



BEST PROXIMITY POINTS FOR GENERALIZED α - ψ -GERAGHTY PROXIMAL CONTRACTION MULTIMAPS

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Abstract. In this paper, we introduce a new class of generalized α - ψ -Geraghty proximal contraction multimaps and establish best proximity point results for such mappings in the context of generalized metric spaces. Our results extend, generalize, and enhance those of related and relevant results in the literature.

Keywords. Best proximity point; Fixed point; Generalized metric space; Multimap.

1. INTRODUCTION

It is known that the remarkable Banach contraction principle [8] is one of the most useful and fundamental results in modern nonlinear analysis. It establishes the existence and uniqueness of fixed points for certain self-maps in a complete metric space and provides a constructive method to find those fixed points. Given its wide applicability in theory and applied fields, researchers have explored its generalizations in various settings; see, e.g., [7, 20, 25, 26, 27] and the references therein. Note that all these generalizations are only applicable for self-mappings.

Let \mathbb{U} be a nonempty set and $\mathfrak{p} : \mathbb{U} \times \mathbb{U} \rightarrow [0, \infty)$. Let \mathbb{D}, \mathbb{E} be two non-empty subsets of $(\mathbb{U}, \mathfrak{p})$ and $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ be a mapping. Then $\mathfrak{p}(x, \mathbb{F}x) \geq \mathfrak{p}(\mathbb{D}, \mathbb{E})$ for all $x \in \mathbb{D}$, where $\mathfrak{p}(\mathbb{D}, \mathbb{E}) = \inf\{\mathfrak{p}(x, y) : x \in \mathbb{D}, y \in \mathbb{E}\}$. In general, for nonself-mapping $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$, the fixed point equation $\mathbb{F}x = x$ may not have a solution. In this case, we focus on the possibility of finding an element $x \in \mathbb{D}$, which is an approximate solution such that the error $\mathfrak{p}(x, \mathbb{F}x)$ is minimum, possibly $\mathfrak{p}(x, \mathbb{F}x) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$. In case that $\mathfrak{p}(x, \mathbb{F}x) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, we call that x is a best proximity point of \mathbb{F} in \mathbb{D} . The purpose of best proximity point theory is to address the problem of finding a distance between two closed sets by using nonself-mappings from one set to the other. Best proximity point theory analyzes the existence of an approximate solution which is optimal.

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Some applications of best proximity points in the investigations of market equilibrium in non-competitive markets are presented in [13] and [15].

Recently, best proximity point theory attracted the attention of numerous authors. In 2009, Karpagam and Agrawal [17] proved best proximity point results for p -cyclic Meir-Keeler contractions. In [18], they further introduced p -cyclic contractions in metric spaces. In 2014, Kumam et al. [21] obtained best proximity point results for modified α -proximal C -contraction mappings. Kumam et al. [22] also presented coupled best proximity points in ordered metric spaces. In 2015, De la Sen and Karapinar [12] obtained some best proximity points of cyclic contractions in probabilistic metric spaces. In 2017, Ansari and Nantadilok [4] used the concept of C -class functions to establish the best proximity point results for a certain class of proximal contractive mappings in S -metric spaces. More recently, Ansari et al. [6] obtained best proximity point results for generalized cyclic weak (F, ψ, ϕ) -contractions in ordered metric spaces. Later, Nuntadilok et al. [28] investigated best proximity point results in p -cyclic metric spaces. Some other related and relevant results on best proximity points can be found in [3, 5, 19, 30] and the references therein.

A best proximity point becomes a fixed point if the underlying mapping is a self-mapping. Therefore, it can be concluded that best proximity point theorems generalize fixed point theorems in a natural way. In recent years, the existence and convergence of best proximity points is an interesting aspect of optimization theory which attracted the attention of many authors; see, e.g., [1, 9, 10]. Recently, in 2014, Asadi et al. [7] presented the fixed point theorems for α - ψ -Geraghty type contractions in generalized metric spaces and Kumssa [23] introduced best proximity point of modified Suzuki-Edelstein-Geraghty type proximal contractions in metric spaces. In this paper, we introduce the class of generalized α - ψ -Geraghty proximal contraction multimaps (multi-valued mappings) and establish best proximity points results for those multi-valued nonself mappings in the setting of a generalized metric space, where ψ is an auxiliary function which does not require the subadditive property.

2. PRELIMINARIES

Let \mathbb{D}, \mathbb{E} be non-empty subsets of a metric space $(\mathbb{U}, \mathfrak{p})$. The following notations and definitions are crucial for our main results

- 1). $\mathfrak{p}(\mathbb{D}, \mathbb{E}) = \inf\{\mathfrak{p} : x \in \mathbb{D}, y \in \mathbb{E}\}$;
- 2). $\mathfrak{p}(x, \mathbb{E}) = \inf\{\mathfrak{p}(x, b) : b \in \mathbb{E}\}$;
- 3). $\mathbb{D}_0 = \{x \in \mathbb{D} : \mathfrak{p}(x, b) = \mathfrak{p}(\mathbb{D}, \mathbb{E}), \text{ for some } b \in \mathbb{E}\}$;
- 4). $\mathbb{E}_0 = \{y \in \mathbb{E} : \mathfrak{p}(a, y) = \mathfrak{p}(\mathbb{D}, \mathbb{E}), \text{ for some } a \in \mathbb{D}\}$;
- 5). $2^{\mathbb{U}} \setminus \emptyset$: the set of all nonempty subsets of \mathbb{U} ;
- 6). $\mathfrak{CL}(\mathbb{U})$: the set of all nonempty closed subsets of \mathbb{U} ;
- 7). $\mathbb{R}^+ = [0, \infty)$ and \mathbb{N} = the set of positive integers.

We denote by Θ the class of all functions $\theta : [0, \infty) \rightarrow [0, 1)$ satisfying the following condition:

$$\lim_{n \rightarrow \infty} \theta(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0.$$

Remark 2.1. It is to be noted that, for every $x \in \mathbb{D}_0$, there exists $y \in \mathbb{E}_0$ such that $\mathfrak{p}(x, y) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$. Conversely, for every $y \in \mathbb{E}_0$, there exists $x \in \mathbb{D}_0$ such that $\mathfrak{p}(x, y) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$.

For every $\mathbb{D}, \mathbb{E} \in \mathcal{CL}(\mathbb{U})$, the generalized Hausdorff metric induced by \mathfrak{p} is defined by

$$\mathcal{H}_{\mathfrak{p}}(\mathbb{D}, \mathbb{E}) = \begin{cases} \max \{ \sup_{x \in \mathbb{D}} \mathfrak{p}(x, \mathbb{E}), \sup_{y \in \mathbb{E}} \mathfrak{p}(y, \mathbb{D}) \}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

A point $x^* \in \mathbb{D}$ is said to be the best proximity point of a nonself-mapping $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ if $\mathfrak{p}(x^*, \mathbb{F}x^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$. When $\mathbb{D} = \mathbb{E}$, the best proximity point is essentially the fixed point of the mapping \mathbb{F} .

We next recall the concept of generalized metrics [11], which are essential for the subsequent fixed point theory. Let \mathbb{U} be a nonempty set, and let $\mathfrak{p} : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}^+$ satisfy the following conditions for all $x, y \in \mathbb{U}$ and all distinct $u, v \in \mathbb{U}$ each of which is different from x and y

GM1. $\mathfrak{p}(x, y) = 0$ if and only if $x = y$,

GM2. $\mathfrak{p}(x, y) = d(y, x)$,

GM3. $\mathfrak{p}(x, y) \leq \mathfrak{p}(x, u) + \mathfrak{p}(u, v) + \mathfrak{p}(v, y)$.

The function \mathfrak{p} presented above is called a *generalized metric* and the pair $(\mathbb{U}, \mathfrak{p})$ is called a *generalized metric space* (abbreviated as GMS). Let \mathfrak{p} be a generalized metric on \mathbb{U} and $\varepsilon > 0$. We call $B_{\mathfrak{p}}(x, \varepsilon) = \{y \in \mathbb{U} | \mathfrak{p}(x, y) < \varepsilon\}$ an ε -ball centered at x . In the above definition, the condition (GM3) is called the *quadrilateral inequality*. Some examples of generalized metric spaces can be found in [3, 29, 31], which demonstrate several possible properties of generalized metric, different than in the standard metric case.

Remark 2.2. Note that every metric space is a generalized metric space, but the converse is not true in general (see, e.g., [3, 11, 24]).

In 2014, Asadi et al. [7] gave the following definition. Convergent, Cauchy sequence, completeness, and continuity on a generalized metric space $(\mathbb{U}, \mathfrak{p})$ are defined as follows

- 1). A sequence $\{x_n\}$ in $(\mathbb{U}, \mathfrak{p})$ is GMS convergent to a limit x if and only if $\mathfrak{p}(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
- 2). A sequence $\{x_n\}$ in $(\mathbb{U}, \mathfrak{p})$ is GMS Cauchy if and only if, for every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that $\mathfrak{p}(x_n, x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.
- 3). $(\mathbb{U}, \mathfrak{p})$ is said to be complete if every GMS Cauchy sequence in \mathbb{U} is GMS convergent.
- 4). A mapping $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ is continuous if, for each sequence $\{x_n\}$ in \mathbb{U} such that $\mathfrak{p}(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, we have $\mathfrak{p}(\mathbb{F}x_n, \mathbb{F}x) \rightarrow 0$ as $n \rightarrow \infty$.

We recall the following definition introduced by Ali et al. [2]

Let \mathbb{D} and \mathbb{E} be two nonempty subsets of a metric space $(\mathbb{U}, \mathfrak{p})$. A multi-valued mapping $\mathbb{F} : \mathbb{D} \rightarrow 2^{\mathbb{E}} \setminus \emptyset$ is called α -proximal admissible if there exists a mapping $\alpha : \mathbb{D} \times \mathbb{D} \rightarrow [0, \infty)$ such that

$$\left. \begin{array}{l} \alpha(x_1, y_1) \geq 1 \\ \mathfrak{p}(u_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(u_2, y_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(u_1, u_2) \geq 1,$$

where $x_1, x_2, u_1, u_2 \in \mathbb{D}, y_1 \in \mathbb{F}x_1$ and $y_2 \in \mathbb{F}x_2$.

Let Ψ denote the class of auxiliary functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy the following conditions:

- a). ψ is nondecreasing;
- b). ψ is continuous;
- c). $\psi(t) = 0 \Leftrightarrow t = 0$.

Asadi et al.[7] introduced the following two definitions.

Definition 2.3. Let $(\mathbb{U}, \mathfrak{p})$ be a generalized metric space, and let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function. A map $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ is called an α - ψ -Geraghty contraction mapping if there exists $\theta \in \Theta$ such that, for all $x, y \in \mathbb{U}$, $\alpha(x, y)\psi(\mathfrak{p}(\mathbb{F}x, \mathbb{F}y)) \leq \theta(\psi(\mathfrak{p}(x, y)))\psi(\mathfrak{p}(x, y))$, where $\psi \in \Psi$, \mathbb{R} = set of real numbers.

Remark 2.4. If we take $\psi(t) = t$ in Definition 2.3, then \mathbb{F} is called α -Geraghty contraction mapping. Again, if we take $\alpha(x, y) = 1$ for all $x, y \in \mathbb{U}$ in Definition 2.3, then \mathbb{F} is called a ψ -Geraghty contraction mapping.

Definition 2.5. Let $(\mathbb{U}, \mathfrak{p})$ be a generalized metric space, and let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function. A map $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ is called a generalized α - ψ -Geraghty contraction mapping if there exists $\theta \in \Theta$ such that, for all $x, y \in \mathbb{U}$, $\alpha(x, y)\psi(\mathfrak{p}(\mathbb{F}x, \mathbb{F}y)) \leq \theta(\psi(\mathcal{L}(x, y)))\psi(\mathcal{L}(x, y))$, where $\psi \in \Psi$ and $\mathcal{L}(x, y) = \max\{\mathfrak{p}(x, y), \mathfrak{p}(x, \mathbb{F}x), \mathfrak{p}(y, \mathbb{F}y)\}$.

In [7], Asadi, Karapinar, and Kumar obtained fixed point results for those mappings defined by Definition 2.3 and Definition 2.5. See [7] for more details.

The following lemma was presented in [16].

Lemma 2.6. Suppose $(\mathbb{U}, \mathfrak{p})$ is a generalized metric space and $\{x_n\}$ is a sequence in \mathbb{U} such that $\mathfrak{p}(x_n, x_{n+1}) \rightarrow 0$ and $\mathfrak{p}(x_n, x_{n+2}) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $\mathfrak{p}(x_{m_k}, x_{n_k}) \geq \varepsilon$, $\mathfrak{p}(x_{m_k-1}, x_{n_k}) < \varepsilon$ and

- i). $\lim_{k \rightarrow \infty} \mathfrak{p}(x_{m_k-1}, x_{n_k+1}) = \varepsilon$.
- ii). $\lim_{k \rightarrow \infty} \mathfrak{p}(x_{m_k}, x_{n_k}) = \varepsilon$.
- iii). $\lim_{k \rightarrow \infty} \mathfrak{p}(x_{m_k-1}, x_{n_k}) = \varepsilon$.
- iv). $\lim_{k \rightarrow \infty} \mathfrak{p}(x_{m_k}, x_{n_k+1}) = \varepsilon$.

Definition 2.7. Let $\mathbb{F} : \mathbb{D} \rightarrow \mathcal{C}\mathcal{L}(\mathbb{E})$ be a multimap. An element $x^* \in \mathbb{D}_0$ is said to be the best proximity point of a multi-valued nonself mapping \mathbb{F} if $\mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, for some $y^* \in \mathbb{F}x^*$.

In this paper, inspired and motivated by Asadi et al. [7], we establish best proximity point results for generalized α - ψ -Geraghty proximal contraction multimaps in the setting of generalized metric spaces. Our results extend and generalize the results presented by Asadi et al.[7] as well as many comparable results obtained in the literature.

3. MAIN RESULTS

In this section, we introduce the following definitions.

Definition 3.1. Let \mathbb{D}, \mathbb{E} be two non-empty subsets of a generalized metric space $(\mathbb{U}, \mathfrak{p})$, and let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function. A nonself-map $\mathbb{F} : \mathbb{D} \rightarrow 2^{\mathbb{E}} \setminus \emptyset$ is called a generalized α - ψ -Geraghty proximal contraction multimap if there exists $\theta \in \Theta$ such that, for all $x, y \in \mathbb{D}$,

$$\alpha(x, y)\psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x, \mathbb{F}y)) \leq \theta(\psi(\mathcal{L}(x, y)))\psi(\mathcal{L}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \quad (3.1)$$

where $\mathcal{L}(x, y) = \max\{\mathfrak{p}(x, y), \mathfrak{p}(x, \mathbb{F}x), \mathfrak{p}(y, \mathbb{F}y)\}$.

Remark 3.2. If we take $\mathbb{D} = \mathbb{E} = \mathbb{U}$ and $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ in Definition 3.1, then \mathbb{F} is called a generalized α - ψ -Geraghty contraction mapping (see Definition 2.5). If we take $\mathbb{D} = \mathbb{E} = \mathbb{U}$ and $\mathcal{L}(x, y) = \mathfrak{p}(x, y)$ for all $x, y \in \mathbb{U}$ and $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ in Definition 3.1, then \mathbb{F} is called an α - ψ -Geraghty contraction mapping (see Definition 2.3). Again, if we take $\mathbb{D} = \mathbb{E} = \mathbb{U}$, $\alpha(x, y) = 1$, $\mathcal{L}(x, y) = \mathfrak{p}(x, y)$ for all $x, y \in \mathbb{U}$ and $\mathbb{F} : \mathbb{D} \rightarrow \mathbb{E}$ in Definition 3.1, then \mathbb{F} is called a ψ -Geraghty contraction mapping.

Definition 3.3. Let \mathbb{D} and \mathbb{E} be two non-empty subsets of $(\mathbb{U}, \mathfrak{p})$ and $\mathbb{D}_0 \neq \emptyset$. We say that the pair (\mathbb{D}, \mathbb{E}) has weak P -property if, for all $x_1, x_2 \in \mathbb{D}$ and $y_1, y_2 \in 2^{\mathbb{E}}$,

$$\left. \begin{array}{l} \mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(x_2, y_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \mathfrak{p}(x_1, x_2) \leq d(y_1, y_2),$$

Following Jleli and Samet [14], we present the following useful definition.

Definition 3.4. Let \mathbb{D} and \mathbb{E} be two non-empty subsets of $(\mathbb{U}, \mathfrak{p})$ and $\alpha : \mathbb{D} \times \mathbb{D} \rightarrow [0, \infty)$ be a function. We say that a nonself-mapping $\mathbb{F} : \mathbb{D} \rightarrow 2^{\mathbb{E}} \setminus \emptyset$ is said to be α -proximal admissible if, for all $x, y, u, v \in \mathbb{D}$,

$$\left. \begin{array}{l} \alpha(x, y) \geq 1 \\ \mathfrak{p}(u, w_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(v, w_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(u, v) \geq 1.$$

where $w_1 \in \mathbb{F}x$ and $w_2 \in \mathbb{F}y$.

Following Kumam et al. [21], we introduce the following definition.

Definition 3.5. Let \mathbb{D} and \mathbb{E} be two non-empty subsets of $(\mathbb{U}, \mathfrak{p})$ and $\alpha : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}$ be a function. We say that a nonself-mapping $\mathbb{F} : \mathbb{D} \rightarrow 2^{\mathbb{E}} \setminus \emptyset$ is said to be triangular α -proximal admissible if, for all $x, y, z, x_1, x_2, u_1, u_2 \in \mathbb{D}$, and $v_1 \in \mathbb{F}x_1, v_2 \in \mathbb{F}x_2$,

(T1).

$$\left. \begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ \mathfrak{p}(u_1, v_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(u_2, v_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(u_1, u_2) \geq 1,$$

(T2).

$$\left. \begin{array}{l} \alpha(x, z) \geq 1 \\ \alpha(z, y) \geq 1 \end{array} \right\} \Rightarrow \alpha(x, y) \geq 1.$$

Lemma 3.6. Let $\mathbb{F} : \mathbb{D} \rightarrow \mathcal{CL}(E)$ be a triangular α -admissible map. Assume that there exists $x_1 \in \mathbb{U}$ such that $\alpha(x_1, y_1) \geq 1$, where $y_1 \in \mathbb{F}x_1$. Define a sequence $\{x_n\}$ by $x_{n+1} = \mathbb{F}x_n$. Then, $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.

Proof. Define a sequence $\{x_n\}$ by $x_{n+1} = \mathbb{F}x_n$. Since \mathbb{F} is a triangular α -proximal admissible map, for all $x_1, x_2 \in \mathbb{D}$, and we obtain by (T1) that

$$\left. \begin{array}{l} \mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(x_2, y_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(x_1, x_2) \geq 1,$$

where $y_1 \in \mathbb{F}x_1$ and $y_2 \in \mathbb{F}x_2$. By induction, we have

$$\left. \begin{array}{l} \mathfrak{p}(x_n, y_n) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(x_{n+1}, y_{n+1}) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(x_n, x_{n+1}) \geq 1, \quad (3.2)$$

where $x_n, x_{n+1} \in \mathbb{D}$ and $y_n \in \mathbb{F}x_n, y_{n+1} \in \mathbb{F}x_{n+1}$, and

$$\left. \begin{array}{l} \mathfrak{p}(x_{n+1}, y_{n+1}) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(x_m, y_m) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(x_{n+1}, x_m) \geq 1, \quad (3.3)$$

where $x_{n+1}, x_m \in \mathbb{D}$ and $y_{n+1} \in \mathbb{F}x_{n+1}, y_m \in \mathbb{F}x_m$. From (3.2) and (3.3), we get by (T_2) that $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$. \square

Theorem 3.7. *Let \mathbb{D}, \mathbb{E} two non-empty subsets of a complete generalized metric space $(\mathbb{U}, \mathfrak{p})$. Let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function, and let $\mathbb{F} : \mathbb{D} \rightarrow \mathfrak{C}\mathcal{L}(\mathbb{E})$ be a generalized α - ψ -Geraghty proximal contraction multimap (Definition 3.1). Suppose that the following conditions are satisfied:*

- 1). $\mathbb{F}(\mathbb{D}_0) \subseteq \mathbb{E}_0$ and the pair (\mathbb{D}, \mathbb{E}) satisfies the weak P-property;
- 2). \mathbb{F} is triangular α -proximal admissible;
- 3). \mathbb{F} is continuous;
- 4). there exist $x_0, x_1 \in \mathbb{D}$ and $y_1 \in \mathbb{F}x_0$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) \geq 1$.

Then \mathbb{F} has a unique best proximity x^* in \mathbb{D}_0 , (i.e., $\mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, for some $y^* \in \mathbb{F}x^*$).

Proof. By condition (4), we construct the sequence $\{x_n\}$ as $x_{n+1} = \mathbb{F}x_n$, and there exist $x_0, x_1 \in \mathbb{D}$ and $y_1 \in \mathbb{F}x_0$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) \geq 1$. Since $\mathbb{F}(\mathbb{D}_0) \subseteq \mathbb{E}_0$, we see that there exists $y_2 \in \mathbb{F}x_1 \subseteq \mathbb{E}_0$. Hence, by definition of \mathbb{E}_0 , there exists $x_2 \in \mathbb{D}$ such that $\mathfrak{p}(x_2, y_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$. Since \mathbb{F} is α -proximal admissible, we obtain $\alpha(x_1, x_2) \geq 1$. By continuing this process, there exists $y_{n+1} \in \mathbb{F}x_n$ such that

$$\mathfrak{p}(x_{n+1}, y_{n+1}) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \quad (3.4)$$

for all $n \in \mathbb{N}$. Now, by (3.1), we have

$$\alpha(x_n, x_{n+1}) \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_n, \mathbb{F}x_{n+1})) \leq \theta(\psi(\mathcal{L}(x_n, x_{n+1}))) \psi(\mathcal{L}(x_n, x_{n+1}) - \mathfrak{p}(\mathbb{D}, \mathbb{E})),$$

where

$$\begin{aligned} \mathcal{L}(x_n, x_{n+1}) &= \max\{\mathfrak{p}(x_n, x_{n+1}), \mathfrak{p}(x_n, \mathbb{F}x_n), \mathfrak{p}(x_{n+1}, \mathbb{F}x_{n+1})\} \\ &= \max\{\mathfrak{p}(x_n, x_{n+1}), \mathfrak{p}(x_{n+1}, x_{n+2})\}. \end{aligned} \quad (3.5)$$

Suppose, for some $n_0 \in \mathbb{N}$, that $x_{n_0} = x_{n_0+1}$. If $x_{n_0+1} \neq x_{n_0+2}$, then it follows by (3.5) that

$$\begin{aligned} \psi(\mathfrak{p}(x_{n_0+1}, x_{n_0+2})) &\leq \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n_0}, \mathbb{F}x_{n_0+1})) \\ &\leq \alpha(x_{n_0}, x_{n_0+1}) \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n_0}, \mathbb{F}x_{n_0+1})) \\ &\leq \theta(\psi(\mathcal{L}(x_{n_0}, x_{n_0+1}))) \psi(\mathcal{L}(x_{n_0}, x_{n_0+1}) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \text{ (since } \theta \in \Theta) \\ &< \psi(\mathcal{L}(x_{n_0}, x_{n_0+1}) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\ &= \psi(\max\{\mathfrak{p}(x_{n_0}, x_{n_0+1}), d(x_{n_0+1}, x_{n_0+2})\} - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\ &\leq \psi(\mathfrak{p}(x_{n_0+1}, x_{n_0+2})), \end{aligned}$$

which is a contradiction. Therefore $x_{n_0+1} = x_{n_0+2}$, and $x_{n_0} = x_{n_0+1} = x_{n_0+2}$. From (3.4), there exists $y_{n_0+1} \in \mathbb{F}x_{n_0}$ such that $d(x_{n_0+1}, y_{n_0+1}) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, i.e., $x_{n_0} = x_{n_0+1}$ is a best proximity point of \mathbb{F} , a desired result. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. From

(3.1), we obtain

$$\begin{aligned}
\psi(\mathfrak{p}(x_n, x_{n+1})) &\leq \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n-1}, \mathbb{F}x_n)) \\
&\leq \alpha(x_{n-1}, x_n) \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n-1}, \mathbb{F}x_n)) \\
&\leq \theta(\psi(\mathcal{Z}(x_{n-1}, x_n))) \psi(\mathcal{Z}(x_{n-1}, x_n) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\
&< \psi(\mathcal{Z}(x_{n-1}, x_n) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \text{ (since } \theta \in \Theta) \\
&\leq \psi(\max\{\mathfrak{p}(x_{n-1}, x_n), \mathfrak{p}(x_n, x_{n+1})\}).
\end{aligned} \tag{3.6}$$

We consider $\max\{\mathfrak{p}(x_{n-1}, x_n), \mathfrak{p}(x_n, x_{n+1})\}$. If $\max\{\mathfrak{p}(x_{n-1}, x_n), \mathfrak{p}(x_n, x_{n+1})\} = \mathfrak{p}(x_n, x_{n+1})$, then, we get by (3.6) that $\psi(\mathfrak{p}(x_n, x_{n+1})) < \psi(\mathfrak{p}(x_n, x_{n+1}))$, which is a contradiction. Hence

$$\max\{\mathfrak{p}(x_{n-1}, x_n), \mathfrak{p}(x_n, x_{n+1})\} = \mathfrak{p}(x_{n-1}, x_n).$$

By (3.6), we have $\psi(\mathfrak{p}(x_n, x_{n+1})) < \psi(\mathfrak{p}(x_{n-1}, x_n))$. Since ψ is non-decreasing, it follows that $\mathfrak{p}(x_n, x_{n+1}) < \mathfrak{p}(x_{n-1}, x_n)$, for all $n \geq 1$. Hence, we deduce that $\{\mathfrak{p}(x_n, x_{n+1})\}$ is a decreasing sequence of non-negative real numbers. So there exists $\zeta \geq 0$ such that $\lim_{n \rightarrow \infty} \mathfrak{p}(x_n, x_{n+1}) = \zeta$. Suppose that $\zeta > 0$. From (3.6), we have

$$0 < \frac{\psi(\mathfrak{p}(x_n, x_{n+1}))}{\psi(\mathfrak{p}(x_{n-1}, x_n))} \leq \theta(\psi(\mathcal{Z}(x_{n-1}, x_n))) < 1, \tag{3.7}$$

where

$$\begin{aligned}
\mathcal{Z}(x_{n-1}, x_n) &= \max\{\mathfrak{p}(x_{n-1}, x_n), \mathfrak{p}(x_{n-1}, \mathbb{F}x_{n-1}), \mathfrak{p}(x_n, \mathbb{F}x_n)\} \\
&= \max\{\mathfrak{p}(x_{n-1}, x_n), \mathfrak{p}(x_n, x_{n+1})\} \\
&= \mathfrak{p}(x_{n-1}, x_n).
\end{aligned} \tag{3.8}$$

Taking the limit as $n \rightarrow \infty$ in (3.7) and using (3.8), we obtain $\lim_{n \rightarrow \infty} \theta(\psi(\mathfrak{p}(x_{n-1}, x_n))) = 1$. Since $\theta \in \Theta$, it follows that $\lim_{n \rightarrow \infty} \psi(\mathfrak{p}(x_{n-1}, x_n)) = 0$. By using the continuity of ψ , we arrive at $\psi(\lim_{n \rightarrow \infty} \mathfrak{p}(x_{n-1}, x_n)) = 0$, i.e., $\psi(\zeta) = 0$, so that $\zeta = 0$. That is,

$$\lim_{n \rightarrow \infty} \mathfrak{p}(x_{n-1}, x_n) = 0. \tag{3.9}$$

Hence, all elements of the sequence $\{x_n\}$ are distinct.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Suppose $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$\mathfrak{p}(x_{m_k}, x_{n_k}) \geq \varepsilon \quad \text{and} \quad \mathfrak{p}(x_{m_k-1}, x_{n_k}) < \varepsilon. \tag{3.10}$$

Since \mathbb{F} is triangular α -proximal admissible, we obtain by Lemma 3.6 that

$$\alpha(x_n, x_m) \geq 1. \tag{3.11}$$

for all $n, m \in \mathbb{N}$ with $n < m$. We next prove that $\alpha(x_{n_k}, x_{m_k}) \geq 1$ with $n_k < m_k$. From (3.4), we have

$$\alpha(x_m, x_{m+1}) \geq 1. \tag{3.12}$$

Since T is triangular α -proximal admissible, from (3.11) and (3.12) together with (T_2) , we get $\alpha(x_n, x_{m+1}) \geq 1$ for all $n, m \in \mathbb{N}$ with $n < m$. Hence, for any $m_k, n_k \in \mathbb{N}$ with $n_k < m_k$, we get

$\alpha(x_{n_k}, x_{m_k}) \geq 1$. From (3.9) and (3.10) and by the quadrilateral inequality, we can choose a positive integer $k \in \mathbb{N}$ such that

$$\begin{aligned} \mathfrak{p}(x_{n_k}, x_{m_k}) &\leq \mathfrak{p}(x_{n_k}, x_{n_{k+1}}) + \mathfrak{p}(x_{n_{k+1}}, x_{m_{k+1}}) + \mathfrak{p}(x_{m_{k+1}}, x_{m_k}) \\ &\leq \mathfrak{p}(x_{n_k}, x_{n_{k+1}}) + \mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n_k}, \mathbb{F}x_{m_k}) + \mathfrak{p}(x_{m_{k+1}}, x_{m_k}). \end{aligned}$$

Equivalently,

$$\mathfrak{p}(x_{n_k}, x_{m_k}) - \mathfrak{p}(x_{n_k}, x_{n_{k+1}}) - \mathfrak{p}(x_{m_{k+1}}, x_{m_k}) \leq \mathcal{H}_{\mathfrak{p}}(Tx_{n_k}, \mathbb{F}x_{m_k}).$$

By applying ψ , we see that

$$\begin{aligned} \psi(\mathfrak{p}(x_{n_k}, x_{m_k}) - \mathfrak{p}(x_{n_k}, x_{n_{k+1}}) - \mathfrak{p}(x_{m_{k+1}}, x_{m_k})) &\leq \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n_k}, \mathbb{F}x_{m_k})) \\ &\leq \alpha(x_{n_k}, x_{m_k}) \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_{n_k}, \mathbb{F}x_{m_k})) \\ &\leq \theta(\psi(\mathcal{L}(x_{n_k}, x_{m_k}))) \psi(\mathcal{L}(x_{n_k}, x_{m_k}) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &\mathcal{L}(x_{n_k}, x_{m_k}) - \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ &= \max\{\mathfrak{p}(x_{n_k}, x_{m_k}), \mathfrak{p}(x_{n_k}, \mathbb{F}x_{n_k}), \mathfrak{p}(x_{m_k}, \mathbb{F}x_{m_k})\} - \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ &= \max\{\mathfrak{p}(x_{n_k}, x_{m_k}), \mathfrak{p}(x_{n_k}, x_{n_{k+1}}), \mathfrak{p}(x_{m_k}, x_{m_{k+1}})\} - \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ &\leq \max\{\mathfrak{p}(x_{n_k}, x_{m_k}), \mathfrak{p}(x_{n_k}, x_{n_{k+1}}), \mathfrak{p}(x_{m_k}, x_{m_{k+1}})\}. \end{aligned}$$

Applying Lemma 2.6, we obtain

$$\lim_{k \rightarrow \infty} \mathcal{L}(x_{n_k}, x_{m_k}) - \mathfrak{p}(\mathbb{D}, \mathbb{E}) \leq \lim_{k \rightarrow \infty} \max\{\mathfrak{p}(x_{n_k}, x_{m_k}), \mathfrak{p}(x_{n_k}, x_{n_{k+1}}), \mathfrak{p}(x_{m_k}, x_{m_{k+1}})\} = \varepsilon.$$

Taking the limit as $k \rightarrow \infty$ in (3.13) and using (3.9) yield

$$0 < \frac{\psi(\varepsilon)}{\psi(\varepsilon)} \leq \lim_{k \rightarrow \infty} \theta(\psi(\mathcal{L}(x_{n_k}, x_{m_k}))) \leq 1.$$

Therefore, $\lim_{k \rightarrow \infty} \theta(\psi(\mathcal{L}(x_{n_k}, x_{m_k}))) = 1$. Since $\theta \in \Theta$, we obtain $\lim_{k \rightarrow \infty} \psi(\mathcal{L}(x_{n_k}, x_{m_k})) = 0$, or $\lim_{k \rightarrow \infty} \psi(\max\{\mathfrak{p}(x_{n_k}, x_{m_k}), \mathfrak{p}(x_{n_k}, x_{n_{k+1}}), \mathfrak{p}(x_{m_k}, x_{m_{k+1}})\}) = 0$. By the continuity of ψ , this yields

$$\psi\left(\lim_{k \rightarrow \infty} \max\{\mathfrak{p}(x_{m_k}, x_{n_k}), \mathfrak{p}(x_{m_k}, x_{m_{k+1}}), \mathfrak{p}(x_{n_k}, x_{n_{k+1}})\}\right) = 0,$$

i.e., $\psi(\varepsilon) = 0$ and hence $\varepsilon = 0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in \mathbb{D}_0 . Similarly, one can show that $\{y_n\}$ is a Cauchy sequences in \mathbb{E}_0 . By the completeness of \mathbb{U} and the closed property of \mathbb{D}_0 and \mathbb{E}_0 , one obtains that there exist $x^* \in \mathbb{D}_0$ and $y^* \in \mathbb{E}_0$ such that $\lim_{n \rightarrow \infty} x_n = x^*$ and $\lim_{n \rightarrow \infty} y_n = y^*$. Since \mathbb{F} is continuous, one sees from (3.4) that there exists $y^* \in \mathbb{F}x^*$, we obtain

$$\mathfrak{p}(\mathbb{D}, \mathbb{E}) \leq \lim_{n \rightarrow \infty} \mathfrak{p}(x_{n+1}, \mathbb{F}x_n) = \mathfrak{p}(x^*, \mathbb{F}x^*) \leq \mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E}).$$

That is, $\mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, where $y^* \in \mathbb{F}x^*$. Hence x^* is the best proximity of \mathbb{F} in \mathbb{D}_0 .

Next, we show the uniqueness of the best proximity point. Suppose that x_1^* and x_2^* are the two distinct best proximity points of \mathbb{F} . Since $\mathfrak{p}(x_1^*, y_1^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\mathfrak{p}(x_2^*, y_2^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, where $y_1^* \in \mathbb{F}x_1^*$ and $y_2^* \in \mathbb{F}x_2^*$. By the weak P -property of the pair (\mathbb{D}, \mathbb{E}) , we get

$$\mathfrak{p}(x_1^*, x_2^*) \leq \mathfrak{p}(y_1^*, y_2^*) \leq \mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_1^*, \mathbb{F}x_2^*).$$

It follows by (3.1) that

$$\begin{aligned}
\psi(\mathfrak{p}(x_1^*, x_2^*)) &\leq \alpha(x_1^*, x_2^*) \psi(\mathcal{H}_{\mathfrak{p}}(\mathbb{F}x_1^*, \mathbb{F}x_2^*)) \\
&\leq \theta(\psi(\mathcal{Z}(x_1^*, x_2^*))) \psi(\mathcal{Z}(x_1^*, x_2^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\
&< \psi(\mathcal{Z}(x_1^*, x_2^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \text{ (since } \theta \in \Theta \text{)} \\
&= \psi(\max\{\mathfrak{p}(x_1^*, x_2^*), \mathfrak{p}(x_1^*, \mathbb{F}x_1^*), \mathfrak{p}(x_2^*, \mathbb{F}x_2^*)\} - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\
&\leq \psi(\max\{\mathfrak{p}(x_1^*, x_2^*), \mathfrak{p}(x_1^*, y_1^*), \mathfrak{p}(x_2^*, y_2^*)\}) \\
&= \psi(\mathfrak{p}(x_1^*, x_2^*)).
\end{aligned}$$

Therefore, $\mathfrak{p}(x_1^*, x_2^*) < \mathfrak{p}(x_1^*, x_2^*)$. This is a contradiction. Hence $x_1^* = x_2^*$. This completes our proof. \square

In support of our main result, we give the following illustrative example.

Example 3.8. Let $\mathbb{U} = \{m, n, o, p, q\}$. Let $\mathbb{D} = \{m, n, o\}$ and $\mathbb{E} = \{p, q\}$. Define $\mathfrak{p} : \mathbb{U} \times \mathbb{U} \rightarrow [0, \infty)$ by

$$\begin{aligned}
\mathfrak{p}(m, n) &= 0.4, \quad \mathfrak{p}(m, o) = \mathfrak{p}(n, o) = 1, \quad \mathfrak{p}(n, p) = 2, \quad \mathfrak{p}(p, q) = 0.10, \\
\mathfrak{p}(o, p) &= \mathfrak{p}(o, q) = \mathfrak{p}(m, q) = \mathfrak{p}(n, q) = 0.15, \quad \mathfrak{p}(m, p) = 0.12,
\end{aligned}$$

and

$$\mathfrak{p}(x, x) = 0, \quad \mathfrak{p} = d(y, x), \text{ for all } x, y \in \mathbb{U}.$$

Then it is easy to check that $(\mathbb{U}, \mathfrak{p})$ is a generalized metric space which is not a standard metric space. For

$$\mathfrak{p}(m, o) = 1 \not\leq \mathfrak{p}(m, p) + \mathfrak{p}(p, o) = 0.12 + 0.15 = 0.27.$$

Define $\mathbb{F} : \mathbb{D} \rightarrow CL(\mathbb{E})$ by

$$\mathbb{F}x = \begin{cases} \{p\}, & \text{if } x \in \{m, n\} \\ \{q\}, & \text{otherwise,} \end{cases}$$

and $\alpha : \mathbb{D} \times \mathbb{D} \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in \mathbb{U} \setminus \{o\} \\ \frac{1}{4}, & \text{if } x \text{ or } y = o. \end{cases}$$

We next prove that

- 1). \mathbb{F} is triangular α -proximal admissable;
- 2). there exist $x_0, x_1 \in \mathbb{D}$ and $y_1 \in \mathbb{F}x_1$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) \geq 1$.

Let $\theta(t) = \frac{1}{1+t}$ and $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. Notice that $\mathbb{D}_0 = \mathbb{D}, \mathbb{E}_0 = \mathbb{E}$ and $\mathbb{F}x \subseteq \mathbb{E}_0$ for all $x \in \mathbb{D}_0$. Clearly, $\mathfrak{p}(\mathbb{D}, \mathbb{E}) = 0.12$. Also, the pair (\mathbb{D}, \mathbb{E}) satisfies the weak P -property.

Proof. We show

- 1). (i) Let $x, y \in \mathbb{U}$ such that $\alpha(x, y) \geq 1$. Then, by the definition of α , we have

$$\left. \begin{aligned} \alpha(x, y) &\geq 1 \\ \mathfrak{p}(u, u') &= \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(v, v') &= \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{aligned} \right\} \Rightarrow \alpha(u, v) \geq 1.$$

where $u' \in \mathbb{F}x$ and $v' \in \mathbb{F}y$.

(ii) Let $x, y, z \in \mathbb{D}$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. Again, by the definition of α and $x, y, z \in \mathbb{D}$, $\alpha(x, z) \geq 1$.

Thus (i) and (ii) imply that \mathbb{F} is triangular α -proximal admissible.

2). Let $x_0, x_1 \in \mathbb{D}$. Then $\mathbb{F}x_0, \mathbb{F}x_1 \subseteq \mathbb{E}$. Consider $y_0 \in \mathbb{F}x_0, y_1 \in \mathbb{F}x_1$ and $u_1, u_2 \in \mathbb{D}$ such that $\mathfrak{p}(u_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\mathfrak{p}(u_2, y_2) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$. Then $u_1, u_2 \in \mathbb{D}$ and $\alpha(x_0, x_1) = 1$.

For $x_0 = n \in \mathbb{D}_0$ and $y_1 = p \in \mathbb{F}x_0 \subseteq \mathbb{E}_0$, we have $x_1 = m \in \mathbb{D}_0$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(m, p) = 0.12 = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) = \alpha(n, m) = 1$. Equivalently, taking $x_0 = n, x_1 = m$, we have $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(m, p) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) = 0.12$ and $\alpha(x_0, x_1) = \alpha(n, m) = 1$, where $y_1 = p \in \mathbb{F}x_0 = \mathbb{F}n$.

Next, we show that

$$\theta(\psi(\mathcal{Z}(x, y)))\psi(\mathcal{Z}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) - \alpha(x, y)\psi(\mathcal{H}_p(\mathbb{F}x, \mathbb{F}y)) \geq 0,$$

for all $x, y \in A$ and $\theta \in \Theta$.

1). If $x = m, y = n$, then

$$\begin{aligned} & \theta(\psi(\mathcal{Z}(m, n)))\psi(\mathcal{Z}(m, n) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) - \alpha(m, n)\psi(\mathcal{H}_p(\mathbb{F}m, \mathbb{F}n)) \\ &= \theta(\psi(\max\{\mathfrak{p}(m, n), \mathfrak{p}(m, p), \mathfrak{p}(n, p)\}))\psi(\max\{\mathfrak{p}(m, n), \mathfrak{p}(m, p), \mathfrak{p}(n, p)\} - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\ &\quad - \alpha(m, n)\psi(H(\{p\}, \{p\})) \\ &= \theta(\psi(\max\{0.4, 0.12, 2\}))\psi(\max\{0.4, 0.12, 2\} - 0.12) - 0 \\ &= \theta(\psi(2))\psi(1.88) = \theta(1)(0.94) = \left(\frac{1}{2}\right)(0.94) = 0.47 > 0. \end{aligned}$$

2). If $x = m, y = o$, then

$$\begin{aligned} & \theta(\psi(\mathcal{Z}(m, o)))\psi(\mathcal{Z}(m, o) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) - \alpha(m, o)\psi(\mathcal{H}_p(\mathbb{F}m, \mathbb{F}o)) \\ &= \theta(\psi(\max\{\mathfrak{p}(m, o), \mathfrak{p}(m, p), \mathfrak{p}(o, q)\}))\psi(\max\{\mathfrak{p}(m, o), \mathfrak{p}(m, p), \mathfrak{p}(o, q)\} - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\ &\quad - \alpha(m, o)\psi(H(\{p\}, \{q\})) \\ &= \theta(\psi(\max\{1, 0.12, 0.15\}))\psi(\max\{1, 0.12, 0.15\} - 0.12) - \frac{1}{4}(0.10) \\ &= \theta(\psi(1))\psi(0.88) - 0.025 = \theta(0.5)(0.44) - 0.025 \\ &= \left(\frac{1}{1.5}\right)(0.44) - 0.025 = 0.268 > 0. \end{aligned}$$

3). If $x = n, y = o$, then

$$\begin{aligned} & \theta(\psi(\mathcal{Z}(n, o)))\psi(\mathcal{Z}(n, o) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) - \alpha(n, o)\psi(\mathcal{H}_p(\mathbb{F}n, \mathbb{F}o)) \\ &= \theta(\psi(\max\{\mathfrak{p}(n, o), \mathfrak{p}(n, p), \mathfrak{p}(o, q)\}))\psi(\max\{\mathfrak{p}(n, o), \mathfrak{p}(n, p), \mathfrak{p}(o, q)\} - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\ &\quad - \alpha(n, o)\psi(H(\{p\}, \{q\})) \\ &= \theta(\psi(\max\{1, 2, 1.5\}))\psi(\max\{1, 2, 1.5\} - 0.12) - \frac{1}{4}(0.10) \\ &= \theta(\psi(2))\psi(1.88) - 0.025 = \theta(1)(0.94) - 0.025 \\ &= \left(\frac{1}{2}\right)(0.94) - 0.025 = 0.445 > 0. \end{aligned}$$

For all other cases, one can see that

$$\theta(\psi(\mathcal{Z}(x, y)))\psi(\mathcal{Z}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) - \alpha(x, y)\psi(\mathcal{H}_p(\mathbb{F}x, \mathbb{F}y)) \geq 0,$$

for all $x, y \in \mathbb{D}$. This means that

$$\alpha(x, y)\psi(\mathcal{H}_p(\mathbb{F}x, \mathbb{F}y)) \leq \theta(\psi(\mathcal{Z}(x, y)))\psi(\mathcal{Z}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})),$$

for all $x, y \in \mathbb{D}$. Hence, \mathbb{F} is a generalized α - ψ -Geraghty proximal contraction multimap. Therefore, all the conditions of Theorem 3.7 hold and \mathbb{F} has the best proximity point in \mathbb{D}_0 . Here m is the unique best proximity point of \mathbb{F} in \mathbb{D}_0 . \square

If we take $\psi(t) = t$ in Definition 3.1, we get the following definition.

Definition 3.9. Let \mathbb{D}, \mathbb{E} be two non-empty subsets of a generalized metric space $(\mathbb{U}, \mathfrak{p})$, and let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function. A nonself-map $\mathbb{F} : \mathbb{D} \rightarrow 2^{\mathbb{E}} \setminus \emptyset$ is called a generalized α -Geraghty proximal contraction multimap, if there exists $\theta \in \Theta$ such that, for all $x, y \in \mathbb{D}$,

$$\alpha(x, y)\psi(\mathcal{H}_p(\mathbb{F}x, \mathbb{F}y)) \leq \theta(\mathcal{Z}(x, y))(\mathcal{Z}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \quad (3.14)$$

where $\mathcal{Z}(x, y) = \max\{\mathfrak{p}(x, y), \mathfrak{p}(x, \mathbb{F}x), \mathfrak{p}(y, \mathbb{F}y)\}$.

We have the following corollary.

Corollary 3.10. Let \mathbb{D}, \mathbb{E} two non-empty subsets of a complete generalized metric space $(\mathbb{U}, \mathfrak{p})$. Let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function, and let $\mathbb{F} : \mathbb{D} \rightarrow \mathcal{CL}(\mathbb{E})$ be a generalized α -Geraghty proximal contraction map (3.14). Suppose that the following conditions are satisfied:

- 1). $\mathbb{F}(\mathbb{D}_0) \subseteq \mathbb{E}_0$ and the pair (\mathbb{D}, \mathbb{E}) satisfies the weak P-property;
- 2). \mathbb{F} is triangular α -proximal admissible;
- 3). \mathbb{F} is continuous;
- 4). there exist $x_0, x_1 \in \mathbb{D}$ and $y_1 \in \mathbb{F}x_0$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) \geq 1$.

Then \mathbb{F} has a unique best proximity in \mathbb{D}_0 .

Proof. By taking $\psi(\mathcal{Z}(x, y)) = \mathcal{Z}(x, y)$, i.e., ψ is an identity map, and applying Theorem 3.7, we can obtain our result immediately. \square

It is also interesting to remove the continuity of \mathbb{F} by replacing a weaker condition in the above theorem. The following definition is known.

Definition 3.11. Let $(\mathbb{U}, \mathfrak{p})$ be a complete generalized metric space, and $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function. We say that the sequence $\{x_n\}$ is said to be α -regular, if $\{x_n\}$ is a sequence in \mathbb{U} such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x \in \mathbb{U}$ as $n \rightarrow \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k .

Theorem 3.12. Let \mathbb{D}, \mathbb{E} two non-empty subsets of a complete generalized metric space $(\mathbb{U}, \mathfrak{p})$. Let $\alpha : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function, and let $\mathbb{F} : \mathbb{D} \rightarrow \mathcal{CL}(\mathbb{E})$ be a generalized α - ψ -Geraghty proximal contraction multimap (see Definition 3.1). Suppose that the following conditions are satisfied:

- 1). $\mathbb{F}(\mathbb{D}_0) \subseteq \mathbb{E}_0$ and the pair (\mathbb{D}, \mathbb{E}) satisfies the weak P-property;
- 2). \mathbb{F} is triangular α -proximal admissible;
- 3). $\{x_n\}$ is α -regular;
- 4). there exist $x_0, x_1 \in \mathbb{D}$ and $y_1 \in \mathbb{F}x_0$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) \geq 1$.

Then \mathbb{F} has a unique best proximity $x^* \in \mathbb{D}_0$, i.e. $\mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, for some $y^* \in \mathbb{F}x^*$.

Proof. From the proof of Theorem 3.7, we see that the sequence $\{x_n\}$ defined by $x_{n+1} = \mathbb{F}x_n$ for all $n \geq 0$ converges to some $x^* \in \mathbb{D}$, and $\{x_n\}, \{y_n\}$ are Cauchy sequences such that $x_n \rightarrow x^* \in \mathbb{D}_0$ and $y_n \rightarrow y^* \in \mathbb{E}_0$, as $n \rightarrow \infty$. We next prove that $\mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, for some $y^* \in \mathbb{F}x^*$. Suppose $\mathfrak{p}(x^*, y^*) \neq \mathfrak{p}(\mathbb{D}, \mathbb{E})$ for each $y^* \in \mathbb{F}x^*$. From the proof of Theorem 3.7, we have $\mathfrak{p}(x_{n+1}, x_{n+2}) \leq \mathfrak{p}(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cap \{0\}$. Since $\{x_n\} \in \mathbb{D}_0$ is α -regular, $x_n \rightarrow x^* \in \mathbb{D}_0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all k . Thus, from (3.1), it follows that

$$\begin{aligned} \psi(\mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*)) &\leq \alpha(x_{n_k}, x^*) \psi(\mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*)) \\ &\leq \theta(\psi(\mathcal{L}(x_{n_k}, x^*))) \psi(\mathcal{L}(x_{n_k}, x^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \\ &< \psi(\mathcal{L}(x_{n_k}, x^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})), \quad (\text{since } \theta \in \Theta) \\ &= \psi(\max\{\mathfrak{p}(x_{n_k}, x^*), \mathfrak{p}(x_{n_k}, \mathbb{F}x_{n_k}), \mathfrak{p}(x^*, \mathbb{F}x^*)\} - \mathfrak{p}(\mathbb{D}, \mathbb{E})). \end{aligned} \quad (3.15)$$

Taking the limit as $k \rightarrow \infty$ in inequality (3.15), we obtain

$$\psi\left(\lim_{k \rightarrow \infty} \mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*)\right) < \psi(\mathfrak{p}(x^*, \mathbb{F}x^*) - d(A, B)) \leq \psi(\mathfrak{p}(x^*, y^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})). \quad (3.16)$$

By the quadrilateral inequality, we have

$$\begin{aligned} \mathfrak{p}(x^*, y^*) &\leq \mathfrak{p}(x^*, x_{n_k+1}) + \mathfrak{p}(x_{n_k+1}, \mathbb{F}x_{n_k}) + \mathfrak{p}(\mathbb{F}x_{n_k}, y^*) \\ &\leq \mathfrak{p}(x^*, x_{n_k+1}) + \mathfrak{p}(x_{n_k+1}, x_{n_k+1}) + \mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*) \end{aligned}$$

or $\mathfrak{p}(x^*, y^*) - \mathfrak{p}(x_{n_k+1}, x_{n_k+1}) \leq \mathfrak{p}(x^*, x_{n_k+1}) + \mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*)$. From the property of ψ , it follows that

$$\psi(\mathfrak{p}(x^*, y^*) - \mathfrak{p}(x_{n_k+1}, x_{n_k+1})) \leq \psi(\mathfrak{p}(x^*, x_{n_k+1}) + \mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*)).$$

Taking the limit as $k \rightarrow \infty$, we obtain

$$\psi(\mathfrak{p}(x^*, y^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) \leq \psi(\mathfrak{p}(x^*, y^*)) \leq \psi\left(\lim_{k \rightarrow \infty} \mathcal{H}_p(\mathbb{F}x_{n_k}, \mathbb{F}x^*)\right). \quad (3.17)$$

From (3.16) and (3.17), we obtain

$$\psi(\mathfrak{p}(x^*, y^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) < \psi(\mathfrak{p}(x^*, y^*) - \mathfrak{p}(\mathbb{D}, \mathbb{E})),$$

which is a contradiction. Hence $\mathfrak{p}(x^*, y^*) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$, for some $y^* \in \mathbb{F}x^*$. Therefore, x^* is the best proximity point of \mathbb{F} . Uniqueness follows from the proof of Theorem 3.7. \square

We give the following illustrative example to support our main result.

Example 3.13. Let $\mathbb{U} = \{a, b, c, d, e\}$ and $\mathbb{D} = \{a, c, d\}, \mathbb{E} = \{b, e\}$. Let $\mathfrak{p} : \mathbb{U} \times \mathbb{U} \rightarrow \mathbb{R}$ be a function such that

$$\mathfrak{p}(a, b) = 3, \mathfrak{p}(a, c) = \mathfrak{p}(b, e) = \mathfrak{p}(c, e) = 0.20,$$

$$\mathfrak{p}(a, d) = \mathfrak{p}(a, e) = \mathfrak{p}(b, e) = \mathfrak{p}(c, d) = \mathfrak{p}(b, d) = \mathfrak{p}(d, c) = \mathfrak{p}(d, e) = \mathfrak{p}(b, c) = 2,$$

and

$$\mathfrak{p}(x, x) = 0, \mathfrak{p}(x, y) = \mathfrak{p}(y, x), \forall x, y \in \mathbb{U}.$$

Let $\mathbb{D}_0 = \{c\}, \mathbb{E}_0 = \{b, e\}$. It is easy to check that $(\mathbb{U}, \mathfrak{p})$ is a generalized metric space but not a standard metric space because it lacks the triangular inequality property.

$$\mathfrak{p}(a, b) = 3 \not\leq \mathfrak{p}(a, c) + \mathfrak{p}(c, b) = 0.20 + 2 = 2.20.$$

Let $\mathbb{F} : \mathbb{D} \rightarrow CL(\mathbb{E})$ be defined by

$$\mathbb{F}x = \begin{cases} \{b\}, & \text{if } x \in \{a, c\} \\ \{e\}, & \text{otherwise.} \end{cases}$$

Let $\theta(t) = \frac{1}{1+t}$ and $\psi(t) = \frac{t}{3}$. Let $\alpha : \mathbb{D} \times \mathbb{D} \rightarrow [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in \mathbb{D} \\ \frac{1}{4}, & \text{otherwise.} \end{cases}$$

Clearly, $\mathbb{F}(\mathbb{D}_0) \subseteq \mathbb{E}_0$, $\mathfrak{p}(\mathbb{D}, \mathbb{E}) = 0.20$ and the pair (\mathbb{D}, \mathbb{E}) satisfies the weak P -property.

We shall prove that

- 1). \mathbb{F} is triangular α -proximal admissible;
- 2). $\{x_n\}$ is α -regular;
- 3). there exist $x_0, x_1 \in \mathbb{D}$ and $y_1 \in \mathbb{F}x_0$ such that $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(\mathbb{D}, \mathbb{E})$ and $\alpha(x_0, x_1) \geq 1$.

Proof. To show

- 1). (i) Let $x, y, u, v \in \mathbb{D}$ such that $\alpha(x, y) \geq 1$. Then, by the definition of α , we have

$$\left. \begin{array}{l} \alpha(x, y) \geq 1 \\ \mathfrak{p}(u, u') = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \\ \mathfrak{p}(v, v') = \mathfrak{p}(\mathbb{D}, \mathbb{E}) \end{array} \right\} \Rightarrow \alpha(u, v) \geq 1.$$

where $u' \in \mathbb{F}x$ and $v' \in \mathbb{F}y$.

- (ii) Let $x, y, z \in \mathbb{D}$ such that $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$. Again, by the definition of α and $x, y, z \in \mathbb{D}$, $\alpha(x, z) \geq 1$.

Thus (i) and (ii) imply that \mathbb{F} is triangular α -proximal admissible.

- 2). Let $\{x_n\}$ be a sequence in \mathbb{D} such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x$ as $n \rightarrow \infty$. Since \mathbb{D} is closed, we get that $x \in \mathbb{D}$. Therefore, the definition of α gives $\alpha(x_{n_k}, x) = 1$ for each $k \in \mathbb{N}$.
- 3). Taking $x_0 = d, x_1 = c$, we have $\mathfrak{p}(x_1, y_1) = \mathfrak{p}(c, e) = \mathfrak{p}(\mathbb{D}, \mathbb{E}) = 0.20$ and $\alpha(x_0, x_1) = \alpha(d, c) = 2 > 1$, where $y_1 = e \in \mathbb{F}x_0 = \mathbb{F}d$.

Analogously, one can show that, for all $x, y \in \mathbb{D}$,

$$\theta(\psi(\mathcal{Z}(x, y)))\psi(\mathcal{Z}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})) - \alpha(x, y)\psi(\mathcal{H}_p(\mathbb{F}x, \mathbb{F}y)) \geq 0,$$

which means that

$$\alpha(x, y)\psi(\mathcal{H}_p(\mathbb{F}x, \mathbb{F}y)) \leq \theta(\psi(\mathcal{Z}(x, y)))\psi(\mathcal{Z}(x, y) - \mathfrak{p}(\mathbb{D}, \mathbb{E})),$$

for all $x, y \in \mathbb{D}$. Hence \mathbb{F} is a generalized α - ψ -Geraghty proximal contraction multimap. All the conditions of Theorem 3.12 are satisfied. Therefore, \mathbb{F} has a unique best proximity point in \mathbb{D}_0 . Here c is the unique best proximity point of \mathbb{F} in \mathbb{D}_0 . \square

CONCLUSIONS

1). We introduced a new class of generalized α - ψ -Geraghty proximal contraction multimaps and established best proximity point results for those multimaps in the context of generalized metric spaces.

2). Theorem 3.7 is an extension and a generalization of [7, Theorem 12].

As consequence, we obtained Corollary 3.10, an extension and a generalization of [7, Corollary 13].

3). Theorem 3.12 is an extension and a generalization of [7, Theorem 15].

In support of our main results, we gave two illustrative examples. In conclusion, this work broadens the scope of best proximity point theory by introducing generalized α - ψ -Geraghty proximal contraction multimaps. The results not only extend the classical fixed point theory but also pave the way for applications in non-self mappings and optimization problems across various fields. Future research may explore extensions to probabilistic or fuzzy metric spaces.

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