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A GENERAL SHRINKING PROJECTION ALGORITHM ON HADAMARD MANIFOLDS

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Abstract. In this paper, we introduce a general shrinking projection algorithm for solving a finite family of generalized equilibrium problem, a finite family of quasi-variational inclusion problems and a fixed point problem of quasi-pseudo-contractive mappings on Hadamard manifolds. Under suitable conditions, we prove that the sequence generated by our proposed algorithm converges to a common solution of the three problems. Our results extend some recent results in literature.

Keywords. Equilibrium problem; Hadamard manifolds; Monotone vector fields; Quasi-variational inclusion problem.
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1. INTRODUCTION

Variational inequality problems, equilibrium problems, and inclusion problems were extensively studied numerically due to their wide applications in economics, transportation, imaging and so on; see, e.g., [2, 9, 14, 25, 26, 29] and the references therein. Fixed-point (resolvent) methods are efficient to study these problems. Numerous convergence theorems of solutions were obtained in finite and infinite dimensional spaces; see, e.g, [4, 11, 12, 13, 19, 20, 23] and the references therein. Many issues in nonlinear analysis such as variational inequality problems, equilibrium problems, and inclusion problems were extended from linear spaces to manifolds because these problems and their application cannot be posted in linear spaces and require a manifold structure (not necessary with linear structure). Let $\mathbb{M}$ be a Hadamard manifold, $T\mathbb{M}$ the tangent tangent bundle of $\mathbb{M}$, K a closed and geodesic convex subset of $\mathbb{M}$, and F a bifunction satisfies $F(x,x)=0$, for all $x\in K$.Then the equilibrium problem on the Hadamard manifold is to find $x^*\in K$ such that $F(x,y)\geq 0$ for all y in set K. We denote $EP(F)$ by the solution set of the equilibrium problem. Let $\phi : K\to T\mathbb{M}$ be a single-valued vector field. The

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Generalized Equilibrium problem (in short, GEP) considered in this paper is to find $x \in K$ such that $F(x,y) + \langle \phi x, exp_x^{-1} y\rangle \geq 0$ for all y in K, where exp^{-1} is the inverse of the exponential function $exp : T\mathbb{M} \to \mathbb{M}$. We denote by $GEP(F,\phi)$ the solution set of the GEP.

In [22], Oyewole et al. studied the existence of solutions of the GEP on Hadamard manifold by using the KKM lemma. They introduced a viscosity iterative method and established the strong convergence analysis of their iterative method. In [15], Li et al. studied the following inclusion problem on Hadamard manifolds: Find a point x in K such that $0 \in B(x)$, where $B : \mathbb{M} \longrightarrow 2^{T\mathbb{M}}$ is a multivalued vector field on Hadamard manifold and 0 denotes the zero tangent vector of $T\mathbb{M}$. We denote $B^{-1}(0)$ by the set of singularities of the inclusion problem. In [1], Ansari et al. studied the following quasi-variational inclusion problem with two monotone vector fields on Hadamard manifolds: Find a point x in K such that $0 \in A(x) + B(x)$, where A is a continuous and monotone vector field and B is a maximal monotone vector field on a closed geodesic convex subset of Hadamard manifolds. We denote $(A+B)^{-1}(0)$ by the set of singularities of the quasi-variational inclusion problem. In [6], Chang et al. introduced a general shrinking projection algorithm for solving a finite family of quasi-variational inclusion problems. They obtained a convergence theorem of solutions on Hadamard manifolds. In [17], Liu et al. introduced an iterative algorithm for solving the following common solution problem of equilibrium, quasi-variational inclusion and the fixed point problems on Hadamard manifolds. Find $x^* \in K$ such that $x^* \in EP(F) \bigcap (A+B)^{-1}(0) \bigcap F(S)$, where S is a nonexpansive operator with fixed points. They obtained a solution theorem under some mild conditions.

Motivated and inspired by the results presented above, we consider the following common solution problem of a finite family of generalized equilibrium problems, a finite family of quasi-variational inclusion problems and the fixed point problems on Hadamard manifolds: Fin $x^* \in K$ such that

$$x^* \in \bigcap_{j=1}^{N} GEP(F_j, \phi_j) \bigcap_{i=1}^{N} (A_i + B_i)^{-1}(0) \bigcap F(S).$$

where S is a quasi-pseudo-contractive operator.

In this paper, a general shrinking projection algorithm for finding a commmon solution of the problem above is introduced. Under suitable conditions, strong convergence is proved. Our results extend some recent results in [1, 6, 17] and some other results in literature.

2. Preliminaries

In this section, we recall some fundamental definitions, properties, useful results, and notations of Riemannian geometry. We refer to [8, 24, 27] for more details. Let $\mathbb{M}$ be a connected finite-dimensional manifold. For $p \in \mathbb{M}$, we denote $T_p\mathbb{M}$ the tangent space of $\mathbb{M}$ at p which is a vector space of the same dimension as $\mathbb{M}$, and by $T\mathbb{M} = \bigcup_{p\in\mathbb{M}} T_p\mathbb{M}$ the tangent bundle of $\mathbb{M}$. We always suppose that $\mathbb{M}$ can be endowed with a Riemannian metric $\langle\cdot,\cdot\rangle_p$ with corresponding norm denoted $by \|\cdot\|_p$, to become a Riemannian manifold. If there is no confusion casued, we denote $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_p$, $\|\cdot\| := \|\cdot\|_p$. Let $\gamma : [0,1] \longrightarrow \mathbb{M}$ be a piecewise smooth curve joining $\gamma(0) = p$ to $\gamma(1) = q$. We define the length of the curve γ by using the metric as $L(\gamma) = \int_0^1 \|\gamma'(t)\| dt$, minimizing the length function over the set of all such curves, we obtain a Riemannian distance $d(p,q)$ which induces the original topology on $\mathbb{M}$.

Let ∇ be a Levi-Civita connection associated to $(\mathbb{M}, \langle\cdot,\cdot\rangle)$. Given γ a smooth curve, a smooth vector field X along γ is said to be parallel if $\nabla_{\gamma'} X = 0$. If γ' itself is parallel, we say that γ is a

geodesic, and in this case, $\|\gamma'\|$ is a constant. When $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining p to q in M is said to be a minimal geodesic if its length equals to $d(p,q)$. A Riemannian manifold is complete if, for any $p \in \mathbb{M}$, all geodesic emanating from p are defined for all $t \in \mathbb{R}$. From the Hopf-Rinow theorem, we see that if $\mathbb{M}$ is complete, then any pair of points in $\mathbb{M}$ can be joined by a minimal geodesic. Moreover, $(\mathbb{M}, d)$ is a complete metric space and every bounded closed subsets are compact. Let $\mathbb{M}$ be a complete Riemannian manifold and $p \in \mathbb{M}$. The exponential map $exp_p : T_p\mathbb{M} \longrightarrow \mathbb{M}$ is defined as $exp_p v = \gamma_v(1,p)$ for all $v \in T_p\mathbb{M}$, where $\gamma(\cdot) = \gamma_v(\cdot,p)$ is the geodesic starting at p with velocity v (i.e., $\gamma_v(0,p) = p$ and $\gamma'_v(0,p) = v$). Then, for any value of t, we have $exp_p tv = \gamma_v(t,p)$ and $exp_p 0 = \gamma_v(0,p) = p$. Moreover, for any $p,q \in \mathbb{M}$, we have $d(p,q) = \|exp_p^{-1} q\| = \|exp_q^{-1} p\|$. A complete simply connected Riemannian manifold of non-positive sectional curvature is said to be a Hadamard manifold. Throughout the remainder of the paper, we always assume that $\mathbb{M}$ is a finite-dimensional Hadamard manifold.

Let $\mathbb{M}$ be a Hadamard manifold. Then [24], for any two points $x, y \in \mathbb{M}$, there exists a unique normalized geodesic $\gamma : [0,1] \to \mathbb{M}$ joining $x = \gamma(0)$ to $y = \gamma(1)$ which is in fact a minimal geodesic denoted by $\gamma(t) = exp_x t exp_x^{-1} y$ for all $t \in [0,1]$. For any sequence $\{x_n\} \subset \mathbb{M}$ such that $x_n \to x_0 \in \mathbb{M}$, $exp_{x_n}^{-1} y \to exp_{x_0}^{-1} y$ and $exp_{y_n}^{-1} x_n \to exp_y^{-1} x_0$ for all $y \in \mathbb{M}$.

Let $d : \mathbb{M} \times \mathbb{M} \longrightarrow \mathbb{R}$ be the distance function. Then [24], $d(\cdot,\cdot)$ is a geodesic convex function with respect to the product Riemannian metric, that is, for any pair of geodesics $\gamma_1 : [0,1] \longrightarrow \mathbb{M}$ and $\gamma_2 : [0,1] \longrightarrow \mathbb{M}$ the following inequality holds for all $t \in [0,1]$

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t) d(\gamma_1(0), \gamma_2(0)) + t d(\gamma_1(1), \gamma_2(1)).$$

In particular, for each $y \in \mathbb{M}$, the function $d(\cdot, y) : \mathbb{M} \longrightarrow \mathbb{R}$ is a geodesic convex function.

Let $\mathbb{M}$ be a Hadamard manifold and $x \in \mathbb{M}$. Then [10]

(1) $\Phi_x = d^2(x,y)$ is convex. Indeed, for any geodesic $\gamma : [0,1] \to \mathbb{M}$, the following inequality holds for all $t \in [0,1]$:

$$d^2(x, \gamma(t)) \leq (1-t) d^2(x, \gamma(0)) + t d^2(x, \gamma(1)) - t(t-1) d^2(\gamma(0), \gamma(1)).$$

(2) $\Phi_x = d^2(x,y)$ is smooth. Moreover, $\partial \Phi_x(y) = -2 exp_y^{-1} x$.

Let $S : K \to K$ be a mapping. Then the following statements are equivalent [16]

(1) S is firmly nonexpansive;

(2) For any $x, y \in K and t \in [0,1]$ $d(S(x), S(y)) \leq d(exp_x t exp_x^{-1} Sx, exp_y t exp_y^{-1} Sy)$;

(3) For any $x, y \in K$ $\langle exp_{S(x)}^{-1} S(y), exp_{S(x)}^{-1} x \rangle + \langle exp_{S(y)}^{-1} S(x), exp_{S(y)}^{-1} y \rangle \leq 0$.

Let $\mathbb{M}$ be a finite dimensional Hadamard manifold. (1) Let $\gamma : [0,1] \to \mathbb{M}$ be a geodesic joining x to y. Then $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2| d(x,y)$, $\forall t_1, t_2 \in [0,1]$; (From now on $d(x,y)$ denotes the Riemannian distance). (2) For any $x, y, z, u, \omega \in \mathbb{M}$ and $t \in [0,1]$, the following inequalities hold:

$$d(exp_x t exp_x^{-1} y, z) \leq (1-t) d(x,z) + t d(y,z); \tag{2.1}$$

$$d^2(exp_x t exp_x^{-1} y, z) \leq (1-t) d^2(x,z) + t d^2(y,z) - t(1-t) d^2(x,y);$$

and

$$d(exp_x t exp_x^{-1} y, exp_u^{-1}(t exp_u^{-1} \omega) \leq (1-t) d(x,u) + t d(y,\omega).$$

A subset K is said to be geodesic convex if, for every two points p and q in K, the geodesic joining p to q is contained in K, that is, if $\gamma : [a,b] \longrightarrow \mathbb{M}$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in K$ for all $t \in [0,1]$. A real function $f : \mathbb{M} \longrightarrow \mathbb{R}$ is said to

be geodesic convex if, for any geodesic γ in $\mathbb{M}$, the composition function $f\circ\gamma:[a,b]\longrightarrow\mathbb{R}$ is convex, that is,

$$(f\circ\gamma)(ta+(1-t)b)\leq t(f\circ\gamma)(a)+(1-t)(f\circ\gamma)(b)$$

where $a,b\in\mathbb{R}$, and $0\leq t\leq 1$.

It follows from (2.1) that the Riemannian distance function $y\mapsto d(x,y)$ is geodesic convex.

Let K be a closed subset of $\mathbb{M}$ and $\{x_n\}$ be a sequence in $\mathbb{M}$. Recall that $\{x_n\}$ is said to be Fejér convergent with respect to K if, for all $p\in K$ and $n\in\mathbb{N}$, $d(x_{n+1},p)\leq d(x_n,p)$.

Let $f:K\to\mathbb{R}$ be geodesic convex. Let $p\in K$. A vector $r\in T_P\mathbb{M}$ is said to be a subgradient of f at p if and only if $f(q)\geq f(p)+\langle r,exp_p^{-1}q\rangle$ for all q in K.

Lemma 2.1. [10] *Let K be nonempty, convex, and closed subset of $\mathbb{M}$ and $\{x_n\}\subset\mathbb{M}$ be a sequence such that $\{x_n\}$ is a fejér convergent with respect to K. Then*

(1) for every $p\in K,d(x_n,p)$ converges,

(2) $\{x_n\}$ is bounded,

(3) assume that every cluster point of $\{x_n\}$ belongs to K, Then $\{x_n\}$ converges to a point in K.

Recall that a mapping $S:K\to K$ is said to be

(1) contractive if there exists a constant $k\in(0,1)$ such that $d(Sx,Sy)\leq kd(x,y)$ for all $x,y\in K$. If$k=1$, then S is said to be nonexpansive;

(2) quasinonexpansive if $Fix(S)\neq\emptyset$ and $d(Sx,p)\leq d(x,p)$ for all $p\in Fix(S)$, $x\in K$;

(3) firmly nonexpansive if, for all $x,y\in K$, the function $\psi:[0,1]\to[0,\infty]$ defined by

$$\psi(t):=d(exp_xtexp_x^{-1}Sx,exp_ytexp_y^{-1}Sy),\ \ \forall t\in[0,1]$$

is nonincreasing.

(4) k-demicontracive [16] if $Fix(S)\neq\emptyset$ and there exists a constant $k\in[0,1]$ such that

$$d^2(Sx,p)\leq d^2(x,p)+kd^2(x,Sx),\ \ \forall x\in K,\forall p\in Fix(S);$$

(5) quasi-pseudo-contractive if $Fix(S)\neq\emptyset$ and

$$d^2(Sx,p)\leq d^2(x,p)+d^2(x,Sx),\ \ \forall x\in K,p\in Fix(S).$$

Lemma 2.2. [5] *If $S:K\to K$ is a firmly nonexpansive mapping with $Fix(S)\neq\emptyset$, then, for any $x\in K$ and $p\in Fix(S)$, $d^2(Sx,p)\leq d^2(x,p)-d^2(Sx,x)$.*

Remark 2.3. If $Fix(S)\neq\emptyset$, then the following implications hold:

$$\begin{aligned}S\text{ is firmly nonexpansive }&\Longrightarrow S\text{ is nonexpansive }\Longrightarrow S\text{ is quasinonexpansive}\\&\Longrightarrow S\text{ is demicontractive }\Longrightarrow S\text{ is quasi-pseudo-contractive,}\end{aligned}$$

but thes converses are not true.

Let $\mathbb{M}$ be Hadamard manifold, and let $S:K\to K$ be a L-Lipschitzian mapping with $L\geq 1,(i.e.,d(Sx,Sy)\leq Ld(x,y),\forall x,y\in K)$. Let $G:K\to K$ and $T:K\to K$ be two mappings defined by

$$T(x):=exp_x\zeta exp_x^{-1}S(Gx);\ \ G(x):=exp_x\eta exp_x^{-1}Sx,\ \ x\in K.$$

If $0<\zeta<\eta<\frac{1}{1+\sqrt{1+L^2}}$, then (1) $Fix(S)=Fix(T\circ G)=Fix(T)$; (2) If S is demiclosed at 0, then T is also demiclosed at 0. A mapping $S:K\to K$ is said to be demiclosed at 0 if, for any sequence $\{x_n\}\subset K$ with $x_n\to x^*\in K$ and $d(x_n,Sx_n)\to 0$, $x^*\in Fix(S))$. (3) $T:K\to K$ is

L^2-Lipschitzian; (4) In addition, if $S : K \to K$ is quasi-pseudo-contractive, then $T : K \to K$ is a quasi-pseudo-contractive, i.e., for any $x \in K$ and $p \in Fix(T)(= Fix(S))$

$$d^2(Tx, p) \leq d^2(x, p) - \zeta\eta(1 - 2\eta - L^2\eta^2)d^2(x, Sx) \leq d^2(x, p).$$

Let $S : C \to \mathbb{M}$ be a nonexpansive mapping defined on a closed convex set $C \subseteq \mathbb{M}$. Then the fixed point set $Fix(S)$ is closed and convex [18]. Let $\Omega(\mathbb{M})$ denote the set of all single-valued vector fields $A : \mathbb{M} \longrightarrow T\mathbb{M}$ such that $A(x) \in T_x\mathbb{M}$ for each $x \in \mathbb{M}$, and let the domain $D(A)$ of A be defined by $D(A) = \{x \in \mathbb{M} : A(x) \in T_x\mathbb{M}\}$. Let $\mathscr{X}(\mathbb{M})$ denote the set of all set-valued vector fields $B : \mathbb{M} \longrightarrow 2^{T\mathbb{M}}$ such that $B(x) \subseteq T_x\mathbb{M}$ for all $x \in \mathbb{M}$, and let the domain $D(B)$ of B be defined by $D(B) = \{x \in \mathbb{M} : B(x) \neq \emptyset\}$. Recall that a single-valued vector field $A \in \Omega(\mathbb{M})$ is said to be monotone [21] if $\langle A(x), exp_x^{-1}y\rangle \leq \langle A(y), -exp_y^{-1}x\rangle$ for all $x, y \in \mathbb{M}$. Recall that [7] a set-valued vector field $B \in \mathscr{X}(\mathbb{M})$ is said to be

(1) monotone if, for any $x, y \in D(B)$, $\langle u, exp_x^{-1}y\rangle \leq \langle v, -exp_y^{-1}x\rangle$, for all $u \in B(x)$ and $v \in B(y)$;

(2) maximal monotone if it is monotone and, for all $x \in D(B)$ and $u \in T_x\mathbb{M}$,

$$\langle u, exp_x^{-1}y\rangle \leq \langle v, -exp_y^{-1}x\rangle, \ \ \forall y \in D(B) \ \ and \ \ \forall v \in B(y),$$

implies $u \in B(x)$.

(3) Let $B \in \mathscr{X}(\mathbb{M})$ be a set-valued vector field and $\lambda \in (0, \infty)$. The $\lambda - resolvent$ of B is a set-valued map $J_\lambda^B : \mathbb{M} \longrightarrow 2^{T\mathbb{M}}$ defined by

$$J_\lambda^B(x) := \{z \in M : x \in exp_z\lambda B(z)\}, \ \ \forall x \in \mathbb{M}.$$

Let K be a nonempty closed geodesic convex subset of a Hadamard manifold $\mathbb{M}$. The metric projection onto K, denoted by P_K, is defined by

$$P_K(x) = \{p \in K : d(x, p) \leq d(x, q), \forall q \in K\}, \ \ \forall x \in \mathbb{M}.$$

Let K be a closed geodesic convex subset of a Hadamard $\mathbb{M}$. Then [28] (1) For any $x \in \mathbb{M}$, $P_K(x)$ is a singleton set, and the following inequality holds

$$\langle exp_{P_K(x)}^{-1}x, exp_{P_K(x)}^{-1}q\rangle \leq 0, \ \ \forall q \in K.$$

(2) $P_K : \mathbb{M} \to K$ is a firmly nonexpansive mapping. Thus

$$d^2(P_Kx, p) \leq d^2(x, p) - d^2(P_Kx, x), \ \forall\, x \in \mathbb{M} \text{ and } p \in K.$$

Lemma 2.4. [3] *Let K be a closed subset of $\mathbb{M}$ and a vector field $B \in \mathscr{X}(\mathbb{M})$ be a maximal monotone. Let $\{\lambda_n\} \subset (0, \infty)$ be a real sequence with $\lim_{n\to\infty}\lambda_n = \lambda > 0$ and a sequence $\{x_n\} \subset K$ with $\lim_{n\to\infty}x_n = x \in K$ such that $\lim_{n\to\infty}J_{\lambda_n}^B(x_n) = y$. Then, $y = J_\lambda^B(x)$.*

Lemma 2.5. [1] *Let $A \in \Omega(\mathbb{M})$ be a single-valued monotone vector field and $B \in \mathscr{X}(\mathbb{M})$ be a set-valued maximal monotone. For any $x \in K$, the following assertions are equivalent*

(1) $x \in (A + B)^{-1}(0)$;

(2) $x = J_\lambda^B(exp_x(-\lambda A(x)))$, $\forall\lambda > 0$.

Lemma 2.6. [22] *Let K be nonempty, closed and convex sunset of a Hadamard manifold $\mathbb{M}$. Let $\phi : K \to T\mathbb{M}$ be a single-valued monotone vector field and $F : K \times K \to \mathbb{R}$ be a bifunction with $F(x, x) = 0$ such that*

(L1) F is monotone, $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in K$;

(L2) For all $x \in K, F(x, \cdot)$ is convex.

(L3) There exists a compact subset $C \subset K$ containing $u_0 \in C$ such that

$$F(x,u_0) + \langle \psi x, exp_x^{-1}, u_0 \rangle < 0$$

whenever $x \in K \backslash C$.

Then, the GEP is solvable.

Lemma 2.7. [22] *Let $F : K \times K \to \mathbb{R}$ be a bifunction with (L1)-(L3), and let $\phi : C \to T\mathbb{M}$ be a mapping. For $r > 0$, define a set-valued mapping $T_r^{F,\phi} : K \to 2^K$ by*

$$T_r^{F,\phi}(x) = \{z \in K : F(z,y) + \langle \phi z, exp_z^{-1} y \rangle \frac{1}{r} \langle exp_z^{-1} x, exp_z^{-1} y \geq 0 \rangle, \forall y \in K, x \in \mathbb{M}.$$

Then,

(1) $T_r^{F,\phi}$ is single-valued and firmly nonexpansive;

(2) $Fix(T_r^{F,\phi}) = GEP(F,\phi)$ is closed and convex;

(3) if $0 < r \leq s$, then, for all $x \in K$ $d(x, F_r^{F,\phi} x) \leq 2d(x, T_r^{F,\phi} x)$;

(4) for all $x \in K$ and $p \in Fix(T_r^{F,\phi})$, $d^2(p, T_r^{F,\phi} x) + d^2(x, T_r^{F,\phi} x) \leq d^2(x,p)$.

3. Main Results

Throughout this section we always assume that

(1) $\mathbb{M}$ is finite dimensional Hadamard manifold and K is a noempty closed and bounded geodesic convex subset of $\mathbb{M}$;

(2) For $j = 1,2,\ldots,N$, $\phi_j : K \to T\mathbb{M}$ is a monotone vector field and $F_j : K \times K \to \mathbb{R}$ such that $F_j(x,x) = 0$ for all $x \in K$ is a bifunction satisfying conditions (L1)-(L3);

(3) For each $i = 1,2,\ldots,N$, $B_i : \mathbb{M} \to 2^{T\mathbb{M}}$ is a maximal monotone set-valued vector field, $A_i : \mathbb{M} \to T\mathbb{M}$ is a continuous and monotone single-valued vector field, and $J_{\lambda_n}^{B_i}(exp(-\lambda_n A_i)) : K \to \mathbb{M}$ is the mapping defined by A_i, B_i and $\lambda_n > 0$;

(4) $S : K \to K$ is a L-lipsditzian quasi-pseudo-contractive mapping and demi-closed at 0 with $L \geq 1$;

(5) $G : K \to K$ and $T : K \to K$ are the mapping defined by

$$G(x) = exp_x \eta exp_x^{-1} Sx, \quad T(x) = exp_x \zeta exp_x^{-1} S(G(x)),$$

where $0 < \zeta < \eta < \frac{1}{1+\sqrt{1+L^2}}$. It is obvious from that $Fix(S) = Fix(T)$.

Theorem 3.1. *Let $\mathbb{M}, K, A_i, B_i, J_{\lambda_n}^{B_i}(exp(-\lambda_n A_i)), \phi_j, F_j$, and G, T be the same as above. For any given $x_1 \in \mathbb{M}$ and $C_1 = \mathbb{M}$, let $\{x_n\}$ be generated by*

$$\begin{cases} y_n^j = T_{r_n}^{F_j,\phi_j} x_n, \ \ j = 1,2,\cdots,N; \\ u_n \in \{y_n^j, j = 1,2,\cdots,N\} \text{ such that } d(u_n,x_n) = \max\limits_{1 \leq j \leq N} d(y_n^j, x_n); \\ z_n^i = J_{\lambda_n}^{B_i}(exp_{u_n}(-\lambda_n A_i(u_n))), \ \ i = 1,2,\cdots,N; \\ w_n^i = exp_{z_n^i}(1-\beta_n^i) exp_{z_n^i}^{-1} T z_n^i, \ \ i = 1,2,\cdots,N; \\ v_n \in \{w_n^i, i = 1,2,\cdots,N\} \text{ such that } d(v_n,x_n) = \max\limits_{1 \leq i \leq N} d(w_n^i, x_n); \\ C_{n+1} = \{q \in C_n : d(v_n,q) \leq d(x_n,q)\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ \ \forall n \geq 1. \end{cases} \tag{3.1}$$

where $\{r_n\} \subset (0,\infty), \{\beta_n^i\} \subset (0,1)$ *and* $\{\lambda_n\} \subset (0,\infty)$, *such that* $0 < \widehat{\lambda} \leq \lambda_n \leq \widetilde{\lambda} < \infty$. *Assume that* $\Omega := \cap_{j=1}^N GEP(F_j,\phi_j) \cap_{i=1}^N (A_i + B_i)^{-1}(0) \cap F(S) \neq \emptyset$, *and the following conditions are satisfied*

(i) $0 < r \leq r_n$;

(ii) $\sum_{i=1}^{\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$;

(iii) there exists $\rho \in (0,1)$ *such that, for each* $i = 1,2,\ldots,N$ *and* $\forall x,y \in \mathbb{M}$, $\lambda_n > 0$, *the following inequality holds*

$$d(exp_x(-\lambda_n A_i(x)), exp_y(-\lambda_n A_i(y))) \leq (1-\rho)d(x,y);$$

(iv) $0 < a \leq \beta_n^i \leq b < 1$, *for some* $a,b > 0$ *for all* $n \geq 1$.

Then, $\{x_n\}$ *converges to* $x^* \in \Omega$, *where* $x^* = P_\Omega x_1$.

Proof. We divide the proof into six steps.

Step 1. We show $\{x_n\}$ is well-defined for every $n \in N$.

First, by induction, we prove that, for each $n \geq 1$,C_n is a closed and geodesic convex in $\mathbb{M}$. In fact, by the assumption that $C_1 = \mathbb{M}$, it is closed and geodesic convex. Suppose that, for some $n \geq 2$, C_n is closed and geodesic convex subset in $\mathbb{M}$. We prove that C_{n+1} is closed and geodesic convex subset in $\mathbb{M}$. Since $q \mapsto d(x,q)$ is a geodesic convex function. Thus the set C_{n+1} defined by

$$C_{n+1} := \{q \in C_n : d(\gamma_n, q) \leq d(x_n, q)\}$$

is geodesic convex subset in $\mathbb{M}$.

Next, we prove that Ω is a closed and geodesic convex in $\mathbb{M}$. We have from Lemma 2.7 that $\cap_{j=1}^N GEP(F_j,\phi_j)$ and $Fix(T)$ are closed and geodesic convex in $\mathbb{M}$. From Lemma 2.5, we have $(A_i + B_i)^{-1} = F(J_{\lambda_n}^{B_i}(exp(-\lambda_n A_i)))$. Because $J_{\lambda_n}^{B_i}$ is nonexpansive, this together with condition (iii) yields $J_{\lambda_n}^{B_i}(exp(-\lambda_n A_i))$ is nonexpansive. Thus $\bigcap_{j=1}^N (A_i + B_i)^{-1}(0)$ is a closed and geodesic convex in $\mathbb{M}$.

We observe that $\Omega \subset C_n$ for each $n \geq 1$. In fact, it is obvious that $\Omega \subset C_1 = \mathbb{M}$. We prove that $\Omega \subset C_n$ for all $n \geq 2$. In fact, for any $p \in \Omega$, we can re-write ω_n^i defined in (3.1) as $\omega_n^i = \gamma_{n,i}^1(1-\beta_n^i)$, where $\gamma_{n,i}^1 : [0,1] \to \mathbb{M}$ is a sequence of geodesic joining z_n^i to Tz_n^i. Since T is quasi-nonexpansive, From (3.1), we have

$$\begin{aligned}
d(\omega_n^i, p) &= d(\gamma_{n,i}^1(1-\beta_n^i), p) \\
&= \beta_n^i d(\gamma_{n,i}(0), p) + (1-\beta_n^i) d(\gamma_{n,i}^1(1), p) \\
&= \beta_n^i d(z_n^i, p) + (1-\beta_n^i) d(Tz_n^i, p) \\
&\leq \beta_n^i d(z_n^i, p) + (1-\beta_n^i) d(z_n^i, p) \\
&= d(J_{\lambda_n}^{B_i}(exp_{u_n}(-\lambda_n A_i(u_n))), J_{\lambda_n}^{B_i}(exp_p(-\lambda_n A_i(p)))) \\
&\leq d(exp_{u_n}(-\lambda_n A_i(u_n)), exp_p(-\lambda_n A_i(p))). \\
&\leq (1-\rho) d(u_n, p). \\
&\leq d(u_n, p).
\end{aligned} \tag{3.2}$$

In view of (3.1) and Lemma 2.7, we have $d(y_n^j, p) = d(T_{r_n}^{F_j,\phi_j} x_n, p) \leq (x_n, p)$, which implies that

$$d(u_n, p) = \max_{1 \leq j \leq N} d(y_n^j, p) \leq d(x_n, p). \tag{3.3}$$

Hence, we conclude from (3.2) and (3.3) that $d(\omega_n^i, p) \leq d(x_n, p)$, which implies that

$$d(v_n, p) = \max_{1\leq i\leq N} d(\omega_n^i, p) \leq d(x_n, p)$$

that is, $p \in C_{n+1}$ for all $p \in \Omega$. Therefore $\Omega \subset C_n$ for all $n \geq 1$. So, $\{x_n\}$ is well defined.

Step 2. We prove that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, x_1)$ exists.

Note that Ω is a nonempty, closed, and geodesic convex subset $\mathbb{M}$. There exists a unique $v \in \Omega$ such that $v = P_\Omega x_1$. Since $x_n = P_{C_n} x_1$ and $\Omega \subset C_{n+1} \subset C_n$, We have

$$d(x_n, x_1) \leq d(v, x_1),\ d(x_n, x_1) \leq d(x_{n+1}, x_1), \forall n \geq 1.$$

These show that $\{x_n\}$ is bounded, so are $\{y_n^j\}, \{\omega_n^j\}, j = 1, 2, \ldots, N, \{u_n\}$, and $\{v_n\}$. Hence, $\lim_{n\to\infty} d(x_n, x_1)$ exists.

Step 3. We prove that the sequence $\{x_n\}$ is a cauchy sequence in $\mathbb{M}$ and $x_n \to x^*$.

From the construction of C_n, it is clear that $x_m = P_{C_m} x_1 \in C_m \subset C_n$ for $m > n \geq 1$. Note that

$$d^2(x_n, x_m) = d^2(P_{C_n} x_1, x_m) \leq d^2(x_1, x_m) - d^2(x_n, x_1) \to 0,\ \ n, m \to \infty, \tag{3.4}$$

which implies that $\{x_n\}$ is a cauchy sequence in $\mathbb{M}$.

Step 4. We prove that $\lim_{n\to\infty} d(y_n^j, x_n) = \lim_{n\to\infty} d(z_n^i, Tz_n^i) = 0$ and $\lim_{n\to\infty} d(u_n, x_n) = 0$ and $\lim_{n\to\infty} d(z_n^i, u_n) = 0$. From (3.4), we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Since $x_{n+1} \in C_{n+1} \subset C_n$, we have from Lemma 2.7 that

$$\begin{aligned} d^2(\omega_n^i, p) &= d^2(\gamma_{n,i}^1(1-\beta_n^i), p) \\ &\leq (1-\beta_n^i) d^2(\gamma_{n,i}^1(0), p) + \beta_n^i d^2(\gamma_{n,i}^1(1), p) - \beta_n^i(1-\beta_n^i) d^2(\gamma_{n,i}^1(0), \gamma_{n,i}^1(1)) \\ &= (1-\beta_n^i) d^2(z_n^i, p) + \beta_n^i d^2(Tz_n^i, p) - \beta_n^i(1-\beta_n^i) d^2(z_n^i, Tz_n^i) \\ &\leq d^2(z_n^i, p) - \beta_n^i(1-\beta_n^i) d^2(z_n^i, Tz_n^i). \end{aligned} \tag{3.5}$$

Since $J_{\lambda_n}^{B_i}$ is nonexpansive, we find from (3.1) that

$$\begin{aligned} d^2(z_n^i, p) &= d^2(J_{\lambda_n}^{B_i}(exp_{u_n}(-\lambda_n A_i(u_n))), J_{\lambda_n}^{B_i}(exp_p(-\lambda_n A_i(p)))) \\ &\leq d^2(exp_{u_n}(-\lambda_n A_i(u_n), exp_p(-\lambda_n A_i(p)))) \\ &\leq (1-\rho) d^2(u_n, p), \end{aligned}$$

which together with (3.5) derives that

$$\begin{aligned} d^2(\omega_n^i, p) &\leq (1-\rho) d^2(u_n, p) - \beta_n^i(1-\beta_n^i) d^2(z_n^i, Tz_n^i) \\ &\leq d^2(u_n, p) - \beta_n^i(1-\beta_n^i) d^2(z_n^i, Tz_n^i) \\ &\leq d^2(x_n, p) - d^2(y_n^j, x_n) - \beta_n^i(1-\beta_n^i) d^2(z_n^i, Tz_n^i) \end{aligned} \tag{3.6}$$

In view of the definition of C_{n+1}, we have $d(v_n, x_{n+1}) \leq d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. It follows that $d(v_n, x_n) \leq d(v, x_{n+1}) + d(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Thus $d(\omega_n^i, x_n) \to 0$ as $n \to \infty$. Using (3.6), we have, as $n \to \infty$,

$$\begin{aligned} d^2(y_n^j, x_n) + \beta_n(1-\beta_n) d^2(z_n^i, Tz_n^i) &\leq d^2(x_n, p) - d^2(\omega_n^i, p) \\ &= (d(x_n, p) - d(\omega_n^i, p))(d(x_n, p) + d(\omega_n^i, p)) \\ &\leq d(x_n, \omega_n^i)(d(x_n, p) + d(\omega_n^i, p)) \to 0. \end{aligned}$$

Note that

$$\lim_{n\to\infty} d(y_n^j, x_n) = \lim_{n\to\infty} d(z_n^i, Tz_n^i) = 0, \tag{3.7}$$

which yields that $\lim_{n\to\infty} d(u_n, x_n) = 0$. In view of (3.7), we have, as $n \to \infty$,

$$\begin{aligned} d(\omega_n^i, z_n^i) &= d(exp_{z_n^i}(1-\beta_n^i)exp_{z_n^i}^{-1} Tz_n^i, z_n^i) \\ &\leq \beta_n^i d(z_n^i, z_n^i) + (1-\beta_n^i) d(Tz_n^i, z_n^i), \to 0. \end{aligned}$$

which yields $d(z_n^i, x_n) \leq d(z_n^i, \omega_n^i) + d(\omega_n^i, x_n) \to 0$ as $n \to \infty$. This further yields that

$$d(u_n, z_n^i) \leq d(u_n, x_n) + d(x_n, z_n^i \to 0 \text{ as } n \to \infty.$$

Step 5. We prove that $x^* \in \Omega$.

(1) First, we prove that $x^* \in \bigcap_{j=1}^N GEP(F_j, \phi_j)$. Since $T_{r_n}^{F_j,\phi_j}, j = 1, 2 \ldots, N$ is nonexpansive and demiclosed at 0, we have from (3.6) that $x^* \in Fix(T_{r_n}^{F_j,\phi_j}) = \bigcap_{j=1}^N GEP(F_j, \phi_j)$.

(2) We prove that $x^* \in \bigcap_{i=1}^N (A_i + B_i)(0)$. Since $d(u_n, x^*) \leq d(u_n, x_n) + d(x_n, x^*)$. From Step 3, we find that $\lim_{n\to\infty} d(u_n, x^*) = 0$. Since $J_{\lambda_n}^{B_i}(exp(-\lambda_n A_i))$ is nonexpansive mapping, we see that it is demiclosed at 0. We also have

$$x^* \in Fix(J_{\lambda_n}^{B_i}(exp_{x^*}(-\lambda_n A(x^*)))) = (A_i + B_i)^{-1}(0), \ \ \forall i = 1, 2, \ldots, N,$$

i.e., $x^* \in \bigcap_{i=1}^N (A_i + B_i)^{-1}(0)$.

(3) Prove that $x^* \in Fix(T)$.

Since $d(z_n^i, x^*) \leq d(z_n^i, u_n) + d(u_n, x^*)$, we have $\lim_{n\to\infty} d(z_n^i, x^*) = 0$. From (3.8), we obtain that $x^* \in Fix(T)$. Thus we conclude that $x^* \in \Omega$.

Step 6. Finally we prove that $x^* \in P_\Omega^{x_1}$.

From $x_n = P_{C_n}^{x_1}$ and $\Omega \subset C_n$, we have $\langle exp_{x_n}^{-1} x_1, exp_{x_n}^{-1} p \rangle \leq 0$ for all p in Ω. By taking $n \to \infty$ above, we arrive at $\langle exp_{x^*}^{-1}, exp_{x^*}^{-1} p \rangle \leq 0$ for all p in Ω. Thus $x^* = P_\Omega^{x_1}$. This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $\mathbb{M}, K, \phi_j, F_j$, and G, T be the same as above. For any given $x_1 \in \mathbb{M}$ and $C_1 = \mathbb{M}$, let $\{x_n\}$ be generated by*

$$\begin{cases} y_n^j = T_{r_n}^{F_j,\phi_j} x_n, \ \ j = 1, 2, \cdots, N; \\ u_n \in \{y_n^j, j = 1, 2, \cdots, N\} \text{ such that } d(u_n, x_n) = \max\limits_{1\leq j\leq N} d(y_n^j, x_n); \\ w_n = exp_{u_n}(1-\beta_n) exp_{u_n}^{-1} Tu_n; \\ C_{n+1} = \{q \in C_n : d(w_n, q) \leq d(x_n, q)\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ \ \forall n \geq 1. \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\} \subset (0, 1)$. Assume that $\Omega := \bigcap_{j=1}^N GEP(F_j, \phi_j) \cap F(S) \neq \emptyset$, and the following conditions are satisfied

(i) $0 < r \leq r_n$;

(ii) $0 < a \leq \beta_n \leq b < 1$, *for some* $a, b > 0$ *for all* $n \geq 1$.

Then, $\{x_n\}$ converges to $x^* \in \Omega$, where $x^* = P_\Omega x_1$.

Proof. In Theorem 3.1, taking $A_i = 0, B_i = 0$, we find that the conclusion of the corollary can be obtained from Theorem 3.1 immediately. □

Corollary 3.3. *Let $\mathbb{M}, K, A_i, B_i, J_{\lambda_n}^{B_i}(exp(-\lambda_n A_i))$,, and G, T be the same as above. For any given $x_1 \in \mathbb{M}$ and $C_1 = \mathbb{M}$, let $\{x_n\}$ be generated by*

$$\begin{cases} z_n^i = J_{\lambda_n}^{B_i}(exp_{x_n}(-\lambda_n A_i(x_n))), \ \ i = 1,2,\cdots,N; \\ w_n^i = exp_{z_n^i}(1-\beta_n^i) exp_{z_n^i}^{-1} T z_n^i, \ \ i = 1,2,\cdots,N; \\ v_n \in \{w_n^i, i = 1,2,\cdots,N\} \text{ such that } d(v_n, x_n) = \max\limits_{1\le i\le N} d(w_n^i, x_n); \\ C_{n+1} = \{q \in C_n : d(v_n, q) \le d(x_n, q)\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ \ \forall n \ge 1. \end{cases}$$

where $\{\beta_n^i\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ such that $0 < \widehat{\lambda} \le \lambda_n \le \widetilde{\lambda} < \infty$. Assume that $\Omega := \cap_{i=1}^N (A_i + B_i)^{-1}(0) \cap F(S) \ne \emptyset$, and the following conditions are satisfied

(i) $\sum_{i=1}^{\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$;

(ii) there exists $\rho \in (0,1)$ such that, for each $i = 1,2,\ldots,N$ and for all $x, y \in \mathbb{M}$, $\lambda_n > 0$, the following inequality holds

$$d(exp_x(-\lambda_n A_i(x)), exp_y(-\lambda_n A_i(y))) \le (1-\rho) d(x,y);$$

(iii) $0 < a \le \beta_n^i \le b < 1$, for some $a, b > 0$ and for all $n \ge 1$.

Then, $\{x_n\}$ converges to $x^* \in \Omega$, where $x^* = P_\Omega x_1$.

Proof. In Theorem 3.1, taking $F_i = 0, \phi_j = 0$, we see that the conclusion of this corollary can be obtained from Theorem 3.1 immediately. □

Remark 3.4. Our results enhance the main results presented in [6]. Moreover, the method adopted in the paper is different from that of [6].

Corollary 3.5. *Let $\mathbb{M}, K, A, B, J_{\lambda_n}^B(exp(-\lambda_n A)), \phi, F$, and G, T be the same as above. For any given $x_1 \in \mathbb{M}$ and $C_1 = \mathbb{M}$, let $\{x_n\}$ be generated by*

$$\begin{cases} y_n = T_{r_n}^{F,\phi} x_n; \\ z_n = J_{\lambda_n}^B(exp_{y_n}(-\lambda_n A(y_n))); \\ w_n = exp_{z_n}(1-\beta_n) exp_{z_n}^{-1} T z_n; \\ C_{n+1} = \{q \in C_n : d(w_n, q) \le d(x_n, q)\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ \ \forall n \ge 1. \end{cases}$$

where $\{r_n\} \subset (0,\infty), \{\beta_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,\infty)$ such that $0 < \widehat{\lambda} \le \lambda_n \le \widetilde{\lambda} < \infty$. Assume that $\Omega := GEP(F,\phi) \cap (A+B)^{-1}(0) \cap F(S) \ne \emptyset$, and the following conditions are satisfied

(i) $0 < r \le r_n$;

(ii) $\sum_{i=1}^{\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} < \infty$;

(iii) there exists $\rho \in (0,1)$ such that, for all $x, y \in \mathbb{M}$ and $\lambda_n > 0$, the following inequality holds

$$d(exp_x(-\lambda_n A(x)), exp_y(-\lambda_n(y))) \le (1-\rho) d(x,y);$$

(iv) $0 < a \le \beta_n \le b < 1$ for some $a, b > 0$ and for all $n \ge 1$.

Then, $\{x_n\}$ converges to $x^* \in \Omega$, where $x^* = P_\Omega x_1$.

Proof. In Theorem 3.1, taking $j = 1, i = 1$, we see that the conclusion of this corollary can be obtained from Theorem 3.1 immediately. □

Remark 3.6. Our results enhance the main results presented in [17]. Moreover, the method adopted in the paper is different from that of [17].

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