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MULTIPLICITY RESULTS OF STRICTLY NONDECREASING POSITIVE SOLUTIONS FOR A FRACTIONAL THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. Based on the Avery-Peterson fixed point theorem and Green's function, we establish the existence result of at least triple strictly nondecreasing positive solutions to a three-point boundary value problem of a fractional differential equations with out the concavity or convexity of the unknown function. An example is also provided to illustrate our main results.

Keywords. Boundary value problem; Fractional differential equation; Multiplicity; Positive solution;.

1. Introduction

In this paper, we consider the boundary value problem of nonlinear fractional differential equations

$$D^{\alpha}_{0+}u(t)+f\big(t,\ u(t),\ u'(t)\big)=0,\ t\in[0,\ 1], \tag{1.1}$$

$$u(0)=u'(0)=u''(0)=0,\ u''(1)=\beta u''(\eta), \tag{1.2}$$

where $3<\alpha\leq 4,\ \beta>0,\ 0\leq\eta\leq 1$ with $1-\beta\eta^{\alpha-3}>0$ and D^{α}_{0+} denotes the Riemann-Liouville fractional derivative of order α.

Through this paper, we assume that $f:[0,\ 1]\times[0,\ \infty)\times(\infty,\ +\infty)\to[0,\ +\infty)$ satisfies the following conditions of Caratheodory type:

(i) $f(t,\ u,\ v)$ is Lebesgue measurable with respect to t on $[0,\ 1]$.

(ii) $f(t,\ u,\ v)$ is continuous with respect to u and v on $[0,\ \infty)\times(-\infty,\ \infty)$.

With the development of the theory of fractional equations and their applications, such as physics, Bode's analysis of feedback amplifiers, aerodynamics and polymer rheology, numerous results on the basic theory of fractional calculus and fractional order differential equations were established; see, e.g., [2, 4, 5, 6, 9, 10].

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Recently, the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations(FBVPs) with local boundary conditions and nonlocal boundary conditions were extensively. Liang and Zhang [7] considered positive solutions for the problem of fractional differential equation

$$D^{\alpha}_{0+}u(t) = f(t,\ u(t)),\ t \in (0,\ 1), \tag{1.3}$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \tag{1.4}$$

where $3 < \alpha \leq 4$ and D^{α}_{0+} is the standard Riemann-Liouville fractional derivative. By means of lower and upper solution method and fixed point theorems, they obtained the existence results of positive solutions for problem (1.3-1.4).

For the nonlocal case, Liang and Zhang [8] investigated the boundary value problem of fractional differential equation

$$D^{\alpha}_{0+}u(t) = f(t,\ u(t)),\ t \in (0,\ 1),\ 3 < \alpha \leq 4, \tag{1.5}$$

$$u(0) = u'(0) = u''(0) = 0,\ u''(1) = \beta u''(\eta), \tag{1.6}$$

where $0 \leq \eta \leq 1$ satisfies $0 < \beta\eta^{\alpha-3} < 1$. By using a fixed point theorem in partially ordered sets, sufficient conditions for the existence and uniqueness of positive and nondecreasing solutions were established.

Cabrera et al. [3] discussed the singular boundary value problem (1.5-1.6) under the assumption that f is singular at $t = 0$. By means of a fixed point theorem in partially ordered metric spaces, the authors obtained the existence and uniqueness of positive solutions for problem (1.5-1.6).

Recently, Zhai et al. [11] concerned with the existence and uniqueness of positive solutions for the following the fractional differential equation

$$D^{\alpha}_{0+}u(t) = f(t,\ u(t)) + g(t,\ u(t)),\ t \in (0,\ 1),\ 3 < \alpha \leq 4,$$

subject to the boundary conditions

$$u(0) = u'(0) = u''(0) = u''(1) = 0, \quad or$$

$$u(0) = u'(0) = u''(0) = 0,\ u''(1) = \beta u''(\eta).$$

By means of a fixed point theorem of a sum operator, they obtained the existence of a unique positive solution and constructed an iterative scheme for approximating the solution.

There are also some results concerning with the numerical solution of problem (1.5)-(1.6). For an example, Zhang et al. [12] extended the reproducing kernel space method and presented an efficient numerical algorithm to solve problem (1.5)-(1.6). We noticed that in these works the existence results of positive solutions were all established under the assumption that the derivative of the unknown function was not involved in the nonlinear term explicitly. The main reason is that one can not derive the concavity or convexity of the function by the sign of its fractional derivative. On account of the practical meaning of $u'(t)$, it is interesting to consider the BVps of fractional differential equations which the derivative of the unknown function is involved in the nonlinear term explicitly.

In this paper, by using the careful analysis of the associated Green's function and defining the special cone in a suitable Banach space together with the Avery-Peterson fixed point theorem, we obtain the existence of multiple positive solutions without the concavity or convexity of the unknown function. These results complete and extend the previous works on positive solutions to FBVPS.

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u(t)$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided that the right side is point-wise defined on $(0, \infty)$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u(t)$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s) ds$$

where $n = [\alpha] + 1$, provided that the right side is point-wise defined on $(0, \infty)$.

Lemma 2.3. *Let $\alpha > 0$. Then, fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has a solution*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},\ C_i \in R,\ i = 1,\ 2,\ \cdots,\ n.$$

Lemma 2.4. *Let $u(t)$ be a fractional derivative of order $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},\ C_i \in R,\ i = 1,\ 2,\ \cdots,\ n,$$

where n is the smallest integer greater than or equal to α.

Definition 2.5. The map ϕ is said to be a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\phi : P \to [0, +\infty)$ is continuous and

$$\phi(tx + (1-t)y) \le t\phi(x) + (1-t)\phi(y),\ x,\ y \in P,\ t \in [0,1].$$

Definition 2.6. The map β is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y),\ x,\ y \in P,\ t \in [0,1].$$

Let γ, θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then, for positive numbers $a,\ b,\ c$, and d, we define the following convex sets:

$P(\gamma,\ d) = \{x \in P | \gamma(x) < d\}$,

$P(\gamma,\ \phi,\ b,\ d) = \{x \in P | b \le \phi(x),\ \gamma(x) \le d\}$,

$P(\gamma,\ \theta,\ \phi,\ b,\ c,\ d) = \{x \in P | b \le \phi(x),\ \theta(x) \le c,\ \gamma(x) \le d\}$,

and a closed set

$$R(\gamma,\ \psi,\ a,\ d) = \{x \in P | a \le \psi(x),\ \gamma(x) \le d\}.$$

Lemma 2.7. [1] *Let P be a cone in Banach space E. Let $\gamma,\ \theta$ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P satisfying*

$$\psi(\lambda x) \le \lambda \psi(x),\ for\ 0 \le \lambda \le 1,$$

$$\phi(x) \le \psi(x),\ \|x\| \le l\gamma(x) for\ x \in \overline{P(\gamma,\ d)}, \tag{2.1}$$

where $\overline{P(\gamma,\ d)}$ is the closure of the set $P(\gamma,\ d)$. Suppose that $T:\overline{P(\gamma,\ d)}\to\overline{P(\gamma,\ d)}$ is completely continuous and there exist positive numbers $a,\ b,\ c$ with $a<b$ such that
(S_1) $\{x\in P(\gamma,\ \theta,\ \phi,\ b,\ c,\ d)|\phi(x)>b\}\neq\emptyset$ and $\phi(Tx)>b$ for $x\in P(\gamma,\ \theta,\ \phi,\ b,\ c,\ d)$;
(S_2) $\phi(Tx)>b$ for $x\in P(\gamma,\ \phi,\ b,\ d)$ with $\theta(Tx)>c$;
(S_3) $0\notin R(\gamma,\ \psi,\ a,\ d)$ and $\psi(Tx)<a$ for $x\in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x)=a$.
Then T has at least three fixed points $x_1,\ x_2,\ x_3\in\overline{P(\gamma,\ d)}$ such that

$$\gamma(x_i)\leq d,\ i=1,\ 2,\ 3;\ b<\phi(x_1);\ a<\psi(x_2),\ \phi(x_2)<b;\ \psi(x_3)<a.$$

3. Main Results

Lemma 3.1. *Given $y(t)\in C[0,1]$, boundary value problem*

$$^{C}D_{0+}^{\alpha}u(t)+y(t)=0,\ t\in[0,\ 1], \tag{3.1}$$

$$u(0)=u'(0)=u''(0)=0,\ u''(1)=\beta u''(\eta) \tag{3.2}$$

is equivalent to

$$u(t)=\int_0^1 G(t,\ s)y(s)ds,$$

where

$$G(t,s)=\begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}(1-s)^{\alpha-3}-\beta t^{\alpha-1}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & 0\leq s\leq\eta,\ s\leq t,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-3}-\beta t^{\alpha-1}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & 0\leq s\leq\eta,\ s\geq t,\\ -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & \eta\leq s\leq 1,\ s\leq t,\\ \frac{t^{\alpha-1}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & \eta\leq s\leq 1,\ s\geq t. \end{cases}$$

Proof. From the definition and properties of the fractional derivative and integral, we have

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}+c_4t^{\alpha-4}.$$

The boundary conditions $u(0)=u'(0)=u''(0)=0$ give $c_2=c_3=c_4=0$. Considering the boundary condition $u''(1)=\beta u''(\eta)$, we have

$$c_1=\frac{1}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}\left(\int_0^1(1-s)^{\alpha-3}y(s)ds-\beta\int_0^{\eta}(\eta-s)^{\alpha-3}y(s)ds\right).$$

Thus

$$\begin{aligned} u(t)&=\frac{t^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}\left(\int_0^1(1-s)^{\alpha-3}y(s)ds-\beta\int_0^{\eta}(\eta-s)^{\alpha-3}y(s)ds\right)\\ &\quad-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\ &=\int_0^1 G(t,\ s)y(s)ds. \end{aligned}$$

□

Lemma 3.2. *The function $G(t,s)$ satisfies the following properties:*

(1) $G(t,\ s)\in C\big([0,\ 1]\times[0,\ 1]\big)$ *and* $G(t,\ s)\geq 0$ *for all* $t,\ s\in[0,\ 1]$ *and* $G(t,\ s)>0$ *for all* $t,\ s\in(0,\ 1)$*;*

(2) $G(t,\ s)$ *is increasing with respect to* t*;*

(3) $\min_{\eta\leq t\leq 1}G(t,\ s)\geq\gamma_0\max_{\eta\leq t\leq 1}G(t,\ s)=\eta^{\alpha-1}G(1,\ s),\ s\in[0,1]$.

Proof. (1) It is easy to check that $G(t,s)$ is continuous. For $0\leq s\leq\eta,\ s\leq t$, we have

$$-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}(1-s)^{\alpha-3}-\beta t^{\alpha-1}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\geq-\frac{1}{\Gamma(\alpha)}+\frac{1-\beta\eta^{\alpha-3}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}=0.$$

For $\eta\leq s\leq 1,\ s\leq t$, we find that

$$-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t^{\alpha-1}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\geq\frac{(1-\eta)^{\alpha-3}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}\Big(1-(1-\eta)^2(1-\beta\eta^{\alpha-3})\Big)\geq 0.$$

Thus, $G(t,\ s)\in C\big([0,\ 1]\times[0,\ 1]\big)$ and $G(t,\ s)\geq 0,\ for\ t,\ s\in[0,1]$.
(2) To prove that (2) is true, we begin with

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}-\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{t^{\alpha-2}(1-s)^{\alpha-3}-\beta t^{\alpha-2}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}, & 0\leq s\leq\eta,\ s\leq t\\ \frac{t^{\alpha-2}(1-s)^{\alpha-3}-\beta t^{\alpha-2}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}, & 0\leq s\leq\eta,\ s\geq t\\ -\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{t^{\alpha-2}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}, & \eta\leq s\leq 1,\ s\leq t\\ \frac{t^{\alpha-2}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}, & \eta\leq s\leq 1,\ s\geq t\end{cases}$$

For $0\leq s\leq\eta,\ s\leq t$,

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t}&=-\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{t^{\alpha-2}(1-s)^{\alpha-3}-\beta t^{\alpha-2}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}\\&\geq\frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})}\Big(-(1-\beta\eta^{\alpha-3})(1-s)^{\alpha-2}+(1-s)^{\alpha-3}-\beta(\eta-s)^{\alpha-3}\Big)\\&\geq 0.\end{aligned}$$

For $\eta\leq s\leq 1,\ s\leq t$,

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t}&=-\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{t^{\alpha-2}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}\\&\geq\frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})}\Big(-(1-\beta\eta^{\alpha-3})(1-s)^{\alpha-2}+(1-s)^{\alpha-3}\Big)\\&\geq 0.\end{aligned}$$

Thus $G(t,\ s)$ is increasing with respect to t and

$$\max_{0\leq t\leq 1}G(t,\ s)=G(1,\ s),\ s\in[0,\ 1]$$

(3) From the expression and monotonicity of function $G(t,\ s)$ with respect to t , we have

$$\min_{\eta\le t\le 1} G(t,s)=G(\eta,\ s)=\begin{cases} -\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta^{\alpha-1}(1-s)^{\alpha-3}-\beta\eta^{\alpha-1}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & 0\le s\le\eta,\\ \frac{\eta^{\alpha-1}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & \eta\le s\le 1 \end{cases}$$

and

$$\max_{0\le t\le 1} G(t,s)=G(1,\ s)=\begin{cases} -\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-3}-\beta(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & 0\le s\le\eta,\\ -\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}, & \eta\le s\le 1. \end{cases}$$

Thus, for $0<s\le\eta$,

$$\begin{aligned} \frac{G(\eta,\ s)}{G(1,\ s)} &= \frac{-\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta^{\alpha-1}(1-s)^{\alpha-3}-\beta\eta^{\alpha-1}(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}}{-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-3}-\beta(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}} \\ &\ge \frac{\frac{\eta^{\alpha-1}(1-\eta)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}}{-\frac{(1-\eta)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-\eta)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}} \\ &= \frac{\eta^{\alpha-1}}{1-(1-\eta)^2(1-\beta\eta^{\alpha-3})}. \end{aligned}$$

For $\eta\le s<1$,

$$\frac{G(\eta,\ s)}{G(1,\ s)}=\frac{\frac{\eta^{\alpha-1}(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}}{-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}}\ge\eta^{\alpha-1}.$$

Then, we conclude that

$$\begin{aligned} \min_{\eta\le t\le 1} G(t,\ s) &\ge \min\left\{\frac{\eta^{\alpha-1}}{1-(1-\eta)^2(1-\beta\eta^{\alpha-3})},\eta^{\alpha-1}\right\}G(1,\ s) \\ &= \eta^{\alpha-1}G(1,\ s)=\eta^{\alpha-1}\max_{0\le t\le 1} G(t,s),\ s\in[0,\ 1]. \end{aligned}$$

□

Lemma 3.3. *If $u(t)$ is a solution of problem (3.1)-(3.2), then*

$$\max_{0\le t\le 1}|u(t)|\le\max_{0\le t\le 1}|u'(t)|.$$

Proof. The fact that

$$|u(t)|=\left|u(0)+\int_0^t u'(s)ds\right|=\left|\int_0^t u'(s)ds\right|\le\int_0^1|u'(s)|ds$$

ensures that

$$\max_{0\le t\le 1}|u(t)|\le\max_{0\le t\le 1}|u'(t)|.$$

□

Let the space $X = C^1[0,\ 1]$ be endowed with the norm

$$\|u\| := \max\Big\{\max_{0\le t\le 1}|u(t)|,\ \max_{0\le t\le 1}|u'(t)|\Big\}.$$

It is known that X is a Banach space. Define the cone $K \subset X$ by

$$K = \Big\{u \in X:\ u(t) \ge 0,\ \min_{t\in[\eta,\ 1]} u(t) \ge \gamma_0 \max_{0\le t\le 1} u(t),\ \max_{0\le t\le 1}|u(t)| \le \max_{0\le t\le 1}|u'(t)|\Big\}.$$

Lemma 3.4. *Let $T:\ K \to X$ be the operator defined by*

$$(Tu)(t) := \int_0^1 G(t,\ s) f(s,\ u(s),\ u'(s))ds.$$

Then $T:\ K \to K$ is completely continuous.

Proof. First, we show that the operator T is continuous. For any $u_n,\ u \in K,\ n = 1,\ 2,\ \cdots$, with $\lim_{n\to+\infty}\|u_n - u\| = 0$, we have

$$\lim_{n\to+\infty} u_n = u,\ \lim_{n\to+\infty} u_n' = u',$$

$t \in [0,\ 1]$. From the Caratheodory condition of function f, we obtain

$$\lim_{n\to+\infty} f\big(t,\ u_n(t),\ u_n'(t)\big) = f\big(t,\ u(t),\ u'(t)\big),\ t \in [0,\ 1].$$

Thus

$$\sup_{t\in[0,\ 1]}\big|f\big(t,\ u_n(t),\ u_n'(t)\big) - f\big(t,\ u(t),\ u'(t)\big)\big| \to 0,\ n \to +\infty.$$

Therefore,

$$\begin{aligned}
|(Tu_n)(t) - (Tu)(t)| &= \left|\int_0^1 G(t,\ s)\big(f(s,\ u_n(s),\ u_n'(s)) - f(s,\ u(s),\ u'(s))\big)ds\right| \\
&\le \sup_{t\in[0,\ 1]}\big|f(t,\ u_n(t),\ u_n'(t)) - f(t,\ u(t),\ u'(t))\big| \\
&\times\left|\int_0^\eta\left(-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-3} - \beta(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\right)ds\right. \\
&\quad\left. + \int_\eta^1\left(-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\right)ds\right| \\
&= \left(-\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\Big(\frac{1}{\alpha-2} - \frac{\beta\eta^{\alpha-2}}{\alpha-2}\Big)\right) \\
&\times \sup_{t\in[0,\ 1]}\Big|f\big(t,\ u_n(t),\ u_n'(t)\big) - f\big(t,\ u(t),\ u'(t)\big)\Big|,
\end{aligned}$$

and

$$\begin{aligned}
&|(T'u_n)(t)-(T'u)(t)|\\
&=\Bigg|-\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\big(f(s,\,u_n(s),\,u_n'(s))-f(s,\,u(s),\,u'(s))\big)ds\\
&+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})}\bigg(\int_0^1 (1-s)^{\alpha-3}\big(f(s,\,u_n(s),\,u_n'(s))-f(s,\,u(s),\,u'(s))\big)ds\\
&\quad-\beta\int_0^\eta (\eta-s)^{\alpha-3}\big(f(s,\,u_n(s),\,u_n'(s))-f(s,\,u(s),\,u'(s))\big)ds\bigg)\Bigg|\\
&\le\bigg(\frac{1}{\Gamma(\alpha)}+\frac{1}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha-1)}\Big(\frac{1}{\alpha-2}+\frac{\beta\eta^{\alpha-2}}{\alpha-2}\Big)\bigg)\\
&\quad\times\sup_{t\in[0,\,1]}\Big|f\big(t,\,u_n(t),\,u_n'(t)\big)-f\big(t,\,u(t),\,u'(t)\big)\Big|,
\end{aligned}$$

which implies that

$$\|Tu_n-Tu\|\to 0$$

as $n\to\infty$. These ensures that T is continuous.

Second, we show that T is completely continuous. Let $\Omega\subset K$ be bounded. Then there exist a positive constant $R_1>0$ such that $\|u\|\le R_1,\ u\in\Omega$. Denote

$$R=\max_{0\le t\le 1,\ u\in\Omega}|f(t,\,u(t),\,u'(t))|+1.$$

Then, for $u\in\Omega$, we have

$$\begin{aligned}
|Tu|&\le R\int_0^1 G(1,\,s)ds\\
&=\bigg(-\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}\Big(\frac{1}{\alpha-2}-\frac{\beta\eta^{\alpha-2}}{\alpha-2}\Big)\bigg)R
\end{aligned}$$

and

$$\begin{aligned}
&(Tu)'(t)|\\
&=\Bigg|-\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,\,u(s),\,u'(s))ds+\frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\mu\eta^{\alpha-\beta-1})}\\
&\quad\times\bigg(\int_0^1 (1-s)^{\alpha-\beta-1}f(s,\,u(s),\,u'(s))ds-\mu\int_0^\eta (\eta-s)^{\alpha-\beta-1}f(s,\,u(s),\,u'(s))ds\bigg)\Bigg|\\
&\le\bigg(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})}\frac{1+\beta\eta^{\alpha-2}}{\alpha-2}\bigg)\times R.
\end{aligned}$$

Hence $T(\Omega)$ is bounded. For $u \in \Omega,\ t_1,\ t_2 \in [0,\ 1]$, one has

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&\le \left| \frac{1}{\Gamma(\alpha)} \Big(\int_0^{t_2} (t_2 - s)^{\alpha-1} f(s,\ u(s),\ u'(s))ds - \int_0^{t_1} (t_1 - s)^{\alpha-1} f(s,\ u(s),\ u'(s))ds \Big) \right| \\
&\quad + \frac{\left| t_2^{\alpha-1} - t_1^{\alpha-1} \right|}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})} \Big(\int_0^1 (1-s)^{\alpha-3} f(s,\ u(s),\ u'(s))ds \\
&\quad - \beta \int_0^{\eta} (\eta - s)^{\alpha-3} f(s,\ u(s),\ u'(s))ds \Big) \\
&\le \frac{R}{\Gamma(\alpha+1)} \times |t_2^{\alpha} - t_1^{\alpha}| + \frac{(1-\beta\eta^{\alpha-2})R}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})(\alpha-2)} \times |t_2^{\alpha-1} - t_1^{\alpha-1}|,
\end{aligned}$$

and

$$\begin{aligned}
&|(Tu)'(t_2) - (Tu)'(t_1)| \\
&\le \left| \frac{1}{\Gamma(\alpha-1)} \Big(\int_0^{t_1} (t_1 - s)^{\alpha-2} f(s,\ u(s),\ u'(s))ds - \int_0^{t_2} (t_2 - s)^{\alpha-2} f(s,\ u(s),\ u'(s))ds \Big) \right| \\
&\quad + \frac{\left| t_2^{\alpha-2} - t_1^{\alpha-2} \right|}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})} \Big(\int_0^1 (1-s)^{\alpha-3} f(s,\ u(s),\ u'(s))ds \\
&\quad - \beta \int_0^{\eta} (\eta - s)^{\alpha-3} f(s,\ u(s),\ u'(s))ds \Big) \\
&\le \frac{R}{\Gamma(\alpha)} \times |t_2^{\alpha-1} - t_1^{\alpha-1}| + \frac{(1-\beta\eta^{\alpha-2})R}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})(\alpha-2)} \times |t_2^{\alpha-2} - t_1^{\alpha-2}|,
\end{aligned}$$

Thus,

$$\|Tu(t_2) - Tu(t_1)\| \to 0 \ for\ t_1 \to t_2,\ u \in \Omega.$$

By means of the Arzela-Ascoli theorem, we claim that T is completely continuous. Finally, we see that

$$\begin{aligned}
\min_{\eta \le t \le 1} |Tu(t)| &= \min_{\eta \le t \le 1} \int_0^1 G(t,\ s) f(s,\ u(s),\ u'(s))ds \ge \gamma_0 \int_0^1 G(1,\ s) f(s,\ u(s),\ u'(s))ds \\
&\ge \gamma_0 \max_{0 \le t \le 1} \int_0^1 G(t,\ s) f(s,\ u(s),\ u'(s))ds = \gamma_0 \max_{0 \le t \le 1} (Tu)(t).
\end{aligned}$$

Considering the definition of the operator T together with Lemma 3.3, one can find that

$$\max_{0 \le t \le 1} |Tu(t)| \le \max_{0 \le t \le 1} |Tu'(t)|.$$

Thus, we conclude that $T : K \to K$ is a completely continuous operator.

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals $\gamma,\ \theta$ and the nonnegative continuous functional ψ be defined on the cone by

$$\gamma(u) = \max_{0 \le t \le 1} |u'(t)|,\ \theta(u) = \psi(u) = \max_{0 \le t \le 1} |u(t)|,\ \phi(u) = \min_{\eta \le t \le 1} |u(t)|.$$

By Lemmas 3.2 and 3.3, the functionals defined above satisfy that

$$\gamma_0 \theta(u) \le \phi(u) \le \theta(u) = \psi(u),\ \|u\| \le \gamma(u),\ u \in K.$$

Therefore the condition (2.1) of Lemma 2.7 is satisfied. Assume that there exist constants $0 < a,\ b,\ d$ with $a < b < d,\ c = \dfrac{b}{\gamma_0}$ and

$$d > \frac{\alpha\Big((\alpha-2)(1-\beta\eta^{\alpha-3})+(\alpha-1)(1+\beta\eta^{\alpha-2})\Big)}{\gamma_0\Big(-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2}))\Big)}b$$

such that

$$(A_1)\ \ f(t,u,v) \le \frac{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})(\alpha-2)}{(\alpha-2)(1-\beta\eta^{\alpha-3})+(\alpha-1)(1+\beta\eta^{\alpha-2})}d,$$

$(t,\ u,\ v) \in [0,1]\times[0,\ d]\times[-d,\ d]$,

$$(A_2)\ \ f(t,u,v) > \frac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{\gamma_0\Big(-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})\Big)}b,$$

$(t,\ u,\ v) \in [\eta,\ 1]\times[b,\ b/\gamma_0]\times[-d,\ d]$, and

$$(A_3)\ \ f(t,u,v) < \frac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})}a,$$

$(t,\ u,\ v) \in [0,\ 1]\times[0,\ a]\times[-d,\ d]$. □

Theorem 3.5. *Under assumptions $(A_1)-(A_3)$, problem (1.1)-(1.2) has at least three positive solutions $u_1,\ u_2,\ u_3$ satisfying*

$$\max_{0\le t\le 1}|u_i'(t)| \le d,\ i=1,\ 2,\ 3;\ b < \min_{\eta\le t\le 1}|u_1(t)|;\ a < \max_{0\le t\le 1}|u_2(t)|,\ \min_{\eta\le t\le 1}|u_2(t)| < b,\ \max_{0\le t\le 1}|u_3(t)| \le a.$$

Proof. Problem (1.1)-(1.2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = \int_0^1 G(t,s)f\big(s,\ u(s),\ u'(s)\big)ds = (Tu)(t).$$

For $u \in \overline{K(\gamma,\ d)}$, we have $\gamma(u) = \max_{0\le t\le 1}|u'(t)| < d$. From assumption (A_1), we obtain

$$f\big(t,\ u(t),\ u'(t)\big) \le \frac{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})(\alpha-2)}{(\alpha-2)(1-\beta\eta^{\alpha-3})+(\alpha-1)(1+\beta\eta^{\alpha-2})}d.$$

Thus

$$\begin{aligned}\gamma(Tu) &= \max_{0\le t\le 1}\bigg|-\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,\ u(s),\ u'(s))ds + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})}\\ &\quad\times\bigg(\int_0^1 (1-s)^{\alpha-3}f(s,\ u(s),\ u'(s))ds - \beta\int_0^\eta (\eta-s)^{\alpha-3}f(s,\ u(s),\ u'(s))ds\bigg)\bigg|\\ &\le \bigg(\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)(1-\beta\eta^{\alpha-3})}\frac{1+\beta\eta^{\alpha-2}}{\alpha-2}\bigg)\\ &\quad\times\frac{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})(\alpha-2)}{(\alpha-2)(1-\beta\eta^{\alpha-3})+(\alpha-1)(1+\beta\eta^{\alpha-2})}d = d.\end{aligned}$$

Hence, $T:\overline{K(\gamma,d)}\to\overline{K(\gamma,d)}$. The fact that the constant function $u(t)=\frac{b}{\gamma_0}\in K(\gamma,\ \theta,\ \phi,\ b,\ c,\ d)$ and $\phi\left(\frac{b}{\gamma_0}\right)>b$ imply that

$$\left\{u\in K(\gamma,\ \theta,\ \phi,\ b,\ c,\ d|\phi(u)>b)\right\}\neq\emptyset.$$

For $u\in K(\gamma,\ \theta,\ \phi,\ b,\ c,\ d)$, we have $b\leq u(t)\leq\frac{b}{\gamma_0}$ and $|u'(t)|<d$ for $0\leq t\leq 1$. From assumption (A_2), we see

$$f(t,\ u(t),\ u'(t))>\frac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{\gamma_0\Big(-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})\Big)}b.$$

Thus

$$\begin{aligned}\phi(Tu)&=\min_{\eta\leq t\leq 1}\int_0^1 G(t,\ s)f(s,\ u(s),\ u'(s))ds\geq\gamma_0\int_0^1 G(1,\ s)f(s,\ u(s),\ u'(s))ds\\&\geq\gamma_0\Big(\int_0^\eta\Big(\frac{(1-s)^{\alpha-3}-\beta(\eta-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)ds\\&\quad+\int_\eta^1\Big(\frac{(1-s)^{\alpha-3}}{(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}-\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\Big)ds\Big)\\&\quad\times\frac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{\gamma_0\Big(-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})\Big)}b\\&=\gamma_0\Big(-\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}\Big(\frac{1}{\alpha-2}-\frac{\beta\eta^{\alpha-2}}{\alpha-2}\Big)\Big)\\&\quad\frac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{\gamma_0\Big(-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})\Big)}b=b,\end{aligned}$$

which means $\phi(Tu)>b,\ \forall u\in K\Big(\gamma,\ \theta,\ \phi,\ b,\ \frac{b}{\gamma_0},\ d\Big)$. These ensure that condition $(S1)$ of Lemma 2.7 is satisfied. Secondly, for all $u\in K(\gamma,\ \phi,\ b,\ d)$ with $\theta(Tu)>c$,

$$\phi(Tu)\geq\gamma_0\theta(Tu)>\gamma_0 c=\gamma_0\frac{b}{\gamma_0}=b.$$

Thus, condition (S_2) of Lemma 2.7 holds. Finally, we show that (S_3) also holds. We see that $\psi(0)=0<a$ and $0\notin R(\gamma,\psi,a,d)$. Suppose that $u\in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(x)=a$. Then, by assumption (A_3),

$$\begin{aligned}\psi(Tu)=&\Big(-\frac{1}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})}\Big(\frac{1}{\alpha-2}-\frac{\beta\eta^{\alpha-2}}{\alpha-2}\Big)\Big)\\&\times\frac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})}a=a.\end{aligned}$$

Thus, all conditions of Lemma 2.7 are satisfied. Hence problem (1.1)-(1.2) has at least three positive solutions $u_1,\ u_2,\ u_3$ satisfying $\max_{0\leq t\leq 1}|u_i'(t)|\leq d,\ i=1,2,3,\ b<\min_{\eta\leq t\leq 1}|u_1(t)|$, $a<\max_{0\leq t\leq 1}|u_2(t)|$, $\min_{\eta\leq t\leq 1}|u_2(t)|<b$, and $\max_{0\leq t\leq 1}|u_3(t)|\leq a$. □

4. Example

In this section, we present an example to illustrate the main theorems. Consider the nonlinear boundary value problem

$$D_{0+}^{3.7}u(t)+f(t,\ u(t),\ u'(t))=0,\ t\in[0,\ 1], \tag{4.1}$$

$$u(0)=u'(0)=u''(0)=0,\ u''(1)-\frac{3}{4}u\Big(\frac{1}{2}\Big)=0, \tag{4.2}$$

where $\alpha=3.7,\ \beta=\dfrac{3}{4},\ \eta=\dfrac{1}{2},\ \gamma_0\approx 0.1539$ and

$$f(t,u,v)=\begin{cases}\dfrac{1}{\pi^2}e^t+\dfrac{1}{5}\Big(u+1\Big)^4+\dfrac{1}{100}\sin\Big(\dfrac{v}{1000}\Big), & 0\le u\le 7\\ \dfrac{1}{\pi^2}e^t+\dfrac{4096}{5}+\dfrac{1}{100}\sin\Big(\dfrac{v}{1000}\Big), & u>7\end{cases}$$

We choose positive constants $a=1,\ b=5,\ d=1000$ and check that the nonlinear term $f(t,\ u,\ v)$ satisfies

(1) $f(t,\ u,\ v)<\dfrac{\Gamma(\alpha)(1-\beta\eta^{\alpha-3})(\alpha-2)}{(\alpha-2)(1-\beta\eta^{\alpha-3})+(\alpha-1)(1+\beta\eta^{\alpha-2})}d\approx 900.50,\quad (t,\ u,\ v)\in[0,\ 1]\times[0,\ 1000]\times[-1000,\ 1000]$;

(2) $f(t,\ u,\ v)>\dfrac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{\gamma_0\Big(-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})\Big)}b\approx 229.6301,\ (t,\ u,\ v)\in[0.5,\ 1]\times[5,\ 28.2805]\times[-1000,\ 1000]$;

(3) $f(t,\ u,\ v)<\dfrac{\Gamma(\alpha+1)(1-\beta\eta^{\alpha-3})(\alpha-2)}{-(\alpha-2)(1-\beta\eta^{\alpha-3})+\alpha(1-\beta\eta^{\alpha-2})}a\approx 7.0550,\quad (t,\ u,\ v)\in[0,\ 1]\times[0,\ 1]\times[-1000,\ 1000]$.

Then all assumptions of Theorem 3.5 are satisfied. Thus problem (4.1)-(4.2) has at least three positive solutions $u_1(t),\ u_2(t),\ u_3(t)$ satisfying $\max_{0\le t\le 1}|u_i'(t)|\le 1000,\ i=1,2,3,\ 5<\min_{1/2\le t\le 1}|u_1(t)|,\ 1<\max_{0\le t\le 1}|u_2(t)|,\ \min_{1/2\le t\le 1}|u_2(t)|<5$, and $\max_{0\le t\le 1}|u_3(t)|\le 1$.

Remark 4.1. Note that the first order derivative of function $u(t)$ is involved in the nonlinear term of the problem (4.1-4.2) explicitly. The earlier results for positive solutions of this kind of fractional differential equations (see, e.g., [3, 7, 8, 12]) are not applicable to this problem.

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