



ABUNDANT STOCHASTIC EXACT SOLUTIONS FOR THE GENERALIZED KDV-ZAKHAROV-KUZNETSOV EQUATION PERTURBED BY MULTIPLICATIVE NOISE IN SHALLOW WATER WAVES

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Abstract. The stochastic generalized Korteweg–de Vries–Zakharov–Kuznetsov (GKDV-ZK) equation driven by multiplicative noise is considered in this paper. The stochastic GKDV-ZK equation is transformed into another GKDV-ZK equation with random variable coefficients (GKDV–ZKE-RVCs) by applying a proper transformation. Rational, elliptic, hyperbolic, and trigonometric solutions for GKDV-ZKE-RVCs were obtained. In addition, we present several figures to illustrate how multiplicative noise impacts the exact solutions of the stochastic GKDV-ZK equation.

Keywords. Analytical solutions; Jacobi elliptic functions method; Stability by noise; Stochastic solutions.

1. INTRODUCTION

The generalized Korteweg-de Vries-Zakharov-Kuznetsov (GKDV-ZK) equation is a mathematical model that describes the behavior of waves in shallow water. It is a combination of three different equations, the Korteweg-de Vries, Zakharov, and Kuznetsov equations, each of which contributes to different aspects of wave dynamics. By combining these three equations into the GKDV-ZK equation, researchers are able to study a wider range of wave phenomena and gain a better understanding of the complex interactions between different factors that influence wave behavior. The GKDV-ZK equation is often used to study the behavior of surface waves on bodies of water such as lakes, oceans, and rivers, and has various applications in fields such as oceanography, fluid dynamics, and meteorology.

Incorporating a stochastic term into the generalized KdV-ZK equation introduces randomness into the model. This randomness can represent unpredictable factors that affect wave behavior, such as turbulence in the water or fluctuations in the medium through which the wave travels.

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The inclusion of stochastic elements helps scientists and engineers create more accurate models that can better predict real-world scenarios. For example, in oceanography, waves are affected by countless variables like wind, underwater currents, and varying temperatures, making a deterministic model insufficient. By adding a stochastic term, researchers can simulate how these unpredictable factors influence wave propagation.

In this paper, we consider the $(3 + 1)$ -dimensional stochastic GKDV-ZK (SGKDV-ZK) equation as follows [1, 2, 3]:

$$\mathcal{R}_t + \gamma_1 \mathcal{R}^2 \mathcal{R}_x + \gamma_2 \mathcal{R}_{xxx} + \gamma_3 (\mathcal{R}_{yy} + \mathcal{R}_{zz})_x = \mu \mathcal{R} \mathcal{B}_t, \quad (1.1)$$

where \mathcal{R} is wave profile defined in fluid ions; γ_i for $i = 1, 2, 3$, are arbitrary functions; $\mathcal{B}(t)$ is the standard Wiener process, $\mathcal{B}_t = \frac{\partial \mathcal{B}}{\partial t}$; and $\mathcal{R} \mathcal{B}_t$ is multiplicative noise term and μ is the amplitude of noise.

Due to the importance of the GKDV-ZK equation in ocean dynamics, meteorology, engineering, plasma physics and geophysics, many authors investigated its analytical solutions by utilizing various approaches including the Kudryashov's method [4], the Lie symmetry [5], the generalized Riccati equation method [6], the extended direct algebraic function [7], the auxiliary equation [8], the modified extended direct algebraic [3, 9], the modified auxiliary equation [10], the planar dynamical system method [11], and etc.

The motivation of this study is to derive stochastic solutions of the SGKDV-ZK Eq. (1.1). To this end, original Eq. (1.1) is converted into a GKdV-ZK equation with random variable coefficients (GKDV-ZKE-RVCs) through a suitable transformation. The exact solutions for GKDV-ZKE-RVCs are then obtained by applying the Jacobi elliptic method (JEF-method) and the modified extended tanh function method (METF-method). In all previous results, the solutions of the wave equation were deterministic, and this is the first time that we are aware of that they are stochastic.

Finally, we can attain the stochastic solutions of the SGKDV-ZK by employing the transformation that was utilized. The significance of the SGKDV-ZK Eq. (1.1) in plasma physics, fluid dynamics, and nonlinear optics makes these derived solutions are crucial for understanding a variety of difficult physical phenomena. To examine the influence of the stochastic term on the solutions of the SGKDV-ZK Eq. (1.1), we simulate a few graphs using MATLAB tools. Understanding the GKDV-ZK equation with a stochastic term helps to enhance our knowledge of complex systems. It allows for better forecasting of wave behavior, which could be crucial for navigation, coastal management, and predicting natural disasters like tsunamis. As mathematicians and scientists continue to explore these equations, they can refine their models, thereby improving our comprehension of phenomena that affect our daily lives and aid in disaster preparedness. This intersection of mathematics, physics, and unpredictability exemplifies the importance of continuous learning and adaptation in science and technology.

This is the structure of the remainder of the paper: In Section 2, GKDV-ZKE-RVCs are derived from SGKDV-ZK Eq. (1.1) and their solutions are obtained by utilizing the METF-method and JEF-method. While, the solutions to SGKDV-ZK Eq. (1.1) are obtained in Section 3. Finally, we provide a conclusion, Section 4, to end this paper.

2. GKDV-ZK EQUATION WITH RVCs AND ITS SOLUTIONS

Here, we derive the GKDV-ZKE-RVCs from Eq. (1.1). By using the transformation

$$\mathcal{R}(x, y, z, t) = \mathcal{G}(x, y, z, t)e^{\mu \mathcal{B}(t)}, \quad (2.1)$$

and the Itô derivatives rule, one can obtain GKDV-ZKE-RVCs as follows

$$\mathcal{G}_t + \gamma_2 \mathcal{G}_{xxx} + \mathcal{S}(t) \mathcal{G}^2 \mathcal{G}_x + \gamma_3 (\mathcal{G}_{yy} + \mathcal{G}_{zz})_x + \frac{1}{2} \mu^2 \mathcal{G} = 0, \quad (2.2)$$

where $\mathcal{S}(t) = \gamma_1 e^{2\mu \mathcal{B}(t)}$ and \mathcal{G} is a real deterministic function.

Let us obtain the exact solutions of the GKDV-ZKE-RVCs (2.2) by using two different methods, such as the METF-method [12, 13] and JEF-method [14, 15] as follows.

2.1. METF-method. Here, we use the METF-method. In order to determine the solutions of GKDV-ZKE-RVCs (2.2), we suppose that the solutions of Eq. (2.2) have the type

$$\mathcal{G}(x, y, z, t) = \sum_{k=0}^N \alpha_k(t) \mathcal{X}^k(\eta), \quad \eta = k_1 x + k_2 y + k_3 z + \int_0^t \lambda(\tau) d\tau, \quad (2.3)$$

where

$$\mathcal{X}' = \mathcal{X}^2 + \varpi. \quad (2.4)$$

By balancing \mathcal{G}''' with $\mathcal{G}^2 \mathcal{G}'$, we can determine the parameter N as $N = 1$. Now, Eq. (2.3) is rewritten as

$$\mathcal{G}(x, y, z, t) = \alpha_0(t) + \alpha_1(t) \mathcal{X}(\eta). \quad (2.5)$$

If we differentiate Eq. (2.5) with respects to x , y , z and t , then we have

$$\begin{aligned} \mathcal{G}_x &= k_1 [\alpha_1 \mathcal{X}^2 + \varpi \alpha_1], \\ \mathcal{G}_{xxx} &= k_1^3 [6\alpha_1 \mathcal{X}^4 + 8\varpi \alpha_1 \mathcal{X}^2 + 2\varpi^2 \alpha_1], \\ \mathcal{G}_{yyx} &= k_1 k_2^2 [6\alpha_1 \mathcal{X}^4 + 8\varpi \alpha_1 \mathcal{X}^2 + 2\varpi^2 \alpha_1], \\ \mathcal{G}_{zzx} &= k_1 k_3^2 [6\alpha_1 \mathcal{X}^4 + 8\varpi \alpha_1 \mathcal{X}^2 + 2\varpi^2 \alpha_1], \\ \mathcal{G}^2 \mathcal{G}_x &= k [\alpha_1^3 \mathcal{X}^4 + 2\alpha_0 \alpha_1^2 \mathcal{X}^3 + (\alpha_0^2 \alpha_1 + \varpi \alpha_1^3) \mathcal{X}^2 \\ &\quad + 2\varpi \alpha_0 \alpha_1^2 \mathcal{X} + \varpi \alpha_0^2 \alpha_1], \\ \mathcal{G}_t &= (\dot{\alpha}_0 + \varpi \alpha_1 \lambda) + \dot{\alpha}_1 \mathcal{X} + \lambda \alpha_1 \mathcal{X}^2. \end{aligned} \quad (2.6)$$

Substituting Eqs. (2.5) and (2.6) into Eq. (2.2), we obtain

$$\begin{aligned} & [6\alpha_1 k_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + \mathcal{S} k_1 \alpha_1^3] \mathcal{X}^4 + [2k_1 \mathcal{S} \alpha_0 \alpha_1^2] \mathcal{X}^3 \\ & + [\lambda \alpha_1 + 8\varpi \alpha_1 k_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + k_1 \mathcal{S} \alpha_0^2 \alpha_1 + \varpi k_1 \mathcal{S} \alpha_1^3] \mathcal{X}^2 \\ & + [\dot{\alpha}_1 + 2\varpi \mathcal{S} k_1 \alpha_0 \alpha_1^2 + \frac{1}{2} \mu^2 \alpha_1] \mathcal{X} \\ & + [\dot{\alpha}_0 + \varpi \alpha_1 \lambda + 2\varpi^2 \alpha_1 k_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + \varpi k_1 \mathcal{S} \alpha_0^2 \alpha_1 + \frac{1}{2} \mu^2 \alpha_0] = 0. \end{aligned}$$

Equating each coefficient of \mathcal{X}^k to zero, we see that

$$\begin{aligned} 6\alpha_1 k_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + \mathcal{S} k_1 \alpha_1^3 &= 0, \\ 2k_1 \mathcal{S} \alpha_0 \alpha_1^2 &= 0, \\ \lambda \alpha_1 + 8\varpi \alpha_1 k_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + k_1 \mathcal{S} \alpha_0^2 \alpha_1 + \varpi k_1 \mathcal{S} \alpha_1^3 &= 0, \end{aligned}$$

$$\dot{\alpha}_1 + 2\varpi \mathcal{S} k_1 \alpha_0 \alpha_1^2 + \frac{1}{2} \mu^2 \alpha_1 = 0,$$

and

$$\dot{\alpha}_0 + \varpi \alpha_1 \lambda + 2\varpi^2 \alpha_1 k_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + \varpi k_1 \mathcal{S} \alpha_0^2 \alpha_1 + \frac{1}{2} \mu^2 \alpha_0 = 0.$$

Solving these equations yields

$$\begin{aligned} \alpha_0(t) &= 0, \quad \alpha_1 = \hbar e^{-\frac{1}{2}\mu^2 t}, \quad \gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2 = \frac{-\hbar^2}{6} \mathcal{S}(t) e^{-\mu^2 t}, \\ \lambda(t) &= \frac{\gamma_1 k_1 \varpi \hbar^2}{3} e^{2\mu \mathcal{B}(t) - \mu^2 t}, \end{aligned} \quad (2.7)$$

where \hbar is a constant. Substituting Eq. (2.7) into Eq. (2.5), we obtain the solutions of GKDV-ZKE-RVCs (2.2) as follows:

$$\mathcal{G}(x, y, z, t) = \hbar \mathcal{X}(\eta) e^{-\frac{1}{2}\mu^2 t}, \quad \eta = k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau,$$

where \mathcal{X} is the solutions of Eq. (2.4). To obtain \mathcal{X} , the solutions of Eq. (2.4) are, depending on ϖ , as follows:

Case 1: If $\varpi > 0$, then Eq. (2.4) possess the solutions:

$$\begin{aligned} \mathcal{X}_1(\eta) &= \sqrt{\varpi} \tan(\sqrt{\varpi} \eta), \\ \mathcal{X}_2(\eta) &= -\sqrt{\varpi} \cot(\sqrt{\varpi} \eta), \\ \mathcal{X}_3(\eta) &= \sqrt{\varpi} \left(\tan(\sqrt{4\varpi} \eta) \pm \sec(\sqrt{4\varpi} \eta) \right), \\ \mathcal{X}_4(\eta) &= -\sqrt{\varpi} \left(\cot(\sqrt{4\varpi} \eta) \pm \csc(\sqrt{4\varpi} \eta) \right), \\ \mathcal{X}_5(\eta) &= \frac{1}{2} \sqrt{\varpi} \left(\tan\left(\frac{1}{2} \sqrt{\varpi} \eta\right) - \cot\left(\frac{1}{2} \sqrt{\varpi} \eta\right) \right). \end{aligned}$$

Then, GKDV-ZKE-RVCs (2.2) possess the trigonometric functions solutions:

$$\mathcal{G}_1 = \hbar \sqrt{\varpi} \left(\tan\left(\sqrt{\varpi} (k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.8)$$

$$\mathcal{G}_2 = -\hbar \sqrt{\varpi} \left(\cot\left(\sqrt{\varpi} (k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.9)$$

$$\begin{aligned} \mathcal{G}_3 &= \hbar \sqrt{\varpi} \left(\tan\left(\sqrt{4\varpi} (k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau)\right) \right. \\ &\quad \left. \pm \sec\left(\sqrt{4\varpi} (k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}, \end{aligned} \quad (2.10)$$

$$\begin{aligned} \mathcal{G}_4 &= -\hbar \sqrt{\varpi} \left(\cot\left(\sqrt{4\varpi} (k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau)\right) \right. \\ &\quad \left. \pm \csc\left(\sqrt{4\varpi} (k_1 x + k_2 y + k_3 z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}, \end{aligned} \quad (2.11)$$

$$\begin{aligned} \mathcal{G}_5 = & \hbar\sqrt{\varpi} \left(\tan\left(\frac{1}{2}\sqrt{\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)\right) \right. \\ & \left. - \cot\left(\frac{1}{2}\sqrt{\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}. \end{aligned} \quad (2.12)$$

Case 2: If $\varpi < 0$, then Eq. (2.4) has the solutions:

$$\begin{aligned} \mathcal{X}_6(\eta) &= -\sqrt{-\varpi} \tanh\left(\sqrt{-\varpi}\eta\right), \\ \mathcal{X}_7(\eta) &= -\sqrt{-\varpi} \coth\left(\sqrt{-\varpi}\eta\right), \\ \mathcal{X}_8(\eta) &= -\sqrt{-\varpi} \left(\coth(\sqrt{-4\varpi}\eta) \pm \operatorname{csch}(\sqrt{-4\varpi}\eta) \right), \\ \mathcal{X}_9(\eta) &= \frac{-1}{2}\sqrt{-\varpi} \left(\tanh\left(\frac{1}{2}\sqrt{-\varpi}\eta\right) + \coth\left(\frac{1}{2}\sqrt{-\varpi}\eta\right) \right). \end{aligned}$$

Then, GKDV-ZKE-RVCs (2.2) possess the hyperbolic functions solution:

$$\mathcal{G}_6 = -\hbar\sqrt{-\varpi} \left(\tanh\left(\sqrt{-\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.13)$$

$$\mathcal{G}_7 = -\hbar\sqrt{-\varpi} \left(\coth\left(\sqrt{-\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.14)$$

$$\begin{aligned} \mathcal{G}_8 = & -\hbar\sqrt{-\varpi} \left(\coth(\sqrt{-4\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right. \\ & \left. \pm \operatorname{csch}(\sqrt{-4\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{-\frac{1}{2}\mu^2 t}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} \mathcal{G}_9 = & -\frac{\hbar}{2}\sqrt{-\varpi} \left(\tanh\left(\frac{1}{2}\sqrt{-\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)\right) \right. \\ & \left. + \coth\left(\frac{1}{2}\sqrt{-\varpi}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)\right) \right) e^{-\frac{1}{2}\mu^2 t}. \end{aligned} \quad (2.16)$$

Case 3: If $\varpi = 0$, then Eq. (2.4) has the solution $\mathcal{X}_{10}(\eta) = \frac{-1}{\eta}$. Hence, the GKDV-ZKE-RVCs (2.2) has the rational function solution:

$$\mathcal{G}_{10} = \left(\frac{-\hbar}{(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \varpi \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)} \right) e^{-\frac{1}{2}\mu^2 t}. \quad (2.17)$$

2.2. JEF-method. Assume the solutions of GKDV-ZKE-RVCs (2.2), with $N = 1$, has the type

$$\mathcal{G}(x, y, z, t) = a_0(t) + a_1(t)J(\eta), \quad (2.18)$$

where $J(\eta)$ represents one of the next elliptic functions $cn(\delta\eta, \check{n})$, $sn(\delta\eta, \check{n})$ or $dn(\delta\eta, \check{n})$. Differentiating Eq. (2.18) with respects to x , y , z , and t , we obtain

$$\begin{aligned} \mathcal{G}_{xx} &= k_1^2 a_1(B_1 J + B_2 J^3), \quad \mathcal{G}_{xxx} = \delta k_1^3 a_1(B_1 + 2B_2 J^2)J', \\ \mathcal{G}_{yyx} &= \delta k_1 k_2^2 a_1(B_1 + 2B_2 J^2)J', \quad \mathcal{G}_{zzx} = \delta k_1 k_3^2 a_1(B_1 + 2B_2 J^2)J', \\ \mathcal{G}_x \mathcal{G}^2 &= \delta k_1 a_1(a_1^2 J^2 + 2a_0 a_1 J + a_0^2)J', \\ \mathcal{G}_t &= \dot{a}_0 + \dot{a}_1 J + \delta \lambda a_1 J', \quad \mathcal{G}_x = \delta k_1 a_1 J', \end{aligned} \quad (2.19)$$

where B_1 and B_2 are constants relying on δ , and \check{n} . This will be explained later. Put Eqs. (2.18) and (2.19) into Eq. GKDV-ZKE-RVCs (2.2). Then, setting every coefficient of J^k equal to zero yields

$$\begin{aligned} J^0 &: \dot{a}_0 + \frac{1}{2}\mu^2 a_0 = 0, \\ J^1 &: \dot{a}_1 + \frac{1}{2}\mu^2 a_1 = 0, \\ J^0 J^1 &: \delta a_1 [\lambda + k_1 B_1 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + k a_0^2 \mathcal{S}(t)] = 0, \\ J J^1 &: 2\delta k_1 a_0 a_1^2 \mathcal{S}(t) = 0, \end{aligned}$$

and

$$J^2 J^1: 2\delta k_1 a_1 B_2 (\gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2) + \delta k_1 a_1^3 \mathcal{S}(t) = 0.$$

Solving the above system, we have

$$a_0(t) = 0, \quad a_1 = \hbar e^{-\frac{1}{2}\mu^2 t}, \quad \gamma_2 k_1^2 + \gamma_3 k_2^2 + \gamma_3 k_3^2 = \frac{-\hbar^2}{2B_2} \mathcal{S}(t) e^{-\mu^2 t}, \quad \lambda(t) = \frac{\hbar^2 k_1 B_1}{2B_2} \mathcal{S}(t) e^{-\mu^2 t},$$

where \hbar is a constant. Therefore, the GKDV-ZKE-RVCs (2.2) has the solution:

$$\mathcal{G}(x, y, z, t) = \hbar J(\eta), \quad \eta = k_1 x + k_2 y + k_3 z + \frac{\gamma_1 \hbar^2 k_1 B_1}{2B_2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau. \quad (2.20)$$

The next step is to define $J(\eta)$ as follows:

Set 1: When $J(\eta) = cn(\delta\eta, \check{n})$, Eq. (2.20) has the form

$$\mathcal{G} = \hbar \left(cn(\delta k_1 x + \delta k_2 y + \delta k_3 z + \frac{\delta \gamma_1 \hbar^2 k_1 B_1}{2B_2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.21)$$

where

$$B_1 = \delta^2(1 - 2\check{n}^2) \text{ and } B_2 = -2\delta^2\check{n}^2.$$

Set 2: If $J(\eta) = sn(\delta\eta, \check{n})$, then Eq. (2.20) has the form

$$\mathcal{G} = \hbar \left(sn(\delta k_1 x + \delta k_2 y + \delta k_3 z + \frac{\delta \gamma_1 \hbar^2 k_1 B_1}{2B_2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.22)$$

where

$$B_1 = -\delta^2(1 + \check{n}^2) \text{ and } B_2 = 2\delta^2\check{n}^2.$$

Set 3: If $J(\eta) = dn(\delta\eta, \check{n})$, then Eq. (2.20) turns into

$$\mathcal{G} = \hbar \left(dn(\delta k_1 x + \delta k_2 y + \delta k_3 z + \frac{\delta \gamma_1 \hbar^2 k_1 B_1}{2B_2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{-\frac{1}{2}\mu^2 t}, \quad (2.23)$$

where

$$B_1 = \delta^2(2 - \check{n}^2) \text{ and } B_2 = -2\delta^2.$$

3. EXACT SOLUTIONS OF SGKDV-ZK EQUATION

Using the GKDV-ZKE-RVCs (2.2) solutions from the previous section, we can now obtain the SGKDV-ZK Eq. (1.1) solutions as follows:

3.1. **GREM-Method.** Plugging Eqs (2.8)-(2.17) into Eq. (2.1), one sees that SGKDV-ZK Eq. (1.1) has the solutions:

$$\mathcal{R}_1 = \hbar\sqrt{\omega} \left(\tan(\sqrt{\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \quad (3.1)$$

$$\mathcal{R}_2 = -\hbar\sqrt{\omega} \left(\cot(\sqrt{\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \quad (3.2)$$

$$\begin{aligned} \mathcal{R}_3 &= \hbar\sqrt{\omega} \left(\tan(\sqrt{4\omega s}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right. \\ &\quad \left. \pm \sec(\sqrt{4\omega s}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \mathcal{R}_4 &= -\hbar\sqrt{\omega} \left(\cot(\sqrt{4\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right. \\ &\quad \left. \pm \csc(\sqrt{4\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} \mathcal{R}_5 &= \hbar\sqrt{\omega} \left(\tan(\frac{\sqrt{\omega}}{2}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right. \\ &\quad \left. - \cot(\frac{1}{2}\sqrt{\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \end{aligned} \quad (3.5)$$

for $\omega > 0$,

$$\mathcal{R}_6 = -\hbar\sqrt{-\omega} \left(\tanh(\sqrt{-\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \quad (3.6)$$

$$\mathcal{R}_7 = -\hbar\sqrt{-\omega} \left(\coth(\sqrt{-\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \quad (3.7)$$

$$\begin{aligned} \mathcal{R}_8 &= -\hbar\sqrt{-\omega} \left(\coth(\sqrt{-4\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right. \\ &\quad \left. \pm \operatorname{csch}(\sqrt{-4\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \mathcal{R}_9 &= -\frac{\hbar}{2}\sqrt{-\omega} \left(\tanh(\frac{1}{2}\sqrt{-\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right. \\ &\quad \left. + \coth(\frac{1}{2}\sqrt{-\omega}(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)) \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \end{aligned} \quad (3.9)$$

for $\omega > 0$, and

$$\mathcal{R}_{10} = \left(\frac{-\hbar}{(k_1x + k_2y + k_3z + \frac{\gamma_1 k_1 \omega \hbar^2}{3} \int_0^t e^{2\mu\mathcal{B}(\tau) - \mu^2\tau} d\tau)} \right) e^{\mu\mathcal{B}(t) - \frac{1}{2}\mu^2 t}, \quad (3.10)$$

for $\omega = 0$.

3.2. JEF-method. Substituting Eqs (2.22)-(2.23) into Eq. (2.1), we have the SGKDV-ZK Eq. (1.1):

$$\mathcal{R} = \hbar \left(sn(\delta k_1 x + \delta k_2 y + \delta k_3 z - \frac{\delta \gamma_1 \hbar^2 k_1 (1 + \check{n}^2)}{4 \check{n}^2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{\mu \mathcal{B}(t) - \frac{1}{2} \mu^2 t}, \quad (3.11)$$

$$\mathcal{R} = \hbar \left(cn(\delta k_1 x + \delta k_2 y + \delta k_3 z + \frac{\delta \gamma_1 \hbar^2 k_1 (1 - 2\check{n}^2)}{4 \check{n}^2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{\mu \mathcal{B}(t) - \frac{1}{2} \mu^2 t}, \quad (3.12)$$

and

$$\mathcal{R} = \hbar \left(dn(\delta k_1 x + \delta k_2 y + \delta k_3 z - \frac{\delta \gamma_1 \hbar^2 k_1 (2 - \check{n}^2)}{4} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{\mu \mathcal{B}(t) - \frac{1}{2} \mu^2 t}. \quad (3.13)$$

When $\check{n} \rightarrow 1$, the Eqs (3.11)-(3.13) have the form

$$\mathcal{R} = \hbar \left(\tanh(\delta k_1 x + \delta k_2 y + \delta k_3 z - \frac{\delta \gamma_1 \hbar^2 k_1}{2} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{\mu \mathcal{B}(t) - \frac{1}{2} \mu^2 t}, \quad (3.14)$$

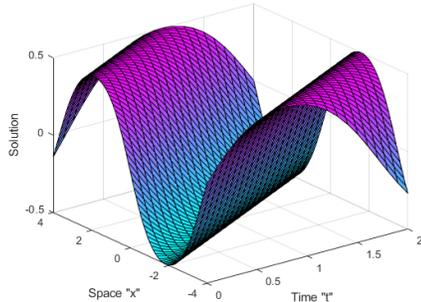
and

$$\mathcal{R} = \hbar \left(\operatorname{sech}(\delta k_1 x + \delta k_2 y + \delta k_3 z - \frac{\delta \gamma_1 \hbar^2 k_1}{4} \int_0^t e^{2\mu \mathcal{B}(\tau) - \mu^2 \tau} d\tau, \check{n}) \right) e^{\mu \mathcal{B}(t) - \frac{1}{2} \mu^2 t}. \quad (3.15)$$

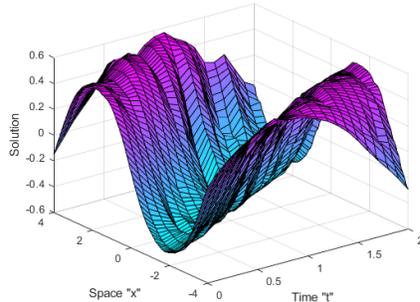
Remark 3.1. Putting $\delta = k_1 = k_2 = k_3 = 1$, and $\mu = 0$ (i.e. no noise) in Eqs (3.11), (3.12), and (3.15), we have the results that were stated in [4].

3.3. Discussion and impacts of noise. Discussion: The solutions to the SGKDV-ZK Eq. (1.1) were derived here. We employed the METF and JEF methods, which generated a wide range of solutions, such as optical trigonometric solutions (3.1)-(3.6), optical hyperbolic solutions (3.6)-(3.9), optical rational solution (3.10), and optical elliptic solutions (3.11)-(3.13). Optical solutions play a key role in the study of the GKDV-ZK equation, allowing researchers to acquire valuable insights into the nonlinear dynamics of waves in a nonlinear medium, predict the behavior of waves, study the stability and dynamics of solutions, and understand the formation of complex wave patterns and structures. By studying the optical properties of materials in nonlinear media, researchers can advance our understanding of wave equations and develop new applications for controlling and manipulating waves in nonlinear media.

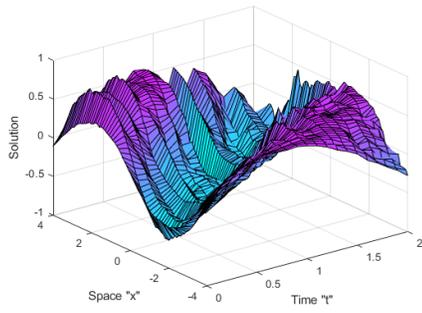
Influence of noise: We examine here how Brownian motion process impacts the exact solution of SGKDV-ZK Eq. (1.1). Many numerical simulations of different solutions with distinct values of noise intensity are shown. Graphs 1, 2 and 3 display the profile of the solutions $\mathcal{R}(x, y, z, t)$ presented respectively in Eqs (3.11), (3.14) and (3.15) for $k_1 = k_2 = k_3 = 1$, $y = z = 0$, $\gamma_1 = 1$, $-4 \leq x \leq 4$, $0 \leq t \leq 2$ and distinct values of noise intensity μ as follows:



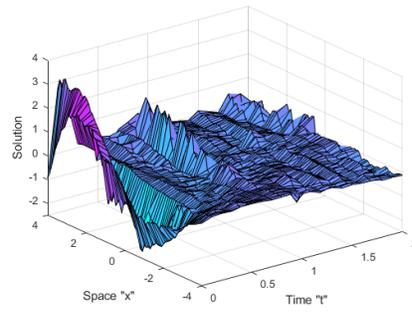
(a) $\mu = 0$



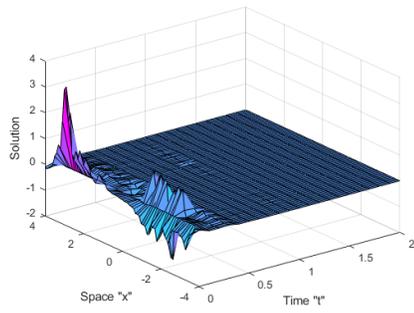
(b) $\mu = 0.1$



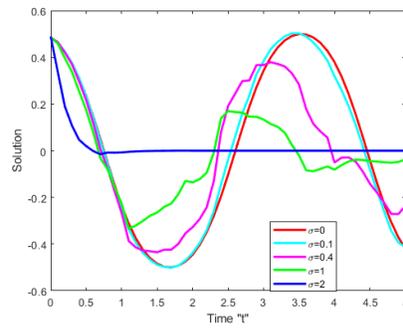
(c) $\mu = 0.4$



(d) $\mu = 1$

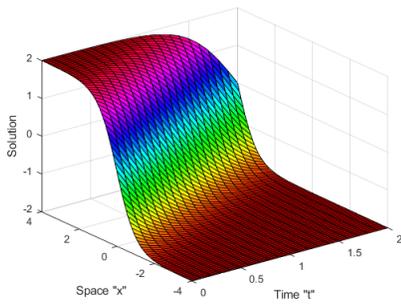


(e) $\mu = 2$

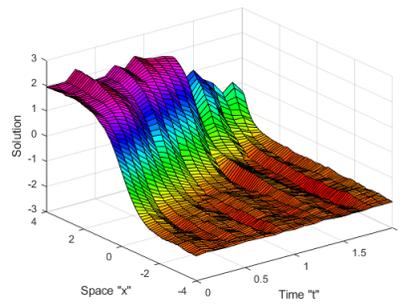


(f) $\mu = 0, 0.1, 0.4, 1, 2$

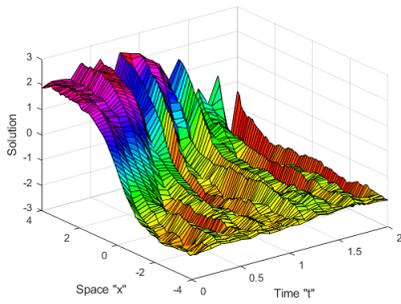
Figure 1. (a-e) show 3D plots of $\mathcal{R}(x, y, z, t)$ stated in Eq. (3.11) with $\check{n} = \check{h} = 0.5$, (f) shows 2D plot of Eq. (3.11) with distinct value μ



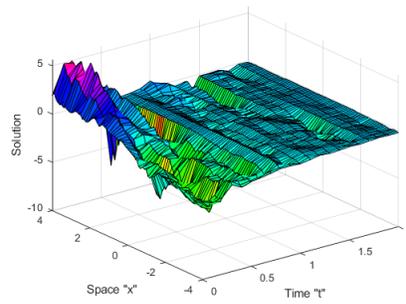
(a) $\mu = 0$



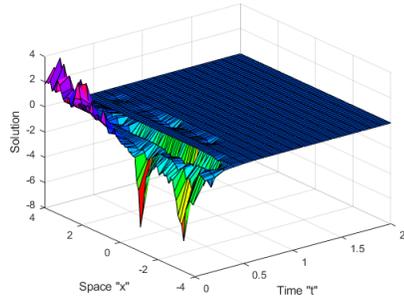
(b) $\mu = 0.1$



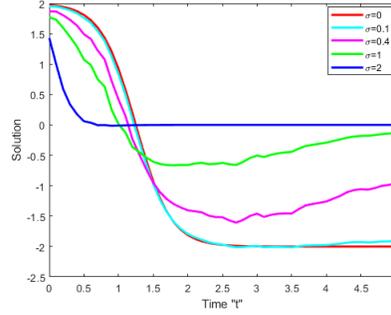
(c) $\mu = 0.4$



(d) $\mu = 1$

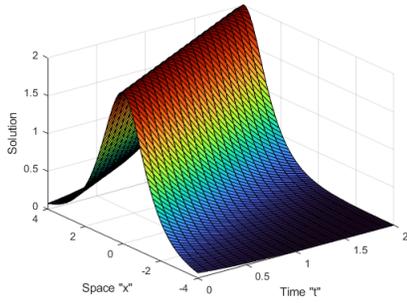


(e) $\mu = 2$

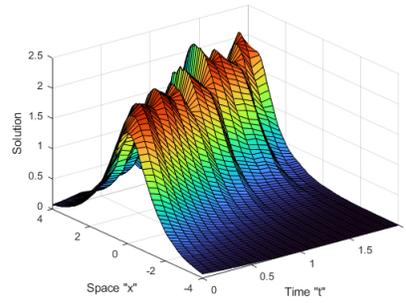


(f) $\mu = 0, 0.1, 0.4, 1, 2$

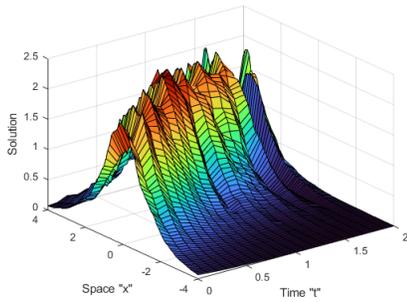
Figure 2. (a-e) show 3D plots of $\mathcal{R}(x,y,z,t)$ stated in Eq. (3.14) with $\delta = \hbar = 1$, (f) shows 2D plot of Eq. (3.14) with distinct value μ



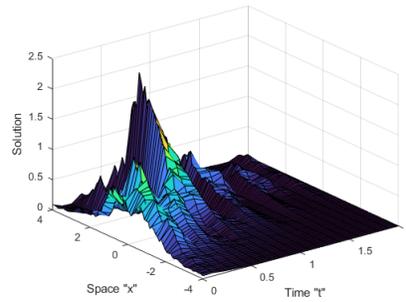
(a) $\mu = 0$



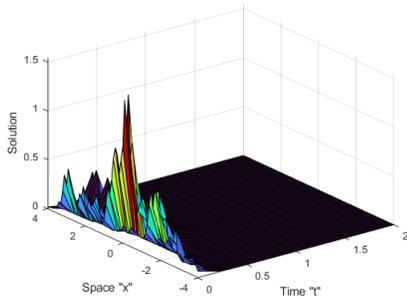
(b) $\mu = 0.1$



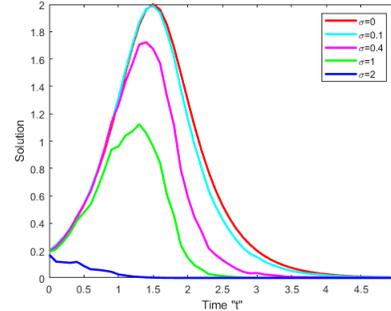
(c) $\mu = 0.4$



(d) $\mu = 1$



(e) $\mu = 2$



(f) $\mu = 0, 0.1, 0.4, 1, 2$

Figure 3. (a-e) show 3D plots of $\mathcal{R}(x,y,z,t)$ stated in Eq. (3.15) with $\delta = \hbar = 1$, (f) shows 2D plot of Eq. (3.15) with distinct value μ

Figures 1(a)–3(a) show that many kinds of solutions for example solitary periodic solutions, solitary kink solutions, solitary bright solutions, appear when noise is ignored (i.e., $\mu = 0$). As illustrated in Figures 1–3(b-e), the surface flattens when noise is introduced at $\mu = 0.1, 0.4, 1, 2$. This results indicates how the solutions of the SGKDV-ZK Eq. (1.1) are affected by multiplicative Brownian motion, stabilizing them around zero.

4. CONCLUSIONS

The stochastic generalized Korteweg–de Vries–Zakharov–Kuznetsov (SGKDV-ZK) Eq. (1.1) perturbed by multiplicative noise was investigated in this paper. The SGKDV-ZK equation was transformed into another GKDV-ZK equation with RVCs (2.2) by utilizing a proper transformation. New exact stochastic solutions for GKDV-ZKE-RVCs in the type of rational, elliptic, trigonometric, and hyperbolic functions were achieved by using the METF and JEF methods. In addition, we acquired the solutions to the SGKDV-ZK Equation (1.1). Furthermore, we extended certain existing answers, such as the results presented in [4]. The importance of the GKDV-ZK equation makes the given solutions crucial for understanding several difficult physical processes, which is utilized in ocean dynamics, meteorology, engineering, plasma physics, and geophysics. Finally, some graphics were presented to highlight how the stochastic term impacts the exact stochastic solutions of the SGKDV-ZK equation.

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