



UNIQUENESS OF POSITIVE SOLUTIONS FOR A FRACTIONAL DIFFERENTIAL EQUATION BOUNDARY VALUE PROBLEM VIA NEW FIXED POINT METHODS

JING LIU, CHENGBO ZHAI*

School of Mathematics and Statistics, Shanxi University, Taiyuan 030006, China

Abstract. This paper studies the existence and uniqueness of positive solutions for a class of differential equation boundary value problems involving Riemann-Liouville fractional derivative. To reach this aim, we first develop several fixed point theorems for sum operators composed of mixed monotone and decreasing operators, which extend some previous fixed point results in ordered spaces. Then, we formulate sufficient conditions that ensure the existence and uniqueness of positive solutions to the considered Riemann-Liouville fractional problem. Finally, two examples are provided to show our main results.

Keywords. Fixed point; Fractional differential equation; Mixed monotone operator; Riemann-Liouville derivative; Sum operator.

2020 MSC. 34B15, 34B18, 47H10, 26A33.

1. INTRODUCTION

Nowadays, fractional differential equations (FDEs, for short) have become a hot topic that arises in numerous fields. Riemann-Liouville fractional differential equations are of considerable significance across various fields, including signal processing, control theory, biomedical engineering, and financial engineering; see, e.g., [2, 6, 9, 12, 13]. One hot issue of the Riemann-Liouville fractional boundary value problems (FBVPs, for short) is the existence of positive solutions. Numerous methods were utilized to demonstrate the existence of solutions, including the Avery-Peterson fixed-point theorem [20], fixed-point index theory [8, 19], the Schauder fixed-point theorem [5], and the Guo-Krasnosel'skii fixed-point theorem [14]. For the results on the uniqueness, Banach fixed-point theorem, the partial-order method, Gronwall's inequality, and the Laplace transform technique were frequently employed. In [20], Zhang,

*Corresponding author.

E-mail address: cbzhai@sxu.edu.cn (C. Zhai).

Received April 8, 2026; Accepted September 11, 2026.

O'Regan, and Xu addressed a Riemann-Liouville FBVP with impulses

$$\begin{cases} -D_{0+}^{\alpha} \chi(t) = h_1(t, \chi(t)), & t \in (0, 1) \setminus \{t_k\}_{k=1}^m, \\ \Delta \chi(t_k) = J_k(\chi(t_k)), & k = 1, 2, \dots, m, \\ \chi(0) = \chi'(0) = \dots = \chi^{(n-2)}(0), & \chi'(1) = \int_0^1 h_2(s, \chi(s)) d\eta(s), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative (RLFD, for short) of order $n - 1 < \alpha \leq n$, $n \geq 3$, $n \in \mathbb{N}^+$. Further, h_i, η, J_k, t_k satisfy the following conditions:

- (i) Set $\{t_k\}_{k=1}^m$ meets $\Delta \chi(t_k) = \chi(t_k^+) - \chi(t_k^-)$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = 1$, where $\chi(t_k) = \chi(t_k^-)$, $\lim_{h \rightarrow 0^-} \chi(t_k - h) = \chi(t_k^-)$ and $\lim_{h \rightarrow 0^+} \chi(t_k + h) = \chi(t_k^+)$ are the left- and right-hand limits of $\chi(t)$ at $t = t_k$;
- (ii) $h_i \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$ ($i = 1, 2$), $J_k \in C(\mathbb{R}^+, \mathbb{R}^+)$ ($k = 1, 2, \dots, m$);
- (iii) $\eta(t)$ is a non-decreasing and non-constant function for $t \in [0, 1]$.

Under certain conditions on the spectral radius corresponding to the related linear operator, the authors demonstrated several existence results for the aforementioned problem and derived multiple positive solutions by employing the Avery-Peterson fixed-point theorem.

In [16], by applying a fixed point theorem for generalized concave operators, Zhai and Wang studied the existence of a unique solution for the following FBVP:

$$\begin{cases} D_{0+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = u'(0) = 0, & u(1) = \beta \int_0^1 u(s) ds, \end{cases}$$

where D_{0+}^{α} is the RLFD, $2 < \alpha < 3$, $0 < \beta < \alpha$ and $\lambda > 0$ is a parameter. In [1], Abdelhamid et al. investigated the FBVP with the following variable order derivatives of Riemann-Liouville type:

$$\begin{cases} D_{0+}^{\beta(t, y(t))} y(t) = \psi(t, y(t)), & t \in \Delta := (0, M], \quad 0 < M < \infty, \\ y(0) = 0, \end{cases}$$

where $D_{0+}^{\beta(t, y(t))}$ stands for the RLFD of the variable order $\beta(t, y(t))$, ψ is a given function and β satisfies $0 < \beta_* \leq \beta(t, y(t)) \leq \beta^* < 1$. They studied the existence and uniqueness of solutions in two different Banach spaces and achieved new criteria applied to a Banach-type fixed-point theorem.

In [8], Kong et al. studied positive solutions for a singular FBVP:

$$\begin{cases} -D^{\alpha} x(s) + a(s)x(s) = f(s, x(s)), & s \in (0, 1), \quad n - 1 < \alpha < n, \quad n \geq 3, \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, & x(1) = 0, \end{cases}$$

where $a \in L^1[0, 1]$, $f : (0, 1) \times (0, +\infty) \rightarrow \mathbb{R}^+$ is continuous and may be singular at $s = 0, 1$ and $x = 0$. The existence of positive solutions under sharp conditions was obtained by concerning the spectral radius and the fixed point index theory.

In [14], Wang considered a class of Riemann-Liouville type two-term FBVPs:

$$\begin{cases} -D_{0+}^{\alpha} u(t) + au(t) = y(t), & 0 < t < 1, \\ u(0) = u'(0) = 0, & u(1) = 0, \end{cases}$$

where D_{0+}^{α} is the RLFD, $2 < \alpha < 3$, and $a > 0$. Two applications were given by the Guo-Krasnosel'skii fixed point theorem and the monotone iterative technique.

Motivated by the results mentioned above, this research examines the existence and uniqueness of positive solutions for the following FBVPs:

$$\begin{cases} -D^\alpha x(s) + a(s)x(s) = f(s, x(s), x(s)) + g(s, x(s)), & s \in (0, 1), \quad n-1 < \alpha < n, \quad n \geq 3, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad x(1) = 0, \end{cases} \quad (1.1)$$

where $a \in L^1[0, 1]$, $f : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ and $g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ are continuous.

To solve this problem, we need to establish several new fixed point results. In [17], Zhai and Hao focused on the existence and uniqueness of positive fixed points for mixed monotone operators. Recently, numerous scholars explored such types of operators and derived a series of interesting and significant results [4, 7, 10, 11, 15, 18]. Moreover, the results in [17] dealt with the operator equations with increasing perturbations, while our results deal with decreasing perturbations. Each framework has its own domain of applicability. Therefore, our results are neither a simple variant nor a generalization of [17] in the traditional sense, but rather a new direction for mixed monotone operator theory, substantially extending the applicability of the method to a broader class of operator equations. Here, we study an operator equation in ordered Banach spaces:

$$F_1(x, x) + F_2x = x, \quad (1.2)$$

where F_1 is mixed monotone and F_2 is a decreasing $-\beta$ -convex operator. We discuss the existence of a unique positive solution for (1.2) and then establish some new fixed point results.

This paper mainly consists of the following parts. In Section 2, a review of key definitions and preliminary lemmas is given. Section 3 presents several new fixed point results and their corresponding proofs. In Section 4, we investigate the existence of a unique positive solution to FBVP (1.1) by employing our new fixed point theorems. Finally, Section 5 presents two illustrative examples.

2. PRELIMINARIES AND LEMMAS

For the sake of clarity, we present the pertinent definitions, notations, and previously established results; see [3, 4, 18]. Let E be a Banach space and $P \subset E$ a cone. For $x, y \in E$ with $\theta \leq x \leq y$, if there exists $N > 0$ such that $\|x\| \leq N\|y\|$, then P is normal with normality constant N . Set $\mathring{P} = \{x \in P : x \text{ is an interior point of } P\}$. P is said to be solid if $\mathring{P} \neq \emptyset$. For any two elements $x_1, x_2 \in E$, the order interval between x_1 and x_2 is written by $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$. An operator $F : E \rightarrow E$ is said to be decreasing (increasing) if $x \geq y$ yields $Fx \leq Fy$ ($Fx \geq Fy$). For any elements $x, y \in E$, the notation $x \sim y$ means: there exist positive constants $\lambda > 0$ and $\mu > 0$ such that the inequalities $\lambda x \leq y \leq \mu x$ are satisfied. One can check that $\sim$ defines an equivalence relation. For $h > \theta$, we write $P_h = \{x \in E \mid x \sim h\}$. Obviously, $P_h \subset P$. When $h \in \mathring{P}$, we have $P_h = \mathring{P}$. For operator $F : P \times P \rightarrow P$, if $x_i, y_i \in P$ ($i = 1, 2$), $x_1 \geq y_2$, $x_1 \leq y_2$ imply $F(x_1, y_1) \geq F(x_2, y_2)$, then F is said to be mixed monotone; see [4]. A point $x \in P$ is referred to as a fixed point of F if $F(x, x) = x$. Let $D = P$ or $\mathring{P}$, and $\beta \in [0, 1)$. Operator $F : D \rightarrow D$ is said to be $-\beta$ -convex if

$$F(tx) \leq t^{-\beta}Fx, \quad \forall t \in (0, 1), x \in D. \quad (2.1)$$

Lemma 2.1. [18, Theorem 2.1]. Let $P \subset E$ be a normal cone, and let operator $F : P \times P \rightarrow P$ be mixed monotone and satisfy (T_1) there exists $h \in P$ with $h \neq \theta$ such that $F(h, h) \in P_h$; (T_2) there

exists $\varphi(t) \in (t, 1]$ with $F(tu, t^{-1}v) \geq \varphi(t)F(u, v)$, $u, v \in P$, $t \in (0, 1)$. Then (1) $F : P_h \times P_h \rightarrow P_h$ has a unique fixed point x^* in P_h ; (2) there exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ satisfying $ru_0 \leq u_0 < v_0$, $u_0 \leq F(u_0, v_0) \leq F(v_0, u_0) \leq v_0$; (3) for initial elements $x_0, y_0 \in P_h$, if $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$, $n = 1, 2, \dots$, then $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$.

The RLFD of order $\alpha > 0$ for a function χ is

$$\mathcal{D}_{0+}^{\alpha} \chi(s) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^s (s-t)^{n-\alpha-1} \chi(t) dt,$$

where $[\alpha] = n - 1$ denotes the integer part of α ; see [8]. The Riemann-Liouville integral of order $\alpha > 0$ for a function χ is $I_{0+}^{\alpha} \chi(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} \chi(t) dt$.

Lemma 2.2. [8] Let $b \in C(0, 1) \cap L^1[0, 1]$. Then, the FBVP

$$\begin{cases} -\mathcal{D}^{\alpha} \chi(s) = b(s), & 0 < s < 1, \\ \chi(0) = \chi'(0) = \dots = \chi^{(n-2)}(0) = 0, \chi(1) = 0, \end{cases}$$

admits a unique solution $\chi(s) = \int_0^1 H_0(s, t) b(t) dt$, where

$$H_0(s, t) = \frac{1}{\Gamma(\alpha)} \begin{cases} s^{\alpha-1} (1-t)^{\alpha-1} - (s-t)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \\ s^{\alpha-1} (1-t)^{\alpha-1}, & 0 \leq s \leq t \leq 1. \end{cases} \quad (2.2)$$

Define a function $H : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ by $H(s, t) = \sum_{n=0}^{\infty} (-1)^n H_n(s, t)$, where, $H_0(s, t)$ is defined by (2.2) and $H_n(s, t) = \int_0^1 a(\tau) H_0(s, \tau) H_{n-1}(\tau, t) d\tau$, $n \geq 1$.

Lemma 2.3 [8] Let $(\alpha - 1) \int_0^1 |a(\tau)| \varphi(1 - \tau) d\tau < 1$ and $y \in L^1[0, 1] \cap C(0, 1)$. Then, the solution of the FBVP

$$\begin{cases} -D^{\alpha} x(s) + a(s)x(t) = y(s), s \in (0, 1), \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, x(1) = 0, \end{cases}$$

is $x(s) = \int_0^1 H(s, t) y(t) dt$.

Lemma 2.4 [8] If $\Delta_2 < 1$ and $\Lambda_2 > 0$, then the function $H(s, t)$ has the properties:

- (1) $H(s, t) \geq \Gamma(\alpha) \Lambda_2 \varphi(1 - s) \varphi(t)$ for $s, t \in [0, 1]$;
- (2) $H(s, t) \leq (\alpha - 1) \Lambda_1 \varphi(t)$ for $s, t \in [0, 1]$;
- (3) $H(s, t) \leq (\alpha - 1) \Lambda_1 \varphi(1 - s)$ for $s, t \in [0, 1]$,

in which $\varphi(s) = \frac{(1-s)^{\alpha-1}s}{\Gamma(\alpha)}$, and

$$\begin{aligned} \Delta_1 &= \int_0^1 |a(\tau)| d\tau, \quad \Delta_2 = (\alpha - 1) \int_0^1 |a(\tau)| \varphi(1 - \tau) d\tau, \\ \Delta &= \frac{(\alpha - 1)^2 \Delta_1}{\Gamma(\alpha)(1 - \Delta_2)}, \quad \Lambda_1 = 1 + \Delta, \quad \Lambda_2 = 1 - \Delta. \end{aligned}$$

3. NEW FIXED POINT RESULTS

In what follows, we construct some fixed point theorems in a Banach space E , which are partially ordered by a normal cone $P \subset E$. Let $h \in E$, $h > \theta$, P_h be previously described.

Theorem 3.1. Let $\beta \in (0, 1)$. Assume that $F_1 : P \times P \rightarrow P$ is mixed monotone and satisfies

$$F_1(tx, t^{-1}y) \geq t^{\beta} F_1(x, y), \quad t \in (0, 1), x, y \in P \quad (3.1)$$

and $F_2 : P \rightarrow P$ meets

$$F_2(tx) \leq t^{-1}F_2x, \quad t \in (0, 1), x \in P. \quad (3.2)$$

Assume that (i) an element $h_0 \in P_h$ satisfies $F_1(h_0, h_0) \in P_h$ and $F_2h_0 \in P_h$; (ii) a constant $\delta_0 > 0$ satisfies $F_1(x, y) \geq \delta_0 F_2y, x, y \in P$. Then (1) $F_1 : P_h \times P_h \rightarrow P_h, F_2 : P_h \rightarrow P_h$; (2) There exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $ru_0 \leq u_0 < v_0$ and $u_0 \leq F_1(u_0, v_0) + F_2v_0 \leq F_1(v_0, u_0) + F_2u_0 \leq v_0$; (3) Eq. (1.2) has a unique solution x^* in P_h ; (4) For initial elements $x_0, y_0 \in P_h$, construct $x_n = F_1(x_{n-1}, y_{n-1}) + F_2y_{n-1}$ and $y_n = F_1(y_{n-1}, x_{n-1}) + F_2x_{n-1}, n = 1, 2, \dots$. Then $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$.

Proof. From (3.1) and (3.2), one has

$$F_1\left(\frac{1}{t}x, ty\right) \leq \frac{1}{t^\beta}F_1(x, y) \quad \text{and} \quad F_2\left(\frac{1}{t}y\right) \geq tF_2y \quad \text{for } t \in (0, 1), x, y \in P. \quad (3.3)$$

As $F_2h_0, F_1(h_0, h_0) \in P_h$, there exists constants $v_1, v_2, \lambda_1, \lambda_2 > 0$ such that $v_1h \leq F_2h_0 \leq v_2h$ and $\lambda_1h \leq F_1(h_0, h_0) \leq \lambda_2h$. Furthermore, as $h_0 \in P_h$, there exists a constant $t_0 \in (0, 1)$ such that $t_0h \leq h_0 \leq \frac{1}{t_0}h$. By utilizing (3.1) and (3.3) in conjunction with the mixed monotonicity property of F_1 , one has

$$F_1(h, h) \leq F_1\left(\frac{1}{t_0}h_0, t_0h_0\right) \leq \frac{1}{t_0^\beta}F_1(h_0, h_0) \leq \frac{\lambda_2}{t_0^\beta}h$$

and

$$F_1(h, h) \geq F_1\left(t_0h_0, \frac{1}{t_0}h_0\right) \geq t_0^\beta F_1(h_0, h_0) \geq \lambda_1 t_0^\beta h.$$

Since $\lambda_2/t_0^\beta, \lambda_1 t_0^\beta > 0$, $F_1(h, h) \in P_h$ is clear. Therefore, from Lemma 2.1, $F_1 : P_h \times P_h \rightarrow P_h$. In addition, from (3.2), (3.3), and the monotonicity of operator F_2 , one has

$$F_2h \geq F_2\left(\frac{1}{t_0}h_0\right) \geq t_0F_2h_0 \geq v_1t_0h, \quad F_2h \leq F_2(t_0h_0) \leq \frac{1}{t_0}F_2h_0 \leq \frac{v_2}{t_0}h.$$

Now, we prove that $F_2 : P_h \rightarrow P_h$. For any $x \in P_h$, we can take a sufficiently small constant $\mu \in (0, 1)$ such that $\mu h \leq x \leq \frac{1}{\mu}h$. Thus

$$F_2x \geq F_2\left(\frac{1}{\mu}h\right) \geq \mu F_2h \geq \mu t_0 v_1 h, \quad F_2x \leq F_2(\mu h) \leq \frac{1}{\mu}F_2h \leq \frac{1}{\mu} \cdot \frac{v_2}{t_0}h.$$

Not that $\frac{v_2}{\mu t_0}, \mu t_0 v_1 > 0$. We claim that $F_2x \in P_h$, which means $F_2 : P_h \rightarrow P_h$. Thus conclusion (1) holds.

Next, we construct an operator $F = F_1 + F_2$ by $F(x, y) = F_1(x, y) + F_2y$. Then $F : P \times P \rightarrow P$ is mixed monotone and $F(h, h) \in P_h$. Now, we need to seek $\varphi(t) \in (t, 1]$ with respect to $t \in (0, 1)$ such that $F(tx, t^{-1}y) \geq \varphi(t)F(x, y), x, y \in P$. We need to discuss $l_1(t) = \frac{t^\gamma - t}{t^\beta - t^\gamma}$ for all $t \in (0, 1)$, where $\gamma \in (\beta, 1)$. One can prove that l_1 is increasing in $(0, 1)$, $\lim_{t \rightarrow 0^+} l_1(t) = 0$, and $\lim_{t \rightarrow 1^-} l_1(t) = \frac{1-\gamma}{\gamma-\beta}$. In addition, fixing $t \in (0, 1)$, one has $\lim_{\gamma \rightarrow 1^-} l_1(t) = \lim_{\gamma \rightarrow 1^-} \frac{t^\gamma - t}{t^\beta - t^\gamma} = 0$. So there exists $\gamma_0(t) \in (\beta, 1)$ with regard to t such that $\frac{t^{\gamma_0(t)} - t}{t^\beta - t^{\gamma_0(t)}} \leq \delta_0, t \in (0, 1)$. Thus $F_1(x, y) \geq \delta_0 F_2y \geq \frac{t^{\gamma_0(t)} - t}{t^\beta - t^{\gamma_0(t)}} F_2y, t \in (0, 1), x, y \in P$. Hence, $t^\beta F_1(x, y) + tF_2y \geq t^{\gamma_0(t)}[F_1x, y] + F_2y, t \in (0, 1), x, y \in P$. Then, for $t \in (0, 1)$ and $x, y \in P$, $F(tx, t^{-1}y) \geq t^\beta F_1(x, y) + tF_2y \geq t^{\gamma_0(t)}F(x, y)$. Set, for all $t \in (0, 1)$, $\varphi(t) = t^{\gamma_0(t)}$. Then $\varphi(t) \in (t, 1]$ and $F(tx, t^{-1}y) \geq \varphi(t)F(x, y), t \in (0, 1), x, y \in P$. Therefore, the condition (T_2) in Lemma 2.1 holds. Based on Lemma 2.1, we derive

the following conclusions: (i_1) there exist elements $u_0, v_0 \in P_h$ and a constant $r \in (0, 1)$, where $ru_0 \leq u_0 < v_0$ and $u_0 \leq F(u_0, v_0) \leq F(v_0, u_0) \leq v_0$; (i_2) F has a unique fixed point x^* in P_h ; (i_3) for initial elements $x_0, y_0 \in P_h$, $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$, $n = 1, 2, \dots$. Then $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$. That is, statements (2)-(4) are valid. $\square$

Based on the proof of Theorem 3.1, the following conclusion can be readily derived.

Corollary 3.1. Let $\beta \in (0, 1)$. Assume that $F_1 : P \times P \rightarrow P$ is mixed monotone, $F_2 : P \rightarrow P$ is decreasing, and (3.1)-(3.2) hold. Moreover, (i) there exists $h_0 > \theta$ satisfying $F_1(h_0, h_0) \in P_h$ and $F_2 h_0 \in P_h$; (ii) there exists a constant $\delta_0 > 0$ with $F_1(x, y) \geq \delta_0 F_2 y, x, y \in P$. Then, for any given $\mu > 0$, the operator equation $F_1(x, x) + F_2 x = \mu x$ has a unique solution x_μ^* in P_h . Moreover, define $x_n = \frac{1}{\mu} [F_1(x_{n-1}, y_{n-1}) + F_2 y_{n-1}]$ and $y_n = \frac{1}{\mu} [F_1(y_{n-1}, x_{n-1}) + F_2 x_{n-1}]$, $n = 1, 2, \dots$, for initial elements $x_0, y_0 \in P_h$. Then $x_n \rightarrow x_\mu^*$ and $y_n \rightarrow x_\mu^*$ as $n \rightarrow \infty$.

Theorem 3.2. Let $\beta \in (0, 1)$. Assume that $F_1 : P \times P \rightarrow P$ is mixed monotone and satisfies

$$F_1(tx, t^{-1}y) \geq tF_1(x, y), \quad t \in (0, 1), x, y \in P \quad (3.4)$$

and $F_2 : P \rightarrow P$ is decreasing $-\beta$ -convex. Further, assume that (i) there exists $h_0 \in P_h$ satisfying $F_1(h_0, h_0) \in P_h$ and $F_2 h_0 \in P_h$; (ii) there exists a constant $\delta_0 > 0$ satisfying $F_1(x, y) \leq \delta_0 F_2 y, x, y \in P$. Then (1) $F_1 : P_h \times P_h \rightarrow P_h, F_2 : P_h \rightarrow P_h$; (2) There exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that

$$rv_0 \leq u_0 < v_0, \quad u_0 \leq F_1(u_0, v_0) + F_2 v_0 \leq F_1(v_0, u_0) + F_2 u_0 \leq v_0.$$

(3) Eq. (1.2) has a unique solution x^* in P_h ; (4) For initial elements $x_0, y_0 \in P_h$, if $x_n = F_1(x_{n-1}, y_{n-1}) + F_2 y_{n-1}$ and $y_n = F_1(y_{n-1}, x_{n-1}) + F_2 x_{n-1}$, then $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$.

Proof. From (2.1) and (3.4), one has

$$F_1\left(\frac{1}{t}x, ty\right) \leq \frac{1}{t}F_1(x, y) \quad \text{and} \quad F_2\left(\frac{1}{t}y\right) \geq t^\beta F_2 y \quad \text{for } t \in (0, 1), x, y \in P.$$

An operator $F = F_1 + F_2$ is constructed by $F(x, y) = F_1(x, y) + F_2 y$. Following the proof of Theorem 3.1, one sees that $F_1 : P_h \times P_h \rightarrow P_h, F_2 : P_h \rightarrow P_h$ hold. Furthermore, it can be readily verified that $F : P \times P \rightarrow P$ is mixed monotone and $F(h, h) \in P_h$.

Next, we look for $\varphi(t) \in (t, 1]$ with regard to $t \in (0, 1)$ such that $F(tx, t^{-1}y) \geq \varphi(t)F(x, y)$ for all $x, y \in P$. We need to discuss $l_2(t) = \frac{t^\beta - t^\gamma}{t^\gamma - t}$ for all $t \in (0, 1)$, where $\gamma \in (\beta, 1)$. One can prove that l_2 is decreasing in $(0, 1)$ and $\lim_{t \rightarrow 0^+} l_2(t) = +\infty$ and $\lim_{t \rightarrow 1^-} l_2(t) = \frac{\gamma - \beta}{1 - \gamma}$. In addition, fixing $t \in (0, 1)$, one has $\lim_{\gamma \rightarrow 1^-} l_2(t) = \lim_{\gamma \rightarrow 1^-} \frac{t^\beta - t^\gamma}{t^\gamma - t} = +\infty$. So it must exist $\gamma_0(t) \in (\beta, 1)$ with respect to t such that $\frac{t^\beta - t^{\gamma_0(t)}}{t^{\gamma_0(t)} - t} \geq \delta_0, t \in (0, 1)$. Thus $F_1(x, y) \leq \delta_0 F_2 y \leq \frac{t^\beta - t^{\gamma_0(t)}}{t^{\gamma_0(t)} - t} F_2 y$ for all $t \in (0, 1), x, y \in P$. Hence, $tF_1(x, y) + t^\beta F_2 y \geq t^{\gamma_0(t)} [F_1(x, y) + F_2 y]$ for all $t \in (0, 1), x, y \in P$. Further, for $t \in (0, 1)$ and $x, y \in P$, $F(tx, t^{-1}y) \geq tF_1(x, y) + t^\beta F_2 y \geq t^{\gamma_0(t)} F(x, y)$. Set $\varphi(t) = t^{\gamma_0(t)}, t \in (0, 1)$. Thus $\varphi(t) \in (t, 1]$ and $F(tx, t^{-1}y) \geq \varphi(t)F(x, y), t \in (0, 1), x, y \in P$. Therefore, the condition (A_2) in Lemma 2.1 holds. By virtue of Lemma 2.1, we derive the following results: (e_1) there exist elements $u_0, v_0 \in P_h$ and a constant $r \in (0, 1)$ satisfying $rv_0 \leq u_0 < v_0$ and $u_0 \leq F(u_0, v_0) \leq F(v_0, u_0) \leq v_0$; (e_2) F has a unique fixed point x^* in P_h ; (e_3) for initial elements $x_0, y_0 \in P_h$, let $x_n = F(x_{n-1}, y_{n-1})$ and $y_n = F(y_{n-1}, x_{n-1})$, then $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$. That is, conclusions(2)-(4) are true. $\square$

Following the proof of Theorem 3.2, one has the following result immediately.

Corollary 3.2. Let $\beta \in (0, 1)$. Assume that $F_1 : P \times P \rightarrow P$ is mixed monotone and $F_2 : P \times P \rightarrow P$ is decreasing $-\beta$ -convex. Moreover, assume that (3.4) holds and

- (i) there exists $h_0 > \theta$ with $F_1(h_0, h_0) \in P_h$ and $F_2 h_0 \in P_h$;
- (ii) there exists a constant $\delta_0 > 0$ satisfying $F_1(x, y) \leq \delta_0 F_2 y$, $x, y \in P$.

Then, for any given $\mu > 0$, operator equation $F_1(x, x) + F_2 x = \mu x$ has a unique solution $x_{*\mu}$ in P_h . Moreover, if we set $x_n = \frac{1}{\mu} [F_1(x_{n-1}, y_{n-1}) + F_2 y_{n-1}]$ and $y_n = \frac{1}{\mu} [F_1(y_{n-1}, x_{n-1}) + F_2 x_{n-1}]$, for initial elements $x_0, y_0 \in P_h$, then $x_n \rightarrow x_{*\mu}$ and $y_n \rightarrow x_{*\mu}$ as $n \rightarrow \infty$.

Remark 3.1. If $h \in \mathring{P}$, then $\mathring{P} = P_h$. For a solid cone P , if $F_1 : P_h \times P_h \rightarrow P_h$ or $F_1 : \mathring{P} \times \mathring{P} \rightarrow \mathring{P}$, then $F_1(h_0, h_0) \in P_h$ for $h_0 \in P_h$ (or $\mathring{P}$).

If $S = \mathring{P}$ or P_h , we can also obtain the following results.

Corollary 3.3. Let $F_1 : S \times S \rightarrow S$ be mixed monotone and $F_2 : S \rightarrow S$ be decreasing. Let (3.1) and (3.2) hold. Assume that there exists a constant $\delta_0 > 0$ with $F_1(x, y) \geq \delta_0 F_2 y$, $x, y \in D$. Then, (1) there exist $u_0, v_0 \in S$ and $r \in (0, 1)$ satisfying

$$rv_0 \leq u_0 < v_0, \quad u_0 \leq F_1(u_0, v_0) + F_2 v_0 \leq F_1(v_0, u_0) + F_2 u_0 \leq v_0;$$

(2) Eq. (1.2) has a unique solution x^* in S ; (3) for initial elements $x_0, y_0 \in S$, if we set $x_n = F_1(x_{n-1}, y_{n-1}) + F_2 y_{n-1}$ and $y_n = F_1(y_{n-1}, x_{n-1}) + F_2 x_{n-1}$, then $x_n \rightarrow x^*$ and $y_n \rightarrow x^*$ as $n \rightarrow \infty$.

Corollary 3.4. Let $F_1 : S \times S \rightarrow S$ be mixed monotone and let $F_2 : S \rightarrow S$ be decreasing $-\beta$ -convex. Let (3.4) hold. Moreover, assume that there exists a constant $\delta_0 > 0$ with $F_1(x, y) \geq \delta_0 F_2 y$ for all $x, y \in S$. Then, (1) there exist $u_0, v_0 \in S$ and $r \in (0, 1)$ satisfying $rv_0 \leq u_0 < v_0$ and $u_0 \leq F_1(u_0, v_0) + F_2 v_0 \leq F_1(v_0, u_0) + F_2 u_0 \leq v_0$; (2) Eq. (1.2) has a unique solution x^* in S ; (3) for initial elements $x_0, y_0 \in S$, if $x_n = F_1(x_{n-1}, y_{n-1}) + F_2 y_{n-1}$ and $y_n = F_1(y_{n-1}, x_{n-1}) + F_2 x_{n-1}$, then x_n and y_n converge to x^* as $n \rightarrow \infty$.

4. EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR (1.1)

In this section, we apply Theorems 3.1 and 3.2 to investigate the existence and uniqueness of positive solutions for (1.1). Let $E = C[0, 1]$ and $P = \{x \in E \mid x(t) \geq 0, t \in [0, 1]\}$. It follows that P is a normal cone in E , and a partial order is induced by P such that $x \leq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0, 1]$.

Theorem 4.1. Assume that

- (L₁) $f(t, u, v) : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous, $g : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous with $g(t, v^*) \not\equiv 0$, $t \in [0, 1]$, where $v^* = \frac{1}{\Gamma(\alpha+1)} (1 - \frac{1}{\alpha})^{\alpha-1}$;
- (L₂) For any fixed $t \in [0, 1]$, $f(t, u, v)$ is non-decreasing in $u \in [0, +\infty)$ and non-increasing in $v \in [0, +\infty)$; $g(t, v)$ is non-increasing in $v \in [0, +\infty)$ for fixed $t \in [0, 1]$;
- (L₃) $g(t, \mu v) \leq \mu^{-1} g(t, v)$ for $\mu \in (0, 1)$, $t \in [0, 1]$, $v \in [0, +\infty)$, and there exists a constant $\beta \in (0, 1)$ such that $f(t, \mu u, \mu^{-1} v) \geq \mu^\beta f(t, u, v)$, $t \in [0, 1]$, $\mu \in (0, 1)$, $u, v \in [0, +\infty)$.
- (L₄) There exists a constant $\delta_0 > 0$ with $f(t, u, v) \geq \delta_0 g(t, v)$, $t \in [0, 1]$, $u, v \geq 0$.

Then

(1) There exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $ru_0 \leq u_0 < v_0$ and

$$u_0(s) \leq \int_0^1 H(s, t) f(t, u_0(t), v_0(t)) dt + \int_0^1 H(s, t) g(t, v_0(t)) dt, \quad s \in [0, 1],$$

$$v_0(s) \geq \int_0^1 H(s, t) f(t, v_0(t), u_0(t)) dt + \int_0^1 H(s, t) g(t, u_0(t)) dt, \quad s \in [0, 1],$$

where $h(s) = \varphi(1 - s), s \in [0, 1]$;

(2) Problem(1.1) has a unique positive solution u^* in P_h ;

(3) For any $x_0, y_0 \in P_h$, if

$$x_{n+1}(s) = \int_0^1 H(s, t) f(t, x_n(t), y_n(t)) dt + \int_0^1 H(s, t) g(t, y_n(t)) dt, \quad n = 0, 1, 2, \dots,$$

$$y_{n+1}(s) = \int_0^1 H(s, t) f(s, y_n(t), x_n(t)) dt + \int_0^1 H(s, t) g(t, x_n(t)) dt, \quad n = 0, 1, 2, \dots,$$

then $\|x_n - u^*\| \rightarrow 0$ and $\|y_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Lemma 2.3 implies that (1.1) has an integral representation expressed as

$$x(s) = \int_0^1 H(s, t) [f(t, x(t), x(t)) + g(t, x(t))] dt,$$

where $H(s, t)$ is given as in Lemma 2.2. Construct two operators $F_1 : P \times P \rightarrow E$ and $F_2 : P \rightarrow E$ by

$$F_1(x, y)(s) = \int_0^1 H(s, t) f(t, x(t), y(t)) dt, \quad (F_2 y)(s) = \int_0^1 H(s, t) g(t, y(t)) dt.$$

It can be readily verified that x is a solution to (1.1) if and only if $x = F_1(x, x) + F_2 x$. From (L_1) , we know that $F_1 : P \times P \rightarrow P$ and $F_2 : P \rightarrow P$. Moreover, from (L_2) , F_1 is mixed monotone and F_2 is decreasing. For $\mu \in (0, 1)$ and $x, y \in P$, by (L_3) ,

$$F_1(\mu x, \mu^{-1} y)(s) \geq \mu^\beta \int_0^1 H(s, t) f(t, x(t), y(t)) dt = \mu^\beta F_1(x, y)(s).$$

Namely, $F_1(\mu x, \mu^{-1} y) \geq \mu^\beta F_1(x, y)$ for $\mu \in (0, 1)$ and $x, y \in P$. So, F_1 satisfies (3.1). Also, for $\mu \in (0, 1)$ and $y \in P$, (L_3) yields

$$F_2(\mu y)(s) = \int_0^1 H(s, t) g(t, \mu y(t)) dt \leq \mu^{-1} \int_0^1 H(s, t) g(t, y(t)) dt = \mu^{-1} F_2 y(s),$$

that is, $F_2(\mu y) \leq \mu^{-1} F_2 y$ for $\mu \in (0, 1)$, $y \in P$. So, F_2 satisfies (3.2). Next, we show that $F_1(h, h) \in P_h$ and $F_2 h \in P_h$. Indeed, Lemma 2.4 implies

$$F_1(h, h)(s) = \int_0^1 H(s, t) f(t, h(t), h(t)) dt \leq (\alpha - 1) \Lambda_1 h(s) \int_0^1 f(t, h_{\max}, 0) dt,$$

$$\begin{aligned} F_1(h, h)(s) &= \int_0^1 H(s, t) f(t, h(t), h(t)) dt \geq \Gamma(\alpha) \Lambda_2 h(s) \int_0^1 \varphi(t) f(t, 0, h_{\max}) dt \\ &= \Lambda_2 h(s) \int_0^1 (1 - t)^{\alpha-1} t f(t, 0, h_{\max}) dt, \end{aligned}$$

where $h_{\max} = \max\{h(s) : s \in [0, 1]\} = \frac{1}{\Gamma(\alpha+1)} \left(1 - \frac{1}{\alpha}\right)^{\alpha-1} = v^*$, and $h(s) = \varphi(1-s)$. From (L₂) and (L₄), we have $f(t, h_{\max}, 0) \geq f(t, 0, h_{\max}) \geq \delta_0 g(t, h_{\max}) = \delta_0 g(t, v^*) \geq 0$. Since $(1-t)^{\alpha-1} t g(t, v^*) \neq 0, t \in [0, 1]$, we see that

$$\begin{aligned} (\alpha-1)\Lambda_1 \int_0^1 f(t, h_{\max}, 0) dt &\geq \Lambda_2 \int_0^1 (1-t)^{\alpha-1} t f(t, 0, h_{\max}) dt \\ &\geq \delta_0 \Lambda_2 \int_0^1 (1-t)^{\alpha-1} t g(t, v^*) dt > 0. \end{aligned}$$

Thus $F_1(h, h) \in P_h$. Similarly,

$$\Lambda_2 h(s) \int_0^1 (1-t)^{\alpha-1} t g(t, h_{\max}) dt \leq F_2 h(s) \leq (\alpha-1)\Lambda_1 h(s) \int_0^1 g(t, 0) dt.$$

Since $g(t, v^*) \neq 0$, it follows that $F_2 h \in P_h$. Therefore, the condition (i) of Theorem 3.1 is fulfilled. From Lemma 2.1 and (L₄), one has

$$F_1(x, y)(s) = \int_0^1 H(s, t) f(t, x(t), y(t)) dt \geq \delta_0 \int_0^1 H(s, t) g(t, y(t)) dt = \delta_0 F_2 y(s).$$

Then $F_1(x, y) \geq \delta_0 F_2 y, x, y \in P$. Therefore, the conclusion of Theorem 4.1 can be derived from Theorem 3.1 immediately. $\square$

Theorem 4.2. Assume (L₁), (L₁)', and (L₂) hold and satisfy the following conditions:

- (L₅) There exists a constant $\beta \in (0, 1)$ with $g(t, \mu v) \leq \mu^{-\beta} g(t, v), \mu \in (0, 1), t \in [0, 1], v \in [0, +\infty)$, and $f(t, \mu u, \mu^{-1} v) \geq \mu f(t, u, v), t \in [0, 1], \mu \in (0, 1), u, v \in [0, +\infty)$;
- (L₆) There exists a constant $\delta_0 > 0$ with $f(t, u, v) \leq \delta_0 g(t, v), t \in [0, 1], u, v \geq 0$.

Then

- (1) There exist $u_0, v_0 \in P_h$ and $r \in (0, 1)$ satisfying $rv_0 \leq u_0 < v_0$ and

$$u_0(s) \leq \int_0^1 H(s, t) f(t, u_0(t), v_0(t)) dt + \int_0^1 H(s, t) g(t, v_0(t)) dt, \quad s \in [0, 1],$$

$$v_0(s) \geq \int_0^1 H(s, t) f(t, v_0(t), u_0(t)) dt + \int_0^1 H(s, t) g(t, u_0(t)) dt, \quad s \in [0, 1],$$

where $h(s) = \varphi(1-s), s \in [0, 1]$;

- (2) Problem (1.1) has a unique positive solution u^* in P_h ;
- (3) For any $x_0, y_0 \in P_h$, if we set

$$x_{n+1}(s) = \int_0^1 H(s, t) f(t, x_n(t), y_n(t)) dt + \int_0^1 H(s, t) g(t, y_n(t)) dt, \quad n = 0, 1, 2, \dots,$$

$$y_{n+1}(s) = \int_0^1 H(s, t) f(t, y_n(t), x_n(t)) dt + \int_0^1 H(s, t) g(t, x_n(t)) dt, \quad n = 0, 1, 2, \dots,$$

then $\|x_n - u^*\| \rightarrow 0$ and $\|y_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Following the proof of Theorem 4.1, we give two operators $F_1 : P \times P \rightarrow E$ and $F_2 : P \rightarrow E$ by $F_1(x, y)(s) = \int_0^1 H(s, t) f(t, x(t), y(t)) dt$ and $(F_2 y)(s) = \int_0^1 H(s, t) g(t, y(t)) dt$. From (L₁) and (L₂), $F_1 : P \times P \rightarrow P$ is mixed monotone and $F_2 : P \rightarrow P$ is decreasing. For $x, y \in P$ and $\mu \in (0, 1)$, by (L₅), $F_1(\mu x, \mu^{-1} y)(s) \geq \mu \int_0^1 H(s, t) f(t, x(t), y(t)) dt = \mu F_1(x, y)(s)$. In other words,

$F_1(\mu x, \mu^{-1}y) \geq \mu F_1(x, y)$ for $\mu \in (0, 1)$, $x, y \in P$. So F_1 satisfies (3.4). Also, for $\mu \in (0, 1)$ and $y \in P$, from (L₅),

$$F_2(\mu y)(s) = \int_0^1 H(s, t)g(t, \mu y(t))dt \leq \mu^{-\beta} \int_0^1 H(s, t)g(t, y(t))dt = \mu^{-\beta} F_2 y(s),$$

that is, $F_2(\mu y) \leq \mu^{-\beta} F_2 y$ for $\mu \in (0, 1)$, $y \in P$. So F_2 satisfies (2.1). Next, we prove $F_1(h, h) \in P_h$ and $F_2 h \in P_h$. Lemma 2.4 indicates that

$$\Lambda_2 h(s) \int_0^1 (1-t)^{\alpha-1} t f(t, 0, h_{\max}) dt \leq F_1(h, h)(s) \leq (\alpha-1) \Lambda_1 h(s) \int_0^1 f(t, h_{\max}, 0) dt,$$

where $h_{\max} = \max\{h(s) : s \in [0, 1]\} = \frac{1}{\Gamma(\alpha+1)} (1 - \frac{1}{\alpha})^{\alpha-1} = v^*$. Since $(1-t)^{\alpha-1} t f(t, 0, v^*) \not\equiv 0$, $t \in [0, 1]$, we see that $F_1(h, h) \in P_h$. Similarly,

$$\Lambda_2 h(s) \int_0^1 (1-t)^{\alpha-1} t g(t, h_{\max}) dt \leq F_2 h(s) \leq (\alpha-1) \Lambda_1 h(s) \int_0^1 g(t, 0) dt.$$

From $g(t, v^*) \not\equiv 0$, $F_2 h \in P_h$. Hence, the condition (i) of Theorem 3.2 is satisfied. For $x, y \in P$, from (L₆), $F_1(x, y)(s) = \int_0^1 H(s, t) f(t, x(t), y(t)) dt \leq \delta_0 \int_0^1 H(s, t) g(t, y(t)) dt = \delta_0 F_2 y(s)$. Then $F_1(x, y) \leq \delta_0 F_2 y$, $x, y \in P$, which satisfies the condition (ii) of Theorem 3.2. So the conclusion of Theorem 4.2 follows from Theorem 3.2 immediately. $\square$

5. EXAMPLES

Example 5.1 Let $n = 3$ and $\alpha = 2.5$. Consider

$$\begin{cases} -D^{2.5}x(s) + sx(s) = f(s, x(s), x(s)) + g(s, x(s)), & s \in (0, 1), \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases} \quad (5.1)$$

where $f(s, u, v) = s + u^{1/2} + (1+v)^{-1/2}$, $g(s, v) = s(1+v)^{-1}$, and $a(s) = s$. A direct calculation gives $\Lambda_2 \approx 0.0716 < 1$ and $\Lambda_2 \approx 0.9284 > 0$. Clearly, conditions (L1) and (L2) are satisfied. We now verify condition (L3). Take $\beta = \frac{1}{2} \in (0, 1)$. For any $\mu \in (0, 1)$, $s \in [0, 1]$, and $v \geq 0$, we have $g(s, \mu v) = s(1 + \mu v)^{-1} \leq s(\mu(1+v))^{-1} = \mu^{-1}g(s, v)$. Thus, for $\mu \in (0, 1)$ and $u, v \geq 0$,

$$\begin{aligned} f(s, \mu u, \mu^{-1}v) &= s + (\mu u)^{1/2} + (1 + \mu^{-1}v)^{-1/2} \\ &= s + \mu^{1/2}u^{1/2} + \mu^{1/2}(\mu + v)^{-1/2} \\ &\geq \mu^{1/2}f(s, u, v), \end{aligned}$$

Hence (L3) holds with $\beta = \frac{1}{2}$. Next, we verify condition (L4). Taking $\delta_0 = 1$, for any $s \in [0, 1]$ and $u, v \geq 0$, we obtain

$$f(s, u, v) = s + u^{1/2} + (1+v)^{-1/2} \geq (1+v)^{-1/2} \geq s(1+v)^{-1} = g(s, v),$$

Thus (L4) holds with $\delta_0 = 1$. Consequently, all the hypotheses of Theorem 4.1 are satisfied. Therefore, problem (5.1) has a unique positive solution u^* in P_h , where

$$h(s) = \varphi(1-s) = \frac{s^{3/2}(1-s)}{\Gamma(5/2)} = \frac{4}{3\sqrt{\pi}} s^{3/2}(1-s), \quad s \in [0, 1].$$

For any $x_0, y_0 \in P_h$, setting

$$x_{n+1}(s) = \int_0^1 H(s, t) f(t, x_n(t), y_n(t)) dt + \int_0^1 H(s, t) g(t, y_n(t)) dt, \quad n = 0, 1, 2, \dots,$$

$$y_{n+1}(s) = \int_0^1 H(s, t) f(t, y_n(t), x_n(t)) dt + \int_0^1 H(s, t) g(t, x_n(t)) dt, \quad n = 0, 1, 2, \dots,$$

one has $\|x_n - u^*\| \rightarrow 0$ and $\|y_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Example 5.2 Let $n = 3$ and $\alpha = 2.8$. Consider

$$\begin{cases} -D^{2.8}x(s) + 0.3x(s) = f(s, x(s), x(s)) + g(s, x(s)), & s \in (0, 1), \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases} \quad (5.2)$$

where $f(s, u, v) = s(\frac{u}{1+u})^{0.6}(\frac{1}{1+v})^{0.4} + 0.1$, $g(s, v) = s^2(\frac{1}{1+v})^{0.6} + 0.2$, and $a(s) = 0.3$. Note that $\Delta_2 \approx 0.0303 < 1$ and $\Lambda_2 \approx 0.402 > 0$. Clearly, (L_1) , $(L_1)'$ and (L_2) are satisfied. Take $\beta = 0.6 \in (0, 1)$. Since $\frac{1}{1+\mu v} \leq \frac{1}{\mu(1+v)}$, $\mu \in (0, 1)$, then $(\frac{1}{1+\mu v})^{0.6} \leq \mu^{-0.6}(\frac{1}{1+v})^{0.6}$. Therefore,

$$\begin{aligned} g(s, \mu v) &= s^2(\frac{1}{1+\mu v})^{0.6} + 0.2 \leq \mu^{-0.6} s^2(\frac{1}{1+v})^{0.6} + 0.2 \\ &\leq \mu^{-0.6} s^2(\frac{1}{1+v})^{0.6} + \mu^{-0.6} \cdot 0.2 = \mu^{-0.6} g(s, v). \end{aligned}$$

Thus $g(s, \mu v) \leq \mu^{-0.6} g(s, v)$. Moreover, it can be verified

$$(\frac{\mu u}{1+\mu u})^{0.6}(\frac{1}{1+\mu^{-1}v})^{0.4} \geq (\frac{\mu u}{1+u})^{0.6}(\frac{\mu}{1+v})^{0.4} = \mu(\frac{u}{1+u})^{0.6}(\frac{1}{1+v})^{0.4}.$$

Then

$$\begin{aligned} f(s, \mu u, \mu^{-1}v) &= s(\frac{\mu u}{1+\mu u})^{0.6}(\frac{1}{1+\mu^{-1}v})^{0.4} + 0.1 \\ &\geq \mu s(\frac{u}{1+u})^{0.6}(\frac{1}{1+v})^{0.4} + 0.1 \\ &\geq \mu f(s, u, v). \end{aligned}$$

Thus, (L_5) is satisfied. For $s \in [0, 1]$ and $u, v \geq 0$, we have $g(s, v) = s^2(\frac{1}{1+v})^{0.6} + 0.2 \geq 0.2$ and

$$f(s, u, v) = s(\frac{u}{1+u})^{0.6}(\frac{1}{1+v})^{0.4} + 0.1 \leq s \cdot 1^{0.6} \cdot 1^{0.4} + 0.1 \leq 1.1,$$

Taking $\delta_0 = \frac{1.1}{0.2} = 5.5$, one has $f(s, u, v) \leq \delta_0 g(s, v)$. Hence, (L_6) is verified. Consequently, the assumptions required for Theorem 4.2 are met. Then problem (5.2) has a unique positive solution u^* in P_h , where $h(s) = \varphi(1-s) = \frac{s^{\frac{9}{5}}(1-s)}{\Gamma(2.8)}$, $s \in [0, 1]$.

REFERENCES

- [1] H. Abdelhamid, G. Stamov, M.S. Souid, I. Stamova, New results achieved for fractional differential equations with Riemann-Liouville derivatives of nonlinear variable order, *Axioms* 12 (2023) 895.
- [2] S. Baroudi, A. Kassidi, A. El Mfadel, M. Elomari, An accurate and stable numerical scheme for fractional integro-differential equations, *J. Appl. Numer. Optim.* 8 (2026) 373-394.
- [3] D. Guo, Fixed points of mixed monotone operators with application, *Appl. Anal.* 34 (1988) 215-224.
- [4] D. Guo, V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal.* 11 (1987) 623-632.

- [5] J. Henderson, R. Luca, A. Tudorache, On a system of Riemann-Liouville fractional boundary value problems with ρ -Laplacian operators and positive parameters, *Fractal Fract.* 6 (2022) 299.
- [6] M. Kamenskii, V. Obukhovskii, G. Petrosyan, M. Soroka, On topological properties of solution sets for semilinear differential inclusions of fractional order $\alpha \in (1, 2)$ with non-convex right-hand sides, *Commun. Optim. Theory* 2026 (2026) 26.
- [7] M. Khazou, M.A. Taoudi, Existence and uniqueness of fixed points for monotone operators in partially ordered Banach spaces and applications, *J. Fixed Point Theory Appl.* 23 (2021) 12.
- [8] L. Kong, Y. Wang, Solvability of singular fractional-order differential equations with a perturbation term, *Axioms* 14 (2025) 143.
- [9] Y. Liu, Y. Wei, Optimal control problem of fractional Keller-Segel equations, *Ann. Math. Sci. Appl.* 10 (2025) 553-578.
- [10] P. Lv, On fixed points of nonexpansive operators on Banach spaces, *J. Nonlinear Convex Anal.* 27 (2026) 1-10.
- [11] H.R. Marasi, H. Afshari, M. Daneshbastam, C. Zhai, Fixed points of mixed monotone operators for existence and uniqueness of nonlinear fractional differential equations, *J. Contemp. Math. Anal.* 52 (2017) 8-13.
- [12] A. Oultou, H. Benaissa, A coupled system of Hilfer fractional differential and history-dependent hemivariational inequalities, *Appl. Set-Valued Anal. Optim.* 8 (2026) 271-292.
- [13] V. Obukhovskii, G. Petrosyan, M. Soroka, J.C. Yao, On topological properties of solution sets of semilinear fractional differential inclusions with non-convex right-hand side, *J. Nonlinear Var. Anal.* 8 (2024) 95-108.
- [14] Y. Wang, The Green's function of a class of two-term fractional differential equation boundary value problem and its applications, *Adv. Differ. Equ.* 2020 (2020) 80.
- [15] H. Wang, L. Fang, One fixed point theorem for mixed monotone operators and applications to nonlinear integral equations, *J. Integr. Equations Appl.* 36 (2024) 371-382.
- [16] C. Zhai, F. Wang, Properties of positive solutions for the operator equation $Ax = \mu x$ and applications to fractional differential equations with integral boundary conditions, *Adv. Differ. Equ.* 2015 (2015) 366.
- [17] C. Zhai, M. Hao, Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems, *Nonlinear Anal.* 75 (2012) 2542-2551.
- [18] C. Zhai, L. Zhang, New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems, *J. Math. Anal. Appl.* 382 (2011) 594-614.
- [19] K. Zhang, Q. Sun, D. O'Regan, J. Xu, Positive solutions for a Riemann-Liouville-type impulsive fractional integral boundary value problem, *AIMS Math.* 9 (2024) 10911-10925.
- [20] K. Zhang, D. O'Regan, J. Xu, Solvability of a Riemann-Liouville-type fractional-impulsive differential equation with a Riemann-Stieltjes integral boundary condition, *Fractal Fract.* 9 (2025) 323.