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GLOBAL NONEXISTENCE OF SOLUTIONS FOR A NONLINEAR EXTENSIBLE BEAM EQUATIONS IN A CLASS OF MODIFIED WOINOWSKY-KRIEGER MODELS WITH TIME DELAY

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Abstract. This paper is concerned with a nonlinear extensible beam equations in a class of modified Woinowsky-Krieger models with time delay. We prove the global nonexistence of the solutions. These results generalize and improve some earlier related results in the literature.
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1. Introduction

In this paper, we investigate the nonlinear extensible beam equation with time delay and initial-boundary conditions

$$\begin{cases} u_{tt}+\Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u+\mu_1 |u_t|^{r-1} u_t \\ +\mu_2 |u_t(x,t-\tau)|^{r-1} u_t(x,t-\tau) = |u|^{p-1} u, \quad (x,t)\in\Omega_T, \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega, \\ u_t(x,t-\tau)=f_0(x,t-\tau),\ x\in\Omega,\ t\in(0,\tau), \\ u=\frac{\partial u}{\partial v}=0, \end{cases} \tag{1.1}$$

where $\Omega\subset R^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Omega_T=\Omega\times(0,T)$, the nonlinear term $M(s)$ is defined by

$$M(s)=1+\beta s^{\gamma},\ \gamma\geq 0,\ \beta\geq 0,\ s\geq 0, \tag{1.2}$$

and the exponent p of the source term satisfies

$$1\leq r<p,\ 1<2\gamma+1<p<\infty,\ n\leq 4;\ 1<2\gamma+1<p\leq\frac{n+2}{n-4},\ n\geq 5. \tag{1.3}$$

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In 1950, Woinowsky-Krieger in [17] investigated the original version of the extensible beam equation. This model affords a description when the vibration is the dynamic buckling of a hinged extensible beam under an axial force, while it often depends on the fixing manner and the distance of the two ends of the beam. For the wide applications of this system, we refer to [18, 19, 20, 21, 22].

In the eighteenth century, the first equations with delay were considered by brothers Leonard Euler and Bernoulli. Since 1960, there have been appeared many surveys on the subject. In the middle of 1990s, robust control of systems with uncertain delay was started and led to the *delay bloom* in the beginning of the twenty-first century. Time-delay systems are also named systems with aftereffect or dead-time, equations with deviating argument, hereditary systems, or differential-difference equations. They belong to the class of functional differential equations which are infinite-dimensional, as opposed to ordinary differential equations. Time-delay often seems in many control systems, either in the state, the control input, or the measurements. There can be measurement, transport, or communication delays. Control systems often operate in the presence of delays, primarily due to the time it takes to acquire the information needed for decision-making, to create control decisions, and to execute these decisions. Sensors, actuators and field networks that are involved in feedback loops usually introduce delays. Delays are strongly involved in challenging areas of information technologies and communication: stability of highspeed communication networks or networked control systems. Models with delay: Sampled-data control and networked control systems, congestion control in communication networks, drilling system model, long line with tunnel diode and model of lasers, vehicular traffic flows, neural networks, population dynamics and epidemic models (see, [5, 10]).

In recent years, controlling the behavior of solutions for partial differential equations with delay effects has become an active research area. Generally, delay effects occur in numerous applications and practical problems such as thermal, economics, biological, chemical and physical. In many cases, time delay may effects the instability; see, e.g., [1, 2]. The Timoshenko equation is among the famous wave equation's model which describe extensible beam theory. It was introduced in 1921 by Timoshenko [12]. For detailed information on derivation the equation, we refer to [3, 9]. Datko et al. [2] indicated that a small delay in a boundary control is a source of instability. In [6], Nicaise and Pignotti studied the equation as follows

$$u_{tt} - \Delta u + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0,$$

where a_0 and a are positive real parameters. They obtained that, under the condition $0 \le a \le a_0$, the system is exponentially stable. In the case $a \ge a_0$, they obtained a sequence of delays that shows the solution is instable. In [13], Xu et al. obtained the same result similar to the [6] for the one space dimension by adopting the spectral analysis approach. In [7], Nicaise et al. studied the wave equation in one space dimension in the case of time-varying delay. In that work, they showed an exponential stability result under the condition

$$a \le \sqrt{1-d}\,a_0,$$

where d is a constant such that

$$\tau'(t) \le d < 1, \forall t > 0.$$

In [4], Feng studied the following equation

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

He obtained well-posedness of solutions with $|\mu_2| \leq \mu_1$, and proved decay results under the assumption $|\mu_2| < \mu_1$.

Park [8] investigated the following equation

$$u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \sigma(t)\int_0^t g(t-s)\Delta u(s)\,ds + a_0 u_t + a_1 u_t(t-\tau(t)) = 0.$$

He established decay results under the assumption $|a_1| < \sqrt{1-d}a_0$. In recent years, authors also investigated the related equations; see, e.g., [14, 15, 16].

Our main aim in this paper is to follow the work in [22] to establish the global nonexistence of the solutions for equation (1.1). There is no research, to our best knowledge, related to the nonlinear extensible beam equation with delay term $(\mu_2 |u_t(x,t-\tau)|^{r-1} u_t(x,t-\tau))$. The outline of this paper is as follows. In Section 2, we state some assumptions and the transformation. In Section 3, we prove the global nonexistence results. Section 4 concludes this paper.

2. Preliminaries

In this section, we give some assumptions and the transformation. We show some notations, functionals and partial results given in [21]. Let $\|.\|_p$ indicate the norm in $L^p(\Omega)$ and $(.,.)$ present the inner product in $L^2(\Omega)$. We use the following notations

$$H = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) \mid = \frac{\partial u}{\partial v} = 0 \text{ on } \partial\Omega \right\},\ \|u\|_H^2 = \|\nabla u\|^2 + \|\Delta u\|^2.$$

We indicate by C or C_i a generic constant that may vary from line to line even in the same formula. Denoting by λ_1 the first eigenvalue of the bi-harmonic operator with boundary condition $u|_{\partial\Omega} = \frac{\partial u}{\partial v}|_{\partial\Omega} = 0$, we have $\lambda_1 \|u\|^2 \leq \|\Delta u\|^2$ and $\lambda_1^{\frac{1}{2}} \|\nabla u\|^2 \leq \|\Delta u\|^2$ for all $u \in H$. Similar to [6], we introduce a new function

$$z(x,k,t) = u_t(x,t-\tau k),\ x \in \Omega,\ k \in (0,1).$$

We see that z satisfies $\tau z_t(x,k,t) + z_k(x,k,t) = 0$, $x \in \Omega$, $k \in (0,1)$. Therefore, system (1.1) takes the form

$$\begin{cases} u_{tt} + \Delta^2 u - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 |u_t|^{r-1} u_t \\ \quad + \mu z(x,1,t)^{r-1} |z(x,1,t)| = |u|^{p-1} u, \quad (x,t) \in \Omega_T, \\ \tau z_t(x,k,t) + z_k(x,k,t) = 0, \text{ in } \Omega \times (0,1) \times (0,\tau), \\ z(x,0,t) = u_t(x,t) \qquad\qquad \text{in } \Omega \times (0,\tau), \\ z(x,k,0) = f_0(x,-\tau k),\ x \in \Omega, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega. \end{cases} \tag{2.1}$$

We give the following potential functional and Nehari functional

$$\begin{aligned} J(u) = {} & \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u\|^{2\gamma+2} \\ & + \frac{\zeta}{r+1}\int_\Omega \int_0^1 |z(x,k,t)|^{r+1}\,dk\,dx - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \end{aligned} \tag{2.2}$$

and

$$I(u) = \|\nabla u\|^2 + \|\Delta u\|^2 + \beta\|\nabla u\|^{2\gamma+2} + \zeta\int_\Omega\int_0^1 |z(x,k,t)|^{r+1}\,dk\,dx - \|u\|_{p+1}^{p+1}, \tag{2.3}$$

where ζ is a positive constant defined to satisfy

$$\tau\mu_2 r < \zeta < \tau(\mu_1 - \mu_2). \tag{2.4}$$

We define the energy functional for problem (2.1) as follows

$$\begin{aligned}
E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2 \\
&\quad + \frac{\beta}{2(\gamma+1)}\|\nabla u\|^{2\gamma+2} + \frac{\zeta}{r+1}\int_\Omega\int_0^1 |z(x,k,t)|^{r+1}\,dkdx - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\
&= \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\left(\|\nabla u\|^2 + \|\Delta u\|^2\right) + \left(\frac{1}{2(\gamma+1)} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} \\
&\quad + \zeta\left(\frac{1}{r+1} - \frac{1}{p+1}\right)\int_\Omega\int_0^1 |z(x,k,t)|^{r+1}\,dkdx + \frac{1}{p+1}I(u).
\end{aligned} \tag{2.5}$$

Using (2.4) and the assumption that $\mu_1 > \mu_2 > 0$, we see that

$$\begin{aligned}
E'(t) &= -\left(\mu_1 - \frac{\zeta}{\tau(r+1)} - \frac{\mu_2}{r+1}\right)\|u_t\|_{r+1}^{r+1} - \left(\frac{\zeta}{\tau(r+1)} - \frac{\mu_2 r}{r+1}\right)\int_\Omega |z(x,1,t)|^{r+1}\,dx \\
&\le -C_E\left(\|u_t\|_{r+1}^{r+1} + \int_\Omega |z(x,1,t)|^{r+1}\,dx\right) \le 0,
\end{aligned} \tag{2.6}$$

for some $C_E > 0$. Here, we choose $\mu_1 - \frac{\zeta}{\tau(r+1)} - \frac{\mu_2}{r+1} > 0$ and $\frac{\zeta}{\tau(r+1)} - \frac{\mu_2 r}{r+1} > 0$. From $I(u)$, we give the stable set $W = \{u \in H | I(u) > 0\} \cup \{0\}$ and the unstable set $V = \{u \in |I(u) < 0\}$, respectively. The depth of the potential well is defined by

$$d = \inf_{u\in\mathcal{N}} J(u) = \inf_{u\in H\setminus\{0\}} \sup_{\lambda\ge 0} J(\lambda u)$$

where $\mathcal{N}$ is the Nehari manifold $\mathcal{N} = \{u \in H\setminus\{0\}\,|\, I(u) = 0\}$. It follows by [21, Lemma 2.3] that $d = \frac{p-1}{2(p+1)}\left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}}$, where C is the best embedding constant from H into $L^{p+1}(\Omega)$, that is, $\|u\|_{p+1} \le C\|u\|_H$.

We now state the following lemma.

Lemma 2.1. *Let $J(u) \le d$. Then $I(u) < 0$ if and only if $\|u\|_H > \lambda_* = C^{-\frac{p-1}{p+1}}$.*

Proof. If $\|u\|_H > \lambda_* = C^{-\frac{p-1}{p+1}}$, we see from (2.2), (2.3), and the definition of d that

$$\begin{aligned}
J(u) &= \frac{p-1}{2(p+1)}\left(\|\nabla u\|^2 + \|\Delta u\|^2\right) + \left(\frac{1}{2(\gamma+1)} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} \\
&\quad \left(\frac{\zeta}{r+1} - \frac{\zeta}{p+1}\right)\int_\Omega\int_0^1 |z(x,k,s)|^{r+1}\,dkdx + \frac{1}{p+1}I(u) \\
&\le d = \frac{p-1}{2(p+1)}\left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}} = \frac{p-1}{2(p+1)}\lambda_*^2.
\end{aligned}$$

Therefore, we can obtain $I(u) < 0$ from the fact $p > 2\gamma + 1$ and $\beta \ge 0$. Letting $I(u) < 0$, we obtain

$$\|u\|_H^2 + \beta\|\nabla u\|^{2(\gamma+1)} < \|u\|_{p+1}^{p+1} \le C^{p+1}\|u\|_H^{p+1},$$

which implies that $\|u\|_H > C^{-\frac{p-1}{p+1}} = \lambda_*$. Hence, the proof is completed. □

To achieve our main result, we give some results proved in [21] which are needed later.

Theorem 2.2. [21] *(Local existence). Suppose $u_0(x) \in H$ and $u_1(x) \in H_0^1(\Omega)$. Hence, problem (2.1) has a unique local solution $u = u(x,t) \in C([0,T], H)$ satisfying $u_t \in C\left([0,T]; H_0^1(\Omega)\right) \cap L^{r+1}\left([0,T], L^{r+1}(\Omega)\right)$ for some $T > 0$.*

Combining Theorem 4.3 and Theorem 5.3 in [21], the global nonexistence when $E(0) \le d$ can be stated as follows.

Theorem 2.3. [21] *(Global nonexistence when $E(0) \le d$). Let $u_0(x) \in H$ and $u_1(x) \in H_0^1(\Omega)$ be given functions. Let $E(0) \le d$ and $u_0 \in V$. Then, the solution u to problem (2.1) blows up in finite time.*

Theorem 2.4. [21] *(Global nonexistence when $E(0) > d$ and $r = 1$). Let $u_0(x) \in H$ and $u_1(x) \in H_0^1(\Omega)$ hold. Suppose that $E(0) > 0$, $I(u_0) - \|u_0\|^2 < 0$ and $\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) > \frac{4(p+1)}{(p-1)\widetilde{c}}$, where $\widetilde{c} = \min\{1, C\}$ and C is the best embedding constant from H into $H_0^1(\Omega)$. Then, the solution u to problem (2.1) with $r = 1$ blows up in finite time.*

3. Global Nonexistence

In this section, we establish the global nonexistence results. Let (u,z) be the solution obtained in Theorem 2.2, whose maximal existence time is T_m.

Theorem 3.1. *Let (1.2) and (1.3) hold. Suppose the initial data $u_0(x) \in H$ and $u_1(x) \in H_0^1(\Omega)$ satisfies one of the following conditions*

(i) $E(0) < 0$;

(ii) $0 \le E(0) < \frac{1}{B}\int_\Omega u_0 u_1 dx$,

where B is a positive constant given in (3.1). Then, the solution (u,z) to problem (2.1) blows up in finite time.

Proof. (i) The blow-up result for the case $E(0) < 0$ is a direct conclusion of Theorem 2.3. $E(0) < 0$ and (2.5) satisfies $I(u_0) < 0$, i.e., $u_0 \in V$.

(ii) First, we suppose that the energy $E(0) \ge 0$ for all $t \in [0, T_m)$. On the other hand, there exist a $t_0 \in [0, T_m)$ such that $E(t_0) < 0$. Tking t_0 as the initial time, from case (i), we see that $u(t)$ blows up in finite time, which is a contradiction. We divide the proof of (ii) into two steps as follows:

Step 1. Show the following claim, which is motivated by [11].

Claim: Assume that $u_0(x) \in H$ and $u_1(x) \in H_0^1(\Omega)$ hold, and (u,z) is a weak solution to problem (2.1). We see that there exist positive constants A and B such that

$$\frac{d}{dt}\left(\int_\Omega u u_t dx - BE(t)\right) \ge A\left(\int_\Omega u u_t dx - BE(t)\right) \text{ for all } t \in [0, T_m). \tag{3.1}$$

Proof of the claim. From the first equation of problem (2.1), we see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u u_t dx &= \|u_t\|^2 + \int_\Omega u u_{tt} dx \\
&= \|u_t\|^2 - \|\Delta u\|^2 - \|\nabla u\|^2 - \beta \|\nabla u\|^{2\gamma+2} \\
&- \mu_1 \int_\Omega |u_t|^{r-1} u_t u dx - \mu_2 \int_\Omega |z(1)|^{r-1} z(1) u dx + \|u\|_{p+1}^{p+1}.
\end{aligned}$$

Adding and subtracting $(p+1)(1-\theta)E(t)$ with $\theta\in(0,1)$ in the right hand side of the above equation, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega uu_t dx =& -\frac{(p+1)(1-\theta)+2}{2}\|u_t\|^2+\frac{(p+1)(1-\theta)-2}{2}\left(\|\Delta u\|^2+\|\nabla u\|^2\right)\\ &+\frac{(p+1)(1-\theta)-2(\gamma+1)}{2(\gamma+1)}\beta\|\nabla u\|^{2(\gamma+1)}-\mu_1\int_\Omega |u_t|^{r-1}u_t u dx\\ &-\mu_2\int_\Omega |z(1)|^{r-1}z(1)udx+\frac{(p+1)(1-\theta)\zeta}{r+1}\int_\Omega\int_0^1 |z(x,k,s)|^{r+1}dkdx\\ &-(p+1)(1-\theta)E(t)+\theta\|u\|_{p+1}^{p+1}. \end{aligned}\tag{3.2}$$

Utilizing Young's inequality with $\varepsilon\le 1$, we have

$$\left|\mu_1\int_\Omega |u_t|^{r-1}u_t u dx\right|\le\frac{r}{(r+1)\varepsilon}\|u_t\|_{r+1}^{r+1}+\frac{\varepsilon^r}{r+1}\|u\|_{r+1}^{r+1},\tag{3.3}$$

$$\left|\mu_2\int_\Omega |z(1)|^{r-1}z(1)udx\right|\le\frac{r}{(r+1)\varepsilon}\|z(1)\|_{r+1}^{r+1}+\frac{\varepsilon^r}{r+1}\|u\|_{r+1}^{r+1}.\tag{3.4}$$

By using the interpolation inequality for L^p-norms, we obtain

$$\|u\|_{r+1}^{r+1}\le s\|u\|^2+(1-s)\|u\|_{p+1}^{p+1}\text{ with } s=\frac{p-r}{p-1}\in(0,1].\tag{3.5}$$

Putting (3.3)-(3.5) into (3.2), we see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega uu_t dx\ge& \frac{(p+1)(1-\theta)+2}{2}\|u_t\|^2+\frac{(p+1)(1-\theta)-2}{2}\|\Delta u\|^2\\ &+\frac{(p+1)(1-\theta)-2-\varepsilon}{2}\|\nabla u\|^2+\frac{(p+1)(1-\theta)-2(\gamma+1)}{2(\gamma+1)}\beta\|\nabla u\|^{2(\gamma+1)}\\ &+\frac{(p+1)(1-\theta)\zeta}{r+1}\int_\Omega\int_0^1|z(x,k,s)|^{r+1}dkdx+\left(\theta-\frac{\varepsilon^r(1-s)}{r+1}\right)\|u\|_{p+1}^{p+1}\\ &-\frac{\varepsilon^r s}{r+1}\|u\|^2-\frac{r}{(r+1)\varepsilon}\|u_t\|_{r+1}^{r+1}-\frac{r}{(r+1)\varepsilon}\|z(1)\|_{r+1}^{r+1}\\ &-(p+1)(1-\theta)E(t).\end{aligned}\tag{3.6}$$

By choosing, $\theta=\frac{\varepsilon^r(1-s)}{r+1}$ and $\varepsilon\le\delta_0=\min\left\{1,\left[\frac{(p-2\gamma-1)(r+1)}{(p+1)(1-s)}\right]^{\frac{1}{r}}\right\}$ ($\delta_0=1$ when $r=1$, i.e., $s=1$), we get $(p+1)(1-\theta)-2(\gamma+1)\ge 0$ by $p>2\gamma+1$. If

$$g(\varepsilon)=(p+1)(1-\theta-2-\varepsilon)=(p+1)\left(1-\frac{\varepsilon^r(1-s)}{r+1}\right)-2-\varepsilon\text{ for }\varepsilon\in(0,\delta_0],$$

then

$$g'(\varepsilon)=-\frac{(p+1)r\varepsilon^{r-1}(1-s)}{r+1}-1<0\text{ for }\varepsilon(0,\delta_0],$$

which implies that $g(\varepsilon)$ is strictly decreasing in the interval $(0,\delta_0]$. By $\lim_{\varepsilon\to 0}g(\varepsilon)=p-1>0$ and the continuity of $g(\varepsilon)$, we see that there exists $\delta_1>0$ such that

$$g(\varepsilon)>0\text{ for all }\varepsilon(0,\delta_1).\tag{3.7}$$

For $0<\varepsilon<\delta_1<\delta_0$, by using (2.6), (3.7), and the embedding inequality $\|u\|_2 \le B_1 \|\nabla u\|_2$, and noticing the choice of θ, we can rewrite (3.6)

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega uu_t dx - \frac{r}{(r+1)\varepsilon}E(t)\right) \\ &\ge \frac{(p+1)(1-\theta)+2}{2}\|u_t\|^2 + h(\varepsilon)\|u\|^2 - (p+1)(1-\theta)E(t), \end{aligned} \tag{3.8}$$

where

$$h(\varepsilon)=\left[\frac{(p+1)\left(1-\frac{\varepsilon^r(1-s)}{r+1}\right)-2}{2}-\frac{\varepsilon}{2}\right]\frac{1}{B_1^2}-\frac{\varepsilon^r(1-s)}{r+1} \text{ for } \varepsilon\in(0,\delta_1).$$

Similar to (3.7), we can derive that there exists $\delta_2\in(0,\delta_1)$ such that $h(\varepsilon)>0$ for all $\varepsilon\in(0,\delta_2)$. From Cauchy-Schwarz inequality, (3.8) can be rewritten as

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega uu_t dx - \frac{r}{(r+1)\varepsilon}E(t)\right) \\ &\ge \sqrt{2[(p+1)(1-\theta)+2]h(\varepsilon)}\int_\Omega uu_t dx - (p+1)(1-\theta)E(t) \\ &= A(\varepsilon)\left(\int_\Omega uu_t dx - B(\varepsilon)E(t)\right), \end{aligned} \tag{3.9}$$

where

$$A(\varepsilon)=\sqrt{2[(p+1)(1-\theta)+2]h(\varepsilon)},$$
$$B(\varepsilon)=\frac{(p+1)(1-\theta)}{A(\varepsilon)}=\frac{(p+1)(1-\theta)}{\sqrt{2[(p+1)(1-\theta)+2]h(\varepsilon)}}.$$

It follows that

$$\lim_{\varepsilon\to0^+}B(\varepsilon)=\frac{(p+1)B_1}{\sqrt{(p+3)(p-1)}}<+\infty,\ \lim_{\varepsilon\to0^+}\frac{r}{(r+1)\varepsilon}=+\infty,$$

which implies that there exists a sufficiently small $\delta_3\in(0,\delta_2)$ such that

$$B(\varepsilon)\le\frac{r}{(r+1)\varepsilon} \text{ for any } \varepsilon\in(0,\delta_3).$$

Therefore, for any fixed sufficiently small $\varepsilon_0\in(0,\delta_3)$, (3.9) can be rewritten as

$$\frac{d}{dt}\left(\int_\Omega uu_t dx - \frac{r}{(r+1)\varepsilon_0}E(t)\right)\ge A(\varepsilon_0)\left(\int_\Omega uu_t dx - \frac{r}{(r+1)\varepsilon_0}E(t)\right), \tag{3.10}$$

which implies (3.1) holds with $A=A(\varepsilon_0)$ and $B=\frac{r}{(r+1)\varepsilon_0}$. By (3.10) and the above discussion, we can easily infer that (3.1) also satisfies for the linear damping case ($r=1$).

Step 2. Assume that (u,z) is a global solution to problem (2.1). By (3.1) and Gronwall's inequality, we have

$$\int_\Omega uu_t dx - BE(t)\ge\left(\int_\Omega u_0u_1dx - BE(0)\right)e^{At}>0 \text{ for } t\ge0, \tag{3.11}$$

where the assumption $0\le E(0)<\frac{1}{B}\int_\Omega u_0u_1dx$ is used. From $0\le E(t)\le E(0)$,

$$\frac{d}{dt}\|u_t\|^2=2\int_\Omega uu_t dx,$$

and (3.11), we obtain

$$\begin{aligned}\|u_t\|^2 &= \|u_0\|^2 + 2\int_0^t \int_\Omega u u_\tau dx d\tau \\ &\geq \|u_0\|^2 + 2\int_0^t \left(\int_\Omega u_0 u_1 dx - BE(0)\right) e^{A\tau} d\tau \\ &= \|u_0\|^2 + \frac{2}{A}\left(e^{At}-1\right)\left(\int_\Omega u_0 u_1 dx - BE(0)\right).\end{aligned} \tag{3.12}$$

Moreover, from (2.6) and Hölder's inequality, we have

$$\begin{aligned}\|u\|_2 &= \left\|u_0 + \int_0^t u_t(\tau)\, d\tau\right\|_2 \leq \|u_0\|_2 + \int_0^t \|u_t(\tau)\|_2\, d\tau \\ &\leq \|u_0\|_2 + |\Omega|^{\frac{r-1}{2(r+1)}} \int_0^t \|u_t(\tau)\|_{r+1}\, d\tau \\ &\leq \|u_0\|_2 + |\Omega|^{\frac{r-1}{2(r+1)}} t^{\frac{r}{r+1}} \left(\int_0^t \|u_t(\tau)\|_{r+1}^{r+1}\, d\tau\right)^{\frac{1}{r+1}} \\ &\leq \|u_0\|_2 + |\Omega|^{\frac{r-1}{2(r+1)}} t^{\frac{r}{r+1}} \left(E(0) - E(t)\right)^{\frac{1}{r+1}} \\ &\leq \|u_0\|_2 + |\Omega|^{\frac{r-1}{2(r+1)}} t^{\frac{r}{r+1}} \left(E(0)\right)^{\frac{1}{r+1}},\end{aligned}$$

which is a contradiction to (3.12) for t sufficiently large. We use the assumption that (u,z) is a global solution to problem (2.1) and $E(t) \geq 0$. Therefore, $T_m < \infty$ and (u,z) blows up in finite time. Hence, the proof is completed. □

4. Conclusion

In this paper, we demonstrated that the existence of finite time blow-up solutions with arbitrary initial energy level (including $E(0) > d$). To the best of our knowledge, there were no global nonexistence results for the nonlinear extensible beam equation with time delay and source terms. We also obtained the global nonexistence of solutions, under sufficient conditions.

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