J. Nonlinear Funct. Anal. 2026 (2026) 27 https://doi.org/10.23952/jnfa.2026.27

MATHRES Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

FUNCTIONAL MATRIX INEQUALITIES VIA BLOCK MATRICES

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Abstract. The positivity of block matrices play a vital role in establishing logical relationships among their block entries. Building on this foundation, we delve deeper into this topic by proving several new relations involving these blocks' images under increasing and concave functions. Our findings lead to refined norm inequalities, offering enhanced versions of some known results in the literature.

Keywords. Block matrix; Concave function; Subadditive matrix inequality.
2020 MSC. 47A30, 15A60.

1. INTRODUCTION

The notation $\mathbb{M}_n$ denotes the algebra of all $n \times n$ complex matrices, with zero matrix O and identity I. Using uppercase letters to denote matrices in $\mathbb{M}_n$ is customary. A special class in $\mathbb{M}_n$ is that of positive semidefinite matrices consisting of those $A \in \mathbb{M}_n$ such that $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{C}^n$, where $\langle \cdot, \cdot \rangle$ refers to the standard inner product on $\mathbb{C}^n$. Such A is then denoted as $A \geq O$. This order induces a partial order on the class of Hermitian matrices, where we write $A \geq B$ for Hermitian $A, B \in \mathbb{M}_n$ if $A - B \geq O$. In this context, a matrix A is Hermitian if its numerical range, defined by the set $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$, is a real interval. This is equivalent to $A^* = A$, where A^* is the conjugate transpose of A.

To compare all elements of $\mathbb{M}_n$, we use distinctive positive quantities, such as the spectral norm, spectral radius, and singular values. The spectral norm of $A \in \mathbb{M}_n$ is defined by $\|A\| = \sup_{\|x\|=1} \|Ax\|$, while the spectral radius of A is $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ refers to the spectrum of A, that is the set of eigenvalues of A. However, singular values

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Received February 25, 2026; Accepted September 13, 2026.

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are more technical, as they are the eigenvalues of the positive semidefinite matrix $|A| = (A^*A)^{\frac{1}{2}}$. It is common to write $s_j(A)$ to denote the j-th singular value of A, when these eigenvalues are arranged in nonincreasing order. Thus, $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$. It is known that $\|A\| = s_1(A)$. Furthermore, it is known that $r(A) \leq \|A\|$. Matrix decomposition has been a useful tool in different aspects, where a matrix can be written in a different form that might give further information. Of particular interest, the Cartesian and polar decompositions of $A \in \mathbb{M}_n$ are defined, respectively, via the forms $A = \Re A + \mathrm{i}\Im A$ and $A = U|A|$, where $\Re A = \frac{A+A^*}{2}, \Im A = \frac{A-A^*}{2\mathrm{i}}$ and U is a unitary matrix. A larger class containing the positive semidefinite matrices is the class of accretive matrices consisting of all $A \in \mathbb{M}_n$ whose real part $\Re A$ is positive semidefinite. If, in addition, $\Im A \geq O$, we say that A is accretive-dissipative. If A is Hermitian with eigenvalues $\{\lambda_j\}$, including multiplicities, A can be decomposed via the spectral decomposition $A = UD(\lambda_i)U^*$, where U is unitary and $D(\lambda_i)$ is the diagonal matrix, whose diagonal entries are the λ_i's. Now, if f is a function that is defined on an interval containing the spectrum of the Hermitian A, we define $f(A) = UD(f(\lambda_i))U^*$. The literature has witnessed numerous works treating possible relations among matrix quantities involving $f(A)$, for certain functions f, and certain matrix forms A.

A simple result in this direction is observed in the following lemma.

Lemma 1.1. *Let $A \in \mathbb{M}_n$ be such that $A \geq O$, and let $f : [0,\infty) \to [0,\infty)$ be an increasing function. Then $f\left(s_j(A)\right) = s_j\left(f(A)\right)$, for $j = 1,\ldots,n$.*

Our interest in studying singular values grew partially because of its connection to unitarily invariant norms. We will use the notation $|||\cdot|||$ to denote arbitrary unitarily invariant norms in the sequel. These are matrix norms that absorb unitary matrices in the sense that $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices U, V. For important and recent progress on singular values and unitarily invariant norms, we refer the reader to [3–5, 12, 17–19, 21, 24, 29, 31, 32]. Among those beautiful results on the interaction of a concave function with positive semidefinite matrices, we have the following lemma, which extends the subadditivity behavior of concave functions to the context of the matrices. For the rest of the paper, whenever we write a relation involving $|||\cdot|||$, we mean that this relation is valid for all unitarily invariant norms $|||\cdot|||$. Whenever matrices of different sizes occur in the same unitarily invariant norm inequality, we use the standard zero-padding convention: an $n \times n$ matrix A is identified with $A \oplus O$ when compared with a matrix of larger size.

Lemma 1.2. *[11] Let $A_i \geq O$ $(i = 1,2,\ldots,m)$ and let $f : [0,\infty) \to [0,\infty)$ be a concave function. Then $\left|\left|\left|f\left(\sum_{i=1}^m A_i\right)\right|\right|\right| \leq \left|\left|\left|\sum_{i=1}^m f\left(A_i\right)\right|\right|\right| \leq \sum_{i=1}^m \left|\left|\left|f\left(A_i\right)\right|\right|\right|$.*

Lemma 1.2 was proved as a substantial extension of an earlier result in [2], where the same inequality was shown for nonnegative operator concave functions. For further discussion of the role of concave (or convex) functions and unitarily invariant norms, the reader is referred to [2, 6, 7, 15, 22].

Block matrices are useful in advancing our understanding of the algebra $\mathbb{M}_n$. In this context, if $A, B, C, D \in \mathbb{M}_n$, the form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is an element of $\mathbb{M}_{2n}$. In particular, if $B, C = O$, this block matrix is denoted as $A \oplus D$. Notice that this block form has $2n$ singular values, counting multiplicities. The following lemma gives a possible relation between the singular values of the off-diagonal block and those of the positive block matrix.

Lemma 1.3. *[28] Let $A,B,X \in \mathbb{M}_n$ be such that $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geq O$. Then, for $j = 1,\ldots,2n$,*

$$2s_j\left(\begin{bmatrix} X & O \\ O & O \end{bmatrix}\right) \leq s_j\left(\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right).$$

Equivalently, $2s_j(X) \leq s_j\left(\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right)$, for $j = 1,\ldots,n$.

More interestingly, if the off-diagonal entry of a block matrix is Hermitian, we have the following norm relation.

Lemma 1.4. *[16] Let $A,B,X \in \mathbb{M}_n$ be such that X is Hermitian and $\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq O$. Then*

$$\left|\!\left|\!\left| \begin{bmatrix} A & X \\ X & B \end{bmatrix} \right|\!\right|\!\right| \leq |||A+B|||.$$

Recall that if $A,B \geq O$, and if A is invertible (so that A is positive definite), then the geometric mean of A,B is defined by $A\sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}$. If A is not invertible, we define

$$A\sharp B = \lim_{\varepsilon \downarrow 0}(A+\varepsilon I)\sharp B,$$

where the limit is taken in the strong operator topology. Of course, any norm would give the same limit in $\mathbb{M}_n$.

Lemma 1.5. *[1, Lemma 3.1] Let $A_i,B_i,X \in \mathbb{M}_n$ $(i = 1,2)$ be such that $\begin{bmatrix} A_i & X^* \\ X & B_i \end{bmatrix} \geq O$. Then*

$$\begin{bmatrix} A_1\sharp A_2 & X^* \\ X & B_1\sharp B_2 \end{bmatrix} \geq O.$$

Our goal in this paper is to utilize the positivity of block matrices to produce several new generalized bounds for the image of the off-diagonal entries. This work is motivated, partially, by the result [30, Theorem 4 and Theorem 5]:

$$\left|\!\left|\!\left| f\left(\left|\sum_{i=1}^m A_i\right|\right) \right|\!\right|\!\right| \leq \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{pmatrix} \sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*| \end{pmatrix}\right) \right|\!\right|\!\right| \leq \sum_{i=1}^m |||f(|A_i|)||| \tag{1.1}$$

valid for $A_i \in \mathbb{M}_n$ $(i = 1,2,\ldots,m)$, with $m \geq 2$, and the concave function $f : [0,\infty) \to [0,\infty)$. We can prove that the above interesting inequality is a special case of a result concerning the positivity of block matrices. For further details, see Corollary 2.10 below and the preceding results. We also refer the reader to [10, 13, 20, 25–27] for further progress and background on the role of block matrices in advancing our understanding of matrix inequalities.

Among those applications, we can also show a new numerical radius-norm bound for accretive-dissipative matrices. We recall that if $A \in \mathbb{M}_n$, the numerical radius of A is defined by $\omega(A) = \sup\{|z| : z \in W(A)\}$. For this particular purpose, we recall the following lemma.

Lemma 1.6. *[23] Let A,B be two normal operators. Then $\omega(A+B) \leq \sqrt{2}\,\omega(|A| + \mathrm{i}\,|B|)$.*

Before proceeding to the main results, we show the following interesting functional identity. We recall that if H is Hermitian, $U \in \mathbb{M}_n$ is unitary, and f is a scalar function defined on the spectrum of H, then the spectral functional calculus gives $f(U^*HU) = U^*f(H)U$.

Lemma 1.7. *Let $T \in \mathbb{M}_n$. If $f : [0,\infty) \to \mathbb{R}$ satisfies $f(0) = 0$, then*

$$f\left(\frac{1}{2}\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix}\right) + f\left(\frac{1}{2}\begin{bmatrix} |T| & -T^* \\ -T & |T^*| \end{bmatrix}\right) = f\left(\begin{bmatrix} |T| & O \\ O & |T^*| \end{bmatrix}\right).$$

Proof. Letting $T = U|T|$ be the polar decomposition of T, we have

$$\begin{aligned}
&f\left(\frac{1}{2}\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix}\right) + f\left(\frac{1}{2}\begin{bmatrix} |T| & -T^* \\ -T & |T^*| \end{bmatrix}\right) \\
&= f\left(\frac{\begin{bmatrix} I & -I \\ U & U \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} |T| & O \\ O & O \end{bmatrix}\frac{\begin{bmatrix} I & U^* \\ -I & U^* \end{bmatrix}}{\sqrt{2}}\right) + f\left(\frac{\begin{bmatrix} I & -I \\ -U & -U \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} |T| & O \\ O & O \end{bmatrix}\frac{\begin{bmatrix} I & -U^* \\ -I & -U^* \end{bmatrix}}{\sqrt{2}}\right) \\
&= \frac{\begin{bmatrix} I & -I \\ U & U \end{bmatrix}}{\sqrt{2}} f\left(\begin{bmatrix} |T| & O \\ O & O \end{bmatrix}\right)\frac{\begin{bmatrix} I & U^* \\ -I & U^* \end{bmatrix}}{\sqrt{2}} + \frac{\begin{bmatrix} I & -I \\ -U & -U \end{bmatrix}}{\sqrt{2}} f\left(\begin{bmatrix} |T| & O \\ O & O \end{bmatrix}\right)\frac{\begin{bmatrix} I & -U^* \\ -I & -U^* \end{bmatrix}}{\sqrt{2}} \\
&= \frac{\begin{bmatrix} I & -I \\ U & U \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} f(|T|) & O \\ O & O \end{bmatrix}\frac{\begin{bmatrix} I & U^* \\ -I & U^* \end{bmatrix}}{\sqrt{2}} + \frac{\begin{bmatrix} I & -I \\ -U & -U \end{bmatrix}}{\sqrt{2}}\begin{bmatrix} f(|T|) & O \\ O & O \end{bmatrix}\frac{\begin{bmatrix} I & -U^* \\ -I & -U^* \end{bmatrix}}{\sqrt{2}} \\
&= \frac{1}{2}\begin{bmatrix} f(|T|) & f(|T|)U^* \\ Uf(|T|) & Uf(|T|)U^* \end{bmatrix} + \frac{1}{2}\begin{bmatrix} f(|T|) & -f(|T|)U^* \\ -Uf(|T|) & Uf(|T|)U^* \end{bmatrix} \\
&= \begin{bmatrix} f(|T|) & O \\ O & Uf(|T|)U^* \end{bmatrix} \\
&= f\left(\begin{bmatrix} |T| & O \\ O & |T^*| \end{bmatrix}\right),
\end{aligned}$$

as required. □

Remark 1.8. *If we replace f by $f - f(0)$ in Lemma 1.7, we may restate the result of the lemma for an arbitrary function $f : [0,\infty) \to \mathbb{R}$ in the form:*

$$f\left(\frac{1}{2}\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix}\right) + f\left(\frac{1}{2}\begin{bmatrix} |T| & -T^* \\ -T & |T^*| \end{bmatrix}\right) = f\left(\begin{bmatrix} |T| & O \\ O & |T^*| \end{bmatrix}\right) + f(0)I.$$

We also need the following simple lemma.

Lemma 1.9. *Let $A, B, C \in \mathbb{M}_n$. If $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq O$, then $A + C \geq \pm 2\Re B$.*

Proof. Note that

$$\left\langle \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle \geq 0$$

holds for any vectors $x, y \in \mathbb{C}^n$. That is, $\langle Ax, x\rangle + \langle By, x\rangle + \langle B^*x, y\rangle + \langle Cy, y\rangle \geq 0$ holds for any vector x and y. If we take $y := x$ and $y := -x$, then we obtain the result immediately. □

The main results are organized as follows: First, we discuss m-tuple versions. Then some special cases involving one or two matrices are presented.

2. m-Tuple Versions

The first result shows a general form of the first inequality in (1.1).

Theorem 2.1. *Let $A_i, B_i, X_i \in \mathbb{M}_n$ $(i = 1, 2, \dots, m)$ be such that $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$. If $f : [0,\infty) \to [0,\infty)$ is an increasing function, then, for each $j = 1, \dots, n$,*

$$s_j\left(f\left(\sum_{i=1}^m |X_i|\right)\right) \leq s_j\left(f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m U_i^* B_i U_i \end{bmatrix}\right)\right),$$

where $X_i = U_i|X_i|$ is a polar decomposition of X_i $(i = 1, 2, \dots, m)$.

Proof. Let $X_i = U_i|X_i|$ be a polar decomposition of X_i, $(i = 1, 2, \dots, m)$. Then

$$O \leq \begin{bmatrix} I & O \\ O & U_i^* \end{bmatrix} \begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \begin{bmatrix} I & O \\ O & U_i \end{bmatrix} = \begin{bmatrix} A_i & X_i^* U_i \\ U_i^* X_i & U_i^* B_i U_i \end{bmatrix} = \begin{bmatrix} A_i & |X_i| \\ |X_i| & U_i^* B_i U_i \end{bmatrix}.$$

Summing these inequalities over $i = 1, \dots, m$ gives $O \leq \begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m U_i^* B_i U_i \end{bmatrix}$. Hence, by Lemma 1.3, for $j = 1, \dots, n$,

$$s_j\left(\sum_{i=1}^m |X_i|\right) \leq s_j\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m U_i^* B_i U_i \end{bmatrix}\right).$$

Since f is increasing, Lemma 1.1 yields

$$\begin{aligned} s_j\left(f\left(\sum_{i=1}^m |X_i|\right)\right) = f\left(s_j\left(\sum_{i=1}^m |X_i|\right)\right) &\leq f\left(s_j\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m U_i^* B_i U_i \end{bmatrix}\right)\right) \\ &= s_j\left(f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m U_i^* B_i U_i \end{bmatrix}\right)\right), \end{aligned}$$

which proves the result. □

The singular-value inequalities in Theorem 2.1 imply the following unitarily invariant norm version.

Corollary 2.2. *Let $A_i, B_i, X_i \in \mathbb{M}_n$ $(i = 1, 2, \ldots, m)$ be such that $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$. If $f : [0,\infty) \to [0,\infty)$ is an increasing function, then*

$$\left|\left|\left| f\left(\sum_{i=1}^m |X_i|\right)\right|\right| \leq \left|\left|\left| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m U_i^* B_i U_i \end{bmatrix}\right)\right|\right|\right|,$$

where $X_i = U_i|X_i|$ is a polar decomposition of X_i $(i = 1, 2, \ldots, m)$.

Theorem 2.3. *Let $A_i, B_i, X_i \in \mathbb{M}_n$ $(i = 1, 2, \ldots, m)$ be such that $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$. If $f : [0,\infty) \to [0,\infty)$ is an increasing function, then, for $j = 1, \ldots, 2n$,*

$$s_j\left(f\left(\begin{bmatrix} \left|\sum_{i=1}^m X_i\right| & O \\ O & O \end{bmatrix}\right)\right) \leq s_j\left(f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right).$$

In particular,

$$\left|\left|\left| f\left(\left|\sum_{i=1}^m X_i\right|\right)\right|\right| \leq \left|\left|\left| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right|\right|\right|.$$

Proof. Since $\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix} \geq O$, Lemma 1.3 implies, for $j = 1, \ldots, 2n$,

$$s_j\left(\begin{bmatrix} \left|\sum_{i=1}^m X_i\right| & O \\ O & O \end{bmatrix}\right) \leq s_j\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right).$$

Since f is increasing, Lemma 1.1 yields

$$\begin{aligned} s_j\left(f\left(\begin{bmatrix} \left|\sum_{i=1}^m X_i\right| & O \\ O & O \end{bmatrix}\right)\right) &= f\left(s_j\left(\begin{bmatrix} \left|\sum_{i=1}^m X_i\right| & O \\ O & O \end{bmatrix}\right)\right) \leq f\left(s_j\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right) \\ &= s_j\left(f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right). \end{aligned}$$

Thus the asserted singular-value inequality follows.

For the norm inequality, identifying an $n\times n$ matrix with its direct sum with the zero matrix, we have

$$\left|\!\left|\!\left| f\left(\left|\sum_{i=1}^m X_i\right|\right)\right|\!\right|\!\right| = \left|\!\left|\!\left| \begin{bmatrix} f\left(\left|\sum_{i=1}^m X_i\right|\right) & O \\ O & O \end{bmatrix}\right|\!\right|\!\right| \le \left|\!\left|\!\left| f\left(\begin{bmatrix} \left|\sum_{i=1}^m X_i\right| & O \\ O & O \end{bmatrix}\right)\right|\!\right|\!\right|$$
$$\le \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right|\!\right|\!\right|,$$

which completes the proof. □

Noting that $s_1(\cdot) = \|\cdot\|$, we obtain the following two-sided inequality for the spectral norm. This result will be used to prove the next m-tuple version.

Proposition 2.4. *Let $A,B,X\in\mathbb{M}_n$ be such that $\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}\ge O$. If $f:[0,\infty)\to[0,\infty)$ is an increasing function, then*

$$\|f(|X|)\| \le \left\| f\left(\frac{1}{2}\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}\right)\right\| \le \left\| f\left(\frac{A+U^*BU}{2}\right)\right\|,$$

where $X = U|X|$ is a polar decomposition of X with U unitary.

Proof. Since $X = U|X|$ with U unitary, the positivity of $\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}$ implies that

$$O \le \begin{bmatrix} I & O \\ O & U^*\end{bmatrix}\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}\begin{bmatrix} I & O \\ O & U\end{bmatrix} = \begin{bmatrix} A & |X| \\ |X| & U^*BU\end{bmatrix}.$$

Applying Theorem 2.1 with $m=1$ gives

$$\|f(|X|)\| \le \left\| f\left(\frac{1}{2}\begin{bmatrix} A & |X| \\ |X| & U^*BU\end{bmatrix}\right)\right\|.$$

Since the matrix inside f is positive semidefinite and f is increasing,

$$\left\| f\left(\frac{1}{2}\begin{bmatrix} A & |X| \\ |X| & U^*BU\end{bmatrix}\right)\right\| = f\left(\left\|\frac{1}{2}\begin{bmatrix} A & |X| \\ |X| & U^*BU\end{bmatrix}\right\|\right) \le \left\| f\left(\frac{A+U^*BU}{2}\right)\right\|,$$

where Lemma 1.4 has been used in the inequality.

Finally, the matrices $\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}$ and $\begin{bmatrix} A & |X| \\ |X| & U^*BU\end{bmatrix}$ are unitarily similar. Hence

$$\left\| f\left(\frac{1}{2}\begin{bmatrix} A & |X| \\ |X| & U^*BU\end{bmatrix}\right)\right\| = \left\| f\left(\frac{1}{2}\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}\right)\right\|,$$

and the desired inequalities follow. □

Remark 2.5. If $\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}\ge O$, then $\begin{bmatrix} 2A & 2X^* \\ 2X & 2B\end{bmatrix}\ge O$. Applying Proposition 2.4 to this matrix gives

$$\|f(2|X|)\| \le \left\| f\left(\begin{bmatrix} A & X^* \\ X & B\end{bmatrix}\right)\right\| \le \|f(A+U^*BU)\|,$$

where $X = U|X|$ is a polar decomposition of X with U unitary. If, in addition, $f : [0,\infty) \to [0,\infty)$ is concave, then f is necessarily increasing. Hence, by Lemma 1.2,

$$\begin{aligned}\|f(2|X|)\| \leq \left\| f\left(\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right)\right\| &\leq \|f(A+U^*BU)\| \\ &\leq \|f(A)+U^*f(B)U\| \\ &\leq \|f(A)\|+\|f(B)\|.\end{aligned}$$

Thus, we obtain a refinement of the inequality [8, Corollary 2.2]

$$\left\| f\left(\begin{bmatrix} A & X^* \\ X & B \end{bmatrix}\right)\right\| \leq \|f(A)\|+\|f(B)\|.$$

An m-tuple version of Proposition 2.4 can be stated as follows.

Corollary 2.6. *Let $A_i, B_i, X_i \in \mathbb{M}_n$ $(i = 1,2,\ldots,m)$ be such that $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$. If $f : [0,\infty) \to [0,\infty)$ is an increasing concave function, then*

$$\left\| f\left(\left|\sum_{i=1}^m X_i\right|\right)\right\| \leq \left\| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right\| \leq \sum_{i=1}^m \left\| f\left(\frac{A_i+U_i^*B_iU_i}{2}\right)\right\|,$$

where $X_i = U_i|X_i|$ is a polar decomposition of X_i with U_i unitary $(i = 1,2,\ldots,m)$.

Proof. By Theorem 2.3,

$$\left\| f\left(\left|\sum_{i=1}^m X_i\right|\right)\right\| \leq \left\| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right\|.$$

Since

$$\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix} = \sum_{i=1}^m \frac{1}{2}\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix},$$

Lemma 1.2 gives

$$\left\| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m A_i & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m B_i \end{bmatrix}\right)\right\| \leq \sum_{i=1}^m \left\| f\left(\frac{1}{2}\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix}\right)\right\|.$$

Finally, Proposition 2.4, applied to each i, yields

$$\left\| f\left(\frac{1}{2}\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix}\right)\right\| \leq \left\| f\left(\frac{A_i+U_i^*B_iU_i}{2}\right)\right\|.$$

Combining these inequalities proves the result. □

The above arguments can be extended further by incorporating the geometric mean.

Proposition 2.7. *Let $A_i, B_i, X_i \in \mathbb{M}_n$ $(i = 1,2,\dots,m)$ be such that $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$. If $f : [0,\infty) \to [0,\infty)$ is an increasing function, then, for $j = 1,\dots,n$,*

$$s_j\left(f\left(\sum_{i=1}^m |X_i|\right)\right) \leq s_j\left(f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m (A_i \sharp U_i^* B_i U_i) & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m (A_i \sharp U_i^* B_i U_i) \end{bmatrix}\right)\right),$$

where $X_i = U_i|X_i|$ is a polar decomposition of X_i with U_i unitary $(i = 1,2,\dots,m)$.

Proof. Since $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$, unitary congruence gives $\begin{bmatrix} A_i & |X_i| \\ |X_i| & U_i^* B_i U_i \end{bmatrix} \geq O$. Interchanging the two diagonal blocks also yields

$$\begin{bmatrix} U_i^* B_i U_i & |X_i| \\ |X_i| & A_i \end{bmatrix} \geq O.$$

Therefore, Lemma 1.5, together with the symmetry of the geometric mean, gives

$$\begin{bmatrix} A_i \sharp U_i^* B_i U_i & |X_i| \\ |X_i| & A_i \sharp U_i^* B_i U_i \end{bmatrix} \geq O.$$

Summing over $i = 1,\dots,m$, we obtain

$$\begin{bmatrix} \sum_{i=1}^m (A_i \sharp U_i^* B_i U_i) & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m (A_i \sharp U_i^* B_i U_i) \end{bmatrix} \geq O.$$

Applying Lemma 1.3, followed by Lemma 1.1 and the monotonicity of f, exactly as in the proof of Theorem 2.1, yields the desired inequality. □

The corresponding unitarily invariant norm version of Proposition 2.7 is stated next.

Corollary 2.8. *Let $A_i, B_i, X_i \in \mathbb{M}_n$ $(i = 1,2,\dots,m)$ be such that $\begin{bmatrix} A_i & X_i^* \\ X_i & B_i \end{bmatrix} \geq O$. If $f : [0,\infty) \to [0,\infty)$ is an increasing function, then*

$$\left|\left|\left| f\left(\sum_{i=1}^m |X_i|\right)\right|\right| \leq \left|\left|\left| f\left(\frac{1}{2}\begin{bmatrix} \sum_{i=1}^m (A_i \sharp U_i^* B_i U_i) & \sum_{i=1}^m |X_i| \\ \sum_{i=1}^m |X_i| & \sum_{i=1}^m (A_i \sharp U_i^* B_i U_i) \end{bmatrix}\right)\right|\right|\right|,$$

where $X_i = U_i|X_i|$ is a polar decomposition of X_i with U_i unitary $(i = 1,2,\dots,m)$.

The preceding results were obtained under positivity assumptions on certain block matrices. We next derive a related inequality for arbitrary matrices X_i and invertible matrices A_i, thereby extending (1.1).

Theorem 2.9. *Let $A_i, X_i \in \mathbb{M}_n$ $(i = 1,2,\dots,m)$, and assume that each A_i is invertible. If $f : [0,\infty) \to [0,\infty)$ is a concave function, then*

$$\begin{aligned}\left|\!\left|\!\left| f\left(\left|\sum_{i=1}^m X_i\right|\right)\right|\!\right|\!\right| &\le \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m X_i|A_i|^{-1}X_i^*\end{bmatrix}\right)\right|\!\right|\!\right| \\ &\le \sum_{i=1}^m \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}|A_i| & |X_i| \\ |X_i| & |X_i||A_i|^{-1}|X_i|\end{bmatrix}\right)\right|\!\right|\!\right|.\end{aligned} \tag{2.1}$$

Proof. For each i, write $X_i = U_i|X_i|$ for a polar decomposition of X_i, where U_i is unitary. Since A_i is invertible, $|A_i|$ is positive definite, and

$$\begin{bmatrix} I & O \\ -X_i|A_i|^{-1} & I\end{bmatrix}\begin{bmatrix}|A_i| & X_i^* \\ X_i & X_i|A_i|^{-1}X_i^*\end{bmatrix}\begin{bmatrix} I & -|A_i|^{-1}X_i^* \\ O & I\end{bmatrix} = \begin{bmatrix}|A_i| & O \\ O & O\end{bmatrix} \ge O.$$

Hence, $\begin{bmatrix}|A_i| & X_i^* \\ X_i & X_i|A_i|^{-1}X_i^*\end{bmatrix} \ge O$, $i = 1,\dots,m$. Summing over i gives

$$\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m X_i|A_i|^{-1}X_i^*\end{bmatrix} \ge O.$$

Therefore, by Lemma 1.3, for $j = 1,\dots,n$,

$$s_j\left(\left|\sum_{i=1}^m X_i\right|\right) \le s_j\left(\frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m X_i|A_i|^{-1}X_i^*\end{bmatrix}\right).$$

Since a nonnegative concave function on $[0,\infty)$ is necessarily increasing, Lemma 1.1 yields

$$\begin{aligned}s_j\left(f\left(\left|\sum_{i=1}^m X_i\right|\right)\right) = f\left(s_j\left(\left|\sum_{i=1}^m X_i\right|\right)\right) &\le f\left(s_j\left(\frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m X_i|A_i|^{-1}X_i^*\end{bmatrix}\right)\right) \\ &= s_j\left(f\left(\frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m X_i|A_i|^{-1}X_i^*\end{bmatrix}\right)\right).\end{aligned}$$

The first inequality in (2.1) follows from the singular-value characterization of unitarily invariant norms.

For the second inequality, Lemma 1.2 gives

$$\left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m X_i^* \\ \sum_{i=1}^m X_i & \sum_{i=1}^m X_i|A_i|^{-1}X_i^*\end{bmatrix}\right)\right|\!\right|\!\right| \le \sum_{i=1}^m \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}|A_i| & X_i^* \\ X_i & X_i|A_i|^{-1}X_i^*\end{bmatrix}\right)\right|\!\right|\!\right|.$$

Since

$$\begin{bmatrix} I & O \\ O & U_i^*\end{bmatrix}\begin{bmatrix}|A_i| & X_i^* \\ X_i & X_i|A_i|^{-1}X_i^*\end{bmatrix}\begin{bmatrix} I & O \\ O & U_i\end{bmatrix} = \begin{bmatrix}|A_i| & |X_i| \\ |X_i| & |X_i||A_i|^{-1}|X_i|\end{bmatrix},$$

unitary invariance of the norm and functional calculus imply

$$\left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}|A_i| & X_i^* \\ X_i & X_i|A_i|^{-1}X_i^*\end{bmatrix}\right)\right|\!\right|\!\right| = \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}|A_i| & |X_i| \\ |X_i| & |X_i||A_i|^{-1}|X_i|\end{bmatrix}\right)\right|\!\right|\!\right|.$$

Combining these inequalities proves the second inequality in (2.1). □

At this stage, we are ready to present a new proof of (1.1).

Corollary 2.10. ([30, Theorems 4 and 5]) *Let $A_i \in \mathbb{M}_n$ $(i = 1,2,\dots,m)$, where $m \ge 2$. If $f : [0,\infty) \to [0,\infty)$ is a concave function, then*

$$\left|\!\left|\!\left| f\left(\left|\sum_{i=1}^m A_i\right|\right)\right|\!\right|\!\right| \le \left|\!\left|\!\left| f\left(\frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*|\end{bmatrix}\right)\right|\!\right|\!\right| \le \sum_{i=1}^m |\!|\!| f(|A_i|)|\!|\!|.$$

Proof. For each i, the polar decomposition $A_i = U_i|A_i|$ gives $\begin{bmatrix}|A_i| & A_i^* \\ A_i & |A_i^*|\end{bmatrix} \ge O$. Consequently, Theorem 2.3, applied with $X_i = A_i$, A_i replaced by $|A_i|$, B_i replaced by $|A_i^*|$, yields

$$\left|\!\left|\!\left| f\left(\left|\sum_{i=1}^m A_i\right|\right)\right|\!\right|\!\right| \le |\!|\!| f(H)|\!|\!|,$$

where

$$H = \frac{1}{2}\begin{bmatrix}\sum_{i=1}^m |A_i| & \sum_{i=1}^m A_i^* \\ \sum_{i=1}^m A_i & \sum_{i=1}^m |A_i^*|\end{bmatrix}.$$

This proves the first inequality. We next prove the second inequality. Set

$$H_i = \frac{1}{2}\begin{bmatrix}|A_i| & A_i^* \\ A_i & |A_i^*|\end{bmatrix}, \qquad i = 1,\dots,m,$$

so that $H = \sum_{i=1}^m H_i$. If $A_i = U_i|A_i|$ is a polar decomposition, with U_i chosen unitary, then

$$H_i = W_i\begin{bmatrix}|A_i| & O \\ O & O\end{bmatrix}W_i^*, \qquad W_i = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ U_i & U_i\end{bmatrix}.$$

Assume first that $f(0)=0$. By Lemma 1.2 and unitary invariance,

$$|||f(H)||| \le \sum_{i=1}^{m} |||f(H_i)||| = \sum_{i=1}^{m} \left|\left|\left| f\left(\begin{bmatrix} |A_i| & O \\ O & O \end{bmatrix}\right)\right|\right| = \sum_{i=1}^{m} \left|\left|\left| \begin{bmatrix} f(|A_i|) & O \\ O & O \end{bmatrix}\right|\right| = \sum_{i=1}^{m} |||f(|A_i|)|||.$$

It remains to consider the case $f(0)>0$. Put $c=f(0)$ and $g(t)=f(t)-c$. Since f is nonnegative and concave on $[0,\infty)$, it is increasing. Hence g is a nonnegative concave function satisfying $g(0)=0$. By the preceding case, for every Ky Fan norm and every $k=1,\dots,2n$,

$$\sum_{j=1}^{k} s_j(g(H)) \le \sum_{i=1}^{m}\sum_{j=1}^{r} s_j(g(|A_i|)), \qquad r=\min\{k,n\}.$$

Since $f(H)=g(H)+cI_{2n}$, we have $\sum_{j=1}^{k} s_j(f(H)) = \sum_{j=1}^{k} s_j(g(H)) + kc$. On the other hand,

$$\sum_{j=1}^{r} s_j(f(|A_i|)) = \sum_{j=1}^{r} s_j(g(|A_i|)) + rc.$$

Because $m\ge 2$ and $k\le 2n$, one has $k\le m\min\{k,n\}=mr$. Therefore,

$$\sum_{j=1}^{k} s_j(f(H)) \le \sum_{i=1}^{m}\sum_{j=1}^{r} s_j(g(|A_i|)) + kc \le \sum_{i=1}^{m}\left(\sum_{j=1}^{r} s_j(g(|A_i|)) + rc\right) = \sum_{i=1}^{m}\sum_{j=1}^{r} s_j(f(|A_i|)).$$

Let b_i denote the zero-padded singular-value vector of $f(|A_i|)$. The preceding inequalities show that $s(f(H)) \prec_w \sum_{i=1}^{m} b_i$. Hence, by the Ky Fan dominance theorem and the triangle inequality for the corresponding symmetric gauge function, one has $|||f(H)||| \le \sum_{i=1}^{m} |||f(|A_i|)|||$. This proves the second inequality and completes the proof. □

The following theorem provides a lower bound related to Theorem 2.9 and Corollary 2.10.

Theorem 2.11. *Let $A_i, X_i \in \mathbb{M}_n$ $(i=1,2,\dots,m)$, where each A_i is invertible and each X_i is accretive. If $f:[0,\infty)\to[0,\infty)$ is an increasing function, then*

$$\left|\left|\left| f\left(\sum_{i=1}^{m} \frac{|A_i| + X_i|A_i|^{-1}X_i^*}{2}\right)\right|\right| \ge \left|\left|\left| f\left(\sum_{i=1}^{m} \Re X_i\right)\right|\right|.$$

In particular, if A_i is accretive $(i=1,2,\dots,m)$, then

$$\left|\left|\left| f\left(\sum_{i=1}^{m} \frac{|A_i| + |A_i^*|}{2}\right)\right|\right| \ge \left|\left|\left| f\left(\sum_{i=1}^{m} \Re A_i\right)\right|\right|.$$

Proof. As in the proof of Theorem 2.9, we have

$$\begin{bmatrix} \sum_{i=1}^{m} |A_i| & \sum_{i=1}^{m} X_i^* \\ \sum_{i=1}^{m} X_i & \sum_{i=1}^{m} X_i|A_i|^{-1}X_i^* \end{bmatrix} \ge O.$$

Applying Lemma 1.9 gives $\sum_{i=1}^m \left(|A_i|+X_i|A_i|^{-1}X_i^*\right) \geq \sum_{i=1}^m (X_i+X_i^*) = 2\sum_{i=1}^m \Re X_i$. Since each X_i is accretive,

$$\sum_{i=1}^m \frac{|A_i|+X_i|A_i|^{-1}X_i^*}{2} \geq \sum_{i=1}^m \Re X_i \geq O.$$

Hence, by the monotonicity of eigenvalues for Hermitian matrices,

$$\lambda_j\left(\sum_{i=1}^m \frac{|A_i|+X_i|A_i|^{-1}X_i^*}{2}\right) \geq \lambda_j\left(\sum_{i=1}^m \Re X_i\right), \qquad j=1,\dots,n.$$

Since f is increasing and nonnegative, Lemma 1.1 yields

$$s_j\left(f\left(\sum_{i=1}^m \frac{|A_i|+X_i|A_i|^{-1}X_i^*}{2}\right)\right) \geq s_j\left(f\left(\sum_{i=1}^m \Re X_i\right)\right),$$

for $j=1,\dots,n$. The asserted unitarily invariant norm inequality then follows from the Ky Fan dominance principle. For the particular case, recall that, for every $A_i \in \mathbb{M}_n$, $\begin{bmatrix} |A_i| & A_i^* \\ A_i & |A_i^*| \end{bmatrix} \geq O$. Summing these positive block matrices and applying Lemma 1.9, we obtain

$$\sum_{i=1}^m (|A_i|+|A_i^*|) \geq \sum_{i=1}^m (A_i+A_i^*) = 2\sum_{i=1}^m \Re A_i.$$

Since each A_i is accretive, $\sum_{i=1}^m \frac{|A_i|+|A_i^*|}{2} \geq \sum_{i=1}^m \Re A_i \geq O$. Applying the same eigenvalue-monotonicity argument as above gives

$$\left|\!\left|\!\left| f\left(\sum_{i=1}^m \frac{|A_i|+|A_i^*|}{2}\right)\right|\!\right|\!\right| \geq \left|\!\left|\!\left| f\left(\sum_{i=1}^m \Re A_i\right)\right|\!\right|\!\right|,$$

which completes the proof. □

3. Some Consequences for One and Two Matrices

It was shown in [9, Ineq. (2.18)] that $|T_1+T_2| \leq \frac{1}{2}\left(|T_1|+|T_2|+U^*\left(|T_1^*|+|T_2^*|\right)U\right)$, where $T_1+T_2 = U|T_1+T_2|$ is a polar decomposition of T_1+T_2. We obtain the following refinement involving the spectral norm.

Theorem 3.1. *Let $T_1, T_2 \in \mathbb{M}_n$. Then*

$$\|T_1+T_2\| \leq \frac{1}{2}\left\|\begin{bmatrix} |T_1|+|T_2| & T_1^*+T_2^* \\ T_1+T_2 & |T_1^*|+|T_2^*| \end{bmatrix}\right\| \leq \frac{1}{2}\left\||T_1|+|T_2|+U^*\left(|T_1^*|+|T_2^*|\right)U\right\|, \tag{3.1}$$

where $T_1+T_2 = U|T_1+T_2|$ is a polar decomposition of T_1+T_2. In particular, for every $T \in \mathbb{M}_n$,

$$\|\Re T\| \leq \frac{1}{4}\left\|\begin{bmatrix} |T|+|T^*| & 2\Re T \\ 2\Re T & |T|+|T^*| \end{bmatrix}\right\| \leq \frac{1}{4}\left\||T|+|T^*|+U^*\left(|T|+|T^*|\right)U\right\|,$$

where $\Re T = U|\Re T|$ is a polar decomposition of $\Re T$.

Proof. For every $T \in \mathbb{M}_n$, one has $\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix} \geq O$. Hence

$$\begin{bmatrix} |T_1|+|T_2| & T_1^*+T_2^* \\ T_1+T_2 & |T_1^*|+|T_2^*| \end{bmatrix} \geq O.$$

Applying Proposition 2.4 with $f(t)=t$ gives (3.1). For the particular case, we set $T_1=T$ and $T_2=T^*$ in (3.1). Then $T_1+T_2=2\Re T$, $|T_1|+|T_2|=|T|+|T^*|$, and $|T_1^*|+|T_2^*|=|T^*|+|T|$. Dividing the resulting inequalities by 2 yields

$$\|\Re T\| \le \frac{1}{4}\left\|\begin{bmatrix} |T|+|T^*| & 2\Re T \\ 2\Re T & |T|+|T^*| \end{bmatrix}\right\| \le \frac{1}{4}\left\||T|+|T^*|+U^*\big(|T|+|T^*|\big)U\right\|,$$

where U may be chosen as a unitary polar factor of $\Re T$. □

We obtain another consequence of (3.1) by taking $T_1=\Re T$ and $T_2=i\Im T$.

Corollary 3.2. *Let $T\in\mathbb{M}_n$. Then*

$$\|T\| \le \frac{1}{2}\left\|\begin{bmatrix} |\Re T|+|\Im T| & T^* \\ T & |\Re T|+|\Im T| \end{bmatrix}\right\| \le \frac{1}{2}\left\||\Re T|+|\Im T|+U^*\big(|\Re T|+|\Im T|\big)U\right\|,$$

where $T=U|T|$ is a polar decomposition of T. In particular, if T is accretive-dissipative, then

$$\|T\| \le \frac{1}{2}\left\|\begin{bmatrix} \Re T+\Im T & T^* \\ T & \Re T+\Im T \end{bmatrix}\right\| \le \frac{1}{2}\|\Re T+\Im T+U^*(\Re T+\Im T)U\|.$$

Further, Corollary 3.2 yields the following numerical radius bound.

Corollary 3.3. *Let $T\in\mathbb{M}_n$ be accretive-dissipative. Then $\|T\|\le\frac{\sqrt{2}}{2}\,\omega\left(T+U^*TU\right)$, where $T=U|T|$ is a polar decomposition of T, with U chosen unitary.*

Proof. By the second inequality in Corollary 3.2, $\|T\|\le\frac{1}{2}\|\Re T+U^*(\Re T)U+\Im T+U^*(\Im T)U\|$. Since T is accretive-dissipative, one has $\Re T+U^*(\Re T)U\ge O$ and $\Im T+U^*(\Im T)U\ge O$. Hence their sum is positive semidefinite, and therefore its norm equals its numerical radius. Thus, by Lemma 1.6,

$$\begin{aligned}\|T\| &\le \frac{1}{2}\omega\left(\Re T+U^*(\Re T)U+\Im T+U^*(\Im T)U\right)\\ &\le \frac{\sqrt{2}}{2}\,\omega\left(\Re T+U^*(\Re T)U+\mathrm{i}\big(\Im T+U^*(\Im T)U\big)\right)\\ &= \frac{\sqrt{2}}{2}\,\omega\left(T+U^*TU\right),\end{aligned}$$

which proves the result. □

Remark 3.4. Corollary 3.3 refines the estimate [23, Theorem 2.6] $\|T\|\le\sqrt{2}\,\omega(T)$, valid for accretive-dissipative matrices $T\in\mathbb{M}_n$. Indeed, by the triangle inequality for the numerical radius and its unitary invariance,

$$\frac{\sqrt{2}}{2}\,\omega(T+U^*TU)\le\frac{\sqrt{2}}{2}\left(\omega(T)+\omega(U^*TU)\right)=\sqrt{2}\,\omega(T).$$

We next obtain the following consequence of Corollary 2.6. Recall that every concave function $f:[0,\infty)\to[0,\infty)$ is necessarily increasing.

Corollary 3.5. *Let $T_1,T_2\in\mathbb{M}_n$. If $f:[0,\infty)\to[0,\infty)$ is a concave function, then*

$$\|f\left(|T_1+T_2|\right)\|\le\left\|f\left(\frac{1}{2}\begin{bmatrix} |T_1|+|T_2| & T_1^*+T_2^* \\ T_1+T_2 & |T_1^*|+|T_2^*| \end{bmatrix}\right)\right\|\le\|f(|T_1|)\|+\|f(|T_2|)\|.$$

Proof. For $i=1,2$, we have $\begin{bmatrix} |T_i| & T_i^* \\ T_i & |T_i^*| \end{bmatrix} \geq O$. Therefore, Corollary 2.6 gives

$$\|f(|T_1+T_2|)\| \leq \left\| f\left(\frac{1}{2}\begin{bmatrix} |T_1|+|T_2| & T_1^*+T_2^* \\ T_1+T_2 & |T_1^*|+|T_2^*| \end{bmatrix}\right)\right\| \leq \sum_{i=1}^{2}\left\| f\left(\frac{|T_i|+U_i^*|T_i^*|U_i}{2}\right)\right\|,$$

where $T_i = U_i|T_i|$ is a polar decomposition of T_i, with U_i chosen unitary. Since $U_i^*|T_i^*|U_i = |T_i|$ (see [14, Theorem 4, p. 58]), we obtain

$$\left\| f\left(\frac{|T_i|+U_i^*|T_i^*|U_i}{2}\right)\right\| = \|f(|T_i|)\|,$$

for $i=1,2$. Consequently,

$$\|f(|T_1+T_2|)\| \leq \left\| f\left(\frac{1}{2}\begin{bmatrix} |T_1|+|T_2| & T_1^*+T_2^* \\ T_1+T_2 & |T_1^*|+|T_2^*| \end{bmatrix}\right)\right\| \leq \|f(|T_1|)\|+\|f(|T_2|)\|.$$

This completes the proof. □

Theorem 3.6. *Let $S,T \in \mathbb{M}_n$. If $f:[0,\infty)\to[0,\infty)$ is a concave function with $f(0)=0$, then*

$$\begin{aligned} |||f(|S+T|)||| &\leq \left|\left| f\left(\frac{1}{2}\begin{bmatrix} |S|+|T| & S^*+T^* \\ S+T & |S^*|+|T^*| \end{bmatrix}\right)\right|\right| \\ &\leq \left|\left| f\left(\frac{1}{2}\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix}\right) + f\left(\frac{1}{2}\begin{bmatrix} |S| & S^* \\ S & |S^*| \end{bmatrix}\right)\right|\right| \\ &\leq |||\big(f(|S|)+f(|T|)\big)\oplus\big(f(|S^*|)+f(|T^*|)\big)|||. \end{aligned}$$

Proof. Since $f:[0,\infty)\to[0,\infty)$ is concave, it is necessarily increasing. Therefore, Theorem 2.3 gives

$$|||f(|S+T|)||| \leq \left|\left| f\left(\frac{1}{2}\begin{bmatrix} |S|+|T| & S^*+T^* \\ S+T & |S^*|+|T^*| \end{bmatrix}\right)\right|\right|.$$

Setting $P_T = \frac{1}{2}\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix}$ and $P_S = \frac{1}{2}\begin{bmatrix} |S| & S^* \\ S & |S^*| \end{bmatrix}$, we have

$$P_T+P_S = \frac{1}{2}\begin{bmatrix} |S|+|T| & S^*+T^* \\ S+T & |S^*|+|T^*| \end{bmatrix}.$$

It follows from Lemma 1.2 that $|||f(P_T+P_S)||| \leq |||f(P_T)+f(P_S)|||$. Next, we define

$$Q_T = \frac{1}{2}\begin{bmatrix} |T| & -T^* \\ -T & |T^*| \end{bmatrix}, \qquad Q_S = \frac{1}{2}\begin{bmatrix} |S| & -S^* \\ -S & |S^*| \end{bmatrix}.$$

The four matrices P_T, Q_T, P_S, Q_S are positive semidefinite. Consequently, since f is nonnegative on $[0,\infty)$, $f(P_T),\ f(Q_T),\ f(P_S),\ f(Q_S) \geq O$. It follows that $f(P_T)+f(P_S) \leq f(P_T)+f(Q_T)+f(P_S)+f(Q_S)$, and therefore, by monotonicity of unitarily invariant norms on positive semidefinite matrices, $|||f(P_T)+f(P_S)||| \leq |||f(P_T)+f(Q_T)+f(P_S)+f(Q_S)|||$. By Lemma 1.7, using $f(0)=0$, we have

$$f(P_T)+f(Q_T) = f\left(\begin{bmatrix} |T| & O \\ O & |T^*| \end{bmatrix}\right) = \begin{bmatrix} f(|T|) & O \\ O & f(|T^*|) \end{bmatrix},$$

and similarly,

$$f(P_S)+f(Q_S)=\begin{bmatrix} f(|S|) & O \\ O & f(|S^*|)\end{bmatrix}.$$

Thus

$$\begin{aligned}|||f(P_T)+f(Q_T)+f(P_S)+f(Q_S)|||&=\left|\left|\left|\begin{bmatrix} f(|S|)+f(|T|) & O \\ O & f(|S^*|)+f(|T^*|)\end{bmatrix}\right|\right|\right| \\ &=|||\big(f(|S|)+f(|T|)\big)\oplus\big(f(|S^*|)+f(|T^*|)\big)|||.\end{aligned}$$

Combining the preceding inequalities proves the result. □

Theorem 3.7. *Let $A_i\in\mathbb{M}_n$ $(i=1,\dots,m)$. If $f:[0,\infty)\to[0,\infty)$ is an increasing function, then*

$$\left|\left|\left|f\left(\left|\sum_{i=1}^m A_i\right|\right)\right|\right|\right|\le\left|\left|\left|f\left(\left(m\sum_{i=1}^m|A_i|^2\right)^{1/2}\right)\right|\right|\right|.$$

Proof. For each i, we have $\begin{bmatrix} I & A_i \\ A_i^* & |A_i|^2\end{bmatrix}=\begin{bmatrix} I \\ A_i^*\end{bmatrix}\begin{bmatrix} I & A_i\end{bmatrix}\ge O$. Summing these positive semidefinite matrices gives $\begin{bmatrix} mI & \sum_{i=1}^m A_i \\ \sum_{i=1}^m A_i^* & \sum_{i=1}^m|A_i|^2\end{bmatrix}\ge O$. Since $mI>O$, the Schur complement implies

$$\sum_{i=1}^m|A_i|^2\ge\frac{1}{m}\left(\sum_{i=1}^m A_i^*\right)\left(\sum_{i=1}^m A_i\right)=\frac{1}{m}\left|\sum_{i=1}^m A_i\right|^2. \tag{3.2}$$

Equivalently, $\left|\sum_{i=1}^m A_i\right|^2\le m\sum_{i=1}^m|A_i|^2$. By the Löwner–Heinz inequality,

$$\left|\sum_{i=1}^m A_i\right|\le\left(m\sum_{i=1}^m|A_i|^2\right)^{1/2}.$$

Hence, by the monotonicity of the eigenvalues of positive semidefinite matrices,

$$s_j\left(\left|\sum_{i=1}^m A_i\right|\right)\le s_j\left(\left(m\sum_{i=1}^m|A_i|^2\right)^{1/2}\right),\qquad j=1,\dots,n.$$

Since f is increasing, Lemma 1.1 yields

$$\begin{aligned}s_j\left(f\left(\left|\sum_{i=1}^m A_i\right|\right)\right)=f\left(s_j\left(\left|\sum_{i=1}^m A_i\right|\right)\right)&\le f\left(s_j\left(\left(m\sum_{i=1}^m|A_i|^2\right)^{1/2}\right)\right) \\ &=s_j\left(f\left(\left(m\sum_{i=1}^m|A_i|^2\right)^{1/2}\right)\right).\end{aligned}$$

The desired unitarily invariant norm inequality now follows from the Ky Fan dominance principle. □

Remark 3.8. For $m=2$, taking $A_1=A$ and $A_2=B$ in (3.2) gives $|A|^2+|B|^2 \geq \frac{1}{2}|A+B|^2$, or, equivalently,

$$\frac{|A|^2+|B|^2}{2} \geq \left|\frac{A+B}{2}\right|^2. \tag{3.3}$$

Applying the Löwner–Heinz inequality to (3.3), we obtain $\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2} \geq \frac{|A+B|}{2}$. On the other hand, since the function $t \mapsto t^{1/2}$ is operator concave on $[0,\infty)$, $\left(\frac{|A|^2+|B|^2}{2}\right)^{1/2} \geq \frac{|A|+|B|}{2}$. These two inequalities are generally independent, since $|A+B| \leq |A|+|B|$ does not hold for arbitrary matrices A and B. It is also useful to compare Theorem 3.7 with Lemma 1.2. Suppose that $A,B \geq O$ and that $f:[0,\infty) \to [0,\infty)$ is concave. Since such a function is necessarily increasing, Theorem 3.7 gives $|||f(A+B)||| \leq \left|\left|\left| f\left((2A^2+2B^2)^{1/2}\right)\right|\right|\right|$. On the other hand, Lemma 1.2 gives $|||f(A+B)||| \leq |||f(A)+f(B)|||$. Neither of these two upper bounds dominates the other in general. For example, in the scalar case with $A=B=1$ and $f(t)=t^{1/2}$,

$$f(A+B)=\sqrt{2}, \qquad f\left((2A^2+2B^2)^{1/2}\right)=\sqrt{2}, \qquad f(A)+f(B)=2.$$

Thus, in this case, the bound furnished by Theorem 3.7 is better. In contrast, if $A=0$, $B=2$, and $f(t)=t^{1/2}$, then $f(A+B)=\sqrt{2}$ and $f\left((2A^2+2B^2)^{1/2}\right)=\sqrt{2\sqrt{2}}>\sqrt{2}=f(A)+f(B)$. Hence, in this case, the bound from Lemma 1.2 is better. Therefore, the two estimates provide genuinely different upper bounds.

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