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INFINITE FAMILIES OF HOMOCLINIC SOLUTIONS TO NONSMOOTH DAMPED VIBRATIONS WITH p-LAPLACIAN NONLINEARITY

MOHSEN TIMOUMI

Department of Mathematics, Faculty of Sciences of Monastir, Monastir 5000, Tunisia

Abstract. This paper investigates the existence of infinitely many homoclinic solutions for damped vibration systems involving the p-Laplacian. The systems under consideration are of the form

$$\frac{d}{dt}\Big(|\dot{u}(t)|^{p-2}\dot{u}(t)\Big)+q(t)|\dot{u}(t)|^{p-2}\dot{u}(t)+\nabla V(t,u(t))=0,\ t\in\mathbb{R},$$

where $p>1$, $q\in C(\mathbb{R},\mathbb{R})$, and $V\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$. The potential $V(t,u)$ is a combination of two functions where the associated energy functional is not continuously differentiable and fails to satisfy the Palais-Smale condition. By employing variational methods and the critical point theory, we establish the existence of infinitely many homoclinic solutions. Our results extend previous works on damped vibration systems, highlighting the impact of non-smooth energy functionals. The findings contribute to the understanding of the dynamical behavior of solutions to non-conservative systems modeled by the p-Laplacian.

Keywords. Clark's theorem; Homoclinic solutions; Nonsmooth damped vibration systems, Variational methods.

1. INTRODUCTION

Consider the following damped vibration system with the p-Laplacian

$$(\mathscr{DV}),\ \ \frac{d}{dt}\Big(|\dot{u}(t)|^{p-2}\dot{u}(t)\Big)+q(t)|\dot{u}(t)|^{p-2}\dot{u}(t)-a(t)|u(t)|^{p-2}u(t)+\nabla W(t,u(t))=0,\ t\in\mathbb{R}$$

where $p\geq 2$, $q,a\in C(\mathbb{R},\mathbb{R})$, and $W:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a continuous function which is differentiable with respect to its second argument with the continuous derivative $\nabla W(t,x)=\frac{\partial W}{\partial x}(t,x)$. Systems of second-order differential equations involving the p-Laplacian attracted considerable interest in recent years. Such systems are particularly relevant in areas such as non-Newtonian fluid mechanics and nonlinear filtration theory [5]. More recently, researchers have turned their

E-mail address: mtimoumi12@gmail.com.

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attention to ordinary differential systems driven by the p-Laplacian, utilizing methods from the critical point theory and variational approaches to study their properties and behaviors.

A solution u to system $(\mathscr{DV})$ is said to be homoclinic to 0 if $u(t) \to 0$ as $|t| \to \infty$. If $u \neq 0$, it is called a nontrivial homoclinic solution. Homoclinic solutions are of crucial importance in the analysis of chaotic dynamics. Specifically, if a system has homoclinic solutions that intersect transversely, it indicates the system's chaotic nature. On the other hand, smoothly connected homoclinic orbits suggest that the system is sensitive to perturbations, which may lead to chaotic behavior. Thus studying the existence of homoclinic orbits of system $(\mathscr{DV})$ emanating from 0 is both mathematically intriguing and practically significant.

For the case where $p = 2$ and $q \neq 0$, system $(\mathscr{DV})$ reduces to:

$$\ddot{u}(t) + q(t)\dot{u}(t) - a(t)u(t) + \nabla W(t, u(t)) = 0,$$

which is a special case of the classical damped vibration system:

$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix. Over the past decade, the existence and multiplicity of fast homoclinic solutions for system (1.1) have been the subject of many studies using variational methods and the critical point theory (see, e.g., [1, 3, 6, 11, 12, 13, 14, 15, 16, 17, 19]). A key challenge in obtaining fast homoclinic solutions is the lack of compactness of the embeddings involved. To address this issue, various conditions on matrix L were introduced. In contrast, for cases where $p > 1$ and $q = 0$, system $(\mathscr{DV})$ was studied under specific assumptions on the potential function $V(t, x)$, where

$$V(t, x) = -\frac{1}{p}a(t)\,|x|^p + W(t, x).$$

For such a system, system $(\mathscr{DV})$ becomes

$$\frac{d}{dt}\Big(|\dot{u}(t)|^{p-2}\,\dot{u}(t)\Big) - a(t)\,|u(t)|^{p-2}\,u(t) + \nabla W(t, u(t)) = 0,\ t \in \mathbb{R}. \tag{1.2}$$

Although progress has been made in understanding the existence of homoclinic solutions for such systems, especially under superquadratic and subquadratic growth conditions on potential $W(t, x)$, challenges remain due to issues related to Sobolev embeddings and the Palais-Smale condition. To overcome these obstacles, certain coercivity conditions on $a(t)$ and growth constraints on ∇W have been introduced (see; see, e.g., [7, 8, 18, 20, 21]). These conditions ensure that the energy functional is continuously differentiable in appropriate function spaces, and its critical points correspond to homoclinic solutions.

For the more general case where $p > 1$ and $q \neq 0$, only a few results concerning the existence of homoclinic solutions for $(\mathscr{DV})$ were obtained (see [4, 10]). In [4], the authors explored the existence and multiplicity of fast homoclinic solutions for $(\mathscr{DV})$ when $W(t, x)$ is a combination of functions satisfying the superquadratic and subquadratic Ambrosetti-Rabinowitz conditions, using the mountain pass and symmetric mountain pass theorems. In [10], the monotonicity trick of Jeanjean and the concentration compactness principle were used to prove the existence of infinitely many fast homoclinic solutions for system $(\mathscr{DV})$. These results hold when $W(t, x)$ is periodic in the first variable and superquadratic in the second, without satisfying the Ambrosetti-Rabinowitz condition or exhibiting quadratic asymptotic behavior in the second variable.

In all the aforementioned studies, the associated energy functional of system $(\mathscr{DV})$ is continuously differentiable, with its critical points corresponding to homoclinic solutions. However, in the present work, we consider the following potential function:

$$V(t,x) = -\frac{1}{p}\left[1+\frac{1}{p}\cos(\frac{1}{|x|^{\gamma}})\right]|x|^{p} + d(t)\,|x|^{\sigma}. \tag{1.3}$$

Let $q \in C(\mathbb{R},\mathbb{R})$ be such that $Q(t) = \int_0^t q(s)ds$ is bounded. Then for u belonging to space

$$W_Q^{1,p}(\mathbb{R}) = \left\{u : \mathbb{R} \to \mathbb{R}^N measurable / \int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^p\,dt < \infty, \int_{\mathbb{R}} e^{Q(t)}\,|\dot{u}(t)|^p\,dt < \infty\right\}$$

and v an indefinitely differentiable function from $\mathbb{R}$ into $\mathbb{R}^N$ with compact support, the energy functional J associated to $(\mathscr{DV})$ is given by:

$$J(u) = \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}\,|\dot{u}(t)|^p\,dt + \int_{\mathbb{R}} e^{Q(t)} K(t,u(t))dt - \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt.$$

The derivative $J'(u)$ is

$$\begin{aligned}
J'(u)v = &\int_{\mathbb{R}} e^{Q(t)}\,|\dot{u}(t)|^{p-2}\,\dot{u}(t)\cdot\dot{v}(t)dt + \int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^{p-2}\,u(t)\cdot v(t)dt\\
&+\frac{1}{2}\int_{\mathbb{R}} e^{Q(t)}\cos(|u(t)|^{-\gamma})u(t)\cdot v(t)dt\\
&+\frac{\gamma}{4}\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^{-\gamma}\,sin(|u(t)|^{-\gamma})u(t)\cdot v(t)dt - \sigma\int_{\mathbb{R}} e^{Q(t)} d(t)\,|u(t)|^{\sigma-2}\,u(t)\cdot v(t)dt.
\end{aligned}$$

Let $w(t) = \frac{1}{1+|t|^{\frac{1}{p-\gamma}}}$, $u(t) = (w(t),0,...,0)$ and $v(t) = (w(t)\sin(w^{-\gamma}(t)),0,...,0)$. A straightforward computation shows that $u,v \in W_Q^{1,p}(\mathbb{R})$. On the other hand, we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{Q(t)}\,|u(t)|^{-\gamma}\sin(|u(t)|^{-\gamma})u(t)\cdot v(t)dt &= \int_{\mathbb{R}} e^{Q(t)}\,|w(t)|^{2-\gamma}\sin^2(|w(t)|^{-\gamma})dt\\
&\geq m_0\int_{\mathbb{R}} |w(t)|^{2-\gamma}\sin^2(|w(t)|^{-\gamma})dt\\
&= 2m_0\frac{2-\gamma}{\gamma}\int_1^{\infty} s^{-\frac{1}{\gamma}}\left(s^{\frac{1}{\gamma}}-1\right)^{1-\gamma}\sin^2(s)ds\\
&= +\infty.
\end{aligned}$$

Therefore, J cannot be continuously differentiable on $W_Q^{1,p}(\mathbb{R})$.

In this paper, for the first time, we are interested in the existence of infinitely many pairs of homoclinic solutions for $(\mathscr{DV})$ when $p > 1$, $q(t) \neq 0$, and the function V satisfies some conditions covering the cases as in (1.3). More specifically, we examine scenarios where no growth constraints are imposed on ∇V. Taking $V(t,x) = -K(t,x) + W(t,x)$, where $K,W : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ are continuous functions, differentiable in the second variable with continuous derivatives $\nabla K(t,x)$ and $\nabla W(t,x)$, we obtain the following results:

Theorem 1.1. *Assume the following conditions*

(Q) $\quad$ $q \in C(\mathbb{R},\mathbb{R})$ *and* $Q(t) = \int_0^t q(s)ds$ *is bounded from below with* $m_0 = \inf_{t\in\mathbb{R}} e^{Q(t)}$;

(H_1) *There exist constants* $1 < \nu \leq p$ *and* $a > 0$ *such that* $K(t,x) \geq a|x|^{\nu}$ *for all* $(t,x) \in \mathbb{R}\times\mathbb{R}^N$;
(H_2) *There exist* $\sigma \in]1,\nu[$, $1 \leq \alpha \leq \frac{p}{p-\sigma}$ *and* $d \in L^{\alpha}_{Q}(\mathbb{R},\mathbb{R}^+)$ *such that*

$$|W(t,x)| \leq d(t)\,|x|^{\sigma}, \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N;$$

(H_3) $V(t,-x) = V(t,x)$ *for all* $(t,x) \in \mathbb{R}\times\mathbb{R}^N$;
(H_4) *There exist constants* $\tau \in]1,p[$ *and* $l \in \mathbb{R}^*_+ \cup \{+\infty\}$ *such that* $\lim_{|x|\to 0} \frac{V(t,x)}{|x|^{\tau}} = l$ *uniformly in* $t \in \mathbb{R}$.

Under these conditions, system $(\mathscr{DV})$ *has infinitely many pairs of nontrivial homoclinic solutions.*

Theorem 1.2. *Assume that* (Q), (H_1), (H_2), (H_3) *and the following condition are satisfied*

(H_4')
$$\lim_{|x|\to 0} \frac{V(t,x)}{|x|^p} = +\infty, \text{ uniformly in } t \in \mathbb{R}.$$

Then system $(\mathscr{DV})$ *has infinitely many pairs of nontrivial homoclinic solutions.*

Theorem 1.3. *Assume that* (Q), (H_1), (H_2), (H_3) *and the following conditions are satisfied*
(H_1') *There exist positive constants* b, R *such that* $K(t,x) \leq b|x|^{\nu}$ *for all* $t \in \mathbb{R}$, $|x| \leq R$;
(H_5) *There exist constants* $\tau \in]1,\nu[$, $l \in \mathbb{R}^*_+ \cup \{+\infty\}$, $t_0 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^{\tau}} = l, \text{ uniformly in } t \in]t_0 - r, t_0 + r[.$$

Then system $(\mathscr{DV})$ *possesses infinitely many pairs of nontrivial homoclinic solutions.*

Theorem 1.4. *Assume that* (Q), (H_1), (H_1'), (H_2), (H_3) *and the following condition are satisfied*
(H_5') *There exist constants* $t_0 \in \mathbb{R}$ *and* $r > 0$ *such that*

$$\lim_{|x|\to 0} \frac{W(t,x)}{|x|^{\nu}} = +\infty, \text{ uniformly in } t \in]t_0 - r, t_0 + r[.$$

Then system $(\mathscr{DV})$ *possesses infinitely many pairs of nontrivial homoclinic solutions.*

Remark 1.5. If $Q(t) = \int_0^t q(s)ds \to +\infty$ as $|t| \to \infty$, then a homoclinic solution to $(\mathscr{DV})$ is referred to as a fast homoclinic solution.

Remark 1.6. Under assumptions $(H_1) - (H_5)$, $(H'1)$, $(H'4)$, and (H_5'), the nonlinearity ∇V does not adhere to any growth constraints. Consequently, the energy functional associated with $(\mathscr{DV})$ is continuous but lacks continuous differentiability and does not fulfill the Palais-Smale condition, as previously demonstrated.

The structure of the paper is as follows. Section 2 presents preliminary results that lay the foundation for the subsequent sections. The lasection, Section 3, is dedicated to the proof of the main results.

2. PRELIMINARIES

In order to prove our main results, we recall some definitions and basic results. Let X be a Banach space and X' be its dual space. The weak convergence in X is denoted by " $\rightharpoonup$ ".

Weakly Sequentially Lower Semicontinuous Functional: Let J be a functional defined on X. We say that J is weakly sequentially lower semicontinuous if, for any $u \in X$ and any sequence $(u_n) \subset X$ satisfying $u_n \rightharpoonup u$, $\liminf_{n\to\infty} J(u_n) \geq J(u)$.

E-Differentiable Functional: Let J be a continuous functional defined on X, and let E be a dense subspace of X. We say that J is $E-$differentiable if the following conditions hold:

a) for all $u \in X$ and $v \in E$, the derivative of J at u in the direction v, denoted by $< J'(u), v >$, exists, that is

$$\left\langle J'(u), v\right\rangle = \lim_{s\to 0} \frac{J(u+sv)-J(u)}{s},$$

b) the mapping J' satisfies

(i) $v \mapsto \langle J'(u), v\rangle$ is linear in E for all $u \in X$,

(ii) $u \mapsto \langle J'(u), v\rangle$ is continuous in X for all $v \in E$, that is $\langle J'(u_n), v\rangle \to \langle J'(u), v\rangle$ as $u_n \to u$ in X.

Critical Points: A point $u \in X$ is said to be a critical point of J if $|J'(u)| = 0$, where

$$\left|J'(u)\right| = \sup\left\{\left\langle J'(u), v\right\rangle / v \in E,\ \|v\| = 1\right\}$$

and $\|.\|$ denotes the norm in X.

Now, we are in a position to recall a variant of Clark's Theorem [2].

Theorem 2.1. *Let X be a separable and reflexive Banach space with norm $\|.\|$ and let E be a dense subspace of X. Assume that J is a continuous functional defined on X wich is $E-$differentiable. Suppose that J satisfies the following conditions*

(A_1) *J is an even functional, i.e., $J(-u) = J(u)$ for every $u \in X$, and it is bounded from below;*

(A_2) *If $u \in X$, $(u_n) \subset X$, $|J'(u_n)| \to 0$ and $u_n \rightharpoonup u$ as $n \to \infty$, then $|J'(u)| = 0$;*

(A_3) *J is weakly sequentially lower semicontinuous;*

(A_4) *The set $\{u \in X / J(u) \leq J(0)\}$ is bounded in X;*

(A_5) *For every positive integer k, there exists a $k-$dimentional subspace X_k of X and $\rho_k > 0$ such that $\sup_{X_k \cap S_{\rho_k}} J < J(0)$, where $S_\rho = \{u \in X / \|u\| = \rho\}$.*

Then J has infinitely many pairs of critical points $(\pm u_k)_{k\in\mathbb{N}}$ satisfying $J(\pm u_k) \leq J(0)$, $u_k \neq 0$ for $k \in \mathbb{N}$ and $u_k \rightharpoonup 0$ as $k \to \infty$.

Remark 2.2. Assumption (A_2) can be deduced from the following assumption

(A_2') If $u \in X$, $(u_n) \subset X$, and $u_n \rightharpoonup u$ in X as $n \to \infty$, then

$$\left\langle J'(u_n), v\right\rangle \to \left\langle J'(u), v\right\rangle,\ \forall v \in E.$$

Therefore, the result of Theorem 2.1 is true if assumption (A_2) is replaced by (A_2').

In the following, we use $L^s_Q(\mathbb{R})$ $(1 \leq s < \infty)$ to denote the Banach space of measurable functions from $\mathbb{R}$ into $\mathbb{R}^N$ under the induced norm

$$\|u\|_{L^s_Q} = \left(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s dt\right)^{\frac{1}{s}}$$

and $L^\infty_Q(\mathbb{R})$ denotes the Banach space of functions on $\mathbb{R}$ with values in $\mathbb{R}^N$ under the norm

$$\|u\|_{L^\infty_Q} = ess\sup\left\{e^{\frac{Q(t)}{2}}|u(t)|/t\in\mathbb{R}\right\}.$$

Denote by $W^{1,p}_Q(\mathbb{R})$ the Sobolev's space

$$W^{1,p}_Q(\mathbb{R}) = \left\{u\in L^p_Q(\mathbb{R})/\dot{u}\in L^p_Q(\mathbb{R})\right\}$$

equipped with the usual norm

$$\|u\| = \left(\int_{\mathbb{R}} e^{Q(t)}\left[|\dot{u}(t)|^p + |u(t)|^p\right]dt\right)^{\frac{1}{p}}.$$

Lemma 2.3. *For $u\in X$,*

$$\|u\|_{L^\infty} \le \left(\frac{p}{2m_0}\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}} e^{Q(s)}[|\dot{u}(t)|^p + |u(t)|^p]dt\right)^{\frac{1}{p}}. \tag{2.1}$$

Proof. Letting $u\in X$, we have, for $r\ge 0$,

$$\int_{|t|\ge r}[|\dot{u}(t)|^p + |u(t)|^p]dt \le \frac{1}{m_0}\int_{|t|\ge r} e^{Q(t)}[|\dot{u}(t)|^p + |u(t)|^p]dt < \infty,$$

and

$$\lim_{r\to\infty}\int_{|t|\ge r}[|\dot{u}(t)|^p + |u(t)|^p]dt = 0.$$

It results from [20] that $\lim_{|t|\to\infty} u(t) = 0$. Hence, there exists $t^*\in\mathbb{R}$ such that

$$|u(t^*)| = \max_{t\in\mathbb{R}}|u(t)| = \|u\|_{L^\infty}. \tag{2.2}$$

Consider two real sequences $(t_k)_{k\in\mathbb{N}}$ and $(t_{-k})_{k\in\mathbb{N}}$ such that

$$... < t_{-3} < t_{-2} < t_{-1} < t_1 < t_2 < t_3 < ..., \quad \lim_{k\to\infty} t_k = +\infty,\ \lim_{k\to\infty} t_{-k} = -\infty$$

and $\lim_{k\to\infty} u(t_k) = 0 = \lim_{k\to\infty} u(t_{-k})$. Let us remark that

$$|u(t^*)|^p = |u(t_k)|^p - p\int_{t^*}^{t_k}|u(s)|^{p-2}u(s)\cdot\dot{u}(s)ds \tag{2.3}$$

and

$$|u(t^*)|^p = |u(t_{-k})|^p + p\int_{t_{-k}}^{t^*}|u(s)|^{p-2}u(s)\cdot\dot{u}(s)ds. \tag{2.4}$$

Combining (2.3), (2.4), and Young's inequality yields

$$\begin{aligned}
&|u(t^*)|^p = \frac{1}{2}\Big(|u(t_k)|^p + |u(t_{-k})|^p\Big)\\
&-\frac{p}{2}\int_{t^*}^{t_k}|u(s)|^{p-2}u(s)\cdot\dot{u}(s)ds + \frac{p}{2}\int_{t_{-k}}^{t^*}|u(s)|^{p-2}u(s)\cdot\dot{u}(s)ds\\
&\le \frac{1}{2}\Big(|u(t_k)|^p + |u(t_{-k})|^p\Big) + \frac{p}{2}\int_{t_{-k}}^{t_k}\Big[\frac{1}{p}|\dot{u}(s)|^p + \frac{p-1}{p}|u(s)|^p\Big]ds\\
&\le \frac{1}{2}\Big(|u(t_k)|^p + |u(t_{-k})|^p\Big) + \frac{p}{2m_0}\int_{t_{-k}}^{t_k} e^{Q(s)}\Big[|\dot{u}(s)|^p + |u(s)|^p\Big]ds.
\end{aligned} \tag{2.5}$$

Taking $k \to \infty$ in (2.5), one gets

$$\|u\|_{L^\infty}^p = |u(t^*)|^p \le \frac{p}{2m_0} \int_{\mathbb{R}} e^{Q(s)} \left[|\dot{u}(s)|^p + |u(s)|^p \right] ds,$$

which implies (2.1). □

Remark 2.4. Let $\eta_\infty = \left(\frac{p}{2m_0}\right)^{\frac{1}{p}}$. For $s \ge p$ and $u \in W_Q^{1,p}(\mathbb{R})$, we have

$$\int_{\mathbb{R}} e^{Q(t)} |u(t)|^s \, dt \le \|u\|_{L^\infty}^{s-p} \int_{\mathbb{R}} e^{Q(t)} |u(t)|^p \, dt.$$

Therefore, for all $p \le s \le \infty$, there exists a positive constant η_s such that

$$\|u\|_{L^s} \le \eta_s \|u\|, \; \forall u \in W_Q^{1,p}(\mathbb{R}). \tag{2.6}$$

3. Proof of Theorems

Consider the functional J associated with the system $(\mathscr{DV})$ defined on the space $X = W_Q^{1,p}(\mathbb{R})$ introduced in Section 2 by

$$J(u) = \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt + \int_{\mathbb{R}} e^{Q(t)} K(t, u(t)) dt - \int_{\mathbb{R}} e^{Q(t)} W(t, u(t)) dt.$$

Let $E = \mathscr{D}(\mathbb{R})$ be the space of indefinitely differentiable functions from $\mathbb{R}$ into $\mathbb{R}^N$ with compact support. J is $E-$differentiable and

$$< J'(u), v >= \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^{p-2} \dot{u}(t) \cdot \dot{v}(t) dt - \int_{\mathbb{R}} e^{Q(t)} \nabla V(t, u(t)) \cdot v(t) dt, \; \forall u \in X, \; v \in E.$$

Lemma 3.1. *Assume that* (Q), (H_1), *and* (H_2) *are satisfied. Then J is coercive and bounded from below.*

Proof. If $\alpha = 1$, then

$$\begin{aligned} J(u) &\ge \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt + a \int_{\mathbb{R}} e^{Q(t)} |u(t)|^\nu \, dt - \|u\|_{L^\infty}^\sigma \int_{\mathbb{R}} e^{Q(t)} d(t) dt \\ &\ge \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt + a \Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{p-\nu} \|u\|^{\nu-p} \int_{\mathbb{R}} e^{Q(t)} |u(t)|^p \, dt \\ &\quad - \Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^\sigma \|u\|^\sigma \int_{\mathbb{R}} e^{Q(t)} d(t) dt \\ &\ge \min \left\{ \frac{1}{p}, a \Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{p-\nu} \|u\|^{\nu-p} \right\} \|u\|^p - \Big(\frac{\eta_\infty}{\sqrt{m_0}}\Big)^\sigma \int_{\mathbb{R}} e^{Q(t)} d(t) dt \, \|u\|^\sigma. \end{aligned} \tag{3.1}$$

If $1 < \alpha \le \frac{p}{p-\sigma}$, then $\frac{\sigma\alpha}{\alpha-1} \ge p$ and

$$\begin{aligned} J(u) &\ge \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt + a \int_{\mathbb{R}} e^{Q(t)} |u(t)|^\nu \, dt \\ &\quad - \Big(\int_{\mathbb{R}} e^{Q(t)} d^\alpha(t) dt\Big)^{\frac{1}{\alpha}} \Big(\int_{\mathbb{R}} e^{Q(t)} |u(t)|^{\frac{\sigma\alpha}{\alpha-1}} dt\Big)^{\frac{\alpha-1}{\alpha}} \\ &\ge \frac{1}{p} \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt + a \Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{p-\nu} \|u\|^{\nu-p} \int_{\mathbb{R}} e^{Q(t)} |u(t)|^p \, dt - \|d\|_{L_Q^\alpha} \eta_{\frac{\sigma\alpha}{\alpha-1}}^\sigma \|u\|^\sigma \\ &\ge \min \left\{ \frac{1}{p}, a \Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{p-\nu} \|u\|^{\nu-p} \right\} \|u\|^p - \Big(\int_{\mathbb{R}} e^{Q(t)} d^\alpha(t) dt\Big)^{\frac{1}{\alpha}} \eta_{\frac{\sigma\alpha}{\alpha-1}}^\sigma \|u\|^\sigma. \end{aligned} \tag{3.2}$$

For $\|u\| \geq (2a)^{\frac{1}{p-\nu}} \frac{\sqrt{m_0}}{\eta_\infty}$, inequalities (3.1) and (3.2) imply for a positive constant c_1 that

$$J(u) \geq a\Big(\frac{\sqrt{m_0}}{\eta_\infty}\Big)^{p-\nu} \|u\|^\nu - c_1 \|u\|^\sigma .$$

Therefore J is coercive and bounded from below because $\sigma < \nu$. □

Lemma 3.2. *If $u_n \rightharpoonup u$ and $v \in E$, then $\langle J'(u_n), v\rangle \to \langle J'(u), v\rangle$.*

Proof. Let $u_n \rightharpoonup u$ in X and $v \in E$. Then

$$\begin{aligned} &\int_{\mathbb{R}} e^{Q(t)} |\dot{u}_n(t)|^{p-2} \dot{u}_n(t)\cdot\dot{v}(t)dt + \int_{\mathbb{R}} e^{Q(t)} |u_n(t)|^{p-2} u_n(t)\cdot v(t)dt \\ &\to \int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^{p-2} \dot{u}(t)\cdot\dot{v}(t)dt + \int_{\mathbb{R}} e^{Q(t)} |u(t)|^{p-2} u(t)\cdot v(t)dt. \end{aligned} \tag{3.3}$$

Since $v \in \mathscr{D}(\mathbb{R})$, by the Lebesgue's convergence theorem, on has

$$\begin{aligned} &-\int_{\mathbb{R}} e^{Q(t)} |u_n(t)|^{p-2} u_n(t)\cdot v(t)dt - \int_{\mathbb{R}} e^{Q(t)} \nabla V(t,u_n(t))\cdot v(t)dt \\ &\to -\int_{\mathbb{R}} e^{Q(t)} |u(t)|^{p-2} u(t)\cdot v(t)dt - \int_{\mathbb{R}} e^{Q(t)} \nabla V(t,u_(t))\cdot v(t)dt. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) yields $\langle J'(u_n), v\rangle \to \langle J'(u), v\rangle$. □

Lemma 3.3. *Assume that (H_1) and (H_2) hold. Then J is weakly sequentially lower semicontinuous.*

Proof. Moreover, if $u_n \rightharpoonup u$ in X, it follows from [9, Theorem 1.6] that

$$\liminf_{n\to\infty} \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)} |\dot{u}_n(t)|^p dt \geq \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p dt.$$

Applying Fatou's lemma and using (H_1) lead to

$$\liminf_{n\to\infty} \int_{\mathbb{R}} e^{Q(t)} K(t,u_n(t))dt \geq \int_{\mathbb{R}} e^{Q(t)} K(t,u(t))dt,$$

while (H_2) implies

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{Q(t)} W(t,u_n(t))dt \geq \int_{\mathbb{R}} e^{Q(t)} W(t,u(t))dt.$$

□

By (H_3), J is even. Lemmas 3.1-3.3 imply that J satisfies $(A_1)-(A_4)$. To complete the proof of our results, it remains to verify condition (A_5).

3.1. **Proof of Theorem 1.1.** Given that X_k is a k-dimensional subspace of $\mathscr{D}(\mathbb{R})$, and all norms in a finite-dimensional space are equivalent, we can state that, for any positive integer k, there exists a positive constant γ_k such that

$$\|u\| \leq \gamma_k \|u\|_{L^p_Q},\ \forall u \in X_k. \tag{3.5}$$

By (H_4), for $0 < l_0 < l$, there exists a constant $R_0 > 0$ such that

$$V(t,x) \geq l_0 |x|^\tau,\ \forall t \in \mathbb{R},\ |x| \leq R_0. \tag{3.6}$$

For $u \in X_k$ with $\|u\| \le \frac{R_0\sqrt{m_0}}{\eta_\infty}$, we have

$$\|u\|_{L^\infty} \le \frac{\|u\|_{L_Q^\infty}}{\sqrt{m_0}} \le \frac{\eta_\infty}{\sqrt{m_0}} \|u\| \le R_0. \tag{3.7}$$

Combining (3.6) and (3.7) yields

$$\begin{aligned} J(u) &= \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt - \int_{\mathbb{R}} e^{Q(t)} V(t,u(t))dt \\ &\le \frac{1}{p}\|u\|^p - l_0 \int_{\mathbb{R}} e^{Q(t)} |u(t)|^\tau \, dt \\ &\le \frac{1}{p}\|u\|^p - l_0 \frac{1}{\gamma_k^p} (\frac{\sqrt{m_0}}{\sqrt{\eta_\infty}})^{p-\tau} \|u\|^\tau . \end{aligned}$$

Choosing $\rho_k = \min\left\{R_0, (\frac{l_0}{\gamma_k^p})^{\frac{1}{p-\tau}}\right\} \frac{\sqrt{m_0}}{\eta_\infty}$, we obtain $J(u) < 0$ for $u \in X_k$, $\|u\| = \rho_k$. Therefore (A_5) is satisfied. According to Theorem 2.1, J possesses infinitely many pairs of critical points $\pm u_k, k \in \mathbb{N}$ satisfying

$$J(\pm u_k) \le J(0),\ u_k \ne 0 \ for \ k \in \mathbb{N} \ and \ u_k \rightharpoonup 0 \ as \ k \to \infty.$$

Thus system $(\mathscr{DV})$ has infinitely many pairs of nontrivial homoclinic solutions.

3.2. **Proof of Theorem 1.2.** For any $k \in \mathbb{N}$, let X_k be as above and $M > \frac{\gamma_k^p}{p}$. By assumption (H_4'), there exists a constant $R_k > 0$ such that

$$V(t,x) \ge M |x|^p, \ \forall t \in \mathbb{R}, \ |x| \le R_k. \tag{3.8}$$

Letting $u \in X_k$ with $\|u\| \le \frac{R_k}{\eta_\infty}\sqrt{m_0} = \rho_k$, we have $\|u\|_{L^\infty} \le R_k$. Hence (3.8) yields for $\|u\| = \rho_k$ that

$$\begin{aligned} J(u) &= \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)} |\dot{u}(t)|^p \, dt - \int_{\mathbb{R}} e^{Q(t)} V(t,u(t))dt \\ &\le \frac{1}{p}\|u\|^p - M\|u\|_{L_Q^p}^p \le \frac{1}{p}\|u\|^p - \frac{M}{\gamma_k^p}\|u\|^p \\ &\le (\frac{1}{p} - \frac{M}{\gamma_k^p})\rho_k^p < 0, \end{aligned}$$

which implies that $\sup_{X_k \cap S_{\tau_k}} J < 0$, where $S_{\rho_k} = \{u \in X : \|u\| = \rho_k\}$. Thus condition (A_5) is met. Following the reasoning in the proof of Theorem 1.1, we can conclude that system $(\mathscr{DV})$ possesses infinitely many pairs of nontrivial homoclinic solutions.

3.3. **Proof of Theorem 1.3.** For any $k \in \mathbb{N}$, let X_k be a k-dimensional subspace of $\mathscr{D}(]t_0 - r, t_0 + r[)$. As above, for any positive integer k, there exists a positive constant γ_k such that

$$\|u\| \le \gamma_k \|u\|_{L_Q^p}, \ \forall u \in X_k. \tag{3.9}$$

By (H_5), for $0 < l_0 < l$, there exists a constant $0 < R_1 < R$ such that

$$W(t,x) \ge l_0 |x|^\tau, \ \forall t \in]t_0 - r, t_0 + r[, \ |x| \le R_1. \tag{3.10}$$

For $u \in X_k$ with $\|u\| = \min\left\{R_1, \left(\frac{l_0}{p'b}\right)^{\frac{1}{\nu-\tau}}\right\}\frac{\sqrt{m_0}}{\eta_\infty}$, we have $\|u\|_{L^\infty} \le R_1$. Thus (3.10) implies

$$\begin{aligned}
J(u) &= \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^p\,dt + \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt \\
&\le \frac{1}{p}\|u\|^p + b\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt - l_0\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\
&\le \frac{1}{p}\|u\|^p + b\left(\frac{\eta_\infty}{\sqrt{m_0}}\right)^{\nu-\tau}\|u\|^{\nu-\tau}\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt - l_0\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\
&\le \frac{1}{p}\|u\|^p - \left[l_0 - b\left(\frac{\eta_\infty}{\sqrt{m_0}}\right)^{\nu-\tau}\|u\|^{\nu-\tau}\right]\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\tau\,dt \\
&\le \frac{1}{p}\|u\|^p - \left[l_0 - b\left(\frac{\eta_\infty}{\sqrt{m_0}}\right)^{\nu-\tau}\|u\|^{\nu-\tau}\right]\frac{1}{\gamma_k^p}\left(\frac{\sqrt{m_0}}{\eta_\infty}\right)^{p-\tau}\|u\|^\tau \\
&\le \frac{1}{p}\|u\|^p - \frac{l_0}{p}\frac{1}{\gamma_k^p}\left(\frac{\sqrt{m_0}}{\eta_\infty}\right)^{p-\tau}\|u\|^\tau.
\end{aligned}$$

In view of $0 < \tau < p$, we can conclude that there exists a sufficiently small positive constant ρ_k such that $J(u) < 0$ for $u \in X_k$ with $\|u\| = \rho_k$. This satisfies condition (A_5). Therefore, system $(\mathscr{DV})$ has infinitely many pairs of nontrivial homoclinic solutions.

3.4. **Proof of Theorem 1.4.** For any $k \in \mathbb{N}$, let X_k be defined as in the proof of Theorem 1.3 and let $M > b$. By assumption (H_5'), there exists a constant $0 < R_k < R$ such that

$$W(t,x) \ge M|x|^\nu,\ \forall t \in]t_0 - r, t_0 + r[,\ |x| \le R_k. \tag{3.11}$$

Let $u \in X_k$ with $\|u\| = \inf\left\{R_k, \left(\frac{M-b}{\gamma_k^2}\right)^{\frac{1}{2-\nu}}\right\}\frac{\sqrt{m_0}}{\eta_\infty} = \rho_k$. Then $\|u\|_{L^\infty} \le R_k$. Hence (3.11) and (H_2') yields

$$\begin{aligned}
J(u) &= \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^p\,dt + \int_{\mathbb{R}} e^{Q(t)}K(t,u(t))dt - \int_{\mathbb{R}} e^{Q(t)}W(t,u(t))dt \\
&\le \frac{1}{p}\int_{\mathbb{R}} e^{Q(t)}|\dot{u}(t)|^p\,dt + b\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt - M\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt \\
&\le \frac{1}{p}\|u\|^p - (M-b)\int_{\mathbb{R}} e^{Q(t)}|u(t)|^\nu\,dt \\
&\le \frac{1}{p}\|u\|^p - \frac{M-b}{\gamma_k^p}\left(\frac{\sqrt{m_0}}{\eta_\infty}\right)^{p-\nu}\|u\|^\nu \\
&\le -\left(\frac{M-b}{\gamma_k^p}\right)^{\frac{p}{p-\nu}}\left(\frac{\sqrt{m_0}}{\eta_\infty}\right)^p < 0.
\end{aligned}$$

Given that condition (A_5) holds, we can infer, as previously discussed, that system $(\mathscr{DV})$ possesses an infinite number of pairs of nontrivial homoclinic solutions.

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