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INVARIANT TORI FOR A TWO-DIMENSIONAL SCHRÖDINGER EQUATION WITH A GENERAL NONLINEAR TERM AND A LARGE FORCING TERM

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Abstract. This paper is devoted to the study of a two-dimensional Schrödinger equation with a general nonlinear term and a large forcing term. It is proved that the equation admits a Whitney smooth family of small amplitude quasi-periodic solutions which are partially hyperbolic for the given frequency vector (non-external parameters) and the large forcing term. Firstly, by introducing a symplectic change of coordinates, the Hamiltonian of the equation is transformed into a linear autonomous system plus higher order term perturbation (non-small perturbation), that is, the reducibility of the non-autonomous linear part is realized. Secondly, by introducing a symplectic change of coordinates, and action-angle variables, the Hamiltonian is transformed into a small perturbation of nonlinear integrable normal form that depends on angle variables. Then, by introducing a new symplectic change of coordinates, the Hamiltonian is transformed into a small perturbation of linear integrable normal form. Finally, the existence of invariant tori of the Hamiltonian system associated with the equation is proved by constructing an infinite-dimensional Kolmogorov-Arnold-Moser (KAM) theorem.
Keywords. Hamiltonian system; KAM theory; Large forcing term; Quasi-periodic solutions; Schrödinger equation.

1. INTRODUCTION

In this paper, a two-dimensional (2D) Schrödinger equation with a general nonlinear term and a large forcing term

$$\mathrm{i}u_t - \Delta u + f(\widetilde{\omega}t)\,u + f(\widetilde{\omega}t)\,|u|^{2p}u = 0,\ x \in \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2,\ t \in \mathbb{R},\ p \in \mathbb{Z}^+ \tag{1.1}$$

under the periodic boundary conditions

$$u(t,x_1+2\pi,x_2) = u(t,x_1,x_2+2\pi) = u(t,x_1,x_2) \tag{1.2}$$

is considered, where $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \cdots, \widetilde{\omega}_m) \in \mathbb{R}^m$ is a fixed frequency vector, and $f(\widetilde{\omega}t) = f(\widetilde{\theta})$ is real analytic in $\widetilde{\theta} = \widetilde{\omega}t \in \mathbb{T}^m$ and quasi-periodic in t. Furthermore, $f(\widetilde{\omega}t)$ presents a large

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forcing term rather than a small perturbation. It is proven that the boundary value problem (1.1) along with (1.2) admits a Whitney smooth family of small-amplitude quasi-periodic solutions that are partially hyperbolic.

Equation (1.1) is studied as an infinite-dimensional Hamiltonian system. In physics models, many partial differential equations (PDEs) are infinite-dimensional dynamical systems. In order to explore the stability and other dynamical properties of PDEs, in the late 1980s, Wayne [21], Kuksin [15], and Pöschel [17] extended the classical KAM theory to infinite-dimensional spaces and applied it to the study of invariant tori of PDE. Since then, many scholars have constructed quasi-periodic solutions of different types of one-dimensional Hamiltonian PDEs by improving the infinite-dimensional KAM theory; see, e.g., [3, 16, 18, 26] and the references therein.

When the spatial variable is in a high-dimensional space, it becomes more difficult to solve the homology equation and measure estimation because the normal frequency multiplicities tend to infinity. Yuan [25], Geng and You [11, 12] firstly discussed the infinite dimensional KAM theory of Hamiltonian PDEs in high dimensional space. Eliasson and Kuksin [6] made a breakthrough in properly classifying normal frequencies by introducing the Toplitz-Lipschitz property, thus solving the problem caused by the multiplicity of normal frequencies. And a KAM theorem was given which can be applied to a general multidimensional Schrödinger equation with a convolution operator $V(x)$. However, compared with one-dimensional PDEs, there are fewer results about high-dimensional PDEs, which can be referred to [1, 8, 14, 19, 20]. In particular, when the equation itself does not contain parameters, even fewer results are available.

In 2011, Geng, Xu and You [9] combined the Toplitz-Lipschitz idea in [6] and the idea of solving homology equations dependent on angular variables originally proposed by Xu and You in [22], proposed a new KAM theorem, and applied it to study a two-dimensional completely resonant Schrödinger equation

$$\mathrm{i}u_t - \Delta u + |u|^2 u = 0, \quad x \in \mathbb{T}^2, \quad t \in \mathbb{R} \tag{1.3}$$

with periodic boundary conditions. To overcome the resonance, the authors selected an appropriate set of tangential points S (the admissible set). In the case that only the Fourier exponents of tangential points S are excited, it is made the Hamiltonian system corresponding to the Birkhoff normal form recognize the quasi-periodic solutions of the bifurcated equation without introducing external parameters. By using the similar approach in [9], the present authors, Zhang and Si [28] recently studied the existence of quasi-periodic solutions to the two-dimensional completely resonant Schrödinger equation with the general nonlinearity

$$\mathrm{i}u_t - \Delta u + |u|^{2p} u = 0,\ x \in \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2,\ t \in \mathbb{R},\ p \in \mathbb{Z}^+ \tag{1.4}$$

with periodic boundary conditions. The corresponding admissible set of the equation (1.3) in [9] contains only four types of resonances, while the corresponding admissible set of the equation (1.4) in [28] contains $3p+1$ kinds of resonances. Correspondingly, conditions such as Melnikov non-degeneracy conditions will increase by several factors. Therefore, KAM iteration needs to be reconstructed. Obviously, the change in the order of the nonlinear term plays a crucial role in KAM theory. In addition, although both the equations (1.1) and (1.4) have general nonlinear terms, the conclusions in [28] do not apply to the equation (1.1) because (1.1) is non-autonomous.

For non-autonomous systems, periodic or quasi-periodic models are common and are often solved by variational methods. There are relatively few studies on KAM theory. As early as 1996, Chow, Lu and Shen [4] studied the normal forms and analytic conjugations of a class of quasi-periodic analytic evolution equations, including the parabolic equation and the Schrödinger equation. While proving the analytic linearization theorem, they adopted and improved the Zehnder method, which is essentially the KAM method. The KAM method, which is related to the reducibility of non-autonomous linear systems, essentially uses infinite symplectic transformation to gradually transform non-autonomous linear systems into an autonomous linear system plus a higher order perturbation system, and finally realizes the reducibility of non-autonomous systems. Since then, this idea has been applied extensively to the reducibility of non-autonomous systems, see references [2, 5, 7, 13] etc. In [27], by using the idea of similarity, Zhang proved the reducibility and the existence of quasi-periodic solutions for the two-dimensional quasi-periodically forced cubic Schrödinger equation

$$\mathrm{i}u_t - \Delta u + \mu u + \varepsilon\phi(t)(u + |u|^2 u) = 0, \quad \mu \geq 0, \quad x \in \mathbb{T}^2, \quad t \in \mathbb{R} \tag{1.5}$$

with periodic boundary conditions. However, most of the above reducibility conclusions pertain to cases with small perturbations(contain a small parameter ε) and taking the frequency of the forced term as the external parameter. When the equation has a large forcing term, the above method needs to be substantially modified. In [10], Geng and Xue studied the two-dimensional cubic Schrödinger equations with large forcing terms

$$\mathrm{i}u_t - \Delta u + \varphi(\widetilde{\omega}t)\,u + \varphi(\widetilde{\omega}t)\,|u|^2 u = 0,\ x \in \mathbb{T}^2,\ t \in \mathbb{R} \tag{1.6}$$

with periodic boundary conditions, where $\widetilde{\omega} \in \mathbb{R}^m$ is a fixed parameter rather than a variable external parameter, and $\varphi(\widetilde{\omega}t)$ is a large forcing term, not a small perturbation. Since $\widetilde{\omega}$ is a fixed parameter, KAM theory cannot be used to achieve the reducibility of linear systems. By attaching conditions to the parameter $\widetilde{\omega}$, they achieved the reducibility through an elegant symplectic coordinate transformation. In [23], Xue discussed the Schrödinger equation with different forcing terms for the linear and nonlinear terms. However, due to the difference in normal forms, the method in [10] and [23] are not applicable to the equation (1.1).

The equation (1.1) is unlike any of the above equations. It is high dimensional, contains a general nonlinear term, has a large forcing term, and has a fixed frequency. This makes the equation complex and difficult. In this paper, the corresponding Hamiltonian of (1.1) is transformed to normal form by three symplectic transformations. The first symplectic transformation implements the reducibility of non-autonomous linear systems. The difficulty here is not only to achieve reducibility, but also to ensure that the order of the nonlinear term of the Hamiltonian obtained after the transformation is still $2p+2$ with respect to u. That is, no more nonlinear terms are added than the original system. The second symplectic transformation transforms the resulting autonomous system into an integrable nonlinear normal form that depends on angle variables. The denominator $\langle \widetilde{k}, \widetilde{\omega}\rangle - \sum_{\widehat{r}=1}^{p+1}(\lambda_{i_{2\widehat{r}-1}} - \lambda_{i_{2\widehat{r}}})$ will appear, which is different from $\langle \widetilde{k}, \widetilde{\omega}\rangle - \sum_{\widehat{r}=1}^{2}(\lambda_{i_{2\widehat{r}-1}} - \lambda_{i_{2\widehat{r}}})$ in [10] and $\sum_{\widehat{r}=1}^{p+1}(\lambda_{i_{2\widehat{r}-1}} - \lambda_{i_{2\widehat{r}}})$ in [28]. When $\langle \widetilde{k}, \widetilde{\omega}\rangle - \sum_{\widehat{r}=1}^{p+1}(\lambda_{i_{2\widehat{r}-1}} - \lambda_{i_{2\widehat{r}}}) = 0$, this condition corresponds to the term in the normal form that depends on the angle variables. In order to reduce the integrable terms that depend on angle variables and make the normal form brief, it is necessary to select a special tangential frequency set, that is the admissible set in the definition 2.1. The third symplectic transformation is used to transform an integrable nonlinear normal form depends on angle variables to a linear normal form. The difficulty lies in how to find the unitary transformation to diagonalize the block symmetric matrix. Then corresponding to the normal form, the corresponding KAM theory is constructed. Finally, the existence of invariant tori of equation (1.1) is proved.

2. The Main Results and Some Notations

Before stating our main theorem, we first introduce the definition of the admissible set.

The operator$-\Delta$, under periodic boundary conditions, has eigenvalues $\{\lambda_{l_*}\}$ and corresponding eigenfunctions, which satisfy specific properties

$$\lambda_{l_*}=|l_*|^2={l_{*,1}}^2+{l_{*,2}}^2 \text{ and } \phi_{l_*}(x)=\sqrt{\frac{1}{4\pi^2}}e^{\mathrm{i}\langle l_*,x\rangle},\ l_*=(l_{*,1},l_{*,2})\in\mathbb{Z}^2.$$

Definition 2.1. A finite set $S=\left\{j_1^*=(x_1,y_1),\cdots,j_b^*=(x_b,y_b)\right\}\subset\mathbb{Z}^2$ is called an admissible set if

(1) For given $1\le\widehat{l}\le p$ and for any $\left\{j_1,j_2,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}},n,m\right\}\subset\mathbb{Z}^2$, if they satisfy $j_1-j_2+j_3-j_4+\cdots+j_{2\widehat{l}-1}-j_{2\widehat{l}}+n-m=0$ and $|j_1|^2-|j_2|^2+|j_3|^2-|j_4|^2+\cdots+|j_{2\widehat{l}-1}|^2-|j_{2\widehat{l}}|^2+|n|^2-|m|^2=0$ or $j_1+j_2+j_3-j_4+\cdots+j_{2\widehat{l}-1}-j_{2\widehat{l}}-n-m=0$ and $|j_1|^2+|j_2|^2+|j_3|^2-|j_4|^2+\cdots+|j_{2\widehat{l}-1}|^2-|j_{2\widehat{l}}|^2-|n|^2-|m|^2=0$, the intersection of $\left\{j_1,j_2,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}},n,m\right\}$ and S contains at most $2\widehat{l}$ elements, i.e.$\sharp\left\{j_1,j_2,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}},n,m\right\}\bigcap S\le 2\widehat{l}$.

(2) For given $1\le\widehat{l}\le p$ and any $n\in\mathbb{Z}^2\setminus S$, there exists at most one array $\left\{j_1,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}},m\right\}$ with $j_1,j_2,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}}\in S, m\in\mathbb{Z}^2\setminus S$ satisfying $j_1-j_2+j_3-j_4+\cdots+j_{2\widehat{l}-1}-j_{2\widehat{l}}+n-m=0$ and $|j_1|^2-|j_2|^2+|j_3|^2-|j_4|^2+\cdots+|j_{2\widehat{l}-1}|^2-|j_{2\widehat{l}}|^2+|n|^2-|m|^2=0$. If such array exists, we say that n,m are resonant of the $2\widehat{l}-1$ type. n,m are mutually uniquely determined. We say that (n,m) is a resonant pair of the $2\widehat{l}-1$ type and denote all such n by $\mathscr{L}_{2\widehat{l}-1}$.

(3) For given $1\le\widehat{l}\le p$ and any $n\in\mathbb{Z}^2\setminus S$, there exists at most one array $\left\{j_1,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}},m\right\}$ with $j_1,j_2,\cdots,j_{2\widehat{l}-1},j_{2\widehat{l}}\in S, m\in\mathbb{Z}^2\setminus S$ satisfying $j_1+j_2+j_3-j_4+\cdots+j_{2\widehat{l}-1}-j_{2\widehat{l}}-n-m=0$ and $|j_1|^2+|j_2|^2+|j_3|^2-|j_4|^2+\cdots+|j_{2\widehat{l}-1}|^2-|j_{2\widehat{l}}|^2-|n|^2-|m|^2=0$. If such array exists, we say that n,m are resonant of the $2\widehat{l}$ type. n,m are mutually uniquely determined. We say that (n,m) is a resonant pair of the $2\widehat{l}$ type and denote all such n by $\mathscr{L}_{2\widehat{l}}$.

(4) Any $n\in\mathbb{Z}^2\setminus S$ is not resonant of any two of the above $2p$ classes. Geometrically, any two of the above defined graphs cannot share vertex in $\mathbb{Z}^2\setminus S$. That is, when $\widehat{r},\widehat{s}=1,2,\cdots,2p$ and $\widehat{r}\neq\widehat{s}$, then $\mathscr{L}_{\widehat{r}}\cap\mathscr{L}_{\widehat{s}}=\emptyset$.

The concrete method for constructing the admissible set is inspired by [28]. We now present the main result of this paper.

Theorem 2.2. *Let $S=\left\{j_1^*=(x_1,y_1),\cdots,j_b^*=(x_b,y_b)\right\}$ be an admissible set, where $b\ge 2p$, $f^0=\int_{\mathbb{T}^m}f(\widetilde{\theta})d\widetilde{\theta}\neq 0$. There exists a Cantor set $\mathscr{I}^*$ of positive measure such that, for any $(\xi_1,\ldots,\xi_b)\in\mathscr{I}^*$, when*

$$\left\{p\Big[\xi_{j_2}^p+\xi_{j_1}^p+\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(\xi_{j_{2\widehat{r}-1}}^p-\xi_{j_{2\widehat{r}}}^p\big)\Big]+g_{p,\widehat{l}_1,n,2}-g_{p,\widehat{l}_1,n,3}\right\}^2-4(p+1)^2{g^*_{\mathscr{B},p,\widehat{l}_1,n}}^2<0,$$

the nonlinear Schrödinger equation (1.1) admits a small-amplitude, quasi-periodic solution of the form

$$u\big(\widetilde{\omega}t,\widehat{\omega}_1t,\widehat{\omega}_2t,\ldots,\widehat{\omega}_bt,x\big)=\sum_{j\in S}\sqrt{\xi_j}e^{\mathrm{i}\widehat{\omega}_jt}\phi_j+O\big(|\xi|^{\frac{3}{2}}\big),\widehat{\omega}_j=\varepsilon^{-3p}\big(|j|^2+f^0\big)+O\big(|\xi|^p\big).$$

Remark 2.3. The variables $g_{p,\widehat{l}_1,n,2}, g_{p,\widehat{l}_1,n,3}$, and $g^*_{\mathscr{B},p,\widehat{l}_1,n}$ appearing in the theorem are clarified in the proof presented later.

Remark 2.4. To ensure that the obtained tori are partially hyperbolic, the following condition must be satisfied

$$\left\{p\Big[\xi_{j_2}^p+\xi_{j_1}^p+\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(\xi_{j_{2\widehat{r}-1}}^p-\xi_{j_{2\widehat{r}}}^p\big)\Big]+g_{p,\widehat{l}_1,n,2}-g_{p,\widehat{l}_1,n,3}\right\}^2-4(p+1)^2{g^*_{\mathscr{B},p,\widehat{l}_1,n}}^2<0.$$

If this condition is not satisfied, i.e., when

$$\left\{p\Big[\xi_{j_2}^p+\xi_{j_1}^p+\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(\xi_{j_{2\widehat{r}-1}}^p-\xi_{j_{2\widehat{r}}}^p\big)\Big]+g_{p,\widehat{l}_1,n,2}-g_{p,\widehat{l}_1,n,3}\right\}^2-4(p+1)^2{g^*_{\mathscr{B},p,\widehat{l}_1,n}}^2\geq 0,$$

the existence of elliptic tori can indeed be proven, however, the proof in this case is more intricate and will be addressed in a forthcoming paper.

Next, in order to present an infinite-dimensional version of the KAM theorem for two-dimensional Schrödinger equations subject to periodic boundary conditions, we first introduce some notations. Throughout this paper, we use the following notation.

We denote the Lebesgue measure by "meas". $S=\{j_1^*,\cdots,j_b^*\}$ and $\mathbb{Z}_1^2=\mathbb{Z}^2\setminus S$, where $j_1^*,\cdots,j_b^*$ are b vectors in $\mathbb{Z}^2$. l^ρ is a space of sequences with finite norm $\|z\|_\rho=\sum_{l_*\in\mathbb{Z}_1^2}|z_{l_*}|e^{|l_*|\rho}<\infty$, $\rho>0$,where $|l_*|=\sqrt{{l_{*,1}}^2+{l_{*,2}}^2}$, $l_*=(l_{*,1},l_{*,2})\in\mathbb{Z}^2$, which is the space of all complex vectors $z=(\ldots,z_{l_*},\ldots)_{l_*\in\mathbb{Z}_1^2}$ with finite norm. $\theta=\big(\theta_i\big)_{i\in S}$, $\widehat{I}=\big(\widehat{I}_i\big)_{i\in S}$, $z=\big(z_i\big)_{i\in\mathbb{Z}_1^2}$ and $\xi=\big(\xi_i\big)_{i\in S}$, and they constitute the phase space $\big(\theta,\widehat{I},z,\bar{z}\big)\in\Upsilon^\rho=\widehat{\mathbb{T}}^b\times\mathbb{C}^b\times l^\rho\times l^\rho$, where $\widehat{\mathbb{T}}^b$ is the complex neighborhood of the usual $\big(b\big)$-torus $\mathbb{T}^b$.

$D_\rho\big(r,l\big):=\big\{\big(\theta,\widehat{I},z,\bar{z}\big)\in\Upsilon^\rho:\big|\mathrm{Im}\,\theta\big|<r,\big|\widehat{I}\big|<l^2,\|z\|_\rho<l,\|\bar{z}\|_\rho<l\big\}$, where the weighted phase norm is given by $\big|\chi\big|_\rho=\big|\theta\big|+\frac{1}{l^2}\big|\widehat{I}\big|+\frac{1}{l}\|z\|_\rho+\frac{1}{l}\|\bar{z}\|_\rho$, when $\chi=\big(\theta,\widehat{I},z,\bar{z}\big)\in\Upsilon^\rho$.

Let $\mathscr{I}$ denote the parameter set, $\xi\in\mathscr{I}$. Let $\alpha\equiv\big(\ldots,\alpha_i,\ldots\big)_{i\in\mathbb{Z}_1^2},\beta\equiv\big(\ldots,\beta_i,\ldots\big)_{i\in\mathbb{Z}_1^2}$, where $\alpha_i,\beta_i\in\mathbb{N}$ and both have finitely many nonzero components. $z^\alpha\bar{z}^\beta$ denotes $\prod_i z_i^{\alpha_i}\bar{z}_i^{\beta_i}$.

Denote

$$\mathscr{G}\big(\theta,\widehat{I},z,\bar{z}\big)=\sum_{\alpha,\beta}\mathscr{G}_{\alpha\beta}\big(\theta,\widehat{I}\big)z^\alpha\bar{z}^\beta, \tag{2.1}$$

where $\mathscr{G}_{\alpha\beta}=\sum_{k,h}\mathscr{G}_{kh\alpha\beta}\widehat{I}^he^{\mathrm{i}\langle k,\theta\rangle}$ is $4p$ order smooth with respect to the parameter ξ in the Whitney sense and

$$\|\mathscr{G}\|_{\mathscr{I}}=\sum_{\alpha,\beta,k,h}\big|\mathscr{G}_{kh\alpha\beta}\big|_{\mathscr{I}}\big|\widehat{I}^h\big|e^{|k||\mathrm{Im}\,\theta|}\big|z^\alpha\big|\big|\bar{z}^\beta\big|, \tag{2.2}$$

where $\left|\mathscr{G}_{kh\alpha\beta}\right|_{\mathscr{I}}$ is short for $\left|\mathscr{G}_{kh\alpha\beta}\right|_{\mathscr{I}} \equiv \sup_{\xi\in\mathscr{I}} \sum_{0\leq\widehat{i}\leq 4p} \left|\partial_{\xi}^{\widehat{i}}\mathscr{G}_{kh\alpha\beta}\right|$. The derivatives with respect to ξ are understood in the Whitney sense. We define the weighted norm of $\mathscr{G}$ by

$$\|\mathscr{G}\|_{D_\rho(r,l),\mathscr{I}} \equiv \sup_{D_\rho(r,l)} \|\,\mathscr{G}\|_{\mathscr{I}}. \tag{2.3}$$

For the symplectic structure $dI\wedge d\theta + \mathrm{i}dz\wedge d\bar{z}$, the corresponding Hamiltonian vector field associated with $\mathscr{G}$ is $X_{\mathscr{G}} = \big(\mathscr{G}_I, -\mathscr{G}_\theta, \{\mathrm{i}\mathscr{G}_{z_n}\}_{n\in\mathbb{Z}_1^2}, \{-\mathrm{i}\mathscr{G}_{\bar{z}_n}\}_{n\in\mathbb{Z}_1^2}\big)$. Its weighted norm is

$$\begin{aligned}\left\|X_{\mathscr{G}}\right\|_{D_\rho(r,l),\mathscr{I}} \equiv{}& \left\|\mathscr{G}_I\right\|_{D_\rho(r,l),\mathscr{I}} + \frac{1}{l^2}\left\|\mathscr{G}_\theta\right\|_{D_\rho(r,l),\mathscr{I}} \\ &+ \sup_{D_\rho(r,l)}\Big[\frac{1}{l}\Big(\sum_{n\in\mathbb{Z}_1^2}\left\|\mathscr{G}_{z_n}\right\|_{\mathscr{I}} e^{|n|\rho} + \sum_{n\in\mathbb{Z}_1^2}\left\|\mathscr{G}_{\bar{z}_n}\right\|_{\mathscr{I}} e^{|n|\rho}\Big)\Big]\end{aligned} \tag{2.4}$$

3. The Hamiltonian Setting

First of all, the equation (1.1) can be written as a Hamiltonian system

$$u_t = \mathrm{i}\frac{\partial H}{\partial \bar{u}} \tag{3.1}$$

and the corresponding Hamiltonian is

$$H = \langle -\Delta u, u\rangle + \langle f(\widetilde{\omega}t)u, u\rangle + \frac{1}{p+1} f(\widetilde{\omega}t)\int_{\mathbb{T}^2} |u|^{2p+2}\,dx,$$

where $\langle\cdot,\cdot\rangle$ represents the inner product defined in L^2. Let $u(x) = \sum_{j\in\mathbb{Z}^2} q_j\phi_j(x)$. Then system (3.1) becomes equivalent to the lattice Hamiltonian equations

$$\dot{q}_n = \mathrm{i}\big(\lambda_n q_n + f(\widetilde{\omega}t)q_n + \frac{\partial G}{\partial \bar{q}_n}\big)$$

and the perturbation G is given by

$$G = \frac{1}{4^p(p+1)\pi^{2p}} f(\widetilde{\omega}t) \sum_{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0} q_{j_1}\bar{q}_{j_2}q_{j_3}\bar{q}_{j_4}\cdots q_{j_{2p+1}}\bar{q}_{j_{2p+2}}.$$

The corresponding Hamiltonian function is given by

$$H = \sum_{n\in\mathbb{Z}^2}\big(\lambda_n + f(\widetilde{\omega}t)\big)|q_n|^2 + G = \Lambda + G \tag{3.2}$$

Lemma 2 in [18] is a classical result. Based on this result, one sees that the perturbation G in equation (3.2) has the following regularity properties.

Lemma 3.1. *For any fixed $\rho > 0$ in a neighborhood of the origin, then the gradient $G_{\bar{q}}$ is real-analytic and $\|G_{\bar{q}}\|_\rho \leq c\|q\|_\rho^{2p+1}$.*

Proof.

$$\begin{aligned}\|G_{\bar{q}}\|_\rho = \sum_{n\in\mathbb{Z}^2} |G_{\bar{q}_n}|e^{|n|\rho} &\le c \sum_{n,\alpha,\beta-e_n,|\alpha|+|\beta-e_n|=2p+1} |q^\alpha \bar{q}^{\beta-e_n}|e^{|n|\rho} \\ &\le c \sum_{\alpha,\beta-e_n,|\alpha|+|\beta-e_n|=2p+1} |q^\alpha \bar{q}^{\beta-e_n}|e^{|\alpha|\rho}e^{|\beta-e_n|\rho} \\ &\le c\|q\|_\rho^{2p+1}.\end{aligned}$$

□

Secondly, we examine the reducibility of the Hamiltonian in equation (3.2).

By introducing a pair of action-angle variables $\big(\widetilde{\theta},\widetilde{I}\big) = \big((\widetilde{\theta}_1,\dots,\widetilde{\theta}_m),(\widetilde{I}_1,\dots,\widetilde{I}_m)\big) \in \mathbb{T}^m \times \mathbb{R}^m$ with $\widetilde{\theta} = \widetilde{\omega}t$. Then (3.2) can be transformed into the following dynamics

$$\dot{\widetilde{\theta}} = \frac{\partial H}{\partial \widetilde{I}} = \widetilde{\omega}, \quad \dot{\widetilde{I}} = -\frac{\partial H}{\partial \widetilde{\theta}}, \quad \dot{q}_n = \mathrm{i}\frac{\partial H}{\partial \bar{q}_n}, \quad \dot{\bar{q}}_n = -\mathrm{i}\frac{\partial H}{\partial q_n}, \quad n\in\mathbb{Z}^2$$

with

$$H = \langle \widetilde{\omega},\widetilde{I}\rangle + \sum_{n\in\mathbb{Z}^2}\big(\lambda_n + f(\widetilde{\theta})\big)q_n\bar{q}_n + R\big(\widetilde{\theta},q,\bar{q}\big) = N_1 + N_2 + R,$$

where

$$N_1 = \langle \widetilde{\omega}, I\rangle + \sum_{n\in\mathbb{Z}^2}\lambda_n q_n\bar{q}_n, \quad N_2 = \sum_{n\in\mathbb{Z}^2} f(\widetilde{\theta})q_n\bar{q}_n,$$

$$R = \frac{1}{4^p(p+1)\pi^{2p}} f(\widetilde{\theta}) \sum_{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0} q_{j_1}\bar{q}_{j_2}q_{j_3}\bar{q}_{j_4}\cdots q_{j_{2p+1}}\bar{q}_{j_{2p+2}}.$$

We expand $f(\widetilde{\theta})$ into its Fourier series

$$f(\widetilde{\theta}) = \sum_{\widetilde{k}\in\mathbb{Z}^m} f^{\widetilde{k}} e^{\mathrm{i}\langle \widetilde{\mathrm{k}},\widetilde{\theta}\rangle}$$

where $\widetilde{k}\in\mathbb{Z}^m$ and $\widetilde{\theta}\in\mathbb{T}^m$.

Finally, we seek a function T, defined in a domain $D_+ = D_p(r_+ = \frac{r}{2}, l_+)$, such that the time-one map ϕ_T^1 of the Hamiltonian vector field X_T defines a map from D_+ to D and transforms H into H_+. More precisely, by applying the second-order Taylor expansion

$$\begin{aligned} H_+ &= H\circ\phi_T^1 = (N_1+N_2+R)\circ\phi_T^1 \\ &= N_1+N_2+\{N_1+N_2,T\}+R+\int_0^1(1-t)\{\{N_1+N_2,T\},T\}\circ\phi_T^t\,dt \\ &+\int_0^1\{R,T\}\circ\phi_T^t\,dt \\ &= N_1^+ + R_+ + \{N_1+N_2,T\} + N_2 - \sum_{n\in\mathbb{Z}^2} f^0 q_n\bar{q}_n \\ &= \Lambda_+ + R_+ \end{aligned} \tag{3.3}$$

with respect to the symplectic structure $d\widetilde{I}\wedge d\widetilde{\theta}+\mathrm{i}\sum\limits_{n\in\mathbb{Z}^2}dq_n\wedge d\bar{q}_n$, where

$$\Lambda_+=N_1^+=N_1+\sum_{n\in\mathbb{Z}^2}f^0q_n\bar{q}_n=\langle\widetilde{\omega},\widetilde{I}\rangle+\sum_{n\in\mathbb{Z}^2}(\lambda_n+f^0)q_n\bar{q}_n$$

and

$$R_+=R+\int_0^1(1-t)\{\{N_1+N_2,T\},T\}\circ\phi_T^t\,dt+\int_0^1\{R,T\}\circ\phi_T^t\,dt=R$$

a function T of the form $T(\widetilde{\theta},q,\bar{q})=\sum\limits_{\widetilde{k}\in\mathbb{Z}^m,\widetilde{k}\neq0,n\in\mathbb{Z}^2}T^{\widetilde{k}}q_n\bar{q}_ne^{\mathrm{i}\langle\widetilde{\mathrm{k}},\widetilde{\theta}\rangle}$ is sought, which satisfies the equation

$$\{N_1+N_2,T\}+N_2-\sum_{n\in\mathbb{Z}^2}f^0q_n\bar{q}_n=0 \tag{3.4}$$

Lemma 3.2. *T satisfies (3.4) if the Fourier coefficients of T are defined as $\langle\widetilde{k},\widetilde{\omega}\rangle T^{\widetilde{k}}=\mathrm{i}f^{\widetilde{k}}$, $\widetilde{k}\neq0$.*

The proof can be found in [10].

Lemma 3.3. *T defined in (3.4) satisfies $\{N_1+N_2,T\}=0$ and $\{R,T\}=0$.*

Proof. Since the proof of $\{N_1+N_2,T\}=0$ is similar to [10], we only give the proof of $\{R,T\}=0$. Note that

$$R=\frac{1}{4^p(p+1)\pi^{2p}}f(\widetilde{\theta})\sum_{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0}q_{j_1}\bar{q}_{j_2}q_{j_3}\bar{q}_{j_4}\cdots q_{j_{2p+1}}\bar{q}_{j_{2p+2}},$$

where $f(\widetilde{\theta})=\sum\limits_{\widetilde{k}\in\mathbb{Z}^m}f^{\widetilde{k}}e^{\mathrm{i}\langle\widetilde{\mathrm{k}},\widetilde{\theta}\rangle}$ and $T(\widetilde{\theta},q,\bar{q})=T(\widetilde{\theta})\sum\limits_{n\in\mathbb{Z}^2}q_n\bar{q}_n$. In view of

$$\frac{\partial R}{\partial\widetilde{I}}=0,\quad\frac{\partial T}{\partial\widetilde{I}}=0,\quad\frac{\partial T}{\partial\bar{q}_m}=T(\widetilde{\theta})q_m,\quad\frac{\partial T}{\partial q_m}=T(\widetilde{\theta})\bar{q}_m,$$

$$\frac{\partial R}{\partial q_m}=\frac{f(\widetilde{\theta})}{4^p(p+1)\pi^{2p}}\sum_{\widehat{s}=1}^{p+1}\sum_{\substack{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0\\ j_{2\widehat{s}-1}=m}}q_{j_1}\bar{q}_{j_2}\cdots\bar{q}_{j_{2\widehat{s}-2}}\bar{q}_{j_{2\widehat{s}}}q_{j_{2\widehat{s}+1}}\bar{q}_{j_{2\widehat{s}+2}}\cdots q_{j_{2\widehat{p}+1}}\bar{q}_{j_{2\widehat{p}+2}},$$

and

$$\frac{\partial R}{\partial\bar{q}_m}=\frac{f(\widetilde{\theta})}{4^p(p+1)\pi^{2p}}\sum_{\widehat{s}=1}^{p+1}\sum_{\substack{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0\\ j_{2\widehat{s}}=m}}q_{j_1}\bar{q}_{j_2}\cdots\bar{q}_{j_{2\widehat{s}-2}}q_{j_{2\widehat{s}-1}}q_{j_{2\widehat{s}+1}}\bar{q}_{j_{2\widehat{s}+2}}\cdots q_{j_{2\widehat{p}+1}}\bar{q}_{j_{2\widehat{p}+2}},$$

we have

$$\begin{aligned}\{R,T\} &= \frac{\partial R}{\partial \widetilde{\theta}}\cdot\frac{\partial T}{\partial \widetilde{I}} - \frac{\partial R}{\partial \widetilde{I}}\cdot\frac{\partial T}{\partial \widetilde{\theta}} + \mathrm{i}\sum_{m\in\mathbb{Z}^2}\Big(\frac{\partial R}{\partial q_m}\cdot\frac{\partial T}{\partial \bar{q}_m} - \frac{\partial R}{\partial \bar{q}_m}\cdot\frac{\partial T}{\partial q_m}\Big)\\
&= \mathrm{i}\sum_{m\in\mathbb{Z}^2}\Big(\frac{\partial R}{\partial q_m}\cdot\frac{\partial T}{\partial \bar{q}_m} - \frac{\partial R}{\partial \bar{q}_m}\cdot\frac{\partial T}{\partial q_m}\Big)\\
&= \mathrm{i}\sum_{m\in\mathbb{Z}^2}\frac{f(\widetilde{\theta})T(\widetilde{\theta})}{4^p(p+1)\pi^{2p}}\Big[\sum_{\widehat{s}=1}^{p+1}\sum_{\substack{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0\\ j_{2\widehat{s}-1}=m}} q_{j_1}\bar{q}_{j_2}\cdots\bar{q}_{j_{2\widehat{s}-2}}q_m q_{j_{2\widehat{s}+1}}\cdots q_{j_{2\widehat{p}+1}}\bar{q}_{j_{2\widehat{p}+2}}\\
&\quad -\sum_{\widehat{s}=1}^{p+1}\sum_{\substack{\sum_{\widehat{r}=1}^{p+1}(j_{2\widehat{r}-1}-j_{2\widehat{r}})=0\\ j_{2\widehat{s}}=m}} q_{j_1}\bar{q}_{j_2}\cdots q_{j_{2\widehat{s}-1}}\bar{q}_m q_{j_{2\widehat{s}+1}}\bar{q}_{j_{2\widehat{s}+2}}\cdots q_{j_{2\widehat{p}+1}}\bar{q}_{j_{2\widehat{p}+2}}\Big]\\
&= 0.\end{aligned}$$

□

Remark 3.4. The first symplectic transformation ϕ_T^1 not only achieves the reducibility of non-autonomous linear systems but also ensures that after the transformation, the nonlinear term in the Hamiltonian remains unchanged.

Lemma 3.5. *For any fixed $\rho > 0$, T is real-analytic as a map in a neighborhood of the origin, satisfying the estimate*

$$\|T\|_{D_\rho(r_+,l_+),\mathscr{I}} \le c\mu^{-2}\Big(\frac{2\nu+m+1}{r}\Big)^{2\nu+m+1}e^{-(2\nu+m+1)}$$

where the definition of μ,ν is shown in (5.4).

For the proof, we refer to [10].

We now arrive at Hamiltonian systems with the Hamiltonian $H_+ = \Lambda_+ + R_+$. For simplicity in notation, we denote H,Λ,R in place of H_+,Λ_+,R_+, so $H = \Lambda + R$.

4. Partial Birkhoff Normal Form

In this section, for an admissible set of tangential sites $S = \{j_1^*,\dots,j_b^*\}\subset\mathbb{Z}^2$, the corresponding Hamiltonian H can be transformed into its normal form through symplectic transformations. Proposition 4.2 presents the results of the second symplectic transformations.

The second symplectic transformation converts the resulting autonomous system into an integrable nonlinear normal form depending on angle variables. In this process, the denominator $\langle\widetilde{k},\widetilde{\omega}\rangle - \sum_{\widehat{r}=1}^{p+1}(\lambda_{i_{2\widehat{r}-1}} - \lambda_{i_{2\widehat{r}}})$ will appear. When $\langle\widetilde{k},\widetilde{\omega}\rangle - \sum_{\widehat{r}=1}^{p+1}(\lambda_{i_{2\widehat{r}-1}} - \lambda_{i_{2\widehat{r}}}) = 0$, the corresponding term in the normal form depends on the angle variables. By selecting the admissible set in Definition 2.1, the integrable terms that depend on angle variables are reduced, making the normal form more concise.

Lemma 4.1. *For $\widetilde{h}\in\mathbb{Z}\setminus\{0\}, \widetilde{k}\in\mathbb{Z}^m$, we have $\big|\langle\widetilde{k},\widetilde{\omega}\rangle + \widetilde{h}\big| \ge \frac{\mu}{|\widetilde{k}|^{m+1}}$.*

Proof. For $\forall\widetilde{\omega}=(\widetilde{\omega}_1,\cdots,\widetilde{\omega}_m)\in\mathbb{R}^m, \exists\rho>0,$ s.t. $\widetilde{\omega}\in\left[\rho,2\rho\right]^m$, let

$$\mathscr{R}^2_{\widetilde{k},\widetilde{h}}=\{\widetilde{\omega}\in\left[\rho,2\rho\right]^m:\left|\langle\widetilde{k},\widetilde{\omega}\rangle+\widetilde{h}\right|\leq\frac{\rho}{C\left|\widetilde{k}\right|^{m+1}}\},$$

which yields

$$meas\mathscr{R}^2_{\widetilde{k},\widetilde{h}}\leq 2\left|\widetilde{k}\right|^{-1}\rho^{m-1}\frac{\rho}{C\left|\widetilde{k}\right|^{m+1}}\leq\frac{2}{C\left|\widetilde{k}\right|^{m+2}}\rho^m.$$

Denote $\mathscr{R}^2=\bigcup_{\substack{0\neq\widetilde{k}\in\mathbb{Z}^m\\ \widetilde{h}\in\mathbb{Z}}}\mathscr{R}^2_{\widetilde{k},\widetilde{h}}$. If $\widetilde{k}=0,\widetilde{h}\in\mathbb{Z}\setminus\{0\}$, then $\left|\langle\widetilde{k},\widetilde{\omega}\rangle+\widetilde{h}\right|=\left|\widetilde{h}\right|\geq 1$. Obviously, $\mathscr{R}^2=\emptyset$. When $\widetilde{k}\neq 0,\widetilde{h}\in\mathbb{Z}\setminus\{0\}$, if $\left|\widetilde{h}\right|>1+\left|\widetilde{k}\right|\left|\widetilde{\omega}\right|$, then $\left|\langle\widetilde{k},\widetilde{\omega}\rangle+\widetilde{h}\right|\geq\left|\widetilde{h}\right|-\left|\langle\widetilde{k},\widetilde{\omega}\rangle\right|\geq\left|\widetilde{h}\right|-\left|\widetilde{k}\right|\left|\widetilde{\omega}\right|>1$, so $\mathscr{R}^2_{\widetilde{k},\widetilde{h}}=\emptyset$. Thus

$$\begin{aligned}meas\mathscr{R}^2&=meas\bigcup_{0\neq\widetilde{k}\in\mathbb{Z}^m}\bigcup_{\left|\widetilde{h}\right|=1}\mathscr{R}^2_{\widetilde{k},\widetilde{h}}\\&=\sum_{0\neq\widetilde{k}\in\mathbb{Z}^m}2\left(1+\left|\widetilde{k}\right|\left|\widetilde{\omega}\right|\frac{2\rho^m}{C\left|\widetilde{k}\right|^{m+2}}\right)\\&\leq C_0\sum_{0\neq\widetilde{k}\in\mathbb{Z}^m}\frac{\rho^m}{C\left|\widetilde{k}\right|^{m+1}}.\end{aligned}$$

Let $\left|\widetilde{k}\right|_\infty=\max\{\left|\widetilde{k}_1\right|,\cdots,\left|\widetilde{k}_m\right|\}$. Then

$$\sum_{\left|\widetilde{k}\right|_\infty=p}1\leq 2m(2p+1)^{m-1}\quad\text{and}\quad\left|\widetilde{k}\right|_\infty\leq\left|\widetilde{k}\right|\leq m\left|\widetilde{k}\right|_\infty,$$

so $meas\mathscr{R}^2\leq C_0\frac{m\rho^m}{C}\sum_{p=1}^{\infty}(2p+1)^m p^{-(m+1)}\leq\mu\rho^m$. Let $\mathscr{R}=\left[\rho,2\rho\right]^m\setminus(\mathscr{R}^2)$. Then $meas\mathscr{R}\geq(1-\mu)\rho^m$, and

$$\forall\widetilde{\omega}\in\mathscr{R}^m,\left|\langle\widetilde{k},\widetilde{\omega}\rangle+\widetilde{h}\right|\geq\frac{\mu}{\left|\widetilde{k}\right|^{m+1}}.$$

□

Proposition 4.2. *Let S be an admissible set. For the Hamiltonian function (3.3), Ψ is a symplectic transformation in a neighborhood of the origin such that*

$$H\circ\Psi=\Lambda+\sum_{\widehat{l}=1}^{p}\left(\mathscr{A}_{\widehat{l}}+\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+R,\tag{4.1}$$

where

$$\Lambda=\langle\varepsilon^{-3p}\widetilde{\omega},\widetilde{I}\rangle+\langle\widehat{\omega},\widehat{I}\rangle+\langle\Omega w,w\rangle=\langle\varepsilon^{-3p}\widetilde{\omega},\widetilde{I}\rangle+\sum_{j\in S}\widehat{\omega}_j(\xi)\widehat{I}_j+\sum_{n\in\mathbb{Z}_1^2}\Omega_n(\xi)w_n\bar{w}_n$$

$$\begin{aligned}\widehat{\omega}_j(\xi)=\varepsilon^{-3p}(\lambda_j+f^0)+\frac{f^0}{4^p(p+1)\pi^{2p}}&\Big\{(p+1)\xi_j^p\\&+\sum_{i\in S\setminus\{j\}}\left[(p+1)^2\xi_i^p+(p+1)^2p\xi_j^{p-1}\xi_i\right]+g_{\omega,p,j}\left(\xi_{j_1^*},\cdots,\xi_{j_b^*}\right)\Big\}\end{aligned}$$

$$\Omega_n(\xi)=\varepsilon^{-3p}(\lambda_n+f^0)+\frac{f^0}{4^p(p+1)\pi^{2p}}\Big\{\sum_{i\in S}\Big[(p+1)^2\xi_i^p\Big]+g_{\Omega,p,n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)\Big\}$$

$$\mathscr{A}_{\widehat{l}}=\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2\widehat{l}-1}}\Bigg\{\Bigg\{\Bigg[\sum_{i\in S}\Big(\xi_i^{p-\widehat{l}}+g_{\mathscr{A},p,\widehat{l},n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)\Big)\Bigg]$$

$$\cdot\frac{\prod_{\widehat{v}=1}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\cdot\prod_{\widehat{v}=1}^{\widehat{l}}\Bigg[\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}\Bigg]\Bigg\}\cdot w_n\bar{w}_m\Bigg\},\ 1\le\widehat{l}\le p-1$$

$$\mathscr{A}_p=\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p-1}}\left[\frac{\prod_{\widehat{v}=1}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\right]w_n\bar{w}_m$$

$$\mathscr{B}_{\widehat{l}}=\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Bigg\{\Bigg\{\Bigg[\sum_{i\in S}\Big(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)\Big)\Bigg]$$

$$\cdot\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\cdot\prod_{\widehat{v}=2}^{\widehat{l}}\Bigg[\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_{2\widehat{v}-1}}+\widehat{\theta}_{j_{2\widehat{v}}}\right)}\Bigg]\Bigg\}$$

$$\cdot\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_1}-\widehat{\theta}_{j_2}\right)}w_nw_m\Bigg\},\ 1\le\widehat{l}\le p-1$$

$$\mathscr{B}_p=\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\left\{\left[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_{2\widehat{v}-1}}+\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\right]\right.$$

$$\left.\cdot\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_1}-\widehat{\theta}_{j_2}\right)}w_nw_m\right\}$$

$$\overline{\mathscr{B}}_{\widehat{l}}=\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Bigg\{\Bigg\{\Bigg[\sum_{i\in S}\Big(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)\Big)\Bigg]$$

$$\cdot\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\cdot\prod_{\widehat{v}=2}^{\widehat{l}}\Bigg[\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}\Bigg]\Bigg\}$$

$$\cdot\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(\widehat{\theta}_{j_1}+\widehat{\theta}_{j_2}\right)}\bar{w}_n\bar{w}_m\Bigg\},\ 1\le\widehat{l}\le p-1$$

$$\overline{\mathscr{B}}_p=\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\left\{\left[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\right]\cdot\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(\widehat{\theta}_{j_1}+\widehat{\theta}_{j_2}\right)}\bar{w}_n\bar{w}_m\right\}$$

$$R=O\left(\sum_{\widehat{s}=2}^{p+1}\varepsilon^{2\widehat{s}-2}\left|\xi\right|^{p+1-\widehat{s}}\left|\widehat{I}\right|^{\widehat{s}}+\sum_{\widehat{s}=1}^{2p}\varepsilon^{\widehat{s}}\left|\xi\right|^{p-\frac{\widehat{s}}{2}}\left|\widehat{I}\right|^{\frac{\widehat{s}}{2}}\|w\|_{\rho}^{2}+\sum_{\widehat{r}=3}^{2p+2}\sum_{\widehat{s}=0}^{p+1-\frac{\widehat{r}}{2}}\left|\xi\right|^{p+1-\frac{\widehat{r}}{2}-\widehat{s}}\left|\widehat{I}\right|^{\widehat{s}}\|w\|_{\rho}^{\widehat{r}}\right). \tag{4.2}$$

The definitions of the homogeneous polynomials $g_{\omega,p,j}(\xi_{j_1^*},\dots,\xi_{j_b^*})$, $g_{\Omega,p,n}(\xi_{j_1^*},\dots,\xi_{j_b^*})$, $g_{\mathscr{A},p,\widehat{l},n}(\xi_{j_1^*},\dots,\xi_{j_b^*})$, and $g_{\mathscr{B},p,\widehat{l},n}(\xi_{j_1^*},\dots,\xi_{j_b^*})$, as well as the positive integers $b_{11}^n,\dots,b_{1\hbar_1^n}^n$, $b_{21}^n,\dots,b_{2\hbar_2^n}^n$ will be provided in detail during the following proof process.

Proof. Let

$$\Psi=\sum_{\substack{\sum_{\widehat{r}=1}^{p+1}(i_{2\widehat{r}-1}-i_{2\widehat{r}})=0,|\widetilde{k}|\neq0,\\ \sum_{\widehat{r}=1}^{p}\left(|i_{2\widehat{r}-1}|^2-|i_{2\widehat{r}}|^2\right)\neq0\\ \sharp(S\cap\{i_{2\widehat{r}-1},i_{2\widehat{r}}|1\le\widehat{r}\le p+1\})\ge2p}}\frac{\mathrm{i}f^{\widetilde{k}}\cdot q_{i_1}\bar{q}_{i_2}q_{i_3}\bar{q}_{i_4}\cdots q_{i_{2p+1}}\bar{q}_{i_{2p+2}}}{4^p\cdot(p+1)\cdot\pi^{2p}\left(\langle\widetilde{k},\widetilde{\omega}\rangle-\sum_{\widehat{r}=1}^{p+1}\left(\lambda_{i_{2\widehat{r}-1}}-\lambda_{i_{2\widehat{r}}}\right)\right)}e^{\mathrm{i}\langle\widetilde{k},\widetilde{\theta}\rangle} \tag{4.3}$$

and X_{Ψ}^1 be the time-1 mapping of the Hamiltonian vector field of Ψ. By Lemma 4.1, the definition of Ψ is reasonable. Set

$$q_i=\begin{cases}q_i, & i\in S,\\ z_i, & i\in\mathbb{Z}_1^2.\end{cases}$$

Then the symplectic change of coordinates X_{Ψ}^1 takes H into

$$\begin{aligned}\widehat{H}&=H\circ X_{\Psi}^1\\ &=H+\{H,\Psi\}+\int_0^1(1-t)\{\{H,\Psi\},\Psi\}\circ X_{\Psi}^t dt\\ &=\langle\widetilde{\omega},\widetilde{I}\rangle+\sum_{j\in S}(\lambda_j+f^0)\left|q_j\right|^2+\sum_{n\in\mathbb{Z}_1^2}(\lambda_n+f^0)|z_n|^2\\ &\quad+\frac{f^0}{4^p(p+1)\pi^{2p}}\left\{\sum_{\widehat{d}=1}^{p+1}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1\le\cdots\le\widehat{k}_{\widehat{d}}\le p+1\\ 1\le\widehat{a}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p+1\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p+1}}\sum_{\substack{i_1,\cdots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}}\ if\ \check{r}\neq\check{s}}}\right.\\ &\quad\left.\left[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\left(C^{\widehat{k}_{\widehat{s}}}_{p+1-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\right)^2\left|q_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\right|^{2\widehat{k}_{\widehat{s}}}\right]\right\}\\ &\quad+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\left\{\sum_{n\in\mathbb{Z}_1^2}\left\{\sum_{\widehat{d}=1}^{p}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1\le\cdots\le\widehat{k}_{\widehat{d}}\le p\\ 1\le\widehat{a}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p}}\sum_{\substack{i_1,\cdots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}},if\check{r}\neq\check{s}}}\right.\right.\\ &\quad\left.\left.\left[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\left(C^{\widehat{k}_{\widehat{s}}}_{p-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\right)^2\left|q_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\right|^{2\widehat{k}_{\widehat{s}}}\right]\right\}\cdot|z_n|^2\right\}\end{aligned}$$

$$+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\sum_{1\le\widehat{l}\le p-1}\left\{\sum_{n\in\mathscr{L}_{2\widehat{l}-1}}\left\{\sum_{\widehat{d}=1}^{p-\widehat{l}}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1<\cdots<\widehat{k}_{\widehat{d}}\le p-\widehat{l}\\ 1\le\widehat{a}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p-\widehat{l}\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p-\widehat{l}}}\sum_{\substack{i_1,\cdots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}},\ if\ \check{r}\neq\check{s}}}\right.\right.$$

$$\cdot\left[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\left(C^{\widehat{k}_{\widehat{s}}}_{p-\widehat{l}-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\right)^2\left|q_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\right|^{2\widehat{k}_{\widehat{s}}}\right]\Bigg\}$$

$$\left.\cdot\left[\frac{\prod_{\widehat{v}=1}^{\widehat{l}}(p+1-\widehat{v})^2}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\cdot\prod_{\widehat{v}=1}^{\widehat{l}}q_{j_{2\widehat{v}-1}}\bar{q}_{j_{2\widehat{v}}}\right]\cdot z_n\bar{z}_m\right\}$$

$$+\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\cdot\sum_{n\in\mathscr{L}_{2p-1}}\left[\frac{1}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\cdot\prod_{\widehat{v}=1}^{p}q_{j_{2\widehat{v}-1}}\bar{q}_{j_{2\widehat{v}}}\right]\cdot z_n\bar{z}_m$$

$$+\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\sum_{1\le\widehat{l}\le p-1}\left\{\sum_{n\in\mathscr{L}_{2\widehat{l}}}\left\{\sum_{\widehat{d}=1}^{p-\widehat{l}}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1<\cdots<\widehat{k}_{\widehat{d}}\le p-\widehat{l}\\ 1\le\widehat{k}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p-\widehat{l}\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p-\widehat{l}}}\sum_{\substack{i_1,\ldots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}},if\check{r}\neq\check{s}}}\right.\right.$$

$$\cdot\left[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\left(C^{\widehat{k}_{\widehat{s}}}_{p-\widehat{l}-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\right)^2\left|q_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\right|^{2\widehat{k}_{\widehat{s}}}\right]\Bigg\}$$

$$\cdot\left\{\left[\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2\bar{q}_{j_{2\widehat{v}-1}}q_{j_{2\widehat{v}}}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\right]\cdot\bar{q}_{j_1}\bar{q}_{j_2}z_nz_m\right.$$

$$\left.\left.+\left[\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2q_{j_{2\widehat{v}-1}}\bar{q}_{j_{2\widehat{v}}}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\right]\cdot q_{j_1}q_{j_2}\bar{z}_n\bar{z}_m\right\}\right\}$$

$$+\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\cdot\sum_{n\in\mathscr{L}_{2p}}\left[\frac{\prod_{\widehat{v}=2}^{p}\bar{q}_{j_{2\widehat{v}-1}}q_{j_{2\widehat{v}}}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\cdot\bar{q}_{j_1}\bar{q}_{j_2}z_nz_m\right.$$

$$\left.+\frac{\prod_{\widehat{v}=2}^{p}q_{j_{2\widehat{v}-1}}\bar{q}_{j_{2\widehat{v}}}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\cdot q_{j_1}q_{j_2}\bar{z}_n\bar{z}_m\right]$$

$$+O\left(|q|^{4p+2}+|q|^{4p+1}\|z\|_\rho+|q|^{4p}\|z\|^2_\rho+\sum_{\widehat{r}=3}^{2p+2}\left(|q|^{2p+2-\widehat{r}}\|z\|^{\widehat{r}}_\rho\right)\right). \tag{4.4}$$

For any $1 \le \widehat{v} \le p+1$, suppose that the set $\{i_{2\widehat{t}-1}, i_{2\widehat{t}} \mid 1 \le \widehat{t} \le \widehat{v}\}$ is contained in S and the condition $\sum\limits_{\widehat{t}=1}^{\widehat{v}} (i_{2\widehat{t}-1} - i_{2\widehat{t}}) = 0$ holds. Then, according to the definition of S, we obtain $\{i_{2\widehat{t}-1} \mid 1 \le \widehat{t} \le \widehat{v}\} = \{i_{2\widehat{t}} \mid 1 \le \widehat{t} \le \widehat{v}\}$. As a consequence, it follows that $\prod_{\widehat{t}=1}^{\widehat{v}} q_{i_{2\widehat{t}-1}} \bar{q}_{i_{2\widehat{t}}} = \prod_{\widehat{t}=1}^{\widehat{v}} |q_{i_{2\widehat{t}-1}}|^2$. In equation (4.4), $\widehat{d}$ denotes the number of distinct exponent values in $\prod_{\widehat{t}=1}^{\widehat{v}} q_{i_{2\widehat{t}-1}}$. The integers $\widehat{k}_1, \ldots, \widehat{k}_{\widehat{d}}$ correspond to all distinct exponents. Additionally, $\widehat{a}_1, \ldots, \widehat{a}_{\widehat{d}}$ represent the count of terms corresponding to the powers $\widehat{k}_1, \ldots, \widehat{k}_{\widehat{d}}$ respectively. This definition is similar to that in [28].

For $1 \le \widehat{l} \le p$, assume that $n \in \mathscr{L}_{2\widehat{l}-1}$ (and similarly for $n \in \mathscr{L}_{2\widehat{l}}$). Then the pair (n,m) are resonant, and the set $(j_1, j_2, \ldots, j_{2\widehat{l}-1}, j_{2\widehat{l}})$ in (4.4) is uniquely determined by (n,m). In this case, $\hbar_1^n$ denotes the number of distinct elements in the set $\{j_{2\widehat{t}-1} \mid 1 \le \widehat{t} \le \widehat{l}\}$, and $b_{11}^n, \ldots, b_{1\hbar_1^n}^n$ are the corresponding multiplicities. Similarly, $\hbar_2^n$ represents the number of distinct elements in the set $\{j_{2\widehat{t}} \mid 1 \le \widehat{t} \le \widehat{l}\}$ and $b_{21}^n, \ldots, b_{2\hbar_2^n}^n$ are the corresponding multiplicities. Here, $C_{\widehat{t}}^{\widehat{s}} = \frac{\widehat{t}!}{\widehat{s}!(\widehat{t}-\widehat{s})!}$ and $b_{\bullet}^n! = 1 \cdot 2 \cdots (b_{\bullet}^n - 1) \cdot b_{\bullet}^n$.

In the second symplectic transformation, the action-angle variables are introduced as

$$
\begin{aligned}
&q_i = \sqrt{\widehat{I}_i + \xi_i}\, e^{\mathrm{i}\widehat{\theta}_i}, \quad \bar{q}_i = \sqrt{\widehat{I}_i + \xi_i}\, e^{-\mathrm{i}\widehat{\theta}_i}, \quad i \in S \\
&z_n = w_n, \quad \bar{z}_n = \bar{w}_n, \quad n \in \mathbb{Z}_1^2.
\end{aligned} \tag{4.5}
$$

The symplectic coordinate transformation (4.4) transforms the Hamiltonian $\widehat{H}$ into the desired form

$$
\widehat{H} = \langle \widetilde{\omega}, \widetilde{I} \rangle + \sum_{j \in S} (\lambda_j + f^0) \widehat{I}_j + \sum_{n \in \mathbb{Z}_1^2} (\lambda_n + f^0) |w_n|^2
$$

$$
+ \frac{f^0}{4^p (p+1) \pi^{2p}} \cdot \Bigg\{ \sum_{j \in S} \Bigg\{ \sum_{\widehat{d}=1}^{p+1} \sum_{\substack{\widehat{k}_0 = 0, \widehat{a}_0 = 0, \\ 1 \le \widehat{a}_1 \le \cdots \le \widehat{a}_{\widehat{d}} \le p+1 \\ 1 \le \widehat{k}_1 \le \cdots \le \widehat{k}_{\widehat{d}} \le p+1 \\ \widehat{a}_1 \widehat{k}_1 + \cdots + \widehat{a}_{\widehat{d}} \widehat{k}_{\widehat{d}} = p+1}} \sum_{\substack{i_1, \cdots, i_{(\sum_{\check{t}=1}^{\widehat{d}} \widehat{a}_{\check{t}})-1} \in S \setminus \{j\} \\ i_{\check{r}} \ne i_{\check{s}} \ if \ \check{r} \ne \check{s}}} \sum_{\widehat{v}=1}^{\widehat{d}}
$$

$$
\Bigg[\Bigg(\prod_{\substack{1 \le \widehat{s} \le \widehat{d} \\ \widehat{s} \ne \widehat{v}}} \prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}} \Big(C^{\widehat{k}_{\widehat{s}}}_{p+1 - \widehat{a}_{\widehat{v}} \widehat{k}_{\widehat{v}} - (\widehat{t}-1) \widehat{k}_{\widehat{s}} - \sum_{\substack{0 \le \widehat{r} \le \widehat{s}-1 \\ \widehat{r} \ne \widehat{v}}} \widehat{a}_{\widehat{r}} \widehat{k}_{\widehat{r}}} \Big)^2 \xi^{\widehat{k}_{\widehat{s}}}_{i_{\widehat{t} + \widehat{a}_{\widehat{v}} - 1 + \sum_{\widehat{r}=0}^{\widehat{s}-1} \widehat{a}_{\widehat{r}}}} \Bigg)
$$

$$
\cdot \Bigg(\sum_{\widehat{u}=1}^{\widehat{a}_{\widehat{v}}} C^{\widehat{k}_{\widehat{v}}}_{p+1-(\widehat{u}-1)\widehat{k}_{\widehat{v}}} \Bigg)^2 \cdot \Bigg(\prod_{\widehat{u}=1}^{\widehat{a}_{\widehat{v}}-1} \xi^{\widehat{k}_{\widehat{v}}}_{i_{\widehat{u}}} \Bigg) \cdot \widehat{k}_{\widehat{v}} \cdot \xi_j^{\widehat{k}_{\widehat{v}}-1} \Bigg] \Bigg\} \cdot \widehat{I}_j \Bigg\}
$$

$$
\begin{aligned}
&+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\Bigg\{\sum_{n\in\mathbb{Z}_1^2}\Bigg\{\sum_{\widehat{d}=1}^{p}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1\le\cdots\le\widehat{k}_{\widehat{d}}\le p\\ 1\le\widehat{a}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p}}\ \sum_{\substack{i_1,\cdots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}},if\,\check{r}\neq\check{s}}}\\
&\Bigg[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\Big(C^{\widehat{k}_{\widehat{s}}}_{p-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\Big)^2\xi^{\widehat{k}_{\widehat{s}}}_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\Bigg]\Bigg\}|w_n|^2\Bigg\}\\
&+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\sum_{1\le\widehat{l}\le p-1}\Bigg\{\sum_{n\in\mathscr{L}_{2\widehat{l}-1}}\Bigg\{\sum_{\widehat{d}=1}^{p-\widehat{l}}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1<\cdots<\widehat{k}_{\widehat{d}}\le p-\widehat{l}\\ 1\le\widehat{a}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p-\widehat{l}\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p-\widehat{l}}}\ \sum_{\substack{i_1,\cdots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}},if\,\check{r}\neq\check{s}}}\\
&\cdot\Bigg[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\Big(C^{\widehat{k}_{\widehat{s}}}_{p-\widehat{l}-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\Big)^2\xi^{\widehat{k}_{\widehat{s}}}_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\Bigg]\Bigg\}\\
&\cdot\left[\frac{\prod_{\widehat{v}=1}^{\widehat{l}}(p+1-\widehat{v})^2}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\right]\cdot\left[\prod_{\widehat{v}=1}^{\widehat{l}}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}\cdot e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}\right]\cdot w_n\bar{w}_m\Bigg\}\\
&+\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\cdot\sum_{n\in\mathscr{L}_{2p-1}}\left[\frac{\prod_{\widehat{v}=1}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}\cdot e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\right]\cdot w_n\bar{w}_m\\
&+\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\cdot\sum_{1\le\widehat{l}\le p-1}\Bigg\{\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Bigg\{\sum_{\widehat{d}=1}^{p-\widehat{l}}\sum_{\substack{\widehat{k}_0=0,\widehat{a}_0=0,\\ 1\le\widehat{k}_1<\cdots<\widehat{k}_{\widehat{d}}\le p-\widehat{l}\\ 1\le\widehat{a}_1\le\cdots\le\widehat{a}_{\widehat{d}}\le p-\widehat{l}\\ \widehat{a}_1\widehat{k}_1+\cdots+\widehat{a}_{\widehat{d}}\widehat{k}_{\widehat{d}}=p-\widehat{l}}}\ \sum_{\substack{i_1,\cdots,i_{\sum_{\check{t}=1}^{\widehat{d}}\widehat{a}_{\check{t}}}\in S\\ i_{\check{r}}\neq i_{\check{s}},if\,\check{r}\neq\check{s}}}\\
&\cdot\Bigg[\prod_{\widehat{s}=1}^{\widehat{d}}\prod_{\widehat{t}=1}^{\widehat{a}_{\widehat{s}}}\Big(C^{\widehat{k}_{\widehat{s}}}_{p-\widehat{l}-(\widehat{t}-1)\widehat{k}_{\widehat{s}}-\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}\widehat{k}_{\widehat{r}}}\Big)^2\xi^{\widehat{k}_{\widehat{s}}}_{i_{\widehat{t}+\sum_{\widehat{r}=0}^{\widehat{s}-1}\widehat{a}_{\widehat{r}}}}\Bigg]\Bigg\}\\
&\cdot\Bigg\{\left[\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}}}-\widehat{\theta}_{j_{2\widehat{v}-1}}\right)}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\right]\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}(-\widehat{\theta}_{j_1}-\widehat{\theta}_{j_2})}w_nw_m\\
&+\left[\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_{2\widehat{v}}}+\widehat{\theta}_{j_{2\widehat{v}-1}}\right)}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\right]\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}(\widehat{\theta}_{j_1}+\widehat{\theta}_{j_2})}\bar{w}_n\bar{w}_m\Bigg\}
\end{aligned}
$$

$$+\frac{(p+1)(p!)^2 f^0}{4^p\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\Bigg[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}}}-\widehat{\theta}_{j_{2\widehat{v}-1}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_1}-\widehat{\theta}_{j_2}\right)}w_nw_m$$

$$+\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(\widehat{\theta}_{j_1}+\widehat{\theta}_{j_2}\right)}\bar{w}_n\bar{w}_m\Bigg]$$

$$+O\left(\sum_{\widehat{s}=2}^{p+1}\left|\xi\right|^{p+1-\widehat{s}}\left|\widehat{I}\right|^{\widehat{s}}+\sum_{\widehat{s}=1}^{2p}\left|\xi\right|^{p-\frac{\widehat{s}}{2}}\left|\widehat{I}\right|^{\frac{\widehat{s}}{2}}\|w\|_\rho^2+\sum_{\widehat{r}=3}^{2p+2}\sum_{\widehat{s}=0}^{p+1-\frac{\widehat{r}}{2}}\left(\left|\xi\right|^{p+1-\frac{\widehat{r}}{2}-\widehat{s}}\left|\widehat{I}\right|^{\widehat{s}}\|w\|_\rho^{\widehat{r}}\right)\right).$$

For further verification, $\widehat{H}$ can be written as follows

$$\widehat{H}=\langle\widetilde{\omega},\widetilde{I}\rangle+\sum_{j\in S}\Bigg\{(\lambda_j+f^0)+\frac{f^0}{4^p(p+1)\pi^{2p}}\Bigg\{(p+1)\xi_j^p$$

$$+\sum_{i\in S\setminus\{j\}}\left[(p+1)^2\xi_i^p+(p+1)^2p\xi_j^{p-1}\xi_i\right]+g_{\omega,p,j}\Bigg\}\Bigg\}\cdot\widehat{I}_j$$

$$+\sum_{n\in\mathbb{Z}_1^2}\left\{(\lambda_n+f^0)+\frac{f^0\left\{\sum_{i\in S}\left[(p+1)^2\xi_i^p\right]+g_{\Omega,p,n}\right\}}{4^p(p+1)\pi^{2p}}\right\}\cdot|w_n|^2$$

$$+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\sum_{1\le\widehat{l}\le p-1}\Bigg\{\sum_{n\in\mathscr{L}_{2\widehat{l}-1}}\Bigg\{\left[\sum_{i\in S}\left(\xi_i^{p-\widehat{l}}+g_{\mathscr{A},p,\widehat{l},n}\right)\right]$$

$$\cdot\frac{\prod_{\widehat{v}=1}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\cdot\prod_{\widehat{v}=1}^{\widehat{l}}\left[\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}\right]\Bigg\}w_n\bar{w}_m\Bigg\}$$

$$+\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p-1}}\left[\frac{\prod_{\widehat{v}=1}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\right]w_n\bar{w}_m$$

$$+\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\sum_{1\le\widehat{l}\le p-1}\Bigg\{\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Bigg\{\left[\sum_{i\in S}\left(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}\right)\right]$$

$$\cdot\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\prod_{\widehat{v}=2}^{\widehat{l}}\left[\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_{2\widehat{v}-1}}+\widehat{\theta}_{j_{2\widehat{v}}}\right)}\right]\Bigg\}\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_1}-\widehat{\theta}_{j_2}\right)}w_nw_m\Bigg\}$$

$$+\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\left[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}}}-\widehat{\theta}_{j_{2\widehat{v}-1}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\right]\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(-\widehat{\theta}_{j_1}-\widehat{\theta}_{j_2}\right)}w_nw_m$$

$$+\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\sum_{1\le\widehat{l}\le p-1}\Bigg\{\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Bigg\{\left[\sum_{i\in S}\left(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}\right)\right]$$

$$\cdot\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\prod_{\widehat{v}=2}^{\widehat{l}}\left[\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}\right]\Big\}\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(\widehat{\theta}_{j_1}+\widehat{\theta}_{j_2}\right)}\bar{w}_n\bar{w}_m\Big\}$$

$$+\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\left[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}e^{\mathrm{i}\left(\widehat{\theta}_{j_{2\widehat{v}-1}}-\widehat{\theta}_{j_{2\widehat{v}}}\right)}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\right]\sqrt{\xi_{j_1}\xi_{j_2}}e^{\mathrm{i}\left(\widehat{\theta}_{j_1}+\widehat{\theta}_{j_2}\right)}\bar{w}_n\bar{w}_m$$

$$+O\left(\sum_{\widehat{s}=2}^{p+1}|\xi|^{p+1-\widehat{s}}\left|\widetilde{I}\right|^{\widehat{s}}+\sum_{\widehat{s}=1}^{2p}|\xi|^{p-\frac{\widehat{s}}{2}}\left|\widetilde{I}\right|^{\frac{\widehat{s}}{2}}\|z\|_\rho^2+\sum_{\widehat{r}=3}^{2p+2}\sum_{\widehat{s}=0}^{p+1-\frac{\widehat{r}}{2}}\left(|\xi|^{p+1-\frac{\widehat{r}}{2}-\widehat{s}}\left|\widetilde{I}\right|^{\widehat{s}}\|z\|_\rho^{\widehat{r}}\right)\right).$$

Denote $\xi=(\xi_{j_1^*},\xi_{j_2^*},\ldots,\xi_{j_b^*}),\widehat{\alpha}=(\widehat{\alpha}_{j_1^*},\widehat{\alpha}_{j_2^*},\ldots,\widehat{\alpha}_{j_b^*})\in\mathbb{Z}^b$, where $\widehat{\alpha}_{j_i^*}\in\mathbb{N},\widehat{i}=1,\cdots,b$, $|\widehat{\alpha}|=\sum_{i=1}^{b}\left|\widehat{\alpha}_{j_i^*}\right|$. The product $\xi^{\widehat{\alpha}}$ denotes $\prod_{i=1}^{b}\xi_{j_i^*}^{\widehat{\alpha}_{j_i^*}}$. Here

$$g_{\omega,p,j}=g_{\omega,p,j}(\xi_{j_1^*},\cdots,\xi_{j_b^*})=\sum_{\substack{|\widehat{\alpha}|=p\\0\le\widehat{\alpha}_{j_i^*}\le p-1,i=1,\cdots,b\\0\le\widehat{\alpha}_j\le p-2}}g_{\omega,p,j,\widehat{\alpha}}\cdot\xi^{\widehat{\alpha}} \tag{4.6}$$

$$g_{\Omega,p,n}=g_{\Omega,p,n}(\xi_{j_1^*},\cdots,\xi_{j_b^*})=\sum_{\substack{|\widehat{\alpha}|=p\\0\le\widehat{\alpha}_{j_i^*}\le p-1,i=1,\cdots,b}}g_{\Omega,p,n,\widehat{\alpha}}\cdot\xi^{\widehat{\alpha}} \tag{4.7}$$

$$g_{\mathscr{A},p,\widehat{l},n}=g_{\mathscr{A},p,\widehat{l},n}(\xi_{j_1^*},\cdots,\xi_{j_b^*})=\sum_{\substack{|\widehat{\alpha}|=p-\widehat{l}\\0\le\widehat{\alpha}_{j_i^*}\le p-\widehat{l}-1,i=1,\cdots,b}}g_{\mathscr{A},p,\widehat{l},n,\widehat{\alpha}}\cdot\xi^{\widehat{\alpha}},1\le\widehat{l}\le p-1 \tag{4.8}$$

$$g_{\mathscr{B},p,\widehat{l},n}=g_{\mathscr{B},p,\widehat{l},n}(\xi_{j_1^*},\cdots,\xi_{j_b^*})=\sum_{\substack{|\widehat{\alpha}|=p-\widehat{l}\\0\le\widehat{\alpha}_{j_i^*}\le p-\widehat{l}-1,i=1,\cdots,b}}g_{\mathscr{B},p,\widehat{l},n,\widehat{\alpha}}\cdot\xi^{\widehat{\alpha}},1\le\widehat{l}\le p-1, \tag{4.9}$$

where $g_{\omega,p,j,\widehat{\alpha}},g_{\Omega,p,n,\widehat{\alpha}},g_{\mathscr{A},p,\widehat{l},n,\widehat{\alpha}},g_{\mathscr{B},p,\widehat{l},n,\widehat{\alpha}}$ are the corresponding integral coefficients. By applying the following time scaling transformation

$$\xi\to\varepsilon^3\xi,\quad\widetilde{I}\to\varepsilon^5\widetilde{I},\quad\widehat{I}\to\varepsilon^5\widehat{I},\quad\widetilde{\theta}\to\widetilde{\theta},\quad\widehat{\theta}\to\widehat{\theta},\quad w\to\varepsilon^{\frac{5}{2}}w,\quad\bar{w}\to\varepsilon^{\frac{5}{2}}\bar{w},$$

we obtain the transformed Hamiltonian

$$H=\varepsilon^{-(3p+5)}\widehat{H}(\varepsilon^3\xi,\varepsilon^5\widetilde{I},\varepsilon^5\widehat{I},\widetilde{\theta},\widehat{\theta},\varepsilon^{\frac{5}{2}}w,\varepsilon^{\frac{5}{2}}\bar{w}).$$

Thus H satisfies equations (4.1) and (4.2), where $\xi\in\mathscr{I}$. □

Subsequently, a unitary transformation will be found to diagonalize the block symmetric matrix, then the normal form described in Proposition 4.2 can be transformed into a more elegant structure, and the unitary transformation is used to transform an integrable nonlinear normal form dependent on angle variables to a linear normal form. To accomplish this, we use the following lemma from [22].

Lemma 4.3. *Let $k_1, k_2, \ldots, k_{\widetilde{m}} \in \mathbb{Z}^{\widetilde{b}}$ be vectors, and let V be a nonsingular $\widetilde{m} \times \widetilde{m}$ matrix with $V^T \bar{V} = E$. Define $\Phi_0 : (\theta, \widehat{I}, w, \bar{w}) \to (\theta_+, \widehat{I}_+, z, \bar{z})$ by*

$$\begin{cases} \theta_+ = \theta \\ \widehat{I}_+ = \widehat{I} - \sum_{i=1}^{\widetilde{m}} w_i \bar{w}_i k_i \\ z = VQw \\ \bar{z} = \bar{V}\bar{Q}\bar{w} \end{cases}$$

where Q is a diagonal matrix given by

$$Q = Q(k_1, k_2, \ldots, k_{\widetilde{m}}) = \mathrm{diag}\left(e^{\mathrm{i}\langle k_1, \theta\rangle}, e^{\mathrm{i}\langle k_2, \theta\rangle}, \ldots, e^{\mathrm{i}\langle k_{\widetilde{m}}, \theta\rangle}\right)$$

then Φ_0 is symplectic.

Proof. Setting $\widetilde{G} = diag(\langle k_1, d\theta\rangle, \ldots, \langle k_{\widetilde{m}}, d\theta\rangle)$, we have $dQ = \mathrm{i}Q\widetilde{G}$ and $d\bar{Q} = -\mathrm{i}\bar{Q}\widetilde{G}$. As

$$\begin{aligned} \sum_{i=1}^{\widetilde{m}} dz_i \wedge d\bar{z}_i &= (VdQw + VQdw)^T \wedge (\bar{V}d\bar{Q}\bar{w} + \bar{V}\bar{Q}d\bar{w}) \\ &= (VdQw)^T \wedge (\bar{V}d\bar{Q}\bar{w}) + (VQdw)^T \wedge (\bar{V}d\bar{Q}\bar{w}) \\ &+ (VdQw)^T \wedge (\bar{V}\bar{Q}d\bar{w}) + (VQdw)^T \wedge (\bar{V}\bar{Q}d\bar{w}) \end{aligned}$$

and

$$\begin{aligned} (VdQw)^T \wedge (\bar{V}d\bar{Q}\bar{w}) &= (\mathrm{i}VQ\widetilde{G}w)^T \wedge (\mathrm{i}\bar{V}\bar{Q}\widetilde{G}\bar{w}) = (\mathrm{i}w^T\widetilde{G}^T Q^T V^T) \wedge (\mathrm{i}\bar{V}\bar{Q}\widetilde{G}\bar{w}) \\ &= \left(w^T \widetilde{G}^T\right) \wedge (\widetilde{G}\bar{w}) = 0 \end{aligned}$$

we have $(VQdw)^T \wedge (\bar{V}d\bar{Q}\bar{w}) = -\mathrm{i}\sum_{i=1}^{\widetilde{m}} dw_i \wedge (\langle k_i, d\theta\rangle \bar{w}_i)$, $(VdQw)^T \wedge (\bar{V}\bar{Q}d\bar{w}) = \mathrm{i}\sum_{i=1}^{\widetilde{m}} (w_i\langle k_i, d\theta\rangle)$ $\wedge d\bar{w}_i$, and $(VQdw)^T \wedge (\bar{V}\bar{Q}d\bar{w}) = \sum_{i=1}^{\widetilde{m}} dw_i \wedge d\bar{w}_i$. Then $\sum_{i=1}^{\widetilde{m}} dz_i \wedge d\bar{z}_i = \mathrm{i}\sum_{i=1}^{\widetilde{m}} (\langle k_i, d\theta\rangle \bar{w}_i) \wedge dw_i +$ $\mathrm{i}\sum_{i=1}^{\widetilde{m}} (w_i\langle k_i, d\theta\rangle) \wedge d\bar{w}_i + \sum_{i=1}^{\widetilde{m}} dw_i \wedge d\bar{w}_i$, which yields

$$\sum_{i=1}^{\widetilde{m}} d\theta_{+i} \wedge d\widehat{I}_{+i} = \sum_{i=1}^{\widetilde{m}} d\theta_i \wedge d\widehat{I}_i + \sum_{i=1}^{\widetilde{m}} (\langle k_i, d\theta\rangle \bar{w}_i) \wedge dw_i + \sum_{i=1}^{\widetilde{m}} (w_i\langle k_i, d\theta\rangle) \wedge d\bar{w}_i$$

so

$$\sum_{i=1}^{\widetilde{m}} (d\theta_{+i} \wedge d\widehat{I}_{+i}) + \mathrm{i}\sum_{i=1}^{\widetilde{m}} (dz_i \wedge d\bar{z}_i) = \sum_{i=1}^{\widetilde{m}} d\theta_i \wedge d\widehat{I}_i + \mathrm{i}\sum_{i=1}^{\widetilde{m}} dw_i \wedge d\bar{w}_i.$$

We can easily conclude that Ψ_0 preserves the symplectic structure, implying that it is a symplectic transformation. □

A nonlinear symplectic coordinate transformation $\Phi(\exists V)$,

$$\begin{cases} \theta_+ = \theta \\ \widehat{I}_+ = \widehat{I} - \sum\limits_{\widehat{l}=1}^{p} \Big\{ \sum\limits_{n\in\mathscr{L}_{2\widehat{l}-1}} \Big[w_n\bar{w}_n \sum\limits_{\widehat{r}=1}^{\widehat{l}} e_{j_{2\widehat{r}-1}} + w_m\bar{w}_m \sum\limits_{\widehat{r}=1}^{\widehat{l}} e_{j_{2\widehat{r}}} \Big] \\ + \sum\limits_{n\in\mathscr{L}_{2\widehat{l}}} \Big[w_n\bar{w}_n (e_{j_2} - \sum\limits_{\widehat{r}=2}^{\widehat{l}} e_{j_{2\widehat{r}}}) + w_m\bar{w}_m \sum\limits_{\widehat{r}=1}^{\widehat{l}} e_{j_{2\widehat{r}-1}} \Big] \Big\} \\ \begin{pmatrix} z_n \\ z_m \end{pmatrix} = V_n \begin{pmatrix} e^{\mathrm{i}\sum\limits_{\widehat{r}=1}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}-1}}} & 0 \\ 0 & e^{\mathrm{i}\sum\limits_{\widehat{r}=1}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}}}} \end{pmatrix} \begin{pmatrix} w_n \\ w_m \end{pmatrix} \\ \begin{pmatrix} \bar{z}_n \\ \bar{z}_m \end{pmatrix} = \bar{V}_n \begin{pmatrix} e^{-\mathrm{i}\sum\limits_{\widehat{r}=1}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}-1}}} & 0 \\ 0 & e^{-\mathrm{i}\sum\limits_{\widehat{r}=1}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}}}} \end{pmatrix} \begin{pmatrix} \bar{w}_n \\ \bar{w}_m \end{pmatrix}, \quad n \in \mathscr{L}_{2\widehat{l}-1} \\ z_m = w_m e^{-\mathrm{i}\sum\limits_{\widehat{r}=1}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}-1}}}, \bar{z}_m = \bar{w}_m e^{\mathrm{i}\sum\limits_{\widehat{r}=1}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}-1}}} \\ z_n = w_n e^{-\mathrm{i}\widehat{\theta}_{j_2} + \mathrm{i}\sum\limits_{\widehat{r}=2}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}}}}, \bar{z}_n = \bar{w}_n e^{\mathrm{i}\widehat{\theta}_{j_2} - \mathrm{i}\sum\limits_{\widehat{r}=2}^{\widehat{l}} \widehat{\theta}_{j_{2\widehat{r}}}}, n \in \mathscr{L}_{2\widehat{l}} \\ z_n = w_n, \bar{z}_n = \bar{w}_n, n \in \mathbb{Z}_1^2 \setminus \Big\{ \bigcup\limits_{\widehat{l}=1}^{p} \big(\mathscr{L}_{2\widehat{l}-1} \cup \mathscr{L}_{2\widehat{l}} \big) \Big\} \end{cases}$$

where $\sum\limits_{\widehat{r}=2}^{1}(\cdot) = 0$ and for $1 \le \widehat{l} \le p$, $V = \begin{pmatrix} \ddots & & \\ & V_{\widehat{l}} & \\ & & \ddots \end{pmatrix}$,where $V_{\widehat{l}} = \begin{pmatrix} \ddots & & \\ & V_n & \\ & & \ddots \end{pmatrix}$, $n \in \mathscr{L}_{2\widehat{l}-1}$, V_n is a 2×2 non-singular unitary matrix,. Then $V_{\widehat{l}}$ is a non-singular block diagonal matrix, so V is a nonsingular block diagonal unitary matrix.

We have Hamiltonian systems with the Hamiltonian

$$\begin{aligned} H\circ\Psi\circ\Phi^{-1} = {} & \big\langle \varepsilon^{-3p}\widetilde{\omega}, \widetilde{I} \big\rangle + \big\langle \widehat{\omega}(\xi), \widehat{I}_+ \big\rangle + \sum_{n\in\mathbb{Z}_1^2 \setminus \big\{ \bigcup\limits_{\widehat{l}=1}^{p} \big(\mathscr{L}_{2\widehat{l}-1} \cup \mathscr{L}_{2\widehat{l}} \big) \big\}} \Omega_n(\xi) z_n \bar{z}_n \\ & + \sum_{\widehat{l}=1}^{p} \sum_{n\in\mathscr{L}_{2\widehat{l}-1}} \Bigg\{ \Bigg\{ \Bigg[\varepsilon^{-3p} \Big(|n|^2 + (p+1) f^0 + \sum_{\widehat{r}=1}^{\widehat{l}} |j_{2\widehat{r}-1}|^2 \Big) \\ & + \frac{f^0}{4^p \cdot \pi^{2p}} \cdot \Big(\big(\widehat{l}+1\big)(p+1) \sum_{\widehat{i}=1}^{b} \xi^p_{j^*_{\widehat{i}}} + g_{p,\widehat{l},n,1} \Big) \Bigg] \end{aligned}$$

$$
\begin{aligned}
&+\frac{f^0}{2^{2p+1}\pi^{2p}}\Big\{-p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi_{j_{2\widehat{r}-1}}^p+\xi_{j_{2\widehat{r}}}^p\right)+g_{p,\widehat{l},n,2}+g_{p,\widehat{l},n,3}\\
&+\sqrt{\left[p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi_{j_{2\widehat{r}}}^p-\xi_{j_{2\widehat{r}-1}}^p\right)+g_{p,\widehat{l},n,2}-g_{p,\widehat{l},n,3}\right]^2+4\left(p+1\right)^2\left(g^*_{\mathscr{A},p,\widehat{l},n}\right)^2}\Big\}\Big\}z_n\bar{z}_n\\
&+\Big\{\Big[\varepsilon^{-3p}\Big(|m|^2+\left(p+1\right)f^0+\sum_{\widehat{r}=1}^{\widehat{l}}|j_{2\widehat{r}}|^2\Big)\\
&+\frac{f^0}{4^p\pi^{2p}}\Big(\left(\widehat{l}+1\right)\left(p+1\right)\sum_{\widehat{i}=1}^{b}\xi_{j^*_{\widehat{i}}}^p+g_{p,\widehat{l},m,1}\Big)\Big]\\
&+\frac{f^0}{2^{2p+1}\pi^{2p}}\Big\{-p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi_{j_{2\widehat{r}-1}}^p+\xi_{j_{2\widehat{r}}}^p\right)+g_{p,\widehat{l},m,2}+g_{p,\widehat{l},m,3}\\
&-\sqrt{\left[p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi_{j_{2\widehat{r}}}^p-\xi_{j_{2\widehat{r}-1}}^p\right)+g_{p,\widehat{l}_1,m,2}-g_{p,\widehat{l}_1,m,3}\right]^2+4\left(p+1\right)^2\left(g^*_{\mathscr{A},p,\widehat{l},m}\right)^2}\Big\}\Big\}z_m\bar{z}_m\Big\}\\
&+\sum_{\widehat{l}=1}^{p}\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Big[\Big(\Omega_n-\widetilde{\omega}_{j_2}+\sum_{\widehat{r}=2}^{\widehat{l}}\widetilde{\omega}_{j_{2\widehat{r}}}\Big)z_n\bar{z}_n+\Big(\Omega_m-\sum_{\widehat{r}=1}^{\widehat{l}}\widetilde{\omega}_{j_{2\widehat{r}-1}}\Big)z_m\bar{z}_m\Big]+\sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+R\\
&=\Lambda+\sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+R,
\end{aligned}
\tag{4.10}
$$

where

$$
\Lambda=\left\langle\varepsilon^{-3p}\widetilde{\omega},\widetilde{I}\right\rangle+\left\langle\widehat{\omega}\left(\xi\right),\widehat{I}_+\right\rangle+\sum_{n\in\mathbb{Z}_1^2\setminus\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\}}\Omega_n\left(\xi\right)z_n\bar{z}_n
$$

$$
+\sum_{\widehat{l}=1}^{p}\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Big[\Big(\Omega_n-\omega_{j_2}+\sum_{\widehat{r}=2}^{\widehat{l}}\omega_{j_{2\widehat{r}}}\Big)z_n\bar{z}_n+\Big(\Omega_m-\sum_{\widehat{r}=1}^{\widehat{l}}\omega_{j_{2\widehat{r}-1}}\Big)z_m\bar{z}_m\Big]
$$

$$
\begin{aligned}
\widehat{\omega}_j\left(\xi\right)=\varepsilon^{-3p}\left(|j|^2+f^0\right)+\frac{f^0}{4^p\left(p+1\right)\pi^{2p}}\Big\{\left(p+1\right)\xi_i^p\\
+\sum_{i\in S\setminus\{j\}}\left[(p+1)^2\xi_i^p+(p+1)^2p\xi_j^{p-1}\xi_i\right]+g_{\omega,p,j}\Big\}
\end{aligned}
$$

$$
\Omega_n=\varepsilon^{-3p}\left(|n|^2+f^0\right)+\frac{f^0}{4^p\left(p+1\right)\pi^{2p}}\sum_{i\in S}\left[(p+1)^2\xi_i^p+g_{\Omega,p,n}\right],n\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}-1}\}
$$

$$
\Omega_n=\varepsilon^{-3p}\left(|n|^2+\left(p+1\right)f^0+\sum_{\widehat{r}=1}^{\widehat{l}}|j_{2\widehat{r}-1}|^2\right)+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\left[\left(\widehat{l}+1\right)\left(p+1\right)\sum_{\widehat{i}=1}^{b}\xi_{j^*_{\widehat{i}}}^p+g_{p,\widehat{l},n,1}\right]
$$

$$+\frac{f^0}{2^{2p+1}\pi^{2p}}\Big\{-p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}}\right)+g_{p,\widehat{l},n,2}+g_{p,\widehat{l},n,3}$$

$$+\sqrt{\Big[p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi^p_{j_{2\widehat{r}}}-\xi^p_{j_{2\widehat{r}-1}}\right)+g_{p,\widehat{l},n,2}-g_{p,\widehat{l},n,3}\Big]^2+4(p+1)^2\left(g^*_{\mathscr{A},p,\widehat{l},n}\right)^2}\Big\},n\in\mathscr{L}_{2\widehat{l}-1}$$

$$\Omega_m=\varepsilon^{-3p}\left(|m|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l}}|j_{2\widehat{r}}|^2\right)+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\Big[\left(\widehat{l}+1\right)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l},m,1}\Big]$$

$$+\frac{f^0}{2^{2p+1}\pi^{2p}}\Big\{-p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi^p_{j'_{2\widehat{r}-1}}+\xi^p_{j'_{2\widehat{r}}}\right)+g_{p,\widehat{l},m,2}+g_{p,\widehat{l},m,3}$$

$$-\sqrt{\Big[p\sum_{\widehat{r}=1}^{\widehat{l}}\left(\xi^p_{j'_{2\widehat{r}}}-\xi^p_{j'_{2\widehat{r}-1}}\right)+g_{p,\widehat{l},m,2}-g_{p,\widehat{l},m,3}\Big]^2+4(p+1)^2\left(g^*_{\mathscr{A},p,\widehat{l},m}\right)^2}\Big\},m\in\mathscr{L}_{2\widehat{l}-1}$$

$$\Omega_n=\varepsilon^{-3p}\left(|n|^2+f^0\right)+\frac{f^0}{4^p(p+1)\pi^{2p}}\sum_{i\in S}\Big[(p+1)^2\xi_i^p+g_{\Omega,p,n}\Big],n\in\mathscr{L}_{2\widehat{l}}$$

$$\Omega_m=\varepsilon^{-3p}\left(|m|^2+f^0\right)+\frac{f^0}{4^p(p+1)\pi^{2p}}\sum_{i\in S}\Big[(p+1)^2\xi_i^p+g_{\Omega,p,m}\Big],m\in\mathscr{L}_{2\widehat{l}} \tag{4.11}$$

$$\mathscr{B}_{\widehat{l}}=\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\Big\{\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Big[\sum_{i\in S}\left(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}\right)\Big]\cdot\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}$$

$$\cdot\prod_{\widehat{v}=2}^{\widehat{l}}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}\Big\}\cdot\sqrt{\xi_{j_1}\xi_{j_2}}z_nz_m,1\le\widehat{l}\le p-1$$

$$\mathscr{B}_p=\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\Big[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\Big]\sqrt{\xi_{j_1}\xi_{j_2}}z_nz_m$$

$$\overline{\mathscr{B}}_{\widehat{l}}=\frac{(p+1)p^2f^0}{4^p\cdot\pi^{2p}}\Big\{\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Big[\sum_{i\in S}\left(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}\right)\Big]\cdot\frac{\prod_{\widehat{v}=2}^{\widehat{l}}(p+1-\widehat{v})^2}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}$$

$$\cdot\prod_{\widehat{v}=2}^{\widehat{l}}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}\Big\}\cdot\sqrt{\xi_{j_1}\xi_{j_2}}\bar{z}_n\bar{z}_m,1\le\widehat{l}\le p-1$$

$$\overline{\mathscr{B}}_p=\frac{(p+1)(p!)^2f^0}{4^p\cdot\pi^{2p}}\sum_{n\in\mathscr{L}_{2p}}\Big[\frac{\prod_{\widehat{v}=2}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}}{b^n_{11}!\cdots b^n_{1\hbar^n_1}!b^n_{21}!\cdots b^n_{2\hbar^n_2}!}\Big]\sqrt{\xi_{j_1}\xi_{j_2}}\bar{z}_n\bar{z}_m,$$

where

$$g^*_{\mathscr{A},p,\widehat{l},n}=g^*_{\mathscr{A},p,\widehat{l},n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)=\Bigg\{\Big[\sum_{i\in S}\Big(\xi_i^{p-\widehat{l}}+g_{\mathscr{A},p,\widehat{l},n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)\Big)\Big]$$
$$\cdot\frac{\prod\limits_{\widehat{v}=1}^{\widehat{l}}(p+1-\widehat{v})^2}{b^n_{11}!\cdots b^n_{1\hbar_1^n}!b^n_{21}!\cdots b^n_{2\hbar_2^n}!}\cdot\prod_{\widehat{v}=1}^{\widehat{l}}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}\Bigg\},1\le\widehat{l}\le p-1 \tag{4.12}$$

$$g^*_{\mathscr{A},p,p,n}=g^*_{\mathscr{A},p,p,n}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)=(p!)^2\cdot\frac{\prod\limits_{\widehat{v}=1}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}}{b^n_{11}!\cdots b^n_{1\hbar_1^n}!b^n_{21}!\cdots b^n_{2\hbar_2^n}!}, \tag{4.13}$$

and $g_{p,\widehat{l},n,\widehat{t}}=g_{p,\widehat{l},n,\widehat{t}}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)=\sum\limits_{\substack{|\widehat{\alpha}|=p\\ 0\le\widehat{\alpha}_{j_i^*}\le p-1,i=1,\cdots,b}} g_{p,\widehat{l},n,\widehat{t},\widehat{\alpha}}\cdot\xi^{\widehat{\alpha}},\quad \widehat{t}=1,2,3.$

To simplify notation, let $\widehat{I},\theta,H$ denote $\widehat{I}_+,\theta_+$ and $H\circ\Psi\circ\Phi^{-1}$, respectively. Additionally, let $\omega=\big(\varepsilon^{-3p}\widetilde{\omega},\widehat{\omega}\big)$ and $\theta=\big(\widetilde{\theta},\widehat{\theta}\big)$. The term R corresponds to G, where the variables $\big(q_{j_1},\ldots,q_{j_b},\bar{q}_{j_1},\ldots,\bar{q}_{j_b},z_n,\bar{z}_n\big)$ are expressed in terms of $\big(\theta,\widehat{I},z_n,\bar{z}_n\big)$. The Hamiltonian H can be written as

$$\begin{aligned}H=&\big\langle\varepsilon^{-3p}\widetilde{\omega},\widetilde{I}\big\rangle+\big\langle\widehat{\omega}(\xi),\widehat{I}\big\rangle+\sum_{n\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\}}\Omega_n(\xi)z_n\bar{z}_n\\ &+\sum_{\widehat{l}=1}^{p}\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Big[\Big(\Omega_n-\omega_{j_2}+\sum_{\widehat{r}=2}^{p}\omega_{j_{2\widehat{r}}}\Big)z_n\bar{z}_n+\Big(\Omega_m-\sum_{\widehat{r}=1}^{p}\omega_{j_{2\widehat{r}-1}}\Big)z_m\bar{z}_m\Big]\\ &+\sum_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big)+R\big(\theta,\widehat{I},z,\bar{z},\xi\big).\end{aligned}$$

5. An Infinite-dimisional KAM Theorem

5.1. **KAM theorem.**

Let $\big(\theta,\widehat{I}\big)$ represent b-dimensional angle-action coordinates, and let $\big(z,\bar{z}\big)$ denote infinite-dimensional coordinates equipped with a symplectic structure $d\widetilde{I}\wedge d\widetilde{\theta}+d\widehat{I}\wedge d\widehat{\theta}+\mathrm{i}\sum\limits_{n\in\mathbb{Z}_1^2}dz_n\wedge d\bar{z}_n$. We consider perturbations of a family of Hamiltonian functions of the following form

$$H_0=\Lambda+\sum_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big), \tag{5.1}$$

where

$$\begin{aligned}\Lambda=&\big\langle\varepsilon^{-3p}\widetilde{\omega},\widetilde{I}\big\rangle+\big\langle\widehat{\omega}(\xi),\widehat{I}\big\rangle+\sum_{n\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\}}\Omega_n(\xi)z_n\bar{z}_n\\ &+\sum_{\widehat{l}=1}^{p}\sum_{n\in\mathscr{L}_{2\widehat{l}}}\Big[\Big(\Omega_n-\omega_{j_2}+\sum_{\widehat{r}=2}^{p}\omega_{j_{2\widehat{r}}}\Big)z_n\bar{z}_n+\Big(\Omega_m-\sum_{\widehat{r}=1}^{p}\omega_{j_{2\widehat{r}-1}}\Big)z_m\bar{z}_m\Big],\end{aligned}$$

$$\mathscr{B}_{\widehat{l}} = \sum_{n\in\mathscr{L}_{2\widehat{l}}} a_n(\xi)\, z_n z_m,$$

$$\overline{\mathscr{B}}_{\widehat{l}} = \sum_{n\in\mathscr{L}_{2\widehat{l}}} \bar{a}_n(\xi)\, \bar{z}_n \bar{z}_m.$$

The tangent frequencies $\omega = (\omega_i)_{i\in S}$ and normal frequencies $\Omega = (\Omega_n)_{n\in\mathbb{Z}_1^2}$ depend on b -dimensional parameter $\xi = (\xi_i)_{i\in S} \in \mathscr{I} \subset \mathbb{R}^b$, where, for every $\xi \in \mathscr{I}$, $\xi_i \neq 0$, $\mathscr{I}$ is a closed and bounded set with positive Lebesgue measure. For each $\xi \in \mathscr{I}$, the Hamiltonian equation for H_0 admits special solutions $(\theta,0,0,0) \to (\theta+\omega t,0,0,0)$, which represents an invariant torus in the phase space.

Now, we consider the perturbed Hamiltonian

$$H = H_0 + R = \Lambda + \sum_{\widehat{l}=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R\left(\theta,\widehat{I},z,\bar{z},\xi\right) \tag{5.2}$$

Our goal is to show that, for almost all parameter values $\xi \in \mathscr{I}$ (in the sense of Lebesgue measure), the Hamiltonian $H = \Lambda + \sum_{\widehat{l}=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R$ still admits invariant tori provided that $\left\|X_R\right\|_{D_\rho(r,l),\mathscr{I}}$ is sufficiently small.

To prove this, we must present the following assumptions.

Assumption I (Non-degeneracy) Assume that, for each $\xi \in \mathscr{I}$,

$$\begin{cases} \operatorname{rank}\left\{\dfrac{\partial\widehat{\omega}_{j_1^*}}{\partial\xi},\cdots,\dfrac{\partial\widehat{\omega}_{j_b^*}}{\partial\xi}\right\} = \kappa, \\ \operatorname{rank}\left\{\dfrac{\partial^{\widehat{\kappa}}\widehat{\omega}}{\partial\xi^{\widehat{\kappa}}}\,|\forall\widehat{\kappa}, 1 \leq |\,\widehat{\kappa}\,| \leq \min\{b-\kappa+1\}\right\} = b, \end{cases} \tag{5.3}$$

where κ is a given integer such that $1 \leq \kappa \leq b$, $\frac{\partial\widehat{\omega}_{j_1^*}}{\partial\xi},\cdots,\frac{\partial\widehat{\omega}_{j_b^*}}{\partial\xi}$ are vectors representing all first-order partial derivatives with respect to ξ, and for a fixed $\widehat{\kappa}$, $\frac{\partial^{\widehat{\kappa}}\widehat{\omega}}{\partial\xi^{\widehat{\kappa}}} = \left(\frac{\partial^{\widehat{\kappa}}\widehat{\omega}_{j_1^*}}{\partial\xi^{\widehat{\kappa}}},\cdots,\frac{\partial^{\widehat{\kappa}}\widehat{\omega}_{j_b^*}}{\partial\xi^{\widehat{\kappa}}}\right)$. Moreover, $\widehat{\omega}(\xi)$ is 4p order smooth in the Whitney sense with respect to ξ on $\mathscr{I}$.

Assumption II (Asymptotics of normal frequencies)

$$\Omega_n = \varepsilon^{-\tau}\left(|n|^2 + C\right) + \widetilde{\Omega}_n,\ \tau \geq 0,$$

where C is a constant, all components of $\widetilde{\Omega}_n$ are C_W^{4p} functions of ξ and their C_W^{4p}-norm is bounded by a small positive constant L.

Assumption III (Melnikov's nonresonance conditions) There exist positive constants μ,ν such that

$$|\langle k,\omega\rangle| \geq \frac{\mu}{|k|^\nu}, k = \left(\widetilde{k},\widehat{k}\right), \widetilde{k}\in\mathbb{Z}^m, \widehat{k}\in\mathbb{Z}^b, |k| = \left|\widetilde{k}\right| + \left|\widehat{k}\right| > 0, \omega = \left(\varepsilon^{-3p}\widetilde{\omega},\widehat{\omega}\right)$$

$$\left|\langle\widetilde{k},\widetilde{\omega}\rangle + \widetilde{h}\right| \geq \frac{\mu}{\left|\widetilde{k}\right|^\nu}, \widetilde{h}\in\mathbb{Z}, \widetilde{k}\neq 0 \tag{5.4}$$

and for $1 \leq \widehat{l} \leq p$

$$|\langle k,\omega\rangle \pm \Omega_n| \geq \frac{\mu}{|k|^\nu}, n \in \mathbb{Z}_1^2 \setminus \left\{\bigcup_{\widehat{l}=1}^{p} \mathscr{L}_{2\widehat{l}}\right\},$$

$$|\langle k,\omega\rangle \pm \Omega_n \pm \Omega_m| \geq \frac{\mu}{|k|^{\nu}}, n,m \in \mathbb{Z}_1^2 \setminus \{\bigcup_{\widehat{l}=1}^{p} \mathscr{L}_{2\widehat{l}}\}.$$

Set

$$\mathscr{C}_n = \begin{pmatrix} \Omega_n - \widehat{\omega}_{j_2} + \sum_{\widehat{r}=2}^{\widehat{l}} \widehat{\omega}_{j_{2\widehat{r}}} & -a_n \\ \bar{a}_m & -\Omega_m + \sum_{\widehat{r}=1}^{\widehat{l}} \widehat{\omega}_{j_{2\widehat{r}-1}} \end{pmatrix}, n \in \mathscr{L}_{2\widehat{l}}.$$

The matrix $\mathscr{C}_n$ may be its own transpose, and the pair (n,m) represents resonant pairs. The sequence $\left(j_1, j_2, j_3, j_4, \cdots, j_{2\widehat{l}-1}, j_{2\widehat{l}}\right)$ is uniquely determined by (n,m). Additionally, $\widehat{\omega}(\xi)$, $\mathscr{C}_n(\xi) \in C_W^{4p}(\mathscr{I})$ and there are $\mu, \nu > 0$ (here E is the identity matrix and its order depends on the case) such that

$$|\det(\langle k,\omega\rangle E_4 \pm \mathscr{C}_n \otimes E_2 \pm E_2 \otimes \mathscr{C}_{n'})| \geq \frac{\mu}{|k|^{\nu}},\ |k| + \left|n-n'\right| \neq 0.$$

Assumption IV (Regularity) $\sum_{\widehat{l}=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R$ is real analytic in the variables $\theta, \widehat{I}, z, \bar{z}$ and is C_W^{4p} in ξ. Moreover, $\sum_{\widehat{l}=1}^{p} \left(\left\|X_{\mathscr{B}_{\widehat{l}}}\right\|_{D_\rho(r,l),\mathscr{I}}\right) < 1$, $\left\|X_R\right\|_{D_\rho(r,l),\mathscr{I}} < \varepsilon$.

Assumption V (Special form) $\sum_{l=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R$ takes the following special form

$$\begin{aligned} \mathscr{U} = \Big\{ &\sum_{\widehat{l}=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R : \sum_{\widehat{l}=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R \\ &= \sum_{k\in\mathbb{Z}^b, h\in\mathbb{N}^b, \alpha,\beta} \Big(\sum_{\widehat{l}=1}^{p} \left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R\Big)_{kh\alpha\beta}(\xi) \widehat{I}^h e^{\mathrm{i}\langle k,\theta\rangle} z^\alpha \bar{z}^\beta \Big\}, \end{aligned}$$

where $k = \left(\widetilde{k}, \widehat{k}\right) \in \mathbb{Z}^{b+m}$ with $\widetilde{k} \in \mathbb{Z}^m, \widehat{k} \in \mathbb{Z}^b, h \in \mathbb{Z}^b$ and α, β have the following relations

$$\sum_{\widehat{s}=1}^{b} \widehat{k}_{\widehat{s}} j_{\widehat{s}}^* + \sum_{n\in\mathbb{Z}_1^2} (\alpha_n - \beta_n) n = 0 \tag{5.5}$$

and the perturbation is described as exhibiting a partial zero-momentum property. Additionally, the following equation holds

$$\sum_{\widehat{s}=1}^{b} \widehat{k}_{\widehat{s}} + \sum_{n\in\mathbb{Z}_1^2} (\alpha_n - \beta_n) = 0 \tag{5.6}$$

and the perturbation is described as having a partial gauge invariance property.

Assumption VI (Töplitz-Lipschitz property) For any given $n,m \in \mathbb{Z}^2$, and $\widetilde{c} \in \mathbb{Z}^2 \setminus \{0\}$, the following limits exist

$$\lim_{t\to\infty} \frac{\partial^2\big(\sum_{\widehat{l}=1}^{p} \mathscr{B}_{\widehat{l}} + R\big)}{\partial z_{n+\widetilde{c}t} \partial z_{m-\widetilde{c}t}}; \lim_{t\to\infty} \frac{\partial^2\big(\sum_{j\in\mathbb{Z}_1^2} \widetilde{\Omega}_j z_j \bar{z}_j + R\big)}{\partial z_{n+\widetilde{c}t} \partial \bar{z}_{m+\widetilde{c}t}}; \lim_{t\to\infty} \frac{\partial^2\big(\sum_{\widehat{l}=1}^{p} \overline{\mathscr{B}}_{\widehat{l}} + R\big)}{\partial \bar{z}_{n+\widetilde{c}t} \partial \bar{z}_{m-\widetilde{c}t}}.$$

Additionally, there exists a positive constant M such that for $|t| > M$, $\Lambda+\sum_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big)+R$ satisfies the following conditions

$$\left\|\frac{\partial^2\big(\sum\limits_{\widehat{l}=1}^{p}\mathscr{B}_{\widehat{l}}+R\big)}{\partial z_{n+\widetilde{c}t}\partial z_{m-\widetilde{c}t}}-\lim_{t\to\infty}\frac{\partial^2\big(\sum\limits_{\widehat{l}=1}^{p}\mathscr{B}_{\widehat{l}}+R\big)}{\partial z_{n+\widetilde{c}t}\partial z_{m-\widetilde{c}t}}\right\|_{D_\rho(r,l),\mathscr{I}}\le\frac{\varepsilon}{t}e^{-|n+m|\rho},$$

$$\left\|\frac{\partial^2\big(\sum\limits_{j\in\mathbb{Z}_1^2}\widetilde{\Omega}_j z_j\bar{z}_j+R\big)}{\partial z_{n+\widetilde{c}t}\partial \bar{z}_{m+\widetilde{c}t}}-\lim_{t\to\infty}\frac{\partial^2\big(\sum\limits_{j\in\mathbb{Z}_1^2}\widetilde{\Omega}_j z_j\bar{z}_j+R\big)}{\partial z_{n+\widetilde{c}t}\partial \bar{z}_{m+\widetilde{c}t}}\right\|_{D_\rho(r,l),\mathscr{I}}\le\frac{\varepsilon}{t}e^{-|n-m|\rho},$$

$$\left\|\frac{\partial^2\big(\sum\limits_{\widehat{l}=1}^{p}\overline{\mathscr{B}}_{\widehat{l}}+R\big)}{\partial \bar{z}_{n+\widetilde{c}t}\partial \bar{z}_{m-\widetilde{c}t}}-\lim_{t\to\infty}\frac{\partial^2\big(\sum\limits_{\widehat{l}=1}^{p}\overline{\mathscr{B}}_{\widehat{l}}+R\big)}{\partial \bar{z}_{n+\widetilde{c}t}\partial \bar{z}_{m-\widetilde{c}t}}\right\|_{D_\rho(r,l),\mathscr{I}}\le\frac{\varepsilon}{t}e^{-|n+m|\rho}.$$

Here is the precise statement of the infinite-dimensional KAM theorem.

Theorem 5.1. *Suppose that the Hamiltonian* $H=H_0+R=\Lambda+\sum\limits_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big)+R$ *satisfies the assumptions (I)-(VI). Let* $\mu>0$ *be sufficiently small. Then there exists a constant* $\varepsilon>0$ *depending on the parameters* b,L,M,ν,μ,l,r *and* ρ *such that if* $\|X_R\|_{D_\rho(r,l),\mathscr{I}}<\varepsilon$, *then there exists a Cantor set* $\mathscr{I}_\mu\subset\mathscr{I}$ *with* meas $\big(\mathscr{I}\setminus\mathscr{I}_\mu\big)=O\big(\mu^{\frac{1}{4p}}\big)$, *and two maps* $\big($*analytic in* θ *and* C_W^{4p} *in* $\xi\big)$

$$\widehat{\Psi}:\mathbb{T}^{b+m}\times\mathscr{I}_\mu\to D_\rho(r,l),\quad \widehat{\omega}:\mathscr{I}_\mu\to\mathbb{R}^{b+m}.$$

The map $\widehat{\Psi}$ *is* $\frac{\varepsilon}{\mu^{4p}}$*-close to the trivial embedding* $\widehat{\Psi}_0:\mathbb{T}^{b+m}\times\mathscr{I}\to\mathbb{T}^{b+m}\times\{0,0,0\}$, *and* $\widehat{\omega}$ *is* ε*-close to the unperturbed frequency* $\omega=(\varepsilon^{-3p}\widetilde{\omega},\widehat{\omega})$. *For any* $\xi\in\mathscr{I}_\mu$ *and* $\theta\in\mathbb{T}^{b+m}$, *the curve* $t\mapsto\widehat{\Psi}(\theta+\widehat{\omega}(\xi)t,\xi)$ *is a quasi-periodic solution of the Hamiltonian system corresponding to* $H=\Lambda+\sum\limits_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big)+R$. *The resulting invariant tori are partially hyperbolic.*

5.2. **Proof of Theorem 2.2.**

Now we prove that $H=\Lambda+\sum\limits_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big)+R$ satisfies assumptions $\text{(I)}-\text{(VI)}$.

Verification of Assumption I: Letting $\widehat{\omega}(\xi)=\big(\widehat{\omega}_{j_1^*}(\xi),\cdots,\widehat{\omega}_{j_b^*}(\xi)\big)^{\mathrm{T}}$, we find by (4.1) that, for $\widehat{l}=1,\cdots,b$,

$$\frac{\partial^p\widehat{\omega}_{j_{\widehat{l}}^*}(\xi)}{\partial\xi_{j_{\widehat{l}}^*}^{p}}=\frac{p!f^0}{4^p\cdot\pi^{2p}}\tag{5.7}$$

$$\frac{\partial^p\widehat{\omega}_{j_l^*}(\xi)}{\partial\xi_{j_l^*}^{p-1}\partial\xi_{j_{\widehat{l}}^*}}=\frac{(p+1)!f^0}{4^p\cdot\pi^{2p}},\ 1\le\widehat{l}\le b,\ \widehat{l}\ne l.\tag{5.8}$$

Denote

$$\mathscr{U}_{\widehat{\omega}}^{p}=\begin{pmatrix}\frac{\partial^p\widehat{\omega}_{j_1^*}}{\partial\xi_{j_1^*}^{p}} & \frac{\partial^p\widehat{\omega}_{j_2^*}}{\partial\xi_{j_2^*}^{p-1}\partial\xi_{j_1^*}} & \cdots & \frac{\partial^p\widehat{\omega}_{j_b^*}}{\partial\xi_{j_b^*}^{p-1}\partial\xi_{j_1^*}}\\ \frac{\partial^p\widehat{\omega}_{j_1^*}}{\partial\xi_{j_1^*}^{p-1}\partial\xi_{j_2^*}} & \frac{\partial^p\widehat{\omega}_{j_2^*}}{\partial\xi_{j_2^*}^{p}} & \cdots & \frac{\partial^p\widehat{\omega}_{j_b^*}}{\partial\xi_{j_b^*}^{p-1}\partial\xi_{j_2^*}}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial^p\widehat{\omega}_{j_1^*}}{\partial\xi_{j_1^*}^{p-1}\partial\xi_{j_b^*}} & \frac{\partial^p\widehat{\omega}_{j_2^*}}{\partial\xi_{j_2^*}^{p-1}\partial\xi_{j_b^*}} & \cdots & \frac{\partial^p\widehat{\omega}_{j_b^*}}{\partial\xi_{j_b^*}^{p}}\end{pmatrix}_{b\times b}.$$

Then $\mathscr{U}_{\widehat{\omega}}^{p}$ is the submatrix of matrix $\left\{\frac{\partial^p\widehat{\omega}}{\partial\xi^p}\right\}$. According to (5.7) and (5.8), one has

$$\mathscr{U}_{\widehat{\omega}}^{p}=\frac{p!f^0}{4^p\cdot\pi^{2p}}\cdot\begin{pmatrix}1 & p+1 & \cdots & p+1\\ p+1 & 1 & \cdots & p+1\\ \cdots & \cdots & \cdots & \cdots\\ p+1 & p+1 & \cdots & 1\end{pmatrix}_{b\times b}.$$

Thus

$$\det\left(\mathscr{U}_{\widehat{\omega}}^{p}\right)=\left(\frac{p!f^0}{4^p\cdot\pi^{2p}}\right)^{b}\cdot\left[1+(p+1)(b-1)\right]\cdot(-p)^{b-1}\neq 0.$$

That is, rank $\left(\mathscr{U}_{\widehat{\omega}}^{p}\right)=b$. Hence, Assumption I is verified.

Verification of Assumption II: Take $\tau=3p$. It is evident that Assumption II is satisfied.

Verification of Assumption III: This part follows the same argument as in [28]. For completeness, we rewrite it here.

By Lemma 4.1, we have (5.4). Below, we provide the proof for the most sophisticated case

$$\det\left[\langle k,\omega(\xi)\rangle E_4\pm\mathscr{C}_n\otimes E_2\pm E_2\otimes\mathscr{C}_{n'}\right].$$

Let

$$\mathscr{C}_n=\begin{pmatrix}\Omega_n-\widehat{\omega}_{j_2}+\sum_{\widehat{r}=2}^{\widehat{l}}\widehat{\omega}_{j_{2\widehat{r}}} & -\frac{(p+1)f^0}{4^p\pi^{2p}}g^*_{\mathscr{B},p,\widehat{l},n}(\xi_{j_1^*},\ldots,\xi_{j_b^*})\\ \frac{(p+1)f^0}{4^p\pi^{2p}}g^*_{\mathscr{B},p,\widehat{l},m}(\xi_{j_1^*},\ldots,\xi_{j_b^*}) & -\Omega_m+\sum_{\widehat{r}=1}^{\widehat{l}}\widehat{\omega}_{j_{2\widehat{r}-1}}\end{pmatrix}$$

where $n\in\mathscr{L}_{2\widehat{l}}$ and $\mathscr{C}_n$ can be its own transpose. The pair (n,m) is resonant, the indices $(j_1,j_2,j_3,\ldots,j_{2\widehat{l}-1},j_{2\widehat{l}})$ are uniquely determined by (n,m), and

$$g^*_{\mathscr{B},p,\widehat{l},n}=g^*_{\mathscr{B},p,\widehat{l},n}(\xi_{j_1^*},\cdots,\xi_{j_b^*})=\left\{\left[\sum_{i\in S}\left(\xi_i^{p-\widehat{l}}+g_{\mathscr{B},p,\widehat{l},n}(\xi_{j_1^*},\cdots,\xi_{j_b^*})\right)\right]\cdot\frac{\prod_{\widehat{v}=1}^{\widehat{l}}(p+1-\widehat{v})^2}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}\cdot\prod_{\widehat{v}=1}^{\widehat{l}}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}\right\},1\leq\widehat{l}\leq p-1 \tag{5.9}$$

$$g^*_{\mathscr{B},p,p,n}=g^*_{\mathscr{B},p,p,n}(\xi_{j_1^*},\cdots,\xi_{j_b^*})=(p!)^2\cdot\frac{\prod_{\widehat{v}=1}^{p}\sqrt{\xi_{j_{2\widehat{v}-1}}\xi_{j_{2\widehat{v}}}}}{b_{11}^n!\cdots b_{1\hbar_1^n}^n!b_{21}^n!\cdots b_{2\hbar_2^n}^n!}. \tag{5.10}$$

Denote

$$\mathscr{M}(\xi)=\langle k,\omega(\xi)\rangle E_4\pm\mathscr{C}_n\otimes E_2\pm E_2\otimes\mathscr{C}_{n'}.$$

We aim to prove that $\left|\det\big(\mathscr{M}(\xi)\big)\right| \geq \frac{\mu}{|k|^{\nu}}$ for $k \neq 0$. To achieve this, we divide the proof into five cases.

Case 1$\big(n,n' \in \mathscr{L}_{2\widehat{l}_1-1}, 1 \leq \widehat{l}_1 \leq p\big)$. Note that $\langle k,\omega\rangle \pm \Omega_n \pm \Omega_{n'}$.

Set $\alpha^* = \varepsilon^{-3p}\big(|j_1^*|^2+f^0, |j_2^*|^2+f^0, \ldots, \big|j_b^*\big|^2+f^0\big), \xi^p = \big(\xi^p_{j_1^*}, \xi^p_{j_2^*}, \ldots, \xi^p_{j_b^*}\big)$ and $g^*_\omega = \big(g^*_{\omega,j_1^*}, \ldots, g^*_{\omega,j_b^*}\big)$ with

$$g^*_{\omega,j} = g^*_{\omega,j}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big) = \sum_{i\in S\setminus\{j\}} (p+1)^2 p \xi_j^{p-1}\xi_i + g_{\omega,p,j}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big),\ \forall j \in S.$$

It eigenvalues are

$$\begin{aligned}
&\langle \widetilde{k}, \varepsilon^{-3p}\widetilde{\omega}\rangle + \langle \widehat{k}, \alpha^*\rangle + \frac{p\cdot f^0}{4^p\cdot\pi^{2p}}\langle \widehat{k}, \xi^p\rangle + \frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\Big(\sum_{d^*\in S}\widehat{k}_{d^*}\Big)\cdot\Big(\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}\Big)\\
&+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\langle \widehat{k}, g^*_{\omega,j}\rangle \pm \Bigg[\varepsilon^{-3p}\cdot\Big(|n|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l}_1}|j_{2\widehat{r}-1}|^2\Big)\\
&+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\Big(\big(\widehat{l}_1+1\big)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}} + g_{p,\widehat{l}_1,n,1}\Big)\Bigg]\\
&\pm\frac{f^0}{2^{2p+1}\pi^{2p}}\Bigg\{-p\sum_{\widehat{r}=1}^{\widehat{l}_1}\big(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}}\big) + g_{p,\widehat{l}_1,n,2} + g_{p,\widehat{l}_1,n,3}\\
&\pm\sqrt{\Big[p\sum_{\widehat{r}=1}^{\widehat{l}_1}\big(\xi^p_{j_{2\widehat{r}}}-\xi^p_{j_{2\widehat{r}-1}}\big)+g_{p,\widehat{l}_1,n,2}-g_{p,\widehat{l}_1,n,3}\Big]^2+4(p+1)^2\big(g_{\mathscr{A},p,\widehat{l}_1,n}\big)^2}\Bigg\}\\
&\pm\Bigg[\varepsilon^{-3p}\Big(\big|n'\big|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l}_1}\big|j'_{2\widehat{r}-1}\big|^2\Big)+\frac{f^0}{4^p\pi^{2p}}\cdot\Big(\big(\widehat{l}_1+1\big)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n',1}\Big)\Bigg]\\
&\pm\frac{f^0}{2^{2p+1}\pi^{2p}}\Bigg\{-p\sum_{\widehat{r}=1}^{\widehat{l}_1}\big(\xi^p_{j'_{2\widehat{r}-1}}+\xi^p_{j'_{2\widehat{r}}}\big)+g_{p,\widehat{l}_1,n',2}+g_{p,\widehat{l}_1,n',3}\\
&\pm\sqrt{\Big[p\sum_{\widehat{r}=1}^{\widehat{l}_1}\big(\xi^p_{j'_{2\widehat{r}}}-\xi^p_{j'_{2\widehat{r}-1}}\big)+g_{p,\widehat{l}_1,n',2}-g_{p,\widehat{l}_1,n',3}\Big]^2+4(p+1)^2\big(g_{\mathscr{A},p,\widehat{l}_1,n'}\big)^2}\Bigg\},
\end{aligned}$$

where $g_{p,\widehat{l},n,\widehat{t}} = g_{p,\widehat{l},n,\widehat{t}}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big) = \sum_{\substack{|\widehat{\alpha}|=p\\ 0\leq\widehat{\alpha}_{j_i^*}\leq p-1, i=1,\cdots,b}} g_{p,\widehat{l},n,\widehat{t},\widehat{\alpha}}\cdot\xi^{\widehat{\alpha}},\quad \widehat{t}=1,2,3.$

Case 1.1. Suppose that $\big\{j_{2\widehat{i}-1} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\} \neq \big\{j'_{2\widehat{i}-1} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\}$ or $\big\{j_{2\widehat{i}} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\} \neq \big\{j'_{2\widehat{i}} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\}$. Then all the eigenvalues are not identically zero because of the presence of the square root terms.

Case 1.2. Suppose that $\big\{j_{2\widehat{i}-1} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\} = \big\{j'_{2\widehat{i}-1} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\}$ and $\big\{j_{2\widehat{i}} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\} = \big\{j'_{2\widehat{i}} \mid 1\leq\widehat{i}\leq\widehat{l}_1\big\}$. Obviously, $g_{\Omega,\omega,p,\widehat{l}_1,n,\widehat{t}}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big) = g_{\Omega,\omega,p,\widehat{l}_1,n',\widehat{t}}\big(\xi_{j_1^*},\cdots,\xi_{j_b^*}\big)$ with $\widehat{t} = 1,2,3$.

Case 1.2.1. Suppose that the eigenvalue is

$$\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^{*}\rangle+\frac{p\cdot f^{0}}{4^{p}\cdot\pi^{2p}}\langle\widehat{k},\xi^{p}\rangle+\frac{(p+1)f^{0}}{4^{p}\cdot\pi^{2p}}\cdot\Big(\sum_{d^{*}\in S}\widehat{k}_{d^{*}}\Big)\cdot\Big(\sum_{\widehat{i}=1}^{b}\xi^{p}_{j^{*}_{\widehat{i}}}\Big)$$
$$+\frac{f^{0}}{4^{p}\cdot(p+1)\cdot\pi^{2p}}\langle\widehat{k},g^{*}_{\omega,j}\rangle+\varepsilon^{-3p}\left(|n|^{2}-|n'|^{2}\right) \tag{5.11}$$

the coefficient of $\xi^{p}_{i^{*}}\,(\forall i^{*}\in S)$ in (5.11) is $\frac{f^{0}}{(2\pi)^{2p}}\big[p\widehat{k}_{i^{*}}+(p+1)\sum\limits_{d^{*}\in S}\widehat{k}_{d^{*}}\big]$.

Denote

$$\begin{cases}\frac{f^{0}}{(2\pi)^{2p}}\big[p\widehat{k}_{j^{*}_{1}}+(p+1)\sum\limits_{d^{*}\in S}\widehat{k}_{d^{*}}\big] & =0\\ \qquad\vdots & \\ \frac{f^{0}}{(2\pi)^{2p}}\big[p\widehat{k}_{j^{*}_{b}}+(p+1)\sum\limits_{d^{*}\in S}\widehat{k}_{d^{*}}\big] & =0.\end{cases} \tag{5.12}$$

Then equations (5.12) has no integer solutions for $\widehat{k}\neq 0$.

Case 1.2.2. Suppose that the eigenvalue is

$$\begin{aligned}&\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^{*}\rangle+\frac{p\cdot f^{0}}{4^{p}\cdot\pi^{2p}}\langle\widehat{k},\xi^{p}\rangle+\frac{(p+1)f^{0}}{4^{p}\cdot\pi^{2p}}\cdot\Big(\sum_{d^{*}\in S}\widehat{k}_{d^{*}}\Big)\cdot\Big(\sum_{\widehat{i}=1}^{b}\xi^{p}_{j^{*}_{\widehat{i}}}\Big)\\
&+\frac{f^{0}}{4^{p}\cdot(p+1)\cdot\pi^{2p}}\langle\widehat{k},g^{*}_{\omega,j}\rangle+\Bigg[\varepsilon^{-3p}\cdot\Big(|n|^{2}+(p+1)f^{0}+\sum_{\widehat{r}=1}^{\widehat{l}_{1}}|j_{2\widehat{r}-1}|^{2}\Big)\\
&+\frac{f^{0}}{4^{p}\cdot\pi^{2p}}\cdot\Big(\big(\widehat{l}_{1}+1\big)(p+1)\sum_{\widehat{i}=1}^{b}\xi^{p}_{j^{*}_{\widehat{i}}}+g_{p,\widehat{l}_{1},n,1}\Big)\Bigg]\\
&+\Bigg[\varepsilon^{-3p}\Big(|n'|^{2}+(p+1)f^{0}+\sum_{\widehat{r}=1}^{\widehat{l}_{1}}|j_{2\widehat{r}-1}|^{2}\Big)\\
&+\frac{f^{0}}{4^{p}\cdot\pi^{2p}}\cdot\Big(\big(\widehat{l}_{1}+1\big)(p+1)\sum_{\widehat{i}=1}^{b}\xi^{p}_{j^{*}_{\widehat{i}}}+g_{p,\widehat{l}_{1},n',1}\Big)\Bigg]\\
&+\frac{f^{0}}{2^{2p}\pi^{2p}}\Bigg[-p\sum_{\widehat{r}=1}^{\widehat{l}_{1}}\Big(\xi^{p}_{j_{2\widehat{r}-1}}+\xi^{p}_{j_{2\widehat{r}}}\Big)+g_{p,\widehat{l}_{1},n,2}+g_{p,\widehat{l}_{1},n,3}\Bigg]\end{aligned} \tag{5.13}$$

the coefficients of $\xi^{p}_{j_{\widehat{i}}}\big(\widehat{i}=1,2,\cdots,2\widehat{l}_{1}\big)$ and $\xi^{p}_{i^{*}}\big(\forall i^{*}\in S, i^{*}\neq j_{1},\cdots j_{2\widehat{l}_{1}}\big)$ in (5.13) are respectively

$$\frac{f^{0}}{(2\pi)^{2p}}\Big[p\widehat{k}_{j_{\widehat{i}}}+(p+1)\sum_{d^{*}\in S}\widehat{k}_{d^{*}}+2\big(\widehat{l}_{1}+1\big)(p+1)-p\Big]$$

and

$$\frac{f^{0}}{(2\pi)^{2p}}\Big[p\widehat{k}_{i^{*}}+(p+1)\sum_{d^{*}\in S}\widehat{k}_{d^{*}}+2\big(\widehat{l}_{1}+1\big)(p+1)\Big].$$

If we set the coefficients of $\xi^p_{j_{\widehat{i}}}(\widehat{i}=1,2,\cdots,2\widehat{l_1})$ and $\xi^p_{i^*}(\forall i^*\in S, i^*\neq j_1,\cdots j_{2\widehat{l_1}})$ to zero, then the following system holds

$$\begin{cases} [b(p+1)+p]\widehat{k}_{j_1}+[b(p+1)-p+2(p+1)]=0 \\ \widehat{k}_{j_1}=\widehat{k}_{j_2}=\ldots=\widehat{k}_{j_{2\widehat{l_1}-1}}=\widehat{k}_{j_{2\widehat{l_1}}}=\widehat{k}_{i^*}+1 \end{cases} \tag{5.14}$$

The system (5.14) lacks integer solutions. Hence none of the eigenvalues are identically zero.

Case 1.2.3. Suppose that the eigenvalue is

$$\begin{aligned}
&\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^*\rangle+\frac{p\cdot f^0}{4^p\cdot\pi^{2p}}\langle\widehat{k},\xi^p\rangle+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\Big(\sum_{d^*\in S}\widehat{k}_{d^*}\Big)\cdot\Big(\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}\Big)\\
&+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\langle\widehat{k},g^*_{\omega,j}\rangle-\Bigg[\varepsilon^{-3p}\cdot\Big(|n|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l_1}}|j_{2\widehat{r}-1}|^2\Big)\\
&+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\Big(\big(\widehat{l_1}+1\big)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l_1},n,1}\Big)\Bigg]\\
&-\Bigg[\varepsilon^{-3p}\big(|n'|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l_1}}|j_{2\widehat{r}-1}|^2\big)+\frac{f^0}{4^p\cdot\pi^{2p}}\Big(\big(\widehat{l_1}+1\big)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l_1},n',1}\Big)\Bigg]\\
&-\frac{f^0}{2^{2p}\pi^{2p}}\Bigg[-p\sum_{\widehat{r}=1}^{\widehat{l_1}}\big(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}}\big)+g_{p,\widehat{l_1},n,2}+g_{p,\widehat{l_1},n,3}\Bigg]
\end{aligned} \tag{5.15}$$

the coefficients of $\xi^p_{j_{\widehat{i}}}(\widehat{i}=1,2,\cdots,2\widehat{l_1})$ and $\xi^p_{i^*}(\forall i^*\in S, i^*\neq j_1,\cdots j_{2\widehat{l_1}})$ in (5.15) are respectively

$$\frac{f^0}{(2\pi)^{2p}}\Big[p\widehat{k}_{j_{\widehat{i}}}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}-2\big(\widehat{l_1}+1\big)(p+1)+p\Big]$$

and

$$\frac{f^0}{(2\pi)^{2p}}\Big[p\widehat{k}_{i^*}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}-2\big(\widehat{l_1}+1\big)(p+1)\Big].$$

Set the coefficients of $\xi^p_{j_{\widehat{i}}}(\widehat{i}=1,2,\cdots,2\widehat{l_1})$ and $\xi^p_{i^*}(\forall i^*\in S, i^*\neq j_1,\cdots j_{2\widehat{l_1}})$ are zero. Then

$$\begin{cases} [b(p+1)+p]\widehat{k}_{j_1}-[b(p+1)-p+2(p+1)]=0 \\ \widehat{k}_{j_1}=\widehat{k}_{j_2}=\ldots=\widehat{k}_{j_{2\widehat{l_1}-1}}=\widehat{k}_{j_{2\widehat{l_1}}}=\widehat{k}_{i^*}-1 \end{cases} \tag{5.16}$$

The system (5.16) lacks integer solutions. Hence none of the eigenvalues are identically zero.

Case 2 $\big(n\in\mathscr{L}_{2\widehat{l_1}-1}, n'\in\mathscr{L}_{2\widehat{l_2}-1}, 1\leq\widehat{l_1}\leq p, 1\leq\widehat{l_2}\leq p, \widehat{l_1}\neq\widehat{l_2}\big)$.

The eigenvalues of $\mathscr{M}(\xi)$ are

$$\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^*\rangle+\frac{p\cdot f^0}{4^p\cdot\pi^{2p}}\langle\widehat{k},\xi^p\rangle+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot\big(\sum_{d^*\in S}\widehat{k}_{d^*}\big)\cdot\big(\sum_{\widehat{i}=1}^{b}\xi^p_{\widehat{i}^*_i}\big)$$
$$+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\langle\widehat{k},g^*_{\omega,j}\rangle\pm\Bigg[\varepsilon^{-3p}\cdot\Big(|n|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l}_1}|j_{2\widehat{r}-1}|^2\Big)$$
$$+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\Big((\widehat{l}_1+1)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n,1}\Big)\Bigg]$$
$$\pm\frac{f^0}{2^{2p+1}\pi^{2p}}\Bigg\{-p\sum_{\widehat{r}=1}^{\widehat{l}_1}(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}})+g_{p,\widehat{l}_1,n,2}+g_{p,\widehat{l}_1,n,3}$$
$$\pm\sqrt{\Bigg[p\sum_{\widehat{r}=1}^{\widehat{l}_1}(\xi^p_{j_{2\widehat{r}}}-\xi^p_{j_{2\widehat{r}-1}})+g_{p,\widehat{l}_1,n,2}-g_{p,\widehat{l}_1,n,3}\Bigg]^2+4(p+1)^2(g_{\mathscr{A},p,\widehat{l}_1,n})^2}\Bigg\}$$
$$\pm\Bigg[\varepsilon^{-3p}\Big(|n'|^2+(p+1)f^0+\sum_{\widehat{r}=1}^{\widehat{l}_2}|j'_{2\widehat{r}-1}|^2\Big)+\frac{f^0}{4^p\cdot\pi^{2p}}\Big((\widehat{l}_2+1)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_2,n',1}\Big)\Bigg]$$
$$\pm\frac{f^0}{2^{2p+1}\pi^{2p}}\Bigg\{-p\sum_{\widehat{r}=1}^{\widehat{l}_2}(\xi^p_{j'_{2\widehat{r}-1}}+\xi^p_{j'_{2\widehat{r}}})+g_{p,\widehat{l}_2,n',2}+g_{p,\widehat{l}_2,n',3}$$
$$\pm\sqrt{\Bigg[p\sum_{\widehat{r}=1}^{\widehat{l}_2}(\xi^p_{j'_{2\widehat{r}}}-\xi^p_{j'_{2\widehat{r}-1}})+g_{p,\widehat{l}_2,n',2}-g_{p,\widehat{l}_2,n',3}\Bigg]^2+4(p+1)^2(g_{\mathscr{A},p,\widehat{l}_2,n'})^2}\Bigg\}.$$

Hence none of the eigenvalues are identically zero due to the presence of the square root terms.

Case 3 $\big(n,n'\in\mathscr{L}_{2\widehat{l}_1},1\le\widehat{l}_1\le p\big)$.

The eigenvalues of $\mathscr{M}(\xi)$ are

$$\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^*\rangle+\frac{p\cdot f^0}{4^p\pi^{2p}}\langle\widehat{k},\xi^p\rangle+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\big(\sum_{d^*\in S}\widehat{k}_{d^*}\big)\big(\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}\big)+\frac{f^0}{4^p(p+1)\cdot\pi^{2p}}\langle\widehat{k},g^*_{\omega,j}\rangle$$
$$\pm\Bigg[\varepsilon^{-3p}\Big(|n|^2+(\widehat{l}_1-1)f^0-|j_2|^2+\sum_{\widehat{r}=2}^{\widehat{l}_1}|j_{2\widehat{r}}|^2\Big)+\frac{f^0}{4^p\pi^{2p}}\Big((\widehat{l}_1-1)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n,1}\Big)\Bigg]$$
$$\pm\Bigg[\varepsilon^{-3p}\Big(|n'|^2+(\widehat{l}_1-1)f^0-|j'_2|^2+\sum_{\widehat{r}=2}^{\widehat{l}_1}|j'_{2\widehat{r}}|^2\Big)+\frac{f^0}{4^p\pi^{2p}}\Big((\widehat{l}_1-1)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n',1}\Big)\Bigg]$$
$$\pm\frac{f^0}{2^{2p+1}\pi^{2p}}\Bigg\{p\Bigg[\xi^p_{j_2}-\xi^p_{j_1}-\sum_{\widehat{r}=2}^{\widehat{l}_1}(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}})\Bigg]+g_{p,\widehat{l}_1,n,2}+g_{p,\widehat{l}_1,n,3}\pm\sqrt{g_{\Lambda,3,p,\widehat{l}_1,n}}\Bigg\}$$
$$\pm\frac{f^0}{2^{2p+1}\pi^{2p}}\Bigg\{p\Bigg[\xi^p_{j'_2}-\xi^p_{j'_1}-\sum_{\widehat{r}=2}^{\widehat{l}_1}(\xi^p_{j'_{2\widehat{r}-1}}+\xi^p_{j'_{2\widehat{r}}})\Bigg]+g_{p,\widehat{l}_1,n',2}+g_{p,\widehat{l}_1,n',3}\pm\sqrt{g_{\Lambda,3,p,\widehat{l}_1,n'}}\Bigg\},$$

where

$$g_{\Lambda,3,p,\widehat{l}_1,n}=\Big\{p\Big[\xi^p_{j_2}+\xi^p_{j_1}+\sum_{\widehat{r}=2}^{\widehat{l}_1}(\xi^p_{j_{2\widehat{r}-1}}-\xi^p_{j_{2\widehat{r}}})\Big]+g_{p,\widehat{l}_1,n,2}-g_{p,\widehat{l}_1,n,3}\Big\}^2-4(p+1)^2{g^*_{\mathscr{B},p,\widehat{l}_1,n}}^2$$

and

$$g_{\Lambda,3,p,\widehat{l}_1,n'}=\Big\{p\Big[\xi^p_{j'_2}+\xi^p_{j'_1}+\sum_{\widehat{r}=2}^{\widehat{l}_1}(\xi^p_{j'_{2\widehat{r}-1}}-\xi^p_{j'_{2\widehat{r}}})\Big]+g_{p,\widehat{l}_1,n',2}-g_{p,\widehat{l}_1,n',3}\Big\}^2-4\big(p+1\big)^2{g^*_{\mathscr{B},p,\widehat{l}_1,n'}}^2.$$

Case 3.1. Suppose $\{j_2,j_{2\widehat{i}-1}\mid 1\le\widehat{i}\le\widehat{l}_1\}\neq\{j'_2,j'_{2\widehat{i}-1}\mid 1\le\widehat{i}\le\widehat{l}_1\}$ or $\{j_{2\widehat{i}}\mid 2\le\widehat{i}\le\widehat{l}_1\}\neq\{j'_{2\widehat{i}}\mid 2\le\widehat{i}\le\widehat{l}_1\}$. Thus none of the eigenvalues are uniformly zero due to the existence of the square root terms.

Case 3.2. Suppose $\{j_2,j_{2\widehat{i}-1}\mid 1\le\widehat{i}\le\widehat{l}_1\}=\{j'_2,j'_{2\widehat{i}-1}\mid 1\le\widehat{i}\le\widehat{l}_1\}$, $\{j_{2\widehat{i}}\mid 2\le\widehat{i}\le\widehat{l}_1\}=\{j'_{2\widehat{i}}\mid 2\le\widehat{i}\le\widehat{l}_1\}$. Obviously, $g_{p,\widehat{l}_1,n,\widehat{t}}(\xi_{j^*_1},\cdots,\xi_{j^*_b})=g_{p,\widehat{l}_1,n',\widehat{t}}(\xi_{j^*_1},\cdots,\xi_{j^*_b})$ with $\widehat{t}=1,2,3$.

Case 3.2.1. Suppose that the eigenvalue is

$$\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^*\rangle+\frac{p\cdot f^0}{4^p\cdot\pi^{2p}}\langle\widehat{k},\xi^p\rangle+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot(\sum_{d^*\in S}\widehat{k}_{d^*})\cdot(\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}})$$
$$+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\langle\widehat{k},g^*_{\omega,j}\rangle+\varepsilon^{-3p}\big(|n|^2-|n'|^2\big).$$

Since equations (5.12) have no integer solutions for $k\neq 0$, the eigenvalue mentioned above is not uniformly zero.

Case 3.2.2. Suppose that the eigenvalue is

$$\begin{aligned}&\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^*\rangle+\frac{p\cdot f^0}{4^p\pi^{2p}}\langle\widehat{k},\xi^p\rangle+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot(\sum_{d^*\in S}\widehat{k}_{d^*})\cdot(\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}})\\&+\frac{f^0}{4^p(p+1)\pi^{2p}}\langle\widehat{k},g^*_{\omega,j}\rangle+\Big[\varepsilon^{-3p}\Big(|n|^2+(\widehat{l}_1-1)f^0-|j_2|^2+\sum_{\widehat{r}=2}^{\widehat{l}_1}|j_{2\widehat{r}}|^2\Big)\\&+\frac{f^0}{4^p\pi^{2p}}\Big((\widehat{l}_1-1)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n,1}\Big)\Big]\\&+\Big[\varepsilon^{-3p}\Big(|n'|^2+(\widehat{l}_1-1)f^0-|j_2|^2+\sum_{\widehat{r}=2}^{\widehat{l}_1}|j_{2\widehat{r}}|^2\Big)\\&+\frac{f^0}{4^p\pi^{2p}}\Big((\widehat{l}_1-1)(p+1)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n',1}\Big)\Big]\\&+\frac{f^0}{2^{2p}\pi^{2p}}\Big\{p\Big[\xi^p_{j_2}-\xi^p_{j_1}-\sum_{\widehat{r}=2}^{\widehat{l}_1}(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}})\Big]+g_{p,\widehat{l}_1,n,2}+g_{p,\widehat{l}_1,n,3}\Big\}\end{aligned}\tag{5.17}$$

the coefficients of $\xi_{j_2}^p, \xi_{j_{\widehat{i}}}^p(\widehat{i}=1,3,4,\cdots,2\widehat{l_1})$ and $\xi_{i^*}^p(\forall i^*\in S, i^*\neq j_1,\cdots,j_{2\widehat{l_1}})$ in (5.17) are respectively

$$\frac{f^0}{(2\pi)^{2p}}\Big[p\widehat{k}_{j_2}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}+2(\widehat{l_1}-1)(p+1)+p\Big],$$

$$\frac{f^0}{(2\pi)^{2p}}\Big[p\widehat{k}_{j_{\widehat{i}}}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}+2(\widehat{l_1}-1)(p+1)-p\Big],$$

and

$$\frac{f^0}{(2\pi)^{2p}}\Big[p\widehat{k}_{i^*}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}+2(\widehat{l_1}-1)(p+1)\Big].$$

Set the coefficients of $\xi_{j_2}^p, \xi_{j_{\widehat{i}}}^p(\widehat{i}=1,3,4,\cdots,2\widehat{l_1})$ and $\xi_{i^*}^p(\forall i^*\in S, i^*\neq j_1,\cdots,j_{2\widehat{l_1}})$ are zero. Then

$$\widehat{k}_{j_2}=1, \widehat{k}_{j_{\widehat{i}}}=-1(\widehat{i}=1,3,4,\cdots,2\widehat{l_1}), \widehat{k}_{i^*}=0(\forall i^*\in S, i^*\neq j_1,\cdots,j_{2\widehat{l_1}}),$$

i.e., $\widehat{k}=e_{j_2}-e_{j_1}-\sum_{\widehat{r}=2}^{\widehat{l_1}}(e_{j_{2\widehat{r}-1}}+e_{j_{2\widehat{r}}})$. When $|n|\neq|m'|$, the eigenvalue (5.17) is

$$\begin{aligned}&\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\varepsilon^{-3p}(|n|^2-|m'|^2+2(\widehat{l_1}-1)f^0)\\&+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\cdot\langle e_{j_2}-e_{j_1}-\sum_{\widehat{r}=2}^{\widehat{l_1}}(e_{j_{2\widehat{r}-1}}+e_{j_{2\widehat{r}}}),g_{\omega,j}^*\rangle\\&+\frac{f^0}{(2\pi)^{2p}}\cdot[2g_{p,\widehat{l_1},n,1}+g_{p,\widehat{l_1},n,2}+g_{p,\widehat{l_1},n,3}].\end{aligned}$$

Denote

$$\begin{aligned}g_{p,\widehat{l_1},322}^*=g_{p,\widehat{l_1},322}^*(\xi_{j_1^*},\cdots,\xi_{j_b^*})=&\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\cdot\langle e_{j_2}-e_{j_1}-\sum_{\widehat{r}=2}^{\widehat{l_1}}(e_{j_{2\widehat{r}-1}}+e_{j_{2\widehat{r}}}),g_{\omega,j}^*\rangle\\&+\frac{f^0}{(2\pi)^{2p}}\cdot[2g_{p,\widehat{l_1},n,1}+g_{p,\widehat{l_1},n,2}+g_{p,\widehat{l_1},n,3}].\end{aligned}$$

Then $g_{p,l_1,322}^*$ is either zero or a homogeneous polynomial of degree p. Moreover, since $|n|^2-|m'|^2\neq0$, all eigenvalues are nonzero.

Case 3.2.3. Suppose that the eigenvalue is

$$\begin{aligned}&\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle+\langle\widehat{k},\alpha^*\rangle+\frac{p\cdot f^0}{4^p\cdot\pi^{2p}}\langle\widehat{k},\xi^p\rangle+\frac{(p+1)f^0}{4^p\cdot\pi^{2p}}\cdot(\sum_{d^*\in S}\widehat{k}_{d^*})\cdot(\sum_{\widehat{i}=1}^{b}\xi_{j_{\widehat{i}}^*}^p)\\&+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\langle\widehat{k},g_{\omega,j}^*\rangle-\Big[\varepsilon^{-3p}\cdot\Big(|n|^2+(\widehat{l_1}-1)f^0-|j_2|^2+\sum_{\widehat{r}=2}^{\widehat{l_1}}|j_{2\widehat{r}}|^2\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\Big(\big(\widehat{l}_1-1\big)\big(p+1\big)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n,1}\Big)\Big]\\
&-\Big[\varepsilon^{-3p}\cdot\Big(\big|n'\big|^2+\big(\widehat{l}_1-1\big)f^0-|j_2|^2+\sum_{\widehat{r}=2}^{\widehat{l}_1}|j_{2\widehat{r}}|^2\Big)\\
&+\frac{f^0}{4^p\cdot\pi^{2p}}\cdot\Big(\big(\widehat{l}_1-1\big)\big(p+1\big)\sum_{\widehat{i}=1}^{b}\xi^p_{j^*_{\widehat{i}}}+g_{p,\widehat{l}_1,n',1}\Big)\Big]\\
&-\frac{f^0}{2^{2p}\pi^{2p}}\Big\{p\Big[\xi^p_{j_2}-\xi^p_{j_1}-\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(\xi^p_{j_{2\widehat{r}-1}}+\xi^p_{j_{2\widehat{r}}}\big)\Big]+g_{p,\widehat{l}_1,n,2}+g_{p,\widehat{l}_1,n,3}\Big\}
\end{aligned}
\tag{5.18}
$$

the coefficients of $\xi^p_{j_2},\xi^p_{j_{\widehat{i}}}\big(\widehat{i}=1,3,\cdots,2\widehat{l}_1\big)$ and $\xi^p_{i^*}\big(\forall i^*\in S, i^*\neq j_1,\cdots,j_{2\widehat{l}_1}\big)$ in (5.18) are respectively

$$
\begin{aligned}
&\frac{f^0}{(2\pi)^{2p}}\big[p\widehat{k}_{j_2}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}-2\big(\widehat{l}_1-1\big)\big(p+1\big)-p\big],\\
&\frac{f^0}{(2\pi)^{2p}}\big[p\widehat{k}_{j_{\widehat{i}}}+\big(p+1\big)\sum_{d^*\in S}\widehat{k}_{d^*}-2\big(\widehat{l}_1-1\big)\big(p+1\big)+p\big],
\end{aligned}
$$

and

$$
\frac{f^0}{(2\pi)^{2p}}\big[p\widehat{k}_{i^*}+(p+1)\sum_{d^*\in S}\widehat{k}_{d^*}-2\big(\widehat{l}_1-1\big)\big(p+1\big)\big].
$$

Set the coefficients of $\xi^p_{j_2},\xi^p_{j_{\widehat{i}}}\big(\widehat{i}=1,3,4,\cdots,2\widehat{l}_1\big)$ and $\xi^p_{i^*}\big(\forall i^*\in S, i^*\neq j_1,\cdots,j_{2\widehat{l}_1}\big)$ are zero. Then

$$
\widehat{k}_{j_2}=-1,\widehat{k}_{j_{\widehat{i}}}=1\big(\widehat{i}=1,3,4,\cdots,2\widehat{l}_1\big),\widehat{k}_{i^*}=0\big(\forall i^*\in S, i^*\neq j_1,\cdots,j_{2\widehat{l}_1}\big)
$$

i.e., $\widehat{k}=e_{j_1}-e_{j_2}+\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(e_{j_{2\widehat{r}-1}}+e_{j_{2\widehat{r}}}\big)$. When $|n|\neq|m'|$, the eigenvalue (5.18) is

$$
\begin{aligned}
&\langle\widetilde{k},\varepsilon^{-3p}\widetilde{\omega}\rangle-\varepsilon^{-3p}\cdot\big(|n|^2-\big|m'\big|^2+2\big(\widehat{l}_1-1\big)f^0\big)\\
&+\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\big\langle e_{j_1}-e_{j_2}+\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(e_{j_{2\widehat{r}-1}}+e_{j_{2\widehat{r}}}\big),g^*_{\omega,j}\big\rangle\\
&-\frac{f^0}{(2\pi)^{2p}}\cdot\big[2g_{p,\widehat{l}_1,n,1}+g_{p,\widehat{l}_1,n,2}+g_{p,\widehat{l}_1,n,3}\big].
\end{aligned}
$$

Denote

$$
\begin{aligned}
g^*_{p,\widehat{l}_1,323}=&\frac{f^0}{4^p\cdot(p+1)\cdot\pi^{2p}}\big\langle e_{j_1}-e_{j_2}+\sum_{\widehat{r}=2}^{\widehat{l}_1}\big(e_{j_{2\widehat{r}-1}}+e_{j_{2\widehat{r}}}\big),g^*_{\omega,j}\big\rangle\\
&-\frac{f^0}{(2\pi)^{2p}}\cdot\big[2g_{p,\widehat{l}_1,n,1}+g_{p,\widehat{l}_1,n,2}+g_{p,\widehat{l}_1,n,3}\big].
\end{aligned}
$$

Hence $g^*_{p,\widehat{l}_1.323}$ is either zero or a homogeneous polynomial of degree p. Furthermore, since $|n|^2-|m'|^2\neq0$, all eigenvalues are not identically zero.

Case 4 $\big(n\in\mathscr{L}_{2\widehat{l}_1}, n'\in\mathscr{L}_{2\widehat{l}_2}, 1\le\widehat{l}_1\le p, 1\le\widehat{l}_2\le p, \widehat{l}_1\neq\widehat{l}_2\big)$.

As in Case 2, it can be concluded that none of the eigenvalues of $\mathscr{M}(\xi)$ are identically zero.

Case 5 $\left(n \in \mathscr{L}_{2\widehat{l}_1-1}, n' \in \mathscr{L}_{2\widehat{l}_2}, 1 \leq \widehat{l}_1 \leq p, 1 \leq \widehat{l}_2 \leq p\right)$.

As in Case 2, it follows that all the eigenvalues of $\mathscr{M}(\xi)$ are not identically zero.

Based on the definition of $\mathscr{C}_n$, both $\det\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right)$ and $\det\left(\mathscr{C}_n\right)$ can be expressed as linear combinations of terms involving $\xi^{\widehat{\alpha}}$ with $|\widehat{\alpha}| \leq 2p$. Similarly, both $\operatorname{tr}\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right)$ and $\operatorname{tr}\left(\mathscr{C}_n\right)$ are linear combinations of terms involving $\xi^{\widehat{\alpha}}$ where $|\widehat{\alpha}| \leq p$. From the Lemma 3.6 in [10], we obtain

$$\begin{aligned}
\det(\mathscr{M}(\xi)) &= \det\left(\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right) \otimes E_2 \pm E_2 \otimes \mathscr{C}_{n'}\right) \\
&= \left(\det\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right) - \det\left(\mathscr{C}_{n'}\right)\right)^2 \\
&+ \det\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right) \cdot \left(\operatorname{tr}\left(\mathscr{C}_{n'}\right)\right)^2 \\
&+ \det\left(\mathscr{C}_{n'}\right) \cdot \left(\operatorname{tr}\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right)\right)^2 \\
&\pm \left(\det\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right) + \det\left(\mathscr{C}_{n'}\right)\right) \operatorname{tr}\left(\langle k, \omega(\xi)\rangle E_2 \pm \mathscr{C}_n\right) \operatorname{tr}\left(\mathscr{C}_{n'}\right).
\end{aligned}$$

From this calculation, $\det\left(\mathscr{M}(\xi)\right)$ can be expressed as a linear combination of terms involving $\xi^{\widehat{\alpha}}$ with $|\widehat{\alpha}| \leq 4p$. In other words, $\det\left(\mathscr{M}(\xi)\right)$ is a polynomial in the components of ξ with a maximum degree of $4p$. Consequently,

$$\left|\partial_{\xi}^{4p}\left(\det\left(\mathscr{M}(\xi)\right)\right)\right| \geq \frac{\left|f^0\right|}{2(p+1)(2\pi)^{2p}}\left|\widehat{k}\right| \neq 0.$$

From Lemma 4.8 in [9], by excluding a parameter set of measure $O\left(\mu^{\frac{1}{4p}}\right)$, we conclude that $\left|\det\left(\mathscr{M}(\xi)\right)\right| \geq \frac{\mu}{|k|^{\nu}}, k \neq 0$. Assumption III is verified.

Verification of Assumption IV: The verification of Assumption IV follows a similar approach to Lemma 7.3 in [11].

Verification of Assumption V:

$$\begin{aligned}
&\sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R \\
&= \sum_{\sum_{\widehat{s}=1}^{b}\left(\alpha_{j_{\widehat{s}}} - \beta_{j_{\widehat{s}}}\right) j_{\widehat{s}}^{*} + \sum_{n \in \mathbb{Z}_1^2}(\alpha_n - \beta_n) n = 0, \widetilde{k}} \frac{f^{\widetilde{k}}}{4^p \cdot (p+1) \cdot \pi^{2p}} \sqrt{\widehat{I}_1 + \xi_1}^{\alpha_{j_1} + \beta_{j_1}} \cdots \sqrt{\widehat{I}_b + \xi_b}^{\alpha_{j_b} + \beta_{j_b}} \\
&\qquad \times e^{\mathrm{i} \sum_{\widehat{s}=1}^{b}\left(\alpha_{j_{\widehat{s}}} - \beta_{j_{\widehat{s}}}\right) \widehat{\theta}_{\widehat{s}}} \times \sum_{\widehat{l}=1}^{p}\left(e^{\mathrm{i} \sum_{n \in \mathscr{L}_{2\widehat{l}-1}}\left[(\alpha_n - \beta_n)\left(\widehat{\theta}_j + \widehat{\theta}_k\right) + (\alpha_m - \beta_m)\left(\widehat{\theta}_i + \widehat{\theta}_l\right)\right]} e^{\mathrm{i}\langle \widetilde{k}, \widetilde{\theta}\rangle}\right. \\
&\qquad \left. \times e^{-\mathrm{i} \sum_{n \in \mathscr{L}_{2\widehat{l}}}\left[(\alpha_n - \beta_n)\left(\widehat{\theta}_k - \widehat{\theta}_j\right) + (\alpha_m - \beta_m)\left(\widehat{\theta}_i + \widehat{\theta}_l\right)\right]}\right) \times z^{\alpha - \sum_{\widehat{s}=1}^{b} \alpha_{j_{\widehat{s}}} e_{j_{\widehat{s}}}} \bar{z}^{\beta - \sum_{\widehat{s}=1}^{b} \beta_{j_{\widehat{s}}} e_{j_{\widehat{s}}}}.
\end{aligned}$$

Letting $\widehat{k}_1 = \left(\widehat{k}_{1_1}, \ldots, \widehat{k}_{1_b}\right) = \left(\alpha_{j_1} - \beta_{j_1}, \ldots, \alpha_{j_b} - \beta_{j_b}\right)$, for $1 \leq \widehat{l} \leq p$, we have

$$e^{\mathrm{i}\sum_{n \in \mathscr{L}_{2\widehat{l}-1}}\left[(\alpha_n - \beta_n)\left(\widehat{\theta}_j + \widehat{\theta}_k\right) + (\alpha_m - \beta_m)\left(\widehat{\theta}_i + \widehat{\theta}_l\right)\right]} \times e^{-\mathrm{i}\sum_{n \in \mathscr{L}_{2\widehat{l}}}\left[(\alpha_n - \beta_n)\left(\widehat{\theta}_k - \widehat{\theta}_j\right) + (\alpha_m - \beta_m)\left(\widehat{\theta}_i + \widehat{\theta}_l\right)\right]} = e^{\mathrm{i}\langle \widehat{k}_2, \widehat{\theta}\rangle}.$$

Then $\sum\limits_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+R=\sum\limits_{k,\alpha,\beta}\big[\sum\limits_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+R\big]_{kh\alpha\beta}(\xi)\widehat{I}^{h}e^{\mathrm{i}\langle k,\theta\rangle}z^{\alpha}\bar{z}^{\beta}$, where $k=\left(\widetilde{k},\widehat{k}\right)\in\mathbb{Z}^{b+m},\widehat{k}=\left(\widehat{k}_1,\widehat{k}_2,\ldots,\widehat{k}_b\right)=\left(\widehat{k}_{1_1}+\widehat{k}_{2_1},\widehat{k}_{1_2}+\widehat{k}_{2_2},\ldots,\widehat{k}_{1_b}+\widehat{k}_{2_b}\right)\in\mathbb{Z}^b,\theta=\left(\widetilde{\theta},\widehat{\theta}\right)\in\mathbb{R}^{b+m},h\in\mathbb{N}^b,\alpha,\beta$ has the following relation $\sum\limits_{\widehat{s}=1}^{b}\widehat{k}_{\widehat{s}}j_{\widehat{s}}+\sum\limits_{n\in\mathbb{Z}_1^2}(\alpha_n-\beta_n)n=0$ and $\sum\limits_{\widehat{s}=1}^{b}\widehat{k}_{\widehat{s}}+\sum\limits_{n\in\mathbb{Z}_1^2}(\alpha_n-\beta_n)=0$. Then Assumption V is verified.

Verification of Assumption VI: We only need to verify that T satisfies Assumption VI. According to (4.3), the terms in T that contains $z_n\bar{z}_m$ are

$$\sum_{\substack{n-m+\sum\limits_{\widehat{r}=1}^{p}(i_{2\widehat{r}-1}-i_{2\widehat{r}})=0,|\widetilde{k}|\neq 0\\ |n|^2-|m|^2+\sum\limits_{\widehat{r}=1}^{p}\left(|i_{2\widehat{r}-1}|^2-|i_{2\widehat{r}}|^2\right)\neq 0\\ \sharp(S\cap\{n,m,i_{2\widehat{r}-1},i_{2\widehat{r}}|1\leq\widehat{r}\leq p\})\geq 2p}}\left\{\frac{\mathrm{i}\cdot f^{\widetilde{k}}}{4^p(p+1)\pi^{2p}}\,\frac{\left(\prod\limits_{\widehat{r}=1}^{p}q_{i_{2\widehat{r}-1}}\bar{q}_{i_{2\widehat{r}}}\right)\cdot z_n\bar{z}_m\cdot e^{\mathrm{i}\langle\widetilde{k},\widetilde{\theta}\rangle}}{\left[\langle\widetilde{k},\widetilde{\omega}\rangle-\lambda_n+\lambda_m-\sum\limits_{\widehat{r}=1}^{p}\left(\lambda_{i_{2\widehat{r}-1}}-\lambda_{i_{2\widehat{r}}}\right)\right]}\right\}.$$

Then, for t sufficient large and $\forall\widetilde{c}\in\mathbb{Z}^2\setminus\{0\}$, we have the terms in T that contains $z_{n+t\widetilde{c}}\bar{z}_{m+t\widetilde{c}}$ are

$$\sum_{\substack{n-m+\sum\limits_{\widehat{r}=1}^{p}(i_{2\widehat{r}-1}-i_{2\widehat{r}})=0,|\widetilde{k}|\neq 0\\ |n|^2-|m|^2+\sum\limits_{\widehat{r}=1}^{p}\left(|i_{2\widehat{r}-1}|^2-|i_{2\widehat{r}}|^2\right)\neq 0\\ \sharp(S\cap\{n,m,i_{2\widehat{r}-1},i_{2\widehat{r}}|1\leq\widehat{r}\leq p\})\geq 2p}}\left\{\frac{\mathrm{i}\cdot f^{\widetilde{k}}}{4^p(p+1)\pi^{2p}}\,\frac{\left(\prod\limits_{\widehat{r}=1}^{p}q_{i_{2\widehat{r}-1}}\bar{q}_{i_{2\widehat{r}}}\right)\cdot z_{n+t\widetilde{c}}\bar{z}_{m+t\widetilde{c}}\cdot e^{\mathrm{i}\langle\widetilde{k},\widetilde{\theta}\rangle}}{\left[\langle\widetilde{k},\widetilde{\omega}\rangle-\lambda_{n+t\widetilde{c}}+\lambda_{m+t\widetilde{c}}-\sum\limits_{\widehat{r}=1}^{p}\left(\lambda_{i_{2\widehat{r}-1}}-\lambda_{i_{2\widehat{r}}}\right)\right]}\right\}$$

$$=\sum_{\substack{n-m+\sum\limits_{\widehat{r}=1}^{p}(i_{2\widehat{r}-1}-i_{2\widehat{r}})=0,|\widetilde{k}|\neq 0\\ |n|^2-|m|^2+\sum\limits_{\widehat{r}=1}^{p}\left(|i_{2\widehat{r}-1}|^2-|i_{2\widehat{r}}|^2\right)\neq 0\\ \sharp(S\cap\{n,m,i_{2\widehat{r}-1},i_{2\widehat{r}}|1\leq\widehat{r}\leq p\})\geq 2p}}\left\{\frac{\mathrm{i}\cdot f^{\widetilde{k}}}{4^p(p+1)\pi^{2p}}\right.$$

$$\left.\frac{\left(\prod\limits_{\widehat{r}=1}^{p}q_{i_{2\widehat{r}-1}}\bar{q}_{i_{2\widehat{r}}}\right)\cdot z_{n+t\widetilde{c}}\bar{z}_{m+t\widetilde{c}}\cdot e^{\mathrm{i}\langle\widetilde{k},\widetilde{\theta}\rangle}}{\left[\langle\widetilde{k},\widetilde{\omega}\rangle-|n|^2+|m|^2-2t\langle n-m,\widetilde{c}\rangle-\sum\limits_{\widehat{r}=1}^{p}\left(\lambda_{i_{2\widehat{r}-1}}-\lambda_{i_{2\widehat{r}}}\right)\right]}\right\}.$$

If $\langle n-m,\widetilde{c}\rangle=0$, then $\frac{\partial^2 T}{\partial z_{n+\widetilde{c}t}\partial\bar{z}_{m+\widetilde{c}t}}=\frac{\partial^2 T}{\partial z_n\partial\bar{z}_m}$; if $\langle n-m,\widetilde{c}\rangle\neq 0$, then $\left\|\frac{\partial^2 T}{\partial z_{n+\widetilde{c}t}\partial\bar{z}_{m+\widetilde{c}t}}-0\right\|_{D_\rho(r,l),\mathscr{I}}\leq\frac{\varepsilon}{|t|}e^{-|n-m|\rho}$. Similarly, we see that

$$\left\|\frac{\partial^2 T}{\partial z_{n+\widetilde{c}t}\partial z_{m-\widetilde{c}t}}-\lim_{t\to\infty}\frac{\partial^2 T}{\partial z_{n+\widetilde{c}t}\partial z_{m-\widetilde{c}t}}\right\|_{D_\rho(r,l),\mathscr{I}}\leq\frac{\varepsilon}{|t|}e^{-|n+m|\rho}$$

and

$$\left\|\frac{\partial^2 T}{\partial\bar{z}_{n+\widetilde{c}t}\partial\bar{z}_{m-\widetilde{c}t}}-\lim_{t\to\infty}\frac{\partial^2 T}{\partial\bar{z}_{n+\widetilde{c}t}\partial\bar{z}_{m-\widetilde{c}t}}\right\|_{D_\rho(r,l),\mathscr{I}}\leq\frac{\varepsilon}{|t|}e^{-|n+m|\rho}.$$

So T satisfies the Töplitz-Lipschitz property. Given the structure of the Hamiltonian (3.2), it is sufficient to verify that $\{G,T\}$ also satisfies the Töplitz-Lipschitz property. According to Lemma 4.10 in [10], the Poisson bracket maintains the Töplitz-Lipschitz property. Therefore, $\Lambda+\sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+R$ satisfies Assumption VI.

Since all the assumptions of Theorem 5.1 are verified for equation (4.10), we can now apply Theorem 5.1 and deduce Theorem 2.2.

5.3. **Solving homology equations.**

Theorem 5.1 will be proven via a KAM iterative procedure that involves an infinite sequence of variable transformations. At each step, the perturbation from the prior iteration is further diminished, while a small set of parameters is excluded and the weight is decreased. It is necessary to prove the convergence of this iterative process and estimate the measure of the excluded set after infinitely many KAM iterations. Since the proof follows a similar procedure as in [10] and [28], we omit it.

Define D as the truncated part of R given by

$$\begin{aligned}D(\theta,\widehat{I},z,\bar{z}) &= D_0+D_1+D_2\\&=\sum_{|k|\le M,|h|\le 1}R_{kh00}\widehat{I}^h e^{\mathrm{i}\langle k,\theta\rangle}+\sum_{|k|\le M,n}\left(R_n^{k10}z_n+R_n^{k01}\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle}\\&\quad+\sum_{|k|\le M,n,m}\left(R_{nm}^{k20}z_nz_m+R_{nm}^{k02}\bar{z}_n\bar{z}_m\right)e^{\mathrm{i}\langle k,\theta\rangle}+\sum_{|k|\le M,n,m}R_{nm}^{k11}z_n\bar{z}_m e^{\mathrm{i}\langle k,\theta\rangle},\end{aligned}$$

where $R_n^{k10}=R_{kh\alpha\beta}$ with $\alpha=e_n$ and $\beta=0$, where e_n represents a vector with 1 in the n-th component and zeros elsewhere. Similarly, $R_n^{k01}=R_{kh\alpha\beta}$ with $\alpha=0$ and $\beta=e_n$, $R_{nm}^{k20}=R_{kh\alpha\beta}$ with $\alpha=e_n+e_m$ and $\beta=0$, $R_{nm}^{k11}=R_{kh\alpha\beta}$ with $\alpha=e_n$ and $\beta=e_m$, $R_{nm}^{k02}=R_{kh\alpha\beta}$ with $\alpha=0$ and $\beta=e_n+e_m$. Rewriting H as $H=\Lambda+\sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right)+D+(R-D)$, we have$\|X_D\|_{D_\rho(r,l),\mathscr{I}}\le\|X_R\|_{D_\rho(r,l),\mathscr{I}}\le\varepsilon$. Furthermore, taking $l_+\ll l$ such that in the domain $D_\rho(r,l_+)$, we have $\|X_{R-D}\|_{D_\rho(r,l_+)}<c\varepsilon_+$.

Next, we present a function F defined on the domain $D_+=D_\rho\left(r_+,l_+\right)$, where the time-one map ϕ_F^1 of the Hamiltonian vector field X_F maps D_+ into D and transforms H into H_+. Let $F\left(\theta,\widehat{I},z,\bar{z}\right)=F_0+F_1+F_2$, where

$$F_0=\sum_{0<|k|\le M,|h|\le 1}F_{kh00}e^{\mathrm{i}\langle k,\theta\rangle}\widehat{I}^h,$$

$$\begin{aligned}F_1=&\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}}}\left(F_n^{k10}z_n+F_m^{k10}z_m+F_n^{k01}\bar{z}_n+F_m^{k01}\bar{z}_m\right)e^{\mathrm{i}\langle k,\theta\rangle}\\&+\sum_{|k|\le M,n\in\mathbb{Z}_1^2\setminus\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\}}\left(F_n^{k10}z_n+F_n^{k01}\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle},\end{aligned}$$

$$F_2=\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|n-n'|\ne 0}\left(F_{n'n}^{k11}z_{n'}\bar{z}_n+F_{nn'}^{k11}z_n\bar{z}_{n'}\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|n-m'|\neq 0}\left(F^{k11}_{m'n}z_{m'}\bar{z}_n+F^{k11}_{nm'}z_n\bar{z}_{m'}\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|m-n'|\neq 0}\left(F^{k11}_{n'm}z_{n'}\bar{z}_m+F^{k11}_{mn'}z_m\bar{z}_{n'}\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|m-m'|\neq 0}\left(F^{k11}_{m'm}z_{m'}\bar{z}_m+F^{k11}_{mm'}z_m\bar{z}_{m'}\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|n-m'|\neq 0(or)|k|+|n'-m|\neq 0}\left(F^{k20}_{n'n}z_{n'}z_n+F^{k02}_{n'n}\bar{z}_{n'}\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|m-m'|\neq 0(or)|k|+|n'-n|\neq 0}\left(F^{k20}_{m'n}z_{m'}z_n+F^{k02}_{m'n}\bar{z}_{m'}\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|m-m'|\neq 0(or)|k|+|n'-n|\neq 0}\left(F^{k20}_{n'm}z_{n'}z_m+F^{k02}_{n'm}\bar{z}_{n'}\bar{z}_m\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|n-m'|\neq 0(\mathrm{or})|k|+|n'-m|\neq 0}\left(F^{k20}_{m'm}z_{m'}z_m+F^{k02}_{m'm}\bar{z}_{m'}\bar{z}_m\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{|k|\le M,n\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\},m\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\}}\left(F^{k20}_{nm}z_nz_m+F^{k02}_{nm}\bar{z}_n\bar{z}_m\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\},m\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\},|k|+|n-m|\neq 0}\left(F^{k11}_{nm}z_n\bar{z}_m+F^{k11}_{mn}z_m\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\},n'\in\mathscr{L}_{2\widehat{l}}}\left(F^{k20}_{nn'}z_nz_{n'}+F^{k20}_{nm'}z_nz_{m'}\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\},n'\in\mathscr{L}_{2\widehat{l}}}\left(F^{k02}_{nn'}\bar{z}_n\bar{z}_{n'}+F^{k02}_{nm'}\bar{z}_n\bar{z}_{m'}\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

$$+\sum_{\widehat{l}=1}^{p}\sum_{|k|\le M,n\in\mathbb{Z}^2_1\backslash\left\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\right\},n'\in\mathscr{L}_{2\widehat{l}}}$$
$$\left(F^{k11}_{nn'}z_n\bar{z}_{n'}+F^{k11}_{nm'}z_n\bar{z}_{m'}+F^{k11}_{n'n}z_{n'}\bar{z}_n+F^{k11}_{m'n}z_{m'}\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle}$$

satisfying the equation

$$\{\Lambda+\sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\right),F\}+D-R_{0000}-\langle\widehat{\omega},\widehat{I}\rangle-\sum_{n\in\mathbb{Z}^2}R^{011}_{nn}z_n\bar{z}_n-\sum_{\widehat{l}=1}^{p}\left(\widehat{\mathscr{B}}_{\widehat{l}}+\widehat{\overline{\mathscr{B}}}_{\widehat{l}}\right)=0. \tag{5.19}$$

The homological equation (5.19) is equivalent to

$$\{\Lambda,F_0\}+D_0-R_{0000}-\langle\widehat{\omega},\widehat{I}\rangle=0, \tag{5.20}$$

$$\{\Lambda+\sum_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big),F_1\}+D_1=0, \tag{5.21}$$

and

$$\{\Lambda+\sum_{\widehat{l}=1}^{p}\big(\mathscr{B}_{\widehat{l}}+\overline{\mathscr{B}}_{\widehat{l}}\big),F_2\}+D_2-\sum_{n\in\mathbb{Z}^2}R_{nn}^{011}z_n\bar{z}_n-\sum_{\widehat{l}=1}^{p}\big(\widehat{\mathscr{B}}_{\widehat{l}}+\widehat{\overline{\mathscr{B}}}_{\widehat{l}}\big)=0. \tag{5.22}$$

Now, let us solve equations (5.20)-(5.22).

Lemma 5.2. *The transformation F satisfies conditions (5.20) and (5.21) if the Fourier coefficients of F_0, F_1 are defined as follows*

$$\begin{aligned}
&(\langle k,\omega\rangle)\,F_{kh00} &&= \mathrm{i}R_{kh00},\ |h|\le 1, 0<|k|\le M,\\
&(\langle k,\omega\rangle-\Omega_n)\,F_n^{k10} &&= \mathrm{i}R_n^{k10},\ |k|\le M, n\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\},\\
&(\langle k,\omega\rangle+\Omega_n)\,F_n^{k01} &&= \mathrm{i}R_n^{k01},\ |k|\le M, n\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\},\\
&\big(\langle k,\omega\rangle E_2-\mathscr{C}_n\big)\big(F_n^{k10},F_m^{k01}\big)^T &&= \mathrm{i}\big(R_n^{k10},R_m^{k01}\big)^T, |k|\le M, n\in\mathscr{L}_{2\widehat{l}}, 1\le\widehat{l}\le p,\\
&\big(\langle k,\omega\rangle E_2+\mathscr{C}_n\big)\big(F_n^{k01},F_m^{k10}\big)^T &&= \mathrm{i}\big(R_n^{k01},R_m^{k10}\big)^T, |k|\le M, n\in\mathscr{L}_{2\widehat{l}}, 1\le\widehat{l}\le p.
\end{aligned} \tag{5.23}$$

The Fourier coefficients of F_2 are given by the following lemmas.

Case 1. $n,m\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\}$.

Lemma 5.3. *The transformation F satisfies conditions (5.22) if the Fourier coefficients of F_2 are defined as follows*

$$\begin{aligned}
&\big(\langle k,\omega\rangle-\Omega_n-\Omega_m\big)F_{nm}^{k20}=\mathrm{i}R_{nm}^{k20},\ |k|\le M,\\
&\big(\langle k,\omega\rangle-\Omega_n+\Omega_m\big)F_{nm}^{k11}=\mathrm{i}R_{nm}^{k11},\ 0<|k|\le M, |k|+|n-m|\ne 0,\\
&\big(\langle k,\omega\rangle+\Omega_n+\Omega_m\big)F_{nm}^{k02}=\mathrm{i}R_{nm}^{k02},\ |k|\le M.
\end{aligned} \tag{5.24}$$

Case 2. $n\in\mathbb{Z}_1^2\setminus\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\}, n'\in\mathscr{L}_{2\widehat{l}}, 1\le\widehat{l}\le p$.

Lemma 5.4. *The transformation F satisfies conditions (5.22) if the Fourier coefficients of F_2 are defined as follows*

$$\begin{aligned}
&[(\langle k,\omega\rangle-\Omega_n)\,E_2-\mathscr{C}_{n'}]\,\big(F_{nn'}^{k20},F_{nm'}^{k11}\big)^T=\mathrm{i}\big(R_{nn'}^{k20},R_{nm'}^{k11}\big)^T,\\
&[(\langle k,\omega\rangle+\Omega_n)\,E_2+\mathscr{C}_{n'}]\,\big(F_{nn'}^{k02},F_{m'n}^{k11}\big)^T=\mathrm{i}\big(R_{nn'}^{k02},R_{m'n}^{k11}\big)^T,\\
&[(\langle k,\omega\rangle-\Omega_n)\,E_2+\mathscr{C}_{n'}]\,\big(F_{nn'}^{k11},F_{nm'}^{k20}\big)^T=\mathrm{i}\big(R_{nn'}^{k11},R_{nm'}^{k20}\big)^T,\\
&[(\langle k,\omega\rangle+\Omega_n)\,E_2-\mathscr{C}_{n'}]\,\big(F_{n'n}^{k11},F_{m'n}^{k02}\big)^T=\mathrm{i}\big(R_{n'n}^{k11},R_{m'n}^{k02}\big)^T.
\end{aligned} \tag{5.25}$$

Case 3. $n\in\mathscr{L}_{2\widehat{l}_1}, n'\in\mathscr{L}_{2\widehat{l}_2}, 1\le\widehat{l}_1,\widehat{l}_2\le p$.

Case 3.1 If $\{n,m\}\ne\{n',m'\}$, we have the following.

Lemma 5.5. *The transformation F satisfies conditions (5.22) if the Fourier coefficients of F_2 are defined as follows*

$$\begin{aligned}
&\left(\langle k,\omega\rangle E_4-\mathscr{C}_n\otimes E_2+E_2\otimes\mathscr{C}_{n'}\right)\left(F^{k11}_{nn'},F^{k20}_{nm'},F^{k02}_{mn'},F^{k11}_{m'm}\right)^T\\
&\qquad\qquad=\mathrm{i}\left(R^{k11}_{nn'},R^{k20}_{nm'},R^{k02}_{mn'},R^{k11}_{m'm}\right)^T,\\
&\left(\langle k,\omega\rangle E_4+\mathscr{C}_n\otimes E_2-E_2\otimes\mathscr{C}_{n'}\right)\left(F^{k11}_{n'n},F^{k02}_{m'n},F^{k20}_{n'm},F^{k11}_{mm'}\right)^T\\
&\qquad\qquad=\mathrm{i}\left(R^{k11}_{n'n},R^{k02}_{m'n},R^{k20}_{n'm},R^{k11}_{mm'}\right)^T,\\
&\left(\langle k,\omega\rangle E_4-\mathscr{C}_n\otimes E_2-E_2\otimes\mathscr{C}_{n'}\right)\left(F^{k20}_{nn'},F^{k11}_{nm'},F^{k11}_{n'm},F^{k02}_{mm'}\right)^T\\
&\qquad\qquad=\mathrm{i}\left(R^{k20}_{nn'},R^{k11}_{nm'},R^{k11}_{n'm},R^{k02}_{mm'}\right)^T,\\
&\left(\langle k,\omega\rangle E_4+\mathscr{C}_n\otimes E_2+E_2\otimes\mathscr{C}_{n'}\right)\left(F^{k02}_{nn'},F^{k11}_{m'n},F^{k11}_{mn'},F^{k20}_{m'm}\right)^T\\
&\qquad\qquad=\mathrm{i}\left(R^{k02}_{nn'},R^{k11}_{m'n},R^{k11}_{mn'},R^{k20}_{m'm}\right)^T.
\end{aligned}$$

Case 3.2 If $\{n,m\}=\{n',m'\}$ and $k\neq 0$, then $\widehat{l}_1=\widehat{l}_2$ and $\{j_2,j_{2\widehat{r}-1}|1\le\widehat{r}\le\widehat{l}_1\}=\{j'_2,j'_{2\widehat{r}-1}|1\le\widehat{r}\le\widehat{l}_2\}$ and $\{j_{2\widehat{r}}|1\le\widehat{r}\le\widehat{l}_1\}=\{j'_{2\widehat{r}}|1\le\widehat{r}\le\widehat{l}_2\}$. Moreover, we have the result.

Lemma 5.6. *The transformation F satisfies conditions (5.22) if the Fourier coefficients of F_2 are defined as follows,*

$$\left(\langle k,\omega\rangle E_4-\mathscr{C}_n\otimes E_2+E_2\otimes\mathscr{C}_n\right)\left(F^{k11}_{nn},F^{k20}_{nm},F^{k02}_{mn},F^{k11}_{mm}\right)^T=\mathrm{i}\left(R^{k11}_{nn},R^{k20}_{nm},R^{k02}_{mn},R^{k11}_{mm}\right)^T.$$

Proof. see [10]. □

Remark 5.7. In the case that (n,m) and (n',m') are resonant pairs within $\mathscr{L}_{2\widehat{l}_1}$ and $\mathscr{L}_{2\widehat{l}_2}$ respectively, with $1\le\widehat{l}_1,\widehat{l}_2\le p$, the indices k, (n,m) and (n',m') satisfy the following expression

$$\begin{aligned}
&\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|n-n'|\neq 0}\left(F^{k11}_{n'n}z_{n'}\bar{z}_n+F^{k11}_{nn'}z_n\bar{z}_{n'}\right)e^{\mathrm{i}\langle k,\theta\rangle}\\
&+\sum_{|k|\le M,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2},|k|+|n-m'|\neq 0\text{ or }|k|+|n'-m|\neq 0}\left(F^{k20}_{n'n}z_{n'}z_n+F^{k02}_{n'n}\bar{z}_{n'}\bar{z}_n\right)e^{\mathrm{i}\langle k,\theta\rangle}+\cdots\cdots.
\end{aligned}$$

To solve these equations, we require a fundamental algebraic result from matrix theory.

Lemma 5.8. *Let $\widehat{A}$, $\widehat{B}$, and $\widehat{C}$ be $p\times p$, $q\times q$, and $p\times q$ matrices, respectively, and let Y be an unknown $p\times q$ matrix. The matrix equation $\widehat{A}Y+Y\widehat{B}=\widehat{C}$ is solvable if and only if the matrix $E_q\otimes\widehat{A}^T+\widehat{B}\otimes E_p$ is nonsingular.*

For a detailed proof, we refer the reader to the Appendix in [24].

Under the small divisor condition, we obtain the following estimates

$$\begin{aligned}
&|F_{kh00}|_{\mathscr{I}}\le\mu^{-5p}M^{5p(\nu+1)}|R_{kh00}|_{\mathscr{I}},0<|k|\le M,\\
&|F^{k10}_n|_{\mathscr{I}},|F^{k01}_n|_{\mathscr{I}}\le\mu^{-5p}M^{5p(\nu+1)}\varepsilon e^{-|k|r}e^{-|n|\rho},\\
&|F^{k11}_{nn'}|_{\mathscr{I}}\le\mu^{-5p}M^{5p(\nu+1)}\varepsilon e^{-|k|r}e^{-|n-n'|\rho},\\
&|F^{k20}_{nn'}|_{\mathscr{I}}+|F^{k02}_{nn'}|_{\mathscr{I}}\le\mu^{-5p}M^{5p(\nu+1)}\varepsilon e^{-|k|r}e^{-|n+n'|\rho}.
\end{aligned}\tag{5.26}$$

Next, we estimate the coordinate transformation, the new normal form and the new perturbation. We also verify that after one KAM iteration step, Assumptions V and VI remain satisfied. This part follows a similar approach to [10] and [28], we omit the details and provide only the proof of the measure estimate. We must emphasize that the process of it is similar to [10] and [28], but the forms of estimation are different.

5.4. **Measure estimate.**

For simplicity of notation, let $\mathscr{I}_{-1}=\mathscr{I}$ and $M_{-1}=0$. During the v-th step of the KAM iteration, the following resonant set is excluded,

$$\mathscr{J}^{v+1}=\bigcup_{M_v<|k|\leq M_{v+1},n,m,n'}\big(\mathscr{J}_k^{v+1}\cup\mathscr{J}_{kj}^{v+1}\cup\mathscr{J}_{kji}^{v+1}\cup\mathscr{M}_{knn'}^{v+1}(\mu)\big),$$

where

$$\mathscr{J}_k^{v+1}=\big\{\xi\in\mathscr{I}_v:|\langle k,\omega_{v+1}\rangle|<\frac{\mu}{M_{v+1}^{\nu}}\big\},|k|\neq 0,$$

$$\mathscr{J}_{kj}^{v+1}=\big\{\xi\in\mathscr{I}_v:\Big|\langle k,\omega_{v+1}\rangle\pm\Omega_j^{v+1}\Big|<\frac{\mu}{M_{v+1}^{\nu}}\big\},$$

$$\mathscr{J}_{kji}^{v+1}=\Big\{\xi\in\mathscr{I}_v:\Big|\langle k,\omega_{v+1}\rangle\pm\Omega_j^{v+1}\pm\Omega_i^{v+1}\Big|<\frac{\mu}{M_{v+1}^{\nu}},j,i\in\mathbb{Z}_1^2\setminus\big\{\bigcup_{\widehat{l}=1}^{p}\mathscr{L}_{2\widehat{l}}\big\}\Big\},$$

$$\mathscr{M}_{knn'}^{v+1}=\big\{\xi\in\mathscr{I}_v:\big|\det\big(\langle k,\omega_{v+1}\rangle E_4\pm\mathscr{C}_n^{v+1}\otimes E_2\pm E_2\otimes\mathscr{C}_{n'}^{v+1}\big)\big|<\frac{\mu}{M_{v+1}^{\nu}}\big\},$$
$$k\neq 0,n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2}.$$

Recall that $\omega_{v+1}(\xi)=\omega(\xi)+\sum_{i=0}^{v}R_{0h00}^i(\xi)$with $\Big|\sum_{i=0}^{v}R_{0h00}^i(\xi)\Big|_{\mathscr{I}_v}<\varepsilon$ and

$$\big|\Omega_n^{v+1}(\xi)-\Omega_n(\xi)\big|_{\mathscr{I}_v}\leq\sum_{i=0}^{v}\big|R_{nn}^{011,i}\big|\leq\varepsilon.$$

Remark 5.9. As stated before, at the $(v+1)$-th step, the small divisor conditions are automatically satisfied for $|k|\leq M_v$. Therefore, it is sufficient to remove the resonant set $\mathscr{J}^{v+1}$.

Below, we present the proof for the most intricate case, when $k\neq 0$, $1\leq\widehat{l}_1,\widehat{l}_2\leq p$,

$$\big\{\xi\in\mathscr{I}_v:\big|\det\big(\langle k,\omega_{v+1}\rangle E_4+\mathscr{C}_n^{v+1}\otimes E_2-E_2\otimes\mathscr{C}_{n'}^{v+1}\big)\big|<\frac{\mu}{M_{v+1}^{\nu}},n\in\mathscr{L}_{2\widehat{l}_1},n'\in\mathscr{L}_{2\widehat{l}_2}\big\}.$$

When $n\in\mathscr{L}_{2\widehat{l}-1}$ and $n'\in\mathscr{L}_{2\widehat{l}}$, there are no small divisors. For simplicity, let $A^{v+1}=\langle k,\omega_{v+1}\rangle E_4+\mathscr{C}_n^{v+1}\otimes E_2-E_2\otimes\mathscr{C}_{n'}^{v+1}$, $A^v=\langle k,\omega_v\rangle E_4+\mathscr{C}_n^v\otimes E_2-E_2\otimes\mathscr{C}_{n'}^v$. For $|k|\leq M_v$,

$$\begin{aligned}\big\|\big(A^{v+1}\big)^{-1}\big\|&=\big\|\big(A^v+\big(A^{v+1}-A^v\big)\big)^{-1}\big\|\\&=\big\|\big(E+\big(A^v\big)^{-1}\big(A^{v+1}-A^v\big)\big)^{-1}\big(A^v\big)^{-1}\big\|\\&\leq 2\big\|\big(A^v\big)^{-1}\big\|\leq 2\frac{M_v^{\nu}}{\mu}\leq\frac{M_{v+1}^{\nu}}{\mu}.\end{aligned}$$

For $M_\nu < |k| \leq M_{\nu+1}$ and $1 \leq \widehat{l_1}, \widehat{l_2} \leq p$, let $n \in \mathscr{L}_{2\widehat{l_1}}$ and $n' \in \mathscr{L}_{2\widehat{l_2}}$. Assume that (n,m) and (n',m') are resonant pairs in $\mathscr{L}_{2\widehat{l_1}}$ and $\mathscr{L}_{2\widehat{l_2}}$ respectively. Then if $|\widehat{k}| = 0$, as the partial gauge invariance property and $\widetilde{\omega}$ being Diophantine ensure this, we must have $|\widetilde{k}| \neq 0$.

We assume $|\widehat{k}| \neq 0$.

Lemma 5.10. *For any given $n, n' \in \mathbb{Z}_1^2$ with $|n-n'| \leq M_{\nu+1}$, either $\left|\det\left(\langle k, \omega_{\nu+1}\rangle E_4 + \mathscr{C}_n^{\nu+1} \otimes E_2 - E_2 \otimes \mathscr{C}_{n'}^{\nu+1}\right)\right| > 1$, or there exist $n_0, n_0', c \in \mathbb{Z}^2$ with $|n_0|, |n_0'|, |c| \leq 3M_{\nu+1}^2$ and $t \in \mathbb{Z}$, such that $n = n_0 + tc, n' = n_0' + tc$.*

Proof. See [10]. □

Lemma 5.11.

$$\bigcup_{n,n' \in \mathbb{Z}_1^2} \mathscr{M}_{knn'}^{\nu+1} \subset \bigcup_{n_0, n_0', c \in \mathbb{Z}^2, t \in \mathbb{Z}} \mathscr{M}_{k, n_0+tc, n_0'+tc}^{\nu+1}$$

where $|n_0|, |n_0'|, |c| \leq 3M_{\nu+1}^2$.

Proof. See [10]. □

Lemma 5.12. *For fixed k, n_0, n_0', c,*

$$\text{meas}\left(\bigcup_{t \in \mathbb{Z}} \mathscr{C}_{k, n_0+tc, n_0'+tc}^{\nu+1}\right) < c\frac{\mu^{\frac{1}{4p}}}{M_{\nu+1}^{\frac{\nu}{4p(4p+1)}}}.$$

Proof. By applying the analysis above and the Töplitz-Lipschitz property of $\Lambda + \sum_{\widehat{l}=1}^{p}\left(\mathscr{B}_{\widehat{l}} + \overline{\mathscr{B}}_{\widehat{l}}\right) + R$, the coefficient matrix $A^{\nu+1}(t)$ converges to a limit as $t \to \infty$,

$$\left\|A^{\nu+1}(t) - \lim_{t\to\infty} A^{\nu+1}(t)\right\| \leq \frac{\varepsilon_0}{|t|}.$$

We define the resonant set

$$\mathscr{M}_{kn_0n_0'c\infty}^{\nu+1} = \left\{\xi \in \mathscr{I}_\nu : \left|\det \lim_{t\to\infty} A^{\nu+1}(t)\right| < \frac{\mu}{M_{\nu+1}^{\frac{\nu}{4p+1}}}\right\}.$$

Then, for $\xi \in \mathscr{I}_\nu \setminus \mathscr{M}_{kn_0n_0'c\infty}^{\nu+1}$, we have $\left\|\left(\lim_{t\to\infty} A^{\nu+1}(t)\right)^{-1}\right\| \leq \frac{M_{\nu+1}^{\frac{\nu}{4p+1}}}{\mu}$. For $|t| > M_{\nu+1}^{\frac{\nu}{4p+1}}$, since

$$\left\|A^{\nu+1}(t) - \lim_{t\to\infty} A^{\nu+1}(t)\right\| \leq \frac{\varepsilon_0}{|t|},$$

we have

$$\left\|\left(A^{\nu+1}(t)\right)^{-1}\right\| \leq 2\frac{M_{\nu+1}^{\frac{\nu}{4p+1}}}{\mu} \leq \frac{M_{\nu+1}^{\nu}}{\mu}.$$

For $|t| \leq M_{\nu+1}^{\frac{\nu}{4p+1}}$, we define the resonant set

$$\mathscr{M}_{kn_0n_0'ct}^{\nu+1} = \left\{\xi \in \mathscr{I}_\nu : \left|\det A^{\nu+1}(t)\right| < \frac{\mu}{M_{\nu+1}^{\nu}}\right\}. \tag{5.27}$$

In addition,

$$\inf_{\xi \in \mathscr{I}} \max_{0<d\leq 4p} \left|\partial_\xi^d\left(\det A^{\nu+1}(t)\right)\right| \geq \frac{f^0}{2(p+1)(2\pi)^{2p}}|\widehat{k}| \geq \frac{f^0}{2(p+1)(2\pi)^{2p}}.$$

For fixed k, n_0, n'_0, c, t, meas $\left(\mathscr{M}^{\nu+1}_{kn_0n'_0ct}\right) < \left(\frac{\mu}{M^{\nu}_{\nu+1}}\right)^{\frac{1}{4p}}$. It follows by Lemma 4.8 in [9] that

$$\text{meas}\left\{\cup_{|t|\le M_{\nu+1}^{\frac{\nu}{4p+1}}}\mathscr{M}^{\nu+1}_{kn_0n'_0ct}\right\} < M_{\nu+1}^{\frac{\nu}{4p+1}}\left(\frac{\mu}{M^{\nu}_{\nu+1}}\right)^{\frac{1}{4p}} \le \frac{\mu^{\frac{1}{4p}}}{M_{\nu+1}^{\frac{\nu}{4p(4p+1)}}}.$$

Thus meas $\left(\cup_{t\in\mathbb{Z}}\mathscr{M}^{\nu+1}_{k,n_0+tc,n'_0+tc}\right) < c\frac{\mu^{\frac{1}{4p}}}{M_{\nu+1}^{\frac{\nu}{4p(p+1)}}}$. □

Lemma 5.13.

$$\text{meas}\Big(\bigcup_{M_\nu<|k|\le M_{\nu+1}}\mathscr{I}_k^{\nu+1}\Big) \le cM^b_{\nu+1}\frac{\mu}{M^{\nu}_{\nu+1}} = c\frac{\mu}{M^{\nu-b}_{\nu+1}},$$

$$\text{meas}\Big(\bigcup_{M_\nu<|k|\le M_{\nu+1},j}\mathscr{I}_{kj}^{\nu}\Big) \le cM^{2+b}_{\nu+1}\frac{\mu}{M^{\nu}_{\nu+1}} = c\frac{\mu}{M^{\nu-2-b}_{\nu+1}},$$

$$\text{meas}\Big(\bigcup_{M_\nu<|k|\le M_{\nu+1},j,i}\mathscr{I}_{kji}^{\nu+1}\Big) \le c\frac{\mu}{M_{\nu+1}^{\frac{\nu}{2}-12-b}},$$

$$\text{meas}\Big(\bigcup_{M_\nu<|k|\le M_{\nu+1},n,n'}\mathscr{M}_{knn'}^{\nu}\Big) \le c\frac{\mu^{\frac{1}{4p}}}{M_{\nu+1}^{\frac{\nu}{4p(4p+1)}-12-b}}.$$

Lemma 5.14. *Let $\nu > 4p(4p+1)(12+b)$. Then the total measure of the excluded resonant set along the KAM iteration satisfies*

$$\begin{aligned}\text{meas}\Big(\bigcup_{\nu\ge 0}\mathscr{I}^{\nu+1}\Big) &= \text{meas}\Big[\bigcup_{\nu\ge 0}\bigcup_{M_\nu<|k|\le M_{\nu+1},j,i}\Big(\mathscr{I}_k^{\nu+1}\cup\mathscr{I}_{kj}^{\nu+1}\cup\mathscr{I}_{kji}^{\nu+1}(\mu)\cup\mathscr{M}^{\nu+1}_{knn'}(\mu)\Big)\Big]\\ &\le c\sum_{\nu\ge 0}\frac{\mu^{\frac{1}{4p}}}{M_{\nu+1}}\\ &\le c\mu^{\frac{1}{4p}}.\end{aligned}$$

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