on Banach algebras. The analysis is conducted by using the measure of non-compactness. Kuratowski's measure of non-compactness α for a bounded subset $\mathscr{S}$ of a Banach space X is defined as

$$\alpha(S)=\inf\{r>0\ ;\ S\subseteq\bigcup_{i=1}^{n}S_i \text{ with } \operatorname{diam}(S_i)\leq r, \text{ for all } 1\leq i\leq n\}.$$

The Hausdorff measure of non-compactness $\beta(S)$ is defined as

$$\beta(S)=\inf\{r>0;\ \text{there exists a finite set } F \text{ with } S\subseteq F+\mathscr{B}(0,r)\},$$

where $\mathscr{B}(0,r)=\{x\in X\ ;\ \|x\|<r\}$. For further details on Kuratowski's and Hausdorff measures of non-compactness, we refer to [1, 2, 7] and the references cited therein.

Before presenting the main fixed point results, we state some useful lemmas. For the proof, we refer to [2].

Lemma 2.1. *Let S_1 and S_2 be two nonempty and bounded subsets of a Banach space X. If $\beta(S_1)>0$ and $\beta(S_2)>0$, then $\beta(S_1\cdot S_2)\leq\beta(S_1)\|S_2\|_{\mathscr{P}}+\beta(S_2)\|S_1\|_{\mathscr{P}}$.*

Lemma 2.2. *For any nonempty and bounded subset S of a Banach space X, it holds that $\beta(S)\leq\alpha(S)\leq 2\beta(S)$.*

We now apply the following fixed point theorem to demonstrate the existence of solutions to problem (1.1).

Theorem 2.3. *Let X be a Banach algebra and let $A,B,C,D:X\longrightarrow X$ and $B':X\longrightarrow\mathscr{P}_{cl,cv}(X)$ be five multi-valued operators satisfying the following conditions:*
(i) A, B and C are Lipschitzians with constants k_1, k_2 and k_3, respectively,
(ii) D is a contraction with a constant k_4 and $(I-D)^{-1}C(X)$ is bounded
(iii) B' is upper semi-continuous and compact, and
(iv) the set $\{x\in X\mid \gamma x\in Ax+B(I-D)^{-1}Cx\cdot B'(I-D)^{-1}Cx,\ \gamma>1\}$ is bounded.
Then, block operator matrix (2.2) has, at least, one fixed point whenever $k_1+\frac{Mk_2k_3}{1-k_4}<\frac{1}{2}$, for each $r>0$, where $M=\|B'(I-D)^{-1}C(X)\|_{\mathscr{P}}$.

Proof. The application of assumption (ii) as well as Remark 4.1 in [12] enables us to conclude that the inverse operator $(I-D)^{-1}C$ exists and is Lipschitz continuous with a constant $\frac{k_3}{1-k_4}$. To establish this, we utilize Martelli's fixed point theorem [16]. Our primary objective is to demonstrate that the mapping defined by $T(x)=Ax+B(I-D)^{-1}Cx\cdot B'(I-D)^{-1}Cx$ is condensing with respect to the Hausdorff measure of noncompactness and is an upper semi-continuous multi-valued mapping on X.

First, we need to show that T has convex and compact values in X. Let z_1,z_2 be any two elements in Tx. Then there exist $u_1,u_2\in B'(I-D)^{-1}Cx$ such that

$$\begin{cases} z_1=Ax+B(I-D)^{-1}Cx\cdot u_1, \text{ and}\\ z_2=Ax+B(I-D)^{-1}Cx\cdot u_2.\end{cases}$$

For any $\lambda \in [0,1]$, we have

$$\begin{array}{rcl}\lambda z_1+(1-\lambda)z_2 & = & \lambda\left(Ax+B(I-D)^{-1}Cx\cdot u_1\right)+(1-\lambda)\left(Ax+B(I-D)^{-1}Cx\cdot u_2\right)\\ & = & Ax+B(I-D)^{-1}Cx\cdot(\lambda u_1+(1-\lambda)u_2).\end{array}.$$

Since $B'(I-D)^{-1}Cx$ is a convex set, it follows that $\lambda z_1+(1-\lambda)z_2 \in Tx$, confirming that T has convex values on X. Next, considering assumption (i), we find that

$$\begin{array}{rcl}\beta(Tx) & \leq & \beta(Ax)+\beta\left(B(I-D)^{-1}Cx\cdot B'(I-D)^{-1}Cx\right)\\ & \leq & \beta(Ax)+\beta\left(B(I-D)^{-1}Cx\right)\|B'(I-D)^{-1}Cx\|_{\mathscr{P}}\\ & \leq & \alpha(Ax)+\alpha\left(B(I-D)^{-1}Cx\right)\|B'(I-D)^{-1}Cx\|_{\mathscr{P}}.\end{array}$$

Thus, we can express this as $\beta(Tx)\leq 2k_1\beta(\{x\})+2k_2\beta(\{\frac{q}{1-k}x\})\|B'(I-D)^{-1}Cx\|_{\mathscr{P}}=0$ whenever $x \in S$. Consequently, T defines a multi-valued mapping $T: X \longrightarrow \mathscr{P}_{cv,cp}(X)$.

Our next task is to shows that T is β-condensing. Let S be a bounded subset in X. Then,

$$\begin{array}{rcl}T(S) & \subset & A(S)+B(I-D)^{-1}C(S)\cdot B'(I-D)^{-1}C(S)\\ & \subset & A(S)+B(I-D)^{-1}C(X)\cdot B'(I-D)^{-1}C(X).\end{array}$$

Taking into account that $\overline{B'(I-D)^{-1}C(X)}$ is compact and using assumption (i), we conclude that $T(S)$ is a bounded subset of X. By considering the relative compactness of the set $\cup B'(I-D)^{-1}C(S)$, and applying the subadditivity of the Hausdorff measure β, we obtain

$$\begin{array}{rcl}\beta(T(S)) & \leq & \beta(A(S))+\beta\left(B(I-D)^{-1}C(S)\cdot B'(I-D)^{-1}C(S)\right)\\ & \leq & \beta(A(S))+\|B(I-D)^{-1}C(S)\|_{\mathscr{P}}\beta\left(B'(I-D)^{-1}C(S)\right)\\ & & +\|B'(I-D)^{-1}C(S)\|_{\mathscr{P}}\beta\left(B(I-D)^{-1}C(S)\right)\\ & \leq & \left[2k_1+\dfrac{2Mk_2k_3}{1-k_4}\right]\beta(S).\end{array}$$

This demonstrates that T is a β-condensing multi-valued mapping on X into itself.

Finally, we establish that T is an upper semi-continuous multi-valued mapping on X. Let $\{x_n,\ n\in\mathbb{N}\}$ be any sequence in X converging to a point x, and let $y_n \in T(x_n)$ such that $y_n \to y$. Then, we have

$$D(y,Tx)\leq d(y,y_n)+D(y_n,Tx_n)+d_H(Tx_n,Tx). \tag{2.3}$$

For any $x,z\in X$, using [9, Lemma 3.2], we have

$$\begin{aligned}d_H(Tx,Tz) &\leq \|Ax-Az\|+d_H\left(B(I-D)^{-1}Cx\cdot B'(I-D)^{-1}Cx, B(I-D)^{-1}Cz\cdot B'(I-D)^{-1}Cz\right)\\ &\leq d(Ax,Az)+\|B'(I-D)^{-1}Cx\|_{\mathscr{P}}d\left(B(I-D)^{-1}Cx, B(I-D)^{-1}Cz\right)\\ &\quad+\|B(I-D)^{-1}Cz\|_{\mathscr{P}}d_H\left(B'(I-D)^{-1}Cx, B'(I-D)^{-1}Cz\right)\\ &\leq K\|x-z\|+\|B(I-D)^{-1}C(X)\|d_H(B'(I-D)^{-1}Cx, B'(I-D)^{-1}Cz),\end{aligned}$$

where $K=k_1+\dfrac{Mk_2k_3}{1-k_4}$. Since B' is upper semi-continuous, one has

$$d_H\left(B'(I-D)^{-1}Cx_n, B'(I-D)^{-1}Cx\right)\to 0$$

whenever $x_n \to x$. From inequality (2.3), it follows that $y\in Tx$, which confirms that multi-valued mapping $T: X\to\mathscr{P}_{cp,cv}(X)$ is upper semi-continuous. By applying Martelli's fixed point theorem [16], we conclude that T has, at least, one fixed point. □

Remark 2.4. *Since every completely continuous mapping is an upper semi-continuous multi-valued mapping, [12, Theorem 4.1] is a direct consequence of the result established above.*

The following definitions and lemmas, as presented in [14], are essential for demonstrating the existence of solutions to problem (1.1).

Definition 2.5. *A multi-valued map $Q : J \times \mathbb{R} \longrightarrow \mathscr{P}_{cp}(\mathbb{R})$ is called Carathéodory if it satisfies the following conditions*
(i) For each $x \in \mathbb{R}$, $t \mapsto Q(t,x)$ is measurable and
(ii) For each $t \in J$, $x \mapsto Q(t,x)$ is upper semi-continuous;
A Carathéodory multi-valued map Q is classified as L^1-Carathéodory if it additionally satisfies
(iii) for each real number $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R})$ such that $\|Q(t,x)\|_{\mathscr{P}} \leq h_r(t)$ a.e $t \in J$ for all $x \in \mathbb{R}$ with $|x| \leq r$.
Furthermore, a Carathéodory multi-valued map Q is referred to as $L^1_{\mathbb{R}}$-Carathéodory if
(iv) There exists a function $h \in L^1(J,\mathbb{R})$, a growth function of Q, such that $\|Q(t,x)\|_{\mathscr{P}} \leq h(t)$ a.e. $t \in J$ and $x \in \mathbb{R}$.

Lemma 2.6. *Let X be a Banach space with finite dimension. If Q be a L^1-Carathéodory multi-valued map with compact values, then $S^1_Q(x)$ is non-empty, for each $x \in X$.*

Lemma 2.7. *Let $Q : J \times X \longrightarrow \mathscr{P}_{cp}(X)$ be a Carathéodory multi-valued map such that $S^1_Q(x) \neq \emptyset$, and let $\mathscr{L} : L^1(J,X) \longrightarrow C(J,X)$ be a continuous linear mapping. Then, $\mathscr{L} \circ S^1_Q : C(J,X) \longrightarrow \mathscr{P}_{cp,cv}(C(J,X))$ has a closed graph operator.*

We also need the following definitions in the sequel.

Definition 2.8. *A function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a $\mathscr{D}$-function if it meets the following criteria*
(i) ψ is continuous,
(ii) ψ is nondecreasing, and
(iii) ψ is scalarly submultiplicative.

The subsequent assumptions will be utilized throughout the discussion.
(H_0) The function $p : J_\varepsilon \times C_{-\varepsilon} \longrightarrow \mathbb{R}$ is an α-contraction operator with respect to the second variable.
(H_1) The functions $f,k : J_\varepsilon \times \mathbb{R} \times C_{-\varepsilon} \longrightarrow \mathbb{R}$ are continuous. Additionally, there exist two positive and bounded functions l_1 and l_2 defined on J_ε such that

$$\left|f(t,x,y) - f(t,x',y')\right| \leq l_1(t) \max\left\{|x-x'|, \|y-y'\|\right\}$$

and

$$\left|k(t,x,y) - k(t,x',y')\right| \leq l_2(t) \max\left\{|x-x'|, \|y-y'\|\right\}.$$

(H_2) $G(t,x,y)$ is compact and convex for each $(t,x,y) \in J_\varepsilon \times \mathbb{R} \times C_{-\varepsilon}$.
(H_3) G is L^1-Carathéodory with a growth function μ.
(H_4) The function $h : J_{-\varepsilon} \times J_{-\varepsilon} \times C_{-\varepsilon} \longrightarrow \mathbb{R}$ is continuous and there exists a function $\alpha \in L^1(J_{-\varepsilon},\mathbb{R}^+)$ such that $\left|\int_0^t h(t,s,x)ds\right| \leq \alpha(t)\|x\|_{C_{-\varepsilon}}$, $t,s \in J_{-\varepsilon}$.
(H_5) There exist a function $q \in L^1(J,\mathbb{R})$ and a $\mathscr{D}$-function ψ such that

$$\|G(t,x,y)\|_{\mathscr{P}} \leq q(t)\psi\left(\|\phi\|_{C(I_0,\mathbb{R})} + \|y\|\right) \quad a.e \ \ t \in J,\ x \in \mathbb{R} \text{ and } y \in C_{-\varepsilon}.$$

Theorem 2.9. *Let* $(H_0)-(H_5)$ *hold and* $\|l_1\|+\|l_2\|\big(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\big)\leq\frac{1}{4}$. *Assume that that the integral condition is satisfied*

$$\int_{\natural}^{\infty}\frac{d\tau}{\psi(\tau)}>\frac{F}{1-\eta}\int_0^{\varepsilon}q(\tau)(1+\alpha(\tau))d\tau, \tag{2.4}$$

where

$$\natural=\frac{K+F\|\varphi\|_{C_{-\varepsilon}}}{1-\|l_1\|-\|l_2\|\left(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\right)},$$

and $F=\max_{t\in J}|f(t,0,\varphi)|$ *and* $K=\max_{t\in J}|k(t,0,\varphi)|$. *Then, nonlinear functional differential inequality (NFDI) (1.1) has, at least, one solution in* $C(J,\mathbb{R})\times C(J,\mathbb{R})$.

Proof. Recall that problem (2.1) can be expressed in the following manner

$$\begin{cases} x(t)\in Ax(t)+By(t)\cdot B'y(t),\\ y(t)=Cx(t)+Dy(t),\end{cases}$$

where the inputs are defined as

$$\begin{cases} (Ax)(t)=\begin{cases} k(t,x(t),x_t) & \text{if } t\in J_{\varepsilon},\\ 0 & \text{if } t\in J_{-\varepsilon},\end{cases}\\ (Bx)(t)=\begin{cases} f(t,x(t),x_t) & \text{if } t\in J_{\varepsilon},\\ 1 & \text{if } t\in J_{-\varepsilon},\end{cases}\\ (Cx)(t)=\begin{cases} \dfrac{1}{1+b(t)|x(\theta(t))|}-p\left(t,\dfrac{1}{1+b(t)|x(\theta(t))|}\right) & \text{if } t\in J_{\varepsilon},\\ 0 & \text{if } t\in J_{-\varepsilon},\end{cases}\\ (Dx)(t)=\begin{cases} p\left(t,x(t)\right),\ t\in J & \text{if } t\in J_{\varepsilon},\\ 0 & \text{if } t\in J_{-\varepsilon},\end{cases}\\ \text{and}\\ (B'x)(t)=\begin{cases} \varphi(0)+\displaystyle\int_0^t G\left(s,y_s,\int_0^t h(s,\rho,y_\rho)d\rho\right)ds & \text{if } t\in J_{\varepsilon},\\ \varphi(t) & \text{if } t\in J_{-\varepsilon}.\end{cases}\end{cases}$$

Thus, solving NFDI (1.1) is equivalent to solving problem (2.1), which can further be transformed into the operator inclusion

$$\begin{cases} x(t)\in Ax(t)+By(t)\cdot B'y(t),\\ y(t)=Cx(t)+Dy(t).\end{cases}$$

To apply Theorem 2.3, we need to verify the following steps.

Claim 1. A,B and C are Lipschitzians. For all $x,y\in C(J,\mathbb{R})$ and $t\in J$, we have

$$\begin{aligned}\|Ax(t)-Ay(t)\| &\leq |k(t,x(t),x_t)-k(t,y(t),y_t)|\\ &\leq l_2(t)\max\left\{|x(t)-y(t)|,\|x_t-y_t\|_{C_{-\varepsilon}}\right\}\\ &\leq \|l_2\|\|x-y\|,\end{aligned}$$

which implies that $\|Ax-Ay\|\leq\|l_2\|\|x-y\|$ for all $x,y\in C(J,\mathbb{R})$. A similar argument shows that B and C are Lipschitzian with Lipschitz constants $\|l_1\|$ and $\|b\|_\infty(1+\alpha)$ respectively.

Claim 2. B' defines a multi-valued map with convex values. Let $u_1, u_2 \in B'x$. Then, there exist functions $v_1, v_2 \in S^1_G(z)$, where $z = \frac{1}{1+b|x|}$, such that

$$u_1(t) = \begin{cases} \varphi(0) + \int_0^t v_1(s)ds & \text{if } t \in J_\varepsilon \\ \varphi(t) & \text{if } t \in J_{-\varepsilon} \end{cases} \quad \text{and} \quad u_2(t) = \begin{cases} \varphi(0) + \int_0^t v_2(s)ds & \text{if } t \in J_\varepsilon \\ \varphi(t) & \text{if } t \in J_{-\varepsilon} \end{cases}.$$

For any $\lambda \in [0,1]$, we have

$$\lambda u_1(t) + (1-\lambda)u_2(t) = \varphi(0) + \int_0^t (\lambda v_1(s) + (1-\lambda)v_2(s))\,ds = \varphi(0) + \int_0^t v(s)ds,$$

where $v(s) = \lambda v_1(s) + (1-\lambda)v_2(s)$ for all $s \in J$. Based on assumption (H_2), it follows that $v \in S^1_G(z)$. Thus $B'x$ is convex for each $x \in C(J,\mathbb{R})$.

Claim 3. We demonstrate that $(I-D)^{-1}C(C(J,\mathbb{R}))$ is bounded. It is straightforward to establish that the inverse operator $(I-D)^{-1}C$ exists and is Lipschtzian based on Banach contraction principle. Let $x \in C(J,\mathbb{R})$ and $t \in J$. We can easily confirm that

$$Cx(t) = (I-D)\Big(\frac{1}{1+b|x|}\Big)(t).$$

Thus $(I-D)^{-1}C(C(J,\mathbb{R}))$ is bounded with a bound 1.

Claim 4. B' is completely continuous on $C(J,\mathbb{R})$. First, we show that B' maps bounded sets into bounded sets. Let $x \in (I-D)^{-1}C(C(J,\mathbb{R}))$ be arbitrary, and let $u \in B'x \subset \cup B'((I-D)^{-1}C(C(J,\mathbb{R})))$. There exists a function $v \in S^1_G(x)$ such that $u(t) = \varphi(0) + \int_0^t v(s)ds$. Since $u \in C(J,\mathbb{R})$ then there exists $t^* \in J$ such that

$$\|u\|_\infty = |u(t^*)| \le |\varphi(0)| + \left|\int_0^{t^*} v(s)ds\right| \le \|\varphi\|_{C_{-\varepsilon}} + \int_0^{t^*} |v(s)|ds \le \|\varphi\|_{C_{-\varepsilon}} + \|\mu\|_{L^1}.$$

Because $u \in B'x$ was arbitrary, it follows that $\cup B'((I-D)^{-1}C(C(J,\mathbb{R})))$ is bounded. Again, let $u \in B'x$, for some $x \in (I-D)^{-1}C(C(J,\mathbb{R}))$ and let $\varepsilon > 0$. If $t, t' \in J_\varepsilon$ with $t \le t'$, $t - t' \le \varepsilon$, we have $|u(t) - u(t')| \le \left| \int_0^t v(s)ds - \int_0^{t'} v(s)ds \right| \le \int_t^{t'} \mu(s)ds$. In the case that $t, t' \in J_{-\varepsilon}$ with $t \le t'$, we have $|u(t) - u(t')| = |\varphi(t) - \varphi(t')|$. Otherwise, using our assumptions, we find

$$|u(t) - u(t')| \le \left|\varphi(t) - \varphi(0) - \int_0^{t'} v(s)ds\right| \le \left|\varphi(t) - \varphi(0)\right| + \int_0^{t'} \mu(s)ds.$$

In all cases, we see that $|u(t) - u(t')| \to 0$ as $\varepsilon \to 0$. Therefore, $\cup B'((I-D)^{-1}C\big(C(J,\mathbb{R}))\big)$ is an equicontinuous set in $C(J,\mathbb{R})$. By applying the Arzelà–Ascoli [19], we conclude that B' is totally bounded on $C(J,\mathbb{R})$. Now, we show that B' is an upper semi-continuous multi-valued mapping on $C(J,\mathbb{R})$. In the proof of Theorem 2.3, we need to show that B' is upper semi-continuous on $(I-D)^{-1}C(C(J,\mathbb{R}))$. Let $\{x_n,\ n \in \mathbb{N}\}$ be any sequence in $(I-D)^{-1}C(C(J,\mathbb{R}))$ which converge to a point x and let $y_n \in B'x_n$ such that $y_n \to y$. Since $y_n \in B'x_n$, we have that there exists a $v_n \in S^1_G(x_n)$ such that $y_n(t) = \varphi(0) + \int_0^t v_n(s)ds$, $t \in J$. We need to prove that there exists $v \in S^1_G(x)$ such that $y(t) = \varphi(0) + \int_0^t v(s)ds$, $t \in J$. Consider the continuous linear operator $\mathfrak{F} : L^1(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ defined by $\mathfrak{F}v(t) = \int_0^t v(s)ds$ for all $t \in J$. By using [10, Lemma 5.4], we see that $\mathfrak{F} \circ S^1_G(x)$ is a closed graph operator. From the definition of $\mathfrak{F}$, it follows that $y_n(t) - \varphi(0) \in \mathfrak{F} \circ S^1_G(x)$. Since $y_n \to y$, there exists a point $v \in S^1_G(x)$ such that $y(t) = \varphi(0) + \int_0^t v(s)ds$ for all $t \in J$. This establishes that B' is upper semi-continuous.

Claim 5. Let $(x,y)\in C(J,\mathbb{R})$ be any solution to problem (1.1). Then, there exists a function $v\in S^1_G(z)$, where $z=\frac{1}{1+b|x|}$ such that, for any $\lambda=\frac{1}{\gamma}\in[0,1]$,

$$x(t)\in\begin{cases}\lambda k(t,x(t),x_t)+\lambda f(t,z(t),z_t)\cdot\left(\varphi(0)+\int_0^t v(s)ds\right), & \text{if } t\in J_\varepsilon,\\ \lambda\varphi(t), & \text{if } t\in J_{-\varepsilon}.\end{cases}$$

If we take $t\in J_{-\varepsilon}$, then $|x(t)|\le\|\varphi\|$. Thus x is bounded on $J_{-\varepsilon}$. For the case that $t\in J_\varepsilon$, we have

$$\begin{aligned}
|x(t)| &\le \lambda\,|k(s,x(t),x_t)|+\lambda\,|f(s,z(t),z_t)|\left(\|\varphi\|_{C_{-\varepsilon}}+\left|\int_0^t v(s)ds\right|\right)\\
&\le \lambda\,|k(s,x(t),x_t)-k(t,0,0)|+\lambda\,|k(t,0,0)|\\
&+ \lambda\left(|f(s,z(t),z_t)-f(t,0,0)|+|f(t,0,0)|\right)\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t|v(s)|ds\right)\\
&\le \left(l_1(t)\max\left\{|u(t)|,\|u\|_{C_{-\varepsilon}}\right\}+K\right)\\
&+ \left(l_2(t)\max\{|u(t)|,\|u\|_{C_{-\varepsilon}}\}+F\right)\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t|v(s)|ds\right)\\
&\le \left(l_1(t)\max\{|u(t)|,\|u\|_{C_{-\varepsilon}}\}+K\right)+F\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t|v(s)|ds\right)\\
&+ \left(l_2(t)\max\{|u(t)|,\|u\|_{C_{-\varepsilon}}\}\right)\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t|v(s)|ds\right)\\
&\le \max\{|u(t)|,\|u\|_{C(I_0,\mathbb{R})}\}\left(\|l_1\|+\|l_2\|\left(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\right)\right)\\
&+ K+F\left(\|\varphi\|_{C(I_0,\mathbb{R})}+\int_0^t\left\|G\left(s,z_s,\int_0^t h(s,\rho,z_\rho)d\rho\right)\right\|_{\mathscr{P}}ds\right)\\
&\le \max\{|u(t)|,\|u_t\|_{C_{-\varepsilon}}\}\left(\|l_1\|+\|l_2\|\left(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\right)\right)\\
&+ K+F\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t q(s)\psi\left(\|u_s\|_{C_{-\varepsilon}}+\alpha(s)\|z_s\|\right)ds\right).
\end{aligned}$$

Since $u\in C(J_\varepsilon,\mathbb{R})$, there exists $t^*\in[0,t]$ such that $m(t)=\max_{s\in[-\varepsilon,t]}|u(s)|=|u(t^*)|$. By using the above inequality, we obtain

$$\begin{aligned}
m(t) &\le \max\{|u(t^*)|,|u(\theta(t^*))|\}\left(\|l_1\|+\|l_2\|\left(\|\varphi\|_{C(I_0,\mathbb{R})}+\|\mu\|_{L^1}\right)\right)+K\\
&+ F\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t q(s)\psi\left(\|u_s\|_{C_{-\varepsilon}}+\alpha(s)\|u_s\|\right)ds\right)\\
&\le m(t)\left(\|l_1\|+\|l_2\|\left(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\right)\right)+K\\
&+ F\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t q(s)\psi\left(m(s)+\alpha(s)m(s)\right)ds\right)\\
&\le m(t)\left(\|l_1\|+\|l_2\|\left(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\right)\right)+K\\
&+ F\left(\|\varphi\|_{C_{-\varepsilon}}+\int_0^t q(s)\psi\left((1+\alpha(s))m(s)\right)ds\right),
\end{aligned}$$

which implies that

$$m(t)\ \le\ \frac{K+F\|\varphi\|_{C_{-\varepsilon}}}{1-\eta}+\frac{F}{1-\eta}\int_0^t q(s)(1+\alpha(s))\psi(m(s))\,ds,$$

where $\eta=\|l_1\|+\|l_2\|\left(\|\varphi\|_{C_{-\varepsilon}}+\|\mu\|_{L^1}\right)$. Define the mapping $g:J\longrightarrow\mathbb{R}$ by

$$g(t)=\frac{K+F\|\varphi\|_{C_{-\varepsilon}}}{1-\eta}+\frac{F}{1-\eta}\int_0^t q(s)(1+\alpha(s))\psi(m(s))\,ds.$$

A direct differentiation of g yields

$$\begin{cases} g'(s)\le\dfrac{F}{1-\eta}q(s)(1+\alpha(s))\psi(m(s)),\\ g(0)=\dfrac{K+F\|\varphi\|_{C_{-\varepsilon}}}{1-\eta}.\end{cases}$$

Note that $m(t) \le g(t)$, for all $t \in J$. Then $\int_0^t \frac{g'(s)}{\psi(g(s))} ds \le \frac{F}{1-\eta} \int_0^t q(s)(1+\alpha(s))ds$. A simple change of variables gives that

$$\int_{g(0)}^{\tau} \frac{d\tau}{\psi(\tau)} \le \frac{F}{1-\eta} \int_0^{\varepsilon} q(\tau)(1+\alpha(\tau))d\tau.$$

Taking into account that the fact $q, \alpha \in L^1(J, \mathbb{R}^+)$ and using the mean value theorem, we see that there exists a constant $M > 0$ such that $|u(t)| \le f(t) \le M$, $t \in J$, which is a contradiction. Using Theorem 2.3, we see that (1.1) has a solution in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$. □

3. The Example

Let $\mathscr{C}([-\frac{\pi}{2}, 0])$ denote the Banach algebra of all real-valued continuous functions on $J_{-\varepsilon} = [-\frac{\pi}{2}, 0]$ with usual supremum norm $\|\cdot\|_C$. Let us consider an example to validate the result

$$\begin{cases} x(t) \in 1 + \alpha_2 t\left[\|x_t\|_C + \cos(x(t))\right] + \\ +\left(1 + \alpha_1 t\left[\|x_t\|_C + \sin(x(t))\right]\right)\left(\varphi(0) + \int_0^t \left[0, \frac{b(s)y_s}{1+\|y_s\|_C}\right] ds\right) \\ y(t) + \dfrac{\pi}{6} = \dfrac{1}{1+b(t)|x(\theta(t))|} + \dfrac{1}{3}\arctan\left(1 + b(t)|x(\theta(t))|\right) + \dfrac{1}{3}\arctan(y(t)) \ , \ t \in J_{\varepsilon} \\ (x(t), y(t)) = (\sin(t), \sin(t)) \ , \ t \in J_{-\varepsilon} \end{cases}$$

Obviously, $f : J_{\varepsilon} \times \mathbb{R} \times C \longrightarrow \mathbb{R} \backslash \{0\}$. Notice that f, k and p are continuous and Lipschitzian with respect to the second variables with Lipschitz constants α_1 and α_2, respectively. Again G is $L^1_{C(J,\mathbb{R})}$-Carathéodory with the bound function $\mu(t) = b(t)$. By using Theorem 2.9, we can deduce that problem NFIDI (1.1) has, at least, one solution, whenever $\alpha_1 + \alpha_2(1 + \|b\|_\infty) < \frac{1}{4}$, because $\psi(r) = \frac{r}{1+r}$ satisfies the condition (2.4) with $q(t) = b(t)$ and $\alpha(r) = \frac{1}{r}, r > 0$.

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